# Some theoretical results concerning diphasic viscoelastic flows of the Oldroyd kind

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**Abstract** - We present several existence and uniqueness results for the equations satisfied by the three-dimensional non-Newtonian mixture of the Oldroyd kind. We study the coupling of the constitutive law, the Navier-Stokes equations and the Cahn-Hilliard equation which stands for a model of a multiphase non-Newtonian fluid. We prove that a strong local solution exists but also that a global solution exists if the data are small enough. This last result is established even if the fluids are strongly elastic.

**Key Words** - Cahn-Hilliard equation, Navier-Stokes equations, Oldroyd model

**AMS Subject Classification** - 35Q30, 76A10, 76T99, 35K55

## Introduction

The aim of this paper is to show existence, uniqueness and regularity results for a system modeling a mixture of non-Newtonian flow. Several results have already been established either in the monophasic case [13, 15, 21, 24] or in the diphasic Newtonian case [4, 5]. In [9], some results are proven when the stress tensor is submitted to small diffusion process but the methods used are not transposable in the present case.

In [13, 15] the authors proved on the monophasic framework that there exists a unique strong solution local in time. C. Guillopé and J.C. Saut [15] also proved that the solution is globally defined if the data are small and if the fluid is not very elastic. E. Fernandez-Cara, F. Guillen and R.R. Ortega [13, 14] obtained a similar result with large elasticity on $[0, T[$, $T$ being arbitrarily large but finite. In this paper we present the diphasic version of these results and we show global existence for small data without hypothesis on the elasticity. The additional difficulty being the intoduction of an order parameter which describes each phase of the alloy. The evolution of this order parameter is given by the Cahn-Hilliard equation.

The results presented here are valid in dimension 2 and 3. In the two-dimensional case, the estimates are not optimal. For example, it is known that the Navier-Stokes equation admits global strong solution in dimension 2. Nevertheless, in the general framework studied here, there does not exist, to our knowledge, results of this type for unspecified initial data. This is due to non-linearities of the Oldroyd constitutive law. It should be noted that a particular choice of this law (choice of a corotational convected derivative for the constraint, see [18]) makes it possible to obtain better results [9, 21].

This paper is divided in five parts. In section 1 we briefly introduce the model, whose derivation was realised in [5, 9]. A system coupling three equations is obtained. The first one corresponds to a Cahn-Hilliard equation with a transport term. It describes the evolution of the order parameter and the thermodynamical properties of the diffusive interface. The second equation is a non-homogeneous Navier-Stokes equation which model the hydrodynamic properties of the flow. The third equation corresponds to the constitutive law of Oldroyd type [22] describing non-Newtonian effects in the mixture. It should be noted that some numerical works was realised with this model, see for instance [5] or [6].

In section 2, we announce the main results: local existence, uniqueness, global existence and regularity. These results require the introduction of several classical physical assumptions and of a mathematical background linked to the equations.

The next sections are devoted to the proof of the theorems. In section 3 we prove the local existence result. We use the methods developed in [15] together with some results on the estimates of the solution of

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the Cahn-Hilliard equation [4]. Roughly speaking, we consider the system as a fixed point equation and we apply the Schauder theorem. This method allows to control the nonlinear terms but a high regularity of the initial data is required. In section 4, we prove the uniqueness result concerning the strong solutions. There is no general result concerning global existence for arbitrarily large data even in the monophasic case. The fifth section is devoted to global existence of solutions for sufficiently small data. Here, the crucial point is to obtain estimates which balance the coupled linear terms.

# 1 Governing equations

Many constitutive laws of differential type have been introduced to model the behavior of viscoelastic fluids. They are derived from molecular or continuum mechanics considerations. We study in this paper fluids satisfying the Oldroyd constitutive law:

$$\mathcal{W}e\frac{D\sigma}{Dt} + \sigma = 2\delta r D(v). \tag{1.1}$$

Here $v$ is the velocity field, $\sigma$ is the non-Newtonian part of the stress tensor $\tau = 2\delta(1-r)D(v) - pId + \sigma$, $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^t)$ is the deformation tensor, $\mathcal{W}e$ is the Weissenberg number describing the elasticity, $\delta$ is the fluid viscosity and $r$ the retardation parameter. We can note that $0 \le r \le 1$ express the ratio between elasticity and viscosity : if $r = 0$ then $\sigma = 0$ and the fluid is a Newtonian fluid, and in the case $r = 1$ the fluid is called a Maxwell fluid. In this paper, we will assume $r < 1$. Moreover, in (1.1), $\frac{D}{Dt}$ denotes the following objective derivative [22]

$$\frac{D\sigma}{Dt} = \frac{\partial\sigma}{\partial t} + v\cdot\nabla\sigma + g(\sigma,\nabla v), \quad g(\sigma,\nabla v) = -W(v).\sigma + \sigma.W(v) + a(D(v).\sigma + \sigma.D(v)),$$

where $a \in [-1;1]$ and $W(v) = \frac{1}{2}(\nabla v - (\nabla v)^t)$ is the vorticity tensor. All the results of this paper hold for any value of the parameter $a$ but better results are known in some particular cases. For instance when $a = 0$ we can use the fact that $Tr(g(\sigma,\nabla v).\sigma) = 0$, see [9, 21].

To the constitutive law (1.1) we must add the motion and continuity equations for incompressible fluids

$$\mathcal{R}e\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) - 2\,\mathrm{div}\,(\delta(1-r)D(v)) + \nabla p = \mathrm{div}\,\sigma, \quad \mathrm{div}\,v = 0, \tag{1.2}$$

where $p$ is the pressure and $\mathcal{R}e$ is the Reynolds number. Usually, the velocity satisfies a no-slip condition $v.n = 0$ on the boundary ($n$ being the outward normal on the boundary of the domain).

Since we want to study the mixture of two viscoelastic fluids, we introduce an order parameter $\varphi$ describing the volumic fraction of one fluid in the flow. More precisely, where $-1 \le \varphi \le 1$ and $\varphi(t,x) = \pm 1$ if only one pure fluid is present at the time $t$ and the point $x$. This parameter allows us to obtain a continuous description of the interface between the two fluids. Considering only the diffusion phenomenon of $\varphi$ at the interface, the evolution of $\varphi$ is given by a Cahn-Hilliard equation (see [11, 23]):

$$\frac{\partial\varphi}{\partial t} - \mathrm{div}\,(B(\varphi)\nabla\mu) = 0, \quad \mu = -\alpha^2\Delta\varphi + F'(\varphi).$$

In this case, $\mu$ represents a chemical potential derived from the study of the free energy of the fluid, $B$ is a positive mobility coefficient and $\alpha$ measures the thickness of the interface. Furthemore, this equation is provided with the no-flux boundary conditions $\partial_n\varphi = \partial_n\mu = 0$. Obviously, this equation is coupled with (1.1) and (1.2). All the parameters depend on the order parameter, the interface is transported by the velocity and capillary forces appear in the Navier-Stokes equation (1.2), see [8]. In the case where the two phases have the same density, after a dimensionless study, we can write the final model in the form (see [9] for a complete derivation of the model):

$$\begin{cases} \dfrac{\partial v}{\partial t} + v\cdot\nabla v - 2\,\mathrm{div}\,(\eta(\varphi)D(v)) + \nabla p - \mathrm{div}\,\sigma = -\alpha^2\Delta\varphi\nabla\varphi, \\ \mathrm{div}\,v = 0, \\ \dfrac{\partial \varphi}{\partial t} + v\cdot\nabla\varphi - \mathrm{div}\,(B(\varphi)\nabla\mu) = 0, \qquad \mu = -\alpha^2\Delta\varphi + F'(\varphi), \\ \dfrac{\partial \sigma}{\partial t} + v\cdot\nabla\sigma + g(\sigma,\nabla v) + l(\varphi)\sigma = \nu(\varphi)D(v), \\ \varphi(0) = \varphi_0,\ v(0) = v_0,\ \sigma(0) = \sigma_0, \quad \dfrac{\partial\varphi}{\partial n}\Big|_\Gamma = \dfrac{\partial\mu}{\partial n}\Big|_\Gamma = 0, \quad v|_\Gamma = h \quad \text{with } h.n = 0. \end{cases} \tag{1.3}$$

We can remark here that we note $\eta(\varphi)$ the viscosity resulting from the Newtonian constraint ($\eta = (1-r)\delta$), and $\nu(\varphi)$ that resulting from the viscoelastic constraint ($\nu = r\delta$). Since we consider the cases where $0 \le r < 1$ then we will always have $\eta(\varphi) > 0$ and $\nu(\varphi) \ge 0$.

# 2 Main results

## 2.1 Mathematical background and notation

Throughout this paper, the same letter $C$ stands for a positive constant which may be different each time it appears. In the sequel, $d \in \{2;3\}$, $\Omega \subset \mathbb{R}^d$ is a bounded connected open set such that $\Gamma = \partial\Omega$ is regular enough. We will use the following notation: The Lebesgue space $L^p(\Omega)$ (or more simply $L^p$), $1 \le p \le +\infty$, with norm $|.|_p$, the Sobolev space $H^k(\Omega)$ $(H^k)$, $k = -1, 0, 1, ...$ with norm $\|.\|_k$. We will frequently use functions with values in $\mathbb{R}^d$ or in the space of real $d\times d$ matrices. In all cases, the same notation will be used.

During the estimates which will follow in the proofs, we will frequently use the following lemma (see [17]):

**Lemma 2.1**
*The application $(f,g) \longrightarrow fg$ is continuous from $H^{s_1}\times H^{s_2}$ in $H^s$ with*

$$s_1 + s_2 \ge 0, \quad s = \min\left\{s_1\ ,\ s_2\ ,\ s_1+s_2-\frac{d}{2}-\varepsilon\right\}, \quad \varepsilon > 0.$$

*We can take $\varepsilon = 0$ if $s_1 \ne d/2$, $s_2 \ne d/2$ and $\min\{s_1\ ,\ s_2\ ,\ s_1+s_2-d/2\} \ne -d/2$.*

We now introduce the natural spaces linked to the problem, taking into account the boundary conditions:

$$\Phi = \left\{\phi \in \mathcal{D}(\overline{\Omega})\ /\ \ \frac{\partial\phi}{\partial n}\Big|_\Gamma = \frac{\partial\Delta\phi}{\partial n}\Big|_\Gamma = 0\right\} \quad \text{and} \quad \Phi_s = \overline{\Phi}^{H^s} \text{ endowed with the norm } \|.\|_s,$$

$$\mathcal{V} = \{w \in \mathcal{D}(\overline{\Omega})\ /\ \ \mathrm{div}\,w = 0,\quad w|_\Gamma = 0\} \quad \text{and} \quad V_s = \overline{\mathcal{V}}^{H^s} \text{ endowed with the norm } \|.\|_s \text{ for } s \le 3/2.$$

For $s' < 0$, we also use the notation $\Phi_{s'}$ and $V_{s'}$ for the dual spaces of $\Phi_{-s'}$ and $V_{-s'}$ respectively. We will use the orthogonal projection $\mathbb{P}$ of $L^2$ onto $V_0$ and the Stokes operator $A = -\mathbb{P}\Delta$ with domain $V_0 \cap H^2$.

In this paper, we usually use the 0, 1 and 2−contracted products, noted respectively $AB$, $A.B$ and $A:B$. For instance the 2−contracted products of two 2−tensors $A$ and $B$ is defined by

$$A : B = \sum_{i,j=1}^{d} A_{i,j}B_{j,i}.$$

## 2.2 Assumptions

The viscosities $\eta$ and $\nu$ and the mobility $B$ are regular functions of $\varphi$, positive and bounded. More generally, it will be said that a real function $f$ will satisfy the assumption (2.1) if

$$f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \text{ with bounded derivatives and there exists } f_1, f_2 > 0 \text{ such that } f_1 \le f \le f_2. \tag{2.1}$$

Concerning the Cahn-Hilliard potential $F$, we first suppose the regularity of the potential (the positivity is not restrictive because a physical-meaningful potential is always bounded from below and adding a constant does not change the equations):

$$F \in \mathcal{C}^5(\mathbb{R},\mathbb{R}^+). \tag{2.2}$$

The other assumptions allows the choice of a classical Cahn-Hilliard potential ($F(\varphi) = \frac{\varphi^4}{4} - \frac{\varphi^2}{2}$ for instance):

$$\begin{gathered}\text{there exists } F_0 > 0 \text{ such that } \forall x \in \mathbb{R}, \begin{cases} |F''(x)| \le F_0(1+|x|^{p-1}), \\ |F^{(5)}(x)| \le F_0(1+|x|^q), \end{cases} \\ \text{where } \begin{cases} 1 \le p \le 3 \\ 0 \le q < +\infty \end{cases} \text{if } d=3 \quad \text{and} \quad 1 \le p < +\infty \text{ if } d=2. \end{gathered} \tag{2.3}$$

$$\begin{gathered}\forall \gamma \in \mathbb{R},\ \exists\ F_1(\gamma) > 0,\ \exists\ F_2(\gamma) \ge 0\ /\ \forall x \in \mathbb{R},\ (x-\gamma)F'(x) \ge F_1(\gamma)F(x) - F_2(x), \\ \exists\ F_3 \ge 0\ /\ \forall x \in \mathbb{R},\ F''(x) \ge -F_3. \end{gathered} \tag{2.4}$$

The last assumption is about the boundary velocity. It is known that all the regular vector field definite on the boundary $\Gamma$ can be prolonged in a divergence free vector field defined on $\Omega$ (see [19]). We consider here such a raising, still noted $h$ and verifying

$$h \in H^3(\Omega), \quad h.n = 0 \text{ on } \Gamma \quad \text{and } \operatorname{div} h = 0. \tag{2.5}$$

## 2.3 Statement of the results

The first stated result relates to the local existence of strong solution:

**Theorem 2.1**
*Assume $u_0 \in V_1 \cap H^2$, $\varphi_0 \in \Phi_4$, $\sigma_0 \in H^2$, $\eta$, $B$, $l$ and $\nu$ satisfy (2.1), $F$ satisfies (2.2), (2.3), (2.4) and $h$ satisfies (2.5). Then there exists $T^* > 0$ and $(u+h, \varphi, \sigma)$ a unique solution of (1.3) with $v_0 = u_0 + h$ such that*

$$\begin{aligned} &u \in L^\infty(0,T^*;V_1 \cap H^2) \cap L^2(0,T^*;V_1 \cap H^3), \quad \frac{\partial u}{\partial t} \in L^\infty(0,T^*;V_0) \cap L^2(0,T^*;V_1), \\ &\varphi \in L^\infty(0,T^*;\Phi_4) \cap L^2(0,T^*;\Phi_6), \quad \frac{\partial \varphi}{\partial t} \in L^\infty(0,T^*;\Phi_0) \cap L^2(0,T^*;\Phi_2), \\ &\sigma \in L^\infty(0,T^*;H^2), \quad \frac{\partial \sigma}{\partial t} \in L^\infty(0,T^*;H^1). \end{aligned}$$

Next, we are concerned with global existence (in time) of the solutions. Such a result is proven if the initial data are small. For the order parameter $\varphi$ this assumption writes "$\varphi_0$ is close to a metastable point $\omega$" (i.e. $F$ is convex near $\omega$, see figure 1). We obtain

**Theorem 2.2**
*Let $w$ lies in a metastable region of $F$. Under the same assumptions than for the preceding theorem, if $h$, $u_0$, $\sigma_0$ and $\varphi_0 - \omega$ are small enough in their space then the results of the theorem 2.1 hold with $T = +\infty$. Moreover, $\|u\|_2$, $\|\sigma\|_2$ and $\|\varphi - \omega\|_3$ remain small.*

For this theorem, the assumptions concerning the mobility $B$ and the potential $F$ are not necessary. We must only suppose that the assumptions are verified in a neighborhood of $\omega$. In particular, this theorem holds in the following case (usual in some physical considerations, see [8, 11])

$$\begin{aligned} &B(x) = (1-x^2)^a, \ a \ge 0, \\ &F(x) = 1 - x^2 + b((1+x)\log(1+x) + (1-x)\log(1-x)), \end{aligned}$$

for $\omega \in \ ]-1;-\sqrt{1-b}[\ \cup\ ]\sqrt{1-b};1[$.

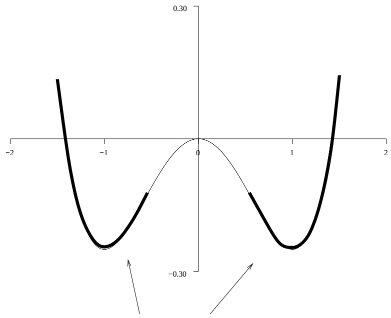


Figure 1: A Cahn-Hilliard potential

# 3 Local existence of a strong solution

In this section, we prove theorem 2.1. We implement a fixed point argument to show the existence of a regular solution to problem (1.3) on a small time interval $[0,T^*[$. The uniqueness of such a solution will be proved in the next part. Let us rewrite the system (1.3) as a fixed point equation

$$\begin{cases} \widetilde{v} = \widetilde{u} + h, \quad \widetilde{\eta} = \eta(\widetilde{\varphi}), \\ \dfrac{\partial u}{\partial t} - 2\,\mathrm{div}\,(\widetilde{\eta} D(u)) + \nabla p = -\alpha^2 \Delta\widetilde{\varphi}\nabla\widetilde{\varphi} + \,\mathrm{div}\,\widetilde{\sigma} - \widetilde{v}\cdot\nabla\widetilde{v} + 2\,\mathrm{div}\,(\widetilde{\eta} D(h)), \\ \mathrm{div}\; u = 0, \\ \dfrac{\partial \varphi}{\partial t} - \,\mathrm{div}\,(B(\varphi)\nabla\mu) = \,\mathrm{div}\,(-\widetilde{\varphi}\,\widetilde{v}), \quad \mu = -\alpha^2\Delta\varphi + F'(\varphi), \\ \dfrac{\partial \sigma}{\partial t} + \widetilde{v}\cdot\nabla\sigma + g_a(\sigma,\widetilde{v}) + l(\widetilde{\varphi})\sigma = \nu(\widetilde{\varphi})D(\widetilde{v}), \\ \varphi(0) = \varphi_0, \quad u(0) = u_0 = v_0 - h, \quad \sigma(0) = \sigma_0, \quad \left.\dfrac{\partial \varphi}{\partial n}\right|_\Gamma = \left.\dfrac{\partial \mu}{\partial n}\right|_\Gamma = 0, \quad u|_\Gamma = 0. \end{cases}$$

Then, it is enough to prove that the mapping $(\widetilde{u}+h,\widetilde{\varphi},\widetilde{\sigma}) \mapsto (u+h,\varphi,\sigma)$ admits a fixed point in a well choosen space. The proof goes through three lemmas, the first one concerning a Stokes problem with variable viscosity, the second one concerning the Cahn-Hilliard equation and the third one about the Oldroyd model.

## 3.1 Linear Stokes equation

For this first lemma, we will consider a Stokes problem with variable viscosity. We study the homogeneous case (null velocity at the boundary) and the non-homogeneous boundary conditions will be treated in the fixed point part, section 3.4:

$$\begin{cases} \dfrac{\partial u}{\partial t} - 2\,\mathrm{div}\,(\widetilde{\eta} D(u)) + \nabla p = G_1, \\ \mathrm{div}\; u = 0, \\ u(0) = u_0, \quad u|_\Gamma = 0. \end{cases} \tag{3.1}$$

The assumptions that we will introduce for $\widetilde{\eta}$ and for the source term $G_1$ will be justified thereafter.

**Lemma 3.1**
*Assume* $G_1 \in L^2(0,T;H^1)$, $\partial_t G_1 \in L^2(0,T;H^{-1})$, $G_1 \cdot \partial_t G_1 \in L^1(0,T;L^1)$, $G_1(0) \in L^2$, $\nabla\widetilde{\eta} \in L^\infty(0,T;H^2)$, $\partial_t\widetilde{\eta} \in L^2(0,T;L^\infty)$, $u_0 \in V_1 \cap H^2$ *and that* $\widetilde{\eta}$ *satisfies* $0 < \widetilde{\eta}_1 \le \widetilde{\eta} \le \widetilde{\eta}_2$.

*Then there exists a unique solution $(u,p)$ of (3.1) such that*

$$\|u\|^2_{L^\infty(0,T;V_1\cap H^2)}+\|u\|^2_{L^2(0,T;V_1\cap H^3)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;V_0)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;V_1)}$$
$$\leq k_1\Bigg(\|u_0\|_2^2, \|G_1\|^2_{L^2(0,T;H^1)}, \left\|\frac{\partial G_1}{\partial t}\right\|^2_{L^2(0,T;H^{-1})}, \|G_1.\partial_t G_1\|_{L^1(0,T;L^1)},$$
$$|G_1(0)|_2^2, \|\nabla\widetilde{\eta}\|^2_{L^\infty(0,T;H^2)}, \left\|\frac{\partial \widetilde{\eta}}{\partial t}\right\|^2_{L^2(0,T;L^\infty)}\Bigg).$$

**Remark 3.1**

- *It is important for the sequel to observe that the function $k_1$ may be chosen to be non decreasing with respect to each of its variables. Moreover this function $k_1$ does not depend on $T$.*
- *Assumption $G_1(0)\in L^2$ can appear superfluous since it is known that if we have $G_1\in L^2(0,T;H^1)$ and $\partial_t G_1\in L^2(0,T;H^{-1})$ then $G_1$ is continuous in time, in values in $[H^{-1},H^1]_{1/2}=L^2$. Nevertheless, this estimate is not independent of $T$ and would thus imply a dependence of $k_1$ compared to time.*
- *Concerning the assumtions, it is interesting to note that $G_1\in L^2(0,T;H^1)$ and $\partial_t G_1\in L^2(0,T;H^{-1})$ do not imply $G_1\cdot\partial_t G_1\in L^1(0,T;L^1)$ since $H^{-1}\times H^1\not\subset L^1$.*

### 3.1.1 Proof of the lemma 3.1

We prove this lemma in several steps. Since the regularity required on velocity is high, we will not be able to directly obtain estimates so strong. We thus start by having weaker estimates on $u$ and $\partial_t u$, then by regularity of the Stokes problem (see [7]), we will be able to increase the regularity.

- **Estimate of $u$ in $L^\infty(0,T;V_1)\cap L^2(0,T;V_1\cap H^2)$**

We use the Galerkin-Fadeo approximation method and we expose only formal calculations here. We multiply the Stokes equation by $Au$ where $A$ is the Stokes operator (let us recall that this operator is defined by $Au=-\Delta u+\nabla\pi\in V_0$, for $u\in V_2$). Developing div $(\widetilde{\eta}D(u))$ we get

$$\int_\Omega \frac{\partial u}{\partial t}.Au-\int_\Omega 2(\nabla\widetilde{\eta}.D(u)).Au-\int_\Omega \widetilde{\eta}\Delta u.Au=\int_\Omega G_1.Au.$$

After integrations by parts, the assumption (2.1) for $\widetilde{\eta}$ shows that we have the following estimate

$$\frac{d}{dt}\left(\frac{|\nabla u|_2^2}{2}\right)+\widetilde{\eta}_1|Au|_2^2\leq\left|\int_\Omega G_1.Au\right|+\left|\int_\Omega 2(\nabla\widetilde{\eta}.D(u)).Au\right|+\left|\int_\Omega\widetilde{\eta}\nabla\pi.Au\right|.$$

For the last term, regularity results for the classical Stokes problem [26] and the Poincaré lemma [1] allow to write

$$\left|\int_\Omega\widetilde{\eta}\nabla\pi.Au\right|=\left|\int_\Omega\pi\nabla\widetilde{\eta}.Au\right|\leq|\pi|_{L^2/\mathbb{R}}|\nabla\widetilde{\eta}|_\infty|Au|_2,$$
$$\leq C|\nabla u|_2|\nabla\widetilde{\eta}|_\infty|Au|_2\leq\frac{\widetilde{\eta}_1}{4}|Au|_2^2+C\|\nabla\widetilde{\eta}\|_2^2|\nabla u|_2^2.$$

The other terms are treated in the same way and we obtain the following inequality:

$$\frac{d}{dt}\left(\frac{|\nabla u|_2^2}{2}\right)+\frac{\widetilde{\eta}_1}{2}|Au|_2^2\leq C(|G_1|_2^2+\|\nabla\widetilde{\eta}\|_2^2|\nabla u|_2^2).$$

The Gronwall lemma implies the first estimate

$$\|\nabla u\|^2_{L^\infty(0,T;V_0)}\leq(|\nabla u_0|_2^2+C\|G_1\|^2_{L^2(0,T;L^2)})e^{C\|\nabla\widetilde{\eta}\|^2_{L^2(0,T;H^2)}},\tag{3.2}$$

and after a time integration

$$\|Au\|^2_{L^2(0,T;V_0)}\leq C(|\nabla u_0|_2^2+\|G_1\|^2_{L^2(0,T;L^2)}+\|\nabla\widetilde{\eta}\|^2_{L^2(0,T;H^2)}\|\nabla u\|^2_{L^\infty(0,T;V_0)}).\tag{3.3}$$

• **Estimate for the time derivative $\partial_t u$ in $L^\infty(0,T;V_0)\cap L^2(0,T;V_1)$**

By deriving in time the Stokes equation (3.1) we obtain a new Stokes equation verified by $w=\partial_t u$:

$$\begin{cases} \dfrac{\partial w}{\partial t} - 2\,\mathrm{div}\,(\tilde\eta D(w)) + \nabla\dfrac{\partial p}{\partial t} = \dfrac{\partial G_1}{\partial t} + 2\,\mathrm{div}\left(\dfrac{\partial\tilde\eta}{\partial t}D(u)\right), \\ \mathrm{div}\; w = 0, \\ w(0) = \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \mathbb{P}G_1\big|_{t=0} + 2\mathbb{P}\,\mathrm{div}\,(\tilde\eta D(u))\big|_{t=0} \in V_0, \\ w|_\Gamma = 0. \end{cases}$$

By formally multiplying this equation by $w$, the Korn inequality [16] allow to write

$$\frac{d}{dt}\left(\frac{|w|_2^2}{2}\right) + \tilde\eta_1|\nabla w|_2^2 \le C\left(\left\|\frac{\partial G_1}{\partial t}\right\|_{-1}\|w\|_1 + \left|\frac{\partial\tilde\eta}{\partial t}\right|_\infty |D(u)|_2|D(w)|_2\right)$$

and finaly by using the Poincaré lemma and the Young inequality, we have

$$\frac{d}{dt}\left(\frac{|w|_2^2}{2}\right) + \frac{\tilde\eta_1}{2}|\nabla w|_2^2 \le C\left(\left\|\frac{\partial G_1}{\partial t}\right\|_{-1}^2 + \left|\frac{\partial\tilde\eta}{\partial t}\right|_\infty^2 |\nabla u|_2^2\right).$$

After time integration,

$$\begin{aligned} \|w\|^2_{L^\infty(0,T;V_0)} + \|\nabla w\|^2_{L^2(0,T;V_0)} \le C\Bigg(|w(0)|_2^2 + \left\|\frac{\partial G_1}{\partial t}\right\|^2_{L^2(0,T;H^{-1})} \\ + \left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{L^2(0,T;L^\infty)}\|\nabla u\|^2_{L^\infty(0,T;V_0)}\Bigg). \end{aligned} \tag{3.4}$$

• **Estimate for the pressure $p$ in $L^2(0,T;H^1/\mathbb{R})\cap L^\infty(0,T;L^2/\mathbb{R})$**

In order to use the regularity of the Stokes operator, the following stage consists in obtaining regularity on the pressure term. In order to isolate this term, let us write the equation (3.1) in the following form

$$\nabla p = G_1 + 2\,\mathrm{div}\,(\tilde\eta D(u)) - w.$$

Since $G_1.\partial_t G_1 \in L^1(0,T;L^1)$, we can write

$$G_1(t)^2 = G_1(0)^2 + 2\int_0^t G_1(s).\frac{\partial G_1}{\partial t}(s)ds.$$

This estimate becomes

$$|G_1|^2_{L^\infty(0,T;L^2)} \le |G_1(0)|_2^2 + 2\left\|G_1.\frac{\partial G_1}{\partial t}\right\|^2_{L^1(0,T;L^1)}.$$

The fact that the constants appearing in this estimate are independant from $T$ is very important and will be used below. The preceeding results concerning the regularity of $u$ and $w$ allow to prove that

$$\nabla p \in L^2(0,T;L^2)\cap L^\infty(0,T;H^{-1}).$$

According to the Poincaré-Wirtinger inequalities (see [26], p. 14-15), we deduce

$$\begin{aligned} \|p\|^2_{L^\infty(0,T;L^2/\mathbb{R})} + \|p\|^2_{L^2(0,T;H^1/\mathbb{R})} \le C\Bigg(|G_1(0)|_2^2 + \left\|G_1.\frac{\partial G_1}{\partial t}\right\|_{L^1(0,T;L^1)} \\ + \|u\|^2_{L^\infty(0,T;V_1)}\|\nabla\tilde\eta\|^2_{L^2(0,T;H^2)} + \|u\|^2_{L^\infty(0,T;V_1)} \\ + \|u\|^2_{L^2(0,T;V_1\cap H^2)} + \|w\|^2_{L^\infty(0,T;V_0)} + \|w\|^2_{L^2(0,T;V_1)}\Bigg), \end{aligned} \tag{3.5}$$

where the constant $C=C(\Omega)$ does not depend on $T$.

• **Estimate of $u$ in $L^\infty(0,T;V_1\cap H^2)\cap L^2(0,T;V_1\cap H^3)$**

We again write the Stokes equation (3.1) in a different form:

$$-\Delta u+\nabla\left(\frac{p}{\widetilde{\eta}}\right)=\frac{G_1}{\widetilde{\eta}}-\frac{1}{\widetilde{\eta}}w-p\,\nabla\left(\frac{1}{\widetilde{\eta}}\right)+\frac{2}{\widetilde{\eta}}\nabla\widetilde{\eta}.D(u).$$

If we denote by $r$ the right member of this equality then we can use the known properties on the Stokes operator [7] and to show in particular that $u\in L^\infty(0,T;V_1\cap H^2)\cap L^2(0,T;V_1\cap H^3)$ with the estimates

$$\|u\|_{L^\infty(0,T;V_1\cap H^2)}\leq\|r\|_{L^\infty(0,T;L^2)},\quad \|u\|_{L^2(0,T;V_1\cap H^3)}\leq\|r\|_{L^2(0,T;H^1)}.$$

Let us evaluate the norm of $r$ in $L^\infty(0,T;L^2)$ and in $L^2(0,T;H^1)$. Noticing that $G_1$, $w$, $p$ and $D(u)$ are in $L^\infty(0,T;L^2)$, the fact of multiplying them by bounded functions like $\widetilde{\eta}$ and its derivative does not deteriorate their regularity. It results from it that $r$ is in $L^\infty(0,T;L^2)$.

To obtain the regularity in $L^2(0,T;H^1)$ of $r$, we can show that the gradient (in space) of $r$ is in $L^2(0,T;L^2)$ by the same methods. We deduce

$$\begin{aligned}&\|u\|^2_{L^\infty(0,T;V_1\cap H^2)}+\|u\|^2_{L^2(0,T;V_1\cap H^3)}\\ &\leq C\Big(|G_1(0)|_2^2+\left\|G_1.\frac{\partial G_1}{\partial t}\right\|_{L^1(0,T;L^1)}+\|G_1\|^2_{L^2(0,T;H^1)}+\|w\|^2_{L^\infty(0,T;V_0)}+\|w\|^2_{L^2(0,T;V_1)}\\ &+\|\nabla\widetilde{\eta}\|^2_{L^\infty(0,T;H^2)}(\|p\|^2_{L^\infty(0,T;L^2)}+\|p\|^2_{L^2(0,T;H^1)})++\|\nabla\widetilde{\eta}\|^4_{L^\infty(0,T;H^2)}\|p\|^2_{L^2(0,T;L^2)}\\ &+\|\nabla\widetilde{\eta}\|^2_{L^\infty(0,T;H^2)}(\|u\|^2_{L^\infty(0,T;V_1)}+\|u\|^2_{L^2(0,T;V_1\cap H^2)})++\|\nabla\widetilde{\eta}\|^4_{L^\infty(0,T;H^2)}\|u\|^2_{L^2(0,T;V_1)}\Big)\end{aligned}\tag{3.6}$$

The estimates (3.2), (3.3), (3.4), (3.5) and (3.6) conclude the proof of the existence part of the Stokes lemma. For the uniqueness, the equation being linear, it is enough to notice that if the initial conditions vanish, and if $G_1=0$ then the preceding inequalities prove that $u$ is identically zero.

## 3.2 Cahn-Hilliard equation

In this second lemma, we will consider a Cahn-Hilliard equation with source term:

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}-\text{div }(B(\varphi)\nabla\mu)=\text{div }(G_2),\\ \mu=-\alpha^2\Delta\varphi+F'(\varphi),\\ \varphi(0)=\varphi_0,\quad \left.\dfrac{\partial\varphi}{\partial n}\right|_\Gamma=\left.\dfrac{\partial\mu}{\partial n}\right|_\Gamma=0.\end{cases}\tag{3.7}$$

As in the “Stokes” lemma 3.1, the assumptions on the data will be justified in the sequel.

**Lemma 3.2**
*Assume $G_2\in L^2(0,T;L^2)$, div $G_2\in L^2(0,T;H^2)$, $G_2.n=0$ on $\Gamma$, div $\partial_tG_2\in L^2(0,T;L^2)$, div $G_2(0)\in L^2$, $\varphi_0\in\Phi_4$, $B$ satisfies (2.1) and $F$ satisfies (2.2), (2.3) and (2.4).*

*Then there exists $T_1>0$ such that the problem (3.7) has a unique solution $\varphi$ on $[0,T_1[$ verifying*

$$\begin{aligned}&\|\varphi\|^2_{L^\infty(0,T_1;\Phi_4)}+\|\varphi\|^2_{L^2(0,T_1;\Phi_6)}+\left\|\frac{\partial\varphi}{\partial t}\right\|^2_{L^\infty(0,T_1;\Phi_0)}+\left\|\frac{\partial\varphi}{\partial t}\right\|^2_{L^2(0,T_1;\Phi_2)}\\ &\leq k_2\left(\|\varphi_0\|_4^2,\|\,div\,G_2\|^2_{L^2(0,T;H^2)},\|G_2\|^2_{L^2(0,T;L^2)},\left\|div\,\frac{\partial G_2}{\partial t}\right\|^2_{L^2(0,T;L^2)},|\,div\,(G_2(0))|_2^2\right).\end{aligned}$$

*Moreover, the time $T_1$ is given by*

$$T_1=\min\left\{T,\frac{1}{\Lambda}\right\},\quad with\quad \Lambda=k_{20}(\|\varphi_0\|_4^2,\|\,div\,G_2\|^2_{L^2(0,T;H^2)}).$$

**Remark 3.2**
*Like in the preceeding lemma, it should be noticed that functions $k_2$ and $k_{20}$ are non decreasing with respect to all variables.*

### 3.2.1 Proof of the lemma 3.2

Exactly as in the proof of the above lemma, strong regularity that this lemma request will impose us to follow several steps.

- **Estimate of $\varphi$ in $L^\infty(0,T;\Phi_1)\cap L^2(0,T;\Phi_3)$**

We choose $\mu$ as a test function in the usual weak formulation of the problem and we obtain directly

$$\int_\Omega \frac{\partial \varphi}{\partial t}\,\mu - \int_\Omega \text{div }(B(\varphi)\nabla\mu)\mu = \int_\Omega \text{div }(G_2)\mu.$$

By definition of $\mu$, integrating by parts and using the assumption $B_1 \le B$:

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \frac{B_1}{2}|\nabla\mu|_2^2 \le \frac{1}{B_1}|G_2|_2^2.$$

We use the following result (see [4]): under the almost-convexity hypotheses (2.4) we have

$$\frac{B_1}{2}|\nabla\mu|_2^2 \ge \frac{B_1}{4}|\nabla\mu|_2^2 + C|\Delta\varphi|_2^2 + C\Big(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\Big) - CF_2(m(\varphi_0)),$$

where the average $m(\varphi) = \int_\Omega \varphi$ is conserved all along the time [12]. We deduce:

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + C_1\left(|\nabla\varphi|_2^2 + \int_\Omega F(\varphi) + |\Delta\varphi|_2^2 + |\nabla\mu|_2^2\right) \le \frac{1}{B_1}|G_2|_2^2 + CF_2(m(\varphi_0)). \tag{3.8}$$

The regularity which we have obtained on $\mu$ indicates that we must have also more regularity on $\varphi$ (since $\mu \approx \Delta\varphi$). This intuitive result requires the additional assumption (2.3) (under-polynomial growth). With this condition, we can show that

$$|\nabla\Delta\varphi|_2^2 \le C_2(|\nabla\mu|_2^2 + |\nabla\varphi|_2^2 + |\nabla\varphi|_2^{2p-2}|\Delta\varphi|_2^2). \tag{3.9}$$

- **Estimate of $\varphi$ in $L^\infty(0,T;\Phi_2)\cap L^2(0,T;\Phi_4)$**

We choose now $\Delta^2\varphi$ as a test function in the Cahn-Hilliard equation. This operation is completely justified because we make a Galerkin method, considering in particular a base of eigenvectors of $-\Delta$ in $\Phi_2$ (see [4, 9]). After integrations by parts, we obtain

$$\frac{d}{dt}\left(\frac{|\Delta\varphi|_2^2}{2}\right) - \int_\Omega \text{div }(B(\varphi)\nabla\mu)\Delta^2\varphi = \int_\Omega \text{div }(G_2)\Delta^2\varphi.$$

Using the definition of the potential $\mu$, we have

$$-\text{ div }(B(\varphi)\nabla\mu) = \alpha^2 B(\varphi)\Delta^2\varphi + \alpha^2 B'(\varphi)\nabla\varphi\cdot\nabla\Delta\varphi - B'(\varphi)\nabla\varphi.\nabla F'(\varphi) - B(\varphi)\Delta F'(\varphi). \tag{3.10}$$

We reveal the quantity $\alpha^2 B_1|\Delta^2\varphi|_2^2$ in the left member and the Young inequality allows us to write

$$\frac{d}{dt}\left(\frac{|\Delta\varphi|_2^2}{2}\right) + \alpha^2 B_1|\Delta^2\varphi|_2^2 \le \frac{\alpha^2 B_1}{2}|\Delta^2\varphi|_2^2 + C(|\text{ div }(G_2)|_2^2 + |\nabla\varphi|_\infty^2|\nabla\Delta\varphi|_2^2 + |\nabla\varphi|_\infty^2|\nabla F'(\varphi)|_2^2 + |\Delta F'(\varphi)|_2^2).$$

Concerning the estimates on the potential $F$, we write

$$\nabla F'(\varphi) = F''(\varphi)\nabla\varphi \quad\text{and}\quad \Delta F'(\varphi) = F'''(\varphi)\nabla\varphi\cdot\nabla\varphi + F''(\varphi)\Delta\varphi,$$

and using the assumptions (2.3), we deduce

$$|\nabla F'(\varphi)|_2^2 \le |F''(\varphi)|_\infty^2|\nabla\varphi|_2^2 \le C(1+|\varphi|_\infty^{2p-2})|\nabla\varphi|_2^2 \le C(1+\|\nabla\varphi\|_1^{2p-2})|\nabla\varphi|_2^2$$
$$|\Delta F'(\varphi)|_2^2 \le C(1+\|\nabla\varphi\|_1^{2q+2})\|\nabla\varphi\|_1^4 + C(1+\|\nabla\varphi\|_1^{2p-2})|\Delta\varphi|_2^2.$$

The energy estimate on $|\Delta\varphi|_2^2$ becomes

$$\frac{d}{dt}\left(|\Delta\varphi|_2^2\right) + \alpha^2 B_1|\Delta^2\varphi|_2^2 \le C_3|\Delta\varphi|_2^2 + C|\,\mathrm{div}\,(G_2)|_2^2 + \|\nabla\varphi\|_2^2\mathcal{K}(\|\nabla\varphi\|_2^2) \tag{3.11}$$

where $\mathcal{K}$ is a continuous non decreasing function such that $\mathcal{K}(0) = 0$.

- **Estimate of $\varphi$ in $L^\infty(0,T;\Phi_3)\cap L^2(0,T;\Phi_5)$**

In this paragraph, we obtain an equation of the same type of a higher order. Taking $-\Delta^3\varphi$ as a test function, we deduce

$$\frac{d}{dt}\left(\frac{|\nabla\Delta\varphi|_2^2}{2}\right) - \int_\Omega \nabla\,\mathrm{div}\,(B(\varphi)\nabla\mu)\cdot\nabla\Delta^2\varphi = \int_\Omega \nabla\,\mathrm{div}\,(G_2)\cdot\nabla\Delta^2\varphi.$$

As previously we develop $\nabla\,\mathrm{div}\,(B(\varphi)\nabla\mu)$ (see (3.10)) to make the term $\alpha^2 B_1|\nabla\Delta^2\varphi|_2^2$ appear:

$$\begin{aligned}-\nabla\,\mathrm{div}\,(B(\varphi)\nabla\mu) = {}& \alpha^2 B(\varphi)\nabla\Delta^2\varphi + \alpha^2 B'(\varphi)\nabla\varphi\Delta^2\varphi + \alpha^2 B''(\varphi)\nabla\varphi\nabla\varphi\cdot\nabla\Delta\varphi + \alpha^2 B'(\varphi)\nabla^2\varphi\cdot\nabla\Delta\varphi\\ &+ \alpha^2 B'(\varphi)\nabla\varphi\cdot\nabla^2\Delta\varphi - B''(\varphi)\nabla\varphi\nabla\varphi.\nabla F'(\varphi) - B'(\varphi)\nabla^2\varphi.\nabla F'(\varphi)\\ &- B'(\varphi)\nabla\varphi.\nabla^2 F'(\varphi) - B'(\varphi)\nabla\varphi\Delta F'(\varphi) - B(\varphi)\nabla\Delta F'(\varphi).\end{aligned}$$

Using Young inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla\Delta\varphi|_2^2}{2}\right) + \frac{\alpha^2 B_1}{2}|\nabla\Delta^2\varphi|_2^2 \le C\ (&|\nabla\,\mathrm{div}\,(G_2)|_2^2 + |\nabla\varphi|_\infty^2|\Delta^2\varphi|_2^2 + |\nabla\varphi|_\infty^4|\nabla\Delta\varphi|_2^2\\ &+ |\nabla^2\varphi|_\infty^2|\nabla\Delta\varphi|_2^2 + |\nabla\varphi|_\infty^2|\nabla^2\Delta\varphi|_2^2 + |\nabla\varphi|_\infty^4|\nabla F'(\varphi)|_2^2\\ &+ \|\nabla^2\varphi\|_1^2\|\nabla F'(\varphi)\|_1^2 + |\nabla\varphi|_\infty^2|\nabla^2 F'(\varphi)|_2^2\\ &+ |\nabla\varphi|_\infty^2|\Delta F'(\varphi)|_2^2 + |\nabla\Delta F'(\varphi)|_2^2).\end{aligned}$$

We use the sobolev injection $H^2 \subset L^\infty$. Like above, we develop the potential and use the assumptions (2.3). The last term is the only one which can bring a linear term. It is estimated by

$$\begin{aligned}&\nabla\Delta F'(\varphi) = F^{(4)}(\varphi)\nabla\varphi(\nabla\varphi\cdot\nabla\varphi) + 2F'''(\varphi)\nabla^2\varphi\cdot\nabla\varphi + F'''(\varphi)\nabla\varphi\Delta\varphi + F''(\varphi)\nabla\Delta\varphi,\\ &|\nabla\Delta F'(\varphi)|_2^2 \le |F^{(4)}(\varphi)|_\infty^2|\nabla\varphi|_\infty^6 + 2|F'''(\varphi)|_\infty^2|\nabla^2\varphi|_2^2|\nabla\varphi|_\infty^2 + |F'''(\varphi)|_\infty^2|\nabla\varphi|_\infty^2|\Delta\varphi|_2^2 + |F''(\varphi)|_\infty^2|\nabla\Delta\varphi|_2^2.\end{aligned}$$

The same type of arguments that for the preceding estimate provides the following estimate

$$\frac{d}{dt}\left(|\nabla\Delta\varphi|_2^2\right) + \alpha^2 B_1|\nabla\Delta^2\varphi|_2^2 \le C_4|\nabla\Delta\varphi|_2^2 + C|\nabla\,\mathrm{div}\,(G_2)|_2^2 + \|\nabla\varphi\|_3^2\mathcal{K}(\|\nabla\varphi\|_2^2) \tag{3.12}$$

where $\mathcal{K}$ is an other continuous non-decreasing function with $\mathcal{K}(0) = 0$.

- **Estimate of $\varphi$ in $L^\infty(0,T;\Phi_4)\cap L^2(0,T;\Phi_6)$**

To obtain more regularity, we take $\Delta^4\varphi$ as a test function, we deduce

$$\frac{d}{dt}\left(\frac{|\Delta^2\varphi|_2^2}{2}\right) - \int_\Omega \Delta\,\mathrm{div}\,(B(\varphi)\nabla\mu).\Delta^3\varphi = \int_\Omega \Delta\,\mathrm{div}\,(G_2).\Delta^3\varphi.$$

As previously we develop $\Delta\,\mathrm{div}\,(B(\varphi)\nabla\mu)$. Writting only the significant term (see [10] for complete computations), we obtain

$$-\Delta\,\mathrm{div}\,(B(\varphi)\nabla\mu) = \alpha^2 B(\varphi)\Delta^3\varphi + \alpha^2 B'(\varphi)\nabla\varphi.\nabla\Delta^2\varphi - B(\varphi)\Delta^2 F'(\varphi) + ... \tag{3.13}$$

After integration, the first term gives $\alpha^2 B_1|\Delta^3\varphi|_2^2$ on the left member of the estimate. Using the Young inequality again, we deduce

$$\frac{d}{dt}\left(\frac{|\Delta^2\varphi|_2^2}{2}\right) + \frac{\alpha^2 B_1}{2}|\Delta^3\varphi|_2^2 \le C\ (|\Delta\,\mathrm{div}\,(G_2)|_2^2 + |\nabla\varphi|_\infty^2|\nabla\Delta^2\varphi|_2^2 + |\Delta^2 F'(\varphi)|_2^2 + ...)$$

We detail here how the significants terms are treated, the other being more regular. We use an integration by parts for the second term

$$|\nabla\varphi|_\infty^2|\nabla\Delta^2\varphi|_2^2 \leq C\|\nabla\varphi\|_2^2|\Delta^2\varphi|_2|\Delta^3\varphi|_2 \leq \frac{\alpha^2 B_1|\Delta^3\varphi|_2^2}{4} + C\|\nabla\varphi\|_3^4$$

whereas the last term is estimated by

$$\begin{aligned}\Delta^2 F'(\varphi) &= F^{(5)}(\varphi)(\nabla\varphi\cdot\nabla\varphi)^2 + 4F^{(4)}(\varphi)\Delta\varphi(\nabla\varphi\cdot\nabla\varphi) + 2F^{(4)}(\varphi)\nabla\varphi.(\nabla^2\varphi\cdot\nabla\varphi)\\ &\quad + 4F'''(\varphi)\nabla\Delta\varphi\cdot\nabla\varphi + 2F'''(\varphi)\nabla^2\varphi:\nabla^2\varphi + F'''(\varphi)(\Delta\varphi)^2 + F''(\varphi)\Delta^2\varphi,\\ |\Delta^2 F'(\varphi)|_2^2 &\leq C(|F^{(5)}(\varphi)|_\infty^2\|\nabla\varphi\|_2^8 + |F^{(4)}(\varphi)|_\infty^2\|\nabla\varphi\|_2^6 + |F'''(\varphi)|_\infty^2\|\nabla\varphi\|_2^4 + |F''(\varphi)|_\infty^2|\Delta^2\varphi|_2^2).\end{aligned}$$

The same type of arguments that for the preceding estimate provides the following estimate

$$\frac{d}{dt}\left(|\Delta^2\varphi|_2^2\right) + \alpha^2 B_1|\Delta^3\varphi|_2^2 \leq C_5|\Delta^2\varphi|_2^2 + C|\Delta \text{ div } (G_2)|_2^2 + \|\nabla\varphi\|_3^2\mathcal{K}(\|\nabla\varphi\|_3^2) \tag{3.14}$$

where $\mathcal{K}$ is an other continuous non-decreasing function such that $\mathcal{K}(0) = 0$.

- **Estimate of $\partial_t\varphi$ in $L^2(0,T;\Phi_2)$**

  It is easy to deduce an estimate on the time derivative $\partial_t\varphi$. We write

$$\frac{\partial\varphi}{\partial t} = \text{div } G_2 + \text{div } (B(\varphi)\nabla\mu)).$$

A classical result on the Cahn-Hilliard equation show that the average of $\partial_t\varphi$ is zero [12]. With the Poincaré inequality and using the equation (3.13), we find:

$$\left\|\frac{\partial\varphi}{\partial t}\right\|_2^2 \leq \left|\Delta\frac{\partial\varphi}{\partial t}\right|_2^2 \leq C_6|\Delta^3\varphi|_2^2 + C_7|\Delta^2\varphi|_2^2 + C|\Delta \text{ div } (G_2)|_2^2 + \|\nabla\varphi\|_3^2\mathcal{K}(\|\nabla\varphi\|_3^2). \tag{3.15}$$

Adding the six estimates on $\varphi$

$$\begin{aligned}\big(2C_2 + 2C_3(C_5 + \alpha^2 B_1 C_7/C_6)\big)/C_1 &\times (3.8)\\ + 2 &\times (3.9)\\ + 2(C_5 + \alpha^2 B_1 C_7/C_6) &\times (3.11)\\ + 1/C_4 &\times (3.12)\\ + 2\alpha^2 B_1 &\times (3.14)\\ + \alpha^4 B_1^2/C_6 &\times (3.15),\end{aligned}$$

and noting that the norms $\|\varphi\|_4^2$ and $\|\varphi\|_6^2$ are equivalent to $|\nabla\varphi|_2^2 + |\Delta\varphi|_2^2 + |\nabla\Delta\varphi|_2^2 + |\Delta^2\varphi|_2^2$ and $|\nabla\varphi|_2^2 + |\Delta\varphi|_2^2 + |\nabla\Delta\varphi|_2^2 + |\Delta^2\varphi|_2^2 + |\nabla\Delta^2\varphi|_2^2 + |\Delta^3\varphi|_2^2$ respectively (use the laplacian regularity, see [27]) we find finally

$$y'(t) + Cz(t) \leq y(t)\mathcal{K}(y(t)) + \|\text{ div } (G_2)\|_2^2 + CF_2(m(\varphi_0)), \tag{3.16}$$

where

$$y(t) = \|\varphi\|_4^2 + \frac{2}{\alpha^2}\int_\Omega F(\varphi) \quad \text{and} \quad z(t) = \|\varphi\|_6^2 + \frac{2}{\alpha^2}\int_\Omega F(\varphi) + \left\|\frac{\partial\varphi}{\partial t}\right\|_2^2.$$

A simply study of this inequation gives the existence of a time $T_1 > 0$ such that $y(t) \leq M$ for all $t \leq T_1$ (i.e. $\varphi$ is bounded in $L^\infty(0,T_1;\Phi_4)$). Moreover $M$ and $T_1$ are given by

$$\begin{aligned}M &= \big(y(0) + \|\text{ div } (G_2)\|_{L^2(0,T;H^2)} + TCF_2(m(\varphi_0))\big)e^1,\\ T_1 &= min\Big\{T, \frac{1}{\mathcal{K}(M)}\Big\}.\end{aligned}$$

To prove this result, consider the set $I = \{s \in \mathbb{R}^+ \ / \ \forall t \in [0,s] \quad y(t) \leq M\}$. Since $y(0) < M$, it's clear that $0 \in I$. Assume that there exists $s < T_1$ such that $s \in I$. For all $t \in [0,s]$, we have $y(t) \leq M$ and since $\mathcal{K}$ increases, we deduce

$$y'(t) \leq y(t)\mathcal{K}(M) + \|\operatorname{div}\,(G_2)\|_2^2 + CF_2(m(\varphi_0)).$$

Using the Gronwall lemma, we obtain

$$y(t) \leq (y(0) + \|\operatorname{div}\,(G_2)\|_{L^2(0,T;H^2)} + TCF_2(m(\varphi_0)))e^{s\mathcal{K}(M)} < M.$$

By continuity, it results that $I$ is open in $[0,T_1[$. It's clear that $I$ is closed, so that $[0,T_1[\subset I$.

Now, integrating in time the inequality (3.16), we deduce an estimate of $\varphi$ in $L^2(0,T_1;\Phi_6)$ and an estimate of $\partial_t\varphi$ in $L^2(0,T_1;\Phi_2)$.

- **Estimate of $\partial_t\varphi$ in $L^\infty(0,T_1;\Phi_0)$**

Since $\partial_t\varphi = \operatorname{div}\,(G_2+B(\varphi)\nabla\mu)$ we prove here that the previous estimates show that $\partial_t\varphi \in L^\infty(0,T_1;\Phi_0)$.Using the estimate (3.10), we have

$$\begin{aligned}\left|\frac{\partial\varphi}{\partial t}\right|_2^2 &\leq C(|\operatorname{div}\,G_2|_2^2 + |\operatorname{div}\,(B(\varphi)\nabla\mu)|_2^2)\\ &\leq C(|\operatorname{div}\,G_2|_2^2 + \|\nabla\varphi\|_3^2 + \|\nabla\varphi\|_2^2\mathcal{K}(\|\nabla\varphi\|_2^2)).\end{aligned}$$

Using the fact that $\operatorname{div}\,G_2 \in L^2(0,T;L^2)$ and that $\partial_t \operatorname{div}\,G_2 \in L^2(0,T;L^2)$ we obtain a bound for $\operatorname{div}\,G_2$ in $L^\infty(0,T;L^2)$. We write:

$$\operatorname{div}\,G_2(t)^2 = \operatorname{div}\,G_2(0)^2 + 2\int_0^t \operatorname{div}\,G_2(s)\operatorname{div}\,\frac{\partial G_2}{\partial t}(s)ds,$$

we integer on $\Omega$ and take the supremum for $t \in [0,T]$:

$$\|\operatorname{div}\,G_2\|_{L^\infty(0,T;L^2)}^2 \leq |\operatorname{div}\,G_2(0)|_2^2 + \|\operatorname{div}\,G_2\|_{L^2(0,T;L^2)}^2 + \left\|\operatorname{div}\,\frac{\partial G_2}{\partial t}\right\|_{L^2(0,T;L^2)}^2.$$

Finaly, we have $\partial_t\varphi \in L^\infty(0,T_1;\Phi_0)$ with the following relation (use the monotony of the function $\mathcal{K}$)

$$\begin{aligned}\left\|\frac{\partial\varphi}{\partial t}\right\|_{L^\infty(0,T_1;\Phi_0)} \leq\ C\ \Bigg(&|\operatorname{div}\,G_2(0)|_2^2 + \|\operatorname{div}\,G_2\|_{L^2(0,T;L^2)}^2 + \left\|\operatorname{div}\,\frac{\partial G_2}{\partial t}\right\|_{L^2(0,T;L^2)}^2\\ &+ \|\nabla\varphi\|_{L^\infty(0,T;\Phi_3)}^2 + \|\nabla\varphi\|_{L^\infty(0,T;\Phi_2)}^2\mathcal{K}(\|\nabla\varphi\|_{L^\infty(0,T;\Phi_2)}^2)\Bigg).\end{aligned} \tag{3.17}$$

where the constant $C$ is independent of $T$.

This inequality concludes the proof of the existence of a regular solution to the problem (3.7). It any more but does not remain to prove the unicity of such a solution:

- **Uniqueness**

Consider $\varphi_1$ and $\varphi_2$ two solutions of the problem (3.7). Let $\varphi = \varphi_1 - \varphi_2$. We make the difference of the two equations then we multiply by $-\Delta\varphi$. After straightforward computations, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla\varphi|_2^2}{2}\right) &+ \alpha^2\int_\Omega (B(\varphi_1) - B(\varphi_2))\nabla\Delta\varphi_1.\nabla\Delta\varphi + \alpha^2\int_\Omega B(\varphi_2)\nabla\Delta\varphi.\nabla\Delta\varphi\\ &- \int_\Omega (BF''(\varphi_1) - BF''(\varphi_2))\nabla\varphi_1.\nabla\Delta\varphi - \int_\Omega BF''(\varphi_2)\nabla\varphi.\nabla\Delta\varphi = 0.\end{aligned}$$

We use the bounds on $B$ and on its derivative (assumption (2.1)):

$$\frac{d}{dt}\left(\frac{|\nabla\varphi|_2^2}{2}\right) + \alpha^2 B_1|\nabla\Delta\varphi|_2^2 \leq C(|\varphi|_\infty|\nabla\Delta\varphi_1|_2|\nabla\Delta\varphi|_2 + |\varphi|_\infty|\nabla\varphi_1|_2|\nabla\Delta\varphi|_2 + |\nabla\varphi|_2|\nabla\Delta\varphi|_2).$$

To treat the first two terms of the right-hand side member, we can use Agmon's inequality [2]:

$$|\varphi|_\infty |\nabla\Delta\varphi|_2 \le |\nabla\varphi|_2^{3/4} |\nabla\Delta\varphi|_2^{5/4} \le \frac{\alpha^2 B_1}{6} |\nabla\Delta\varphi|_2^2 + C|\nabla\varphi|_2^2.$$

Estimates being obvious for the other term (using the Young inequality), we deduce

$$\frac{d}{dt}\left(\frac{|\nabla\varphi|_2^2}{2}\right) + \frac{\alpha^2 B_1}{2} |\nabla\Delta\varphi|_2^2 \le C(\|\nabla\varphi_1\|_2^2 + 1)|\nabla\varphi|_2^2.$$

To conclude, it is enough to notice that $\|\nabla\varphi_1\|_2^2 \in L^1(0,T_1)$ and to apply the Gronwall lemma: the initial conditions on $\varphi$ being zero, we find $\varphi_1 = \varphi_2$.

## 3.3 Constitutive law

We turn now to the study of a problem associated to the constitutive Oldroyd equation for the stress tensor.

$$\begin{cases} \dfrac{\partial\sigma}{\partial t} + (\widetilde{u} + h)\cdot\nabla\sigma + g(\sigma, \nabla(\widetilde{u}+h)) + l(\widetilde{\varphi})\sigma = G_3, \\ \sigma(0) = \sigma_0. \end{cases} \tag{3.18}$$

We obtain the following lemma

**Lemma 3.3**
*Assume $G_3 \in L^\infty(0,T;H^1) \cap L^1(0,T;H^2)$, $\widetilde{u} \in L^\infty(0,T;V_1 \cap H^2) \cap L^1(0,T;V_1 \cap H^3)$, $\widetilde{\varphi} \in L^\infty(0,T;\Phi_3)$, $\sigma_0 \in H^2$, $h$ satisfies (2.5) and $l$ satisfies (2.1).*

*Then the problem (3.18) has a unique solution $\sigma$ such that*

$$\begin{aligned} \|\sigma\|_{L^\infty(0,T;H^2)} + \left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty(0,T;H^1)} &\le k_3(\|\sigma_0\|_2^2, \|G_3\|^2_{L^\infty(0,T;H^1)}, \|G_3\|^2_{L^1(0,T;H^2)}, \|\widetilde{u}\|^2_{L^\infty(0,T;V_1\cap H^2)}, \\ &\qquad \|\widetilde{u}\|^2_{L^1(0,T;V_1\cap H^3)}, \|h\|_3^2, \|\widetilde{\varphi}\|^2_{L^\infty(0,T;\Phi_3)}). \end{aligned} \tag{3.19}$$

**Remark 3.3**
*In fact, the solution $\sigma$ is not only $L^\infty(0,T;H^2)$ but $\mathcal{C}([0,T];H^2)$. Indeed, the assumption on the velocity field $\widetilde{u} \in L^1(0,T;V_1 \cap H^3)$ implies that the solution of the system*

$$\begin{cases} \dfrac{d}{dt}U(t,s;x) = (\widetilde{u}+h)(t, U(t,s;x)) & \text{in } ]0,T[, \\ U(s,s;x) = x & \text{in } \Omega, \end{cases}$$

*is continuous in the two first variables, $H^3$ in space. Futhermore, there is an integral representation (see [3]):*

$$\sigma(t,x) = \sigma_0(U(0,t;x)) + \int_0^t (G_3 - g(\sigma, \nabla(\widetilde{u}+h)) - l(\widetilde{\varphi})\sigma)(s, U(s,t;x))ds.$$

*This formulation enables us to deduce that $\sigma \in \mathcal{C}([0,T];H^2)$. In addition, by adding the following assumptions: $\widetilde{u} \in \mathcal{C}([0,T];H^2)$ and $G_3 \in \mathcal{C}([0,T];H^1)$, it is easy to show that $\partial_t\sigma \in \mathcal{C}([0,T];H^1)$.*

### 3.3.1 Proof of the lemma 3.3

To simplify the notations we note during this proof $\widetilde{v} = \widetilde{u} + h$ so that $\widetilde{v}$ is a vector field defined on $\Omega$, with free divergence and tangent at the boundary.

The existence of a unique solution to (3.18) follows from the application of the characteristics method. It should be noticed that this method is generally used when the velocity field is of $\mathcal{C}^1$ class. Let $\{\widetilde{v}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^1([0,T])$ such $\widetilde{v}_n \to \widetilde{v}$ in $L^1(0,T;V_1 \cap H^3)$ and let $\{\sigma_{0,n}\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^3(\bar{\Omega})$ such that $\sigma_{0,n} \to \sigma_0$ in $H^2$. For each $n \in \mathbb{N}$, the characteristics method give the existence of a unique solution $\sigma_n$

verifying (3.18). It is enough to prove the estimates $(3.19)_n$ (index 'n' being to indicate here that the estimates considered are those relating to $\sigma_n$) then to perform the limit when $n$ goes to $+\infty$.

• **Estimate of $\sigma_n$ in $L^\infty(0,T;H^2)$**

During this paragraph, we note $(.,.)$ the inner product in $L^2$ and $<.,.>$ the one in $H^2$. Taking the inner product in $H^2$ of the equation $(3.18)_n$ with $\sigma_n$, we obtain

$$< \frac{\partial \sigma_n}{\partial t}, \sigma_n > + < \widetilde{v}_n \cdot \nabla \sigma_n, \sigma_n > + < g(\sigma_n, \nabla \widetilde{v}_n), \sigma_n > + < l(\widetilde{\varphi})\sigma_n, \sigma_n > = < G_3, \sigma_n > .$$

Classical computations using the orthogonal property $(\widetilde{v}_n \cdot \nabla \tau, \tau) = 0$ (see [9]) show that:

$$| < \widetilde{v}_n \cdot \nabla \sigma_n, \sigma_n > | \leq C \|\widetilde{v}_n\|_3 \|\sigma_n\|_2^2.$$

In the same way, we obtain (see for instance [9, 15])

$$| < g(\sigma_n, \nabla \widetilde{v}_n), \sigma_n > | \leq C \|\widetilde{v}_n\|_3 \|\sigma_n\|_2^2.$$

Concerning the term $< l(\widetilde{\varphi})\sigma_n, \sigma_n >$, we write

$$\begin{aligned} < l(\widetilde{\varphi})\sigma_n, \sigma_n > &= (l(\widetilde{\varphi})\sigma_n, \sigma_n) + (\nabla(l(\widetilde{\varphi})\sigma_n), \nabla\sigma_n) + (\nabla\nabla(l(\widetilde{\varphi})\sigma_n), \nabla\nabla\sigma_n), \\ &\geq l_1 \|\sigma_n\|_2^2 + (l'(\widetilde{\varphi})\nabla\widetilde{\varphi}\ \sigma_n, \nabla\sigma_n) + (l''(\widetilde{\varphi})(\nabla\widetilde{\varphi})^2 \sigma_n, \nabla\nabla\sigma_n) \\ &\qquad + (l'(\widetilde{\varphi})\nabla\nabla\widetilde{\varphi}\ \sigma_n, \nabla\nabla\sigma_n) + 2(l'(\widetilde{\varphi})\nabla\widetilde{\varphi}\nabla\sigma_n, \nabla\nabla\sigma_n). \end{aligned}$$

Using Sobolev injections, we can obtain the following estimate

$$\frac{d}{dt}\left(\frac{\|\sigma_n\|_2^2}{2}\right) + l_1\|\sigma_n\|_2^2 \leq C(\|\widetilde{v}_n\|_3 + \|\nabla\widetilde{\varphi}\|_2 + \|\nabla\widetilde{\varphi}\|_2^2)\|\sigma_n\|_2^2 + \|G_3\|_2\|\sigma_n\|_2.$$

This can be written

$$\frac{d\|\sigma_n\|_2}{dt} + l_1\|\sigma_n\|_2 \leq C(\|\widetilde{v}_n\|_3 + \|\nabla\widetilde{\varphi}\|_2 + \|\nabla\widetilde{\varphi}\|_2^2)\|\sigma_n\|_2 + \|G_3\|_2.$$

An application of the Gronwall lemma give the result concerning the estimate of $\sigma_n$:

$$\|\sigma_n\|_{L^\infty(0,T;H^2)} \leq (\|\sigma_0\|_2 + \|G_3\|_{L^1(0,T;H^2)}) e^{C(\|\widetilde{u}_n\|_{L^1(0,T;V_1\cap H^3)} + \|h\|_3 + \|\nabla\widetilde{\varphi}\|_{L^1(0,T;\Phi_2)} + \|\nabla\widetilde{\varphi}\|^2_{L^2(0,T;\Phi_2)})}.$$

• **Estimate of $\partial_t\sigma_n$ in $L^\infty(0,T;H^1)$**

For the time derivative estimate, it is enough to isolate this derivative from the equation (3.18):

$$\frac{\partial \sigma_n}{\partial t} = G_3 - \widetilde{v}_n \cdot \nabla\sigma_n - g(\sigma_n, \nabla\widetilde{v}_n) - l(\widetilde{\varphi})\sigma_n$$

then to notice, by using the lemma 2.1, that we have

$$\begin{aligned} &\|\widetilde{v}_n.\nabla\sigma_n\|_{L^\infty(0,T;H^1)} \leq C\|\widetilde{v}_n\|_{L^\infty(0,T;V_1\cap H^2)}\|\sigma_n\|_{L^\infty(0,T;H^2)}, \\ &\|g(\sigma_n, \nabla\widetilde{v}_n)\|_{L^\infty(0,T;H^1)} \leq C\|\widetilde{v}_n\|_{L^\infty(0,T;V_1\cap H^2)}\|\sigma_n\|_{L^\infty(0,T;H^2)}, \\ &\|l(\widetilde{\varphi})\sigma_n\|_{L^\infty(0,T;H^1)} \leq C(1 + \|\nabla\widetilde{\varphi}\|_{L^\infty(0,T;\Phi_0)})\|\sigma_n\|_{L^\infty(0,T;H^2)}, \end{aligned}$$

in order to obtain the estimate on $\partial_t\sigma_n$:

$$\left\|\frac{\partial \sigma_n}{\partial t}\right\|_{L^\infty(0,T;H^1)} \leq \|G_3\|_{L^\infty(0,T;H^1)} + C\Big(1 + \|\widetilde{u}_n\|_{L^\infty(0,T;V_1\cap H^2)} + \|h\|_2 + \|\nabla\widetilde{\varphi}\|_{L^\infty(0,T;\Phi_0)}\Big)\|\sigma_n\|_{L^\infty(0,T;H^2)}. \tag{3.20}$$

• **Perform the limit**

The estimates $(3.19)_n$ prove that a function $\sigma$ exists with

$$\begin{aligned} &\sigma_n \rightharpoonup \sigma \quad \text{in } L^\infty(0,T;H^2) \text{ weak-}*, \\ &\frac{\partial \sigma_n}{\partial t} \rightharpoonup \frac{\partial \sigma}{\partial t} \quad \text{in } L^\infty(0,T;H^1) \text{ weak-}*, \\ &\sigma_n \longrightarrow \sigma \text{ in } L^2(0,T;H^1) \quad \text{p.p..} \end{aligned}$$

These convergences allow us to perform the limits in $(3.18)_n$. To obtain the estimate (3.19) for $\sigma$, we take into account the lower semicontinuity of the norm with respect to the weak-$*$ and weak convergence. The uniqueness assertion stems from the fact that (3.18) is linear in $\sigma$.

### 3.4 Existence proof in theorem 2.1

Assume that the initial data are regular

$$
\begin{aligned}
&v_0 = u_0 + h, \quad u_0 \in V_1 \cap H^2, \quad h \in H^3,\\
&\varphi_0 \in \Phi_4,\\
&\sigma_0 \in H^2.
\end{aligned}
$$

We define the Banach space $X_T = \mathcal{C}([0,T];L^2) \times \mathcal{C}([0,T];L^2) \times \mathcal{C}([0,T];L^2)$ and for arbitrary $T>0$, $M_1>0$, $M_2>0$ and $M_3>0$, let us introduce the subset

$$
\begin{aligned}
R_T(M_1,M_2,M_3) = \Big\{ &(u+h,\varphi,\sigma) \in X_T \text{ such that}\\
&u \in L^\infty(0,T;V_1\cap H^2) \cap L^2(0,T;V_1\cap H^3), \quad \frac{\partial u}{\partial t} \in L^\infty(0,T;V_0) \cap L^2(0,T;V_1),\\
&\varphi \in L^\infty(0,T;\Phi_4) \cap L^2(0,T;\Phi_6), \quad \frac{\partial \varphi}{\partial t} \in L^\infty(0,T;\Phi_0) \cap L^2(0,T;\Phi_2),\\
&\sigma \in L^\infty(0,T;H^2), \quad \frac{\partial \sigma}{\partial t} \in L^\infty(0,T;H^1),\\
&u(0)=u_0, \quad \varphi(0)=\varphi_0, \quad \sigma(0)=\sigma_0,\\
&\|u\|^2_{L^\infty(0,T;V_1\cap H^2)} + \|u\|^2_{L^2(0,T;V_1\cap H^3)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;V_0)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;V_1)} \le M_1,\\
&\|\varphi\|^2_{L^\infty(0,T;\Phi_4)} + \|\varphi\|^2_{L^2(0,T;\Phi_6)} + \left\|\frac{\partial \varphi}{\partial t}\right\|^2_{L^\infty(0,T;\Phi_0)} + \left\|\frac{\partial \varphi}{\partial t}\right\|^2_{L^2(0,T;\Phi_2)} \le M_2,\\
&\|\sigma\|_{L^\infty(0,T;H^2)} + \left\|\frac{\partial \sigma}{\partial t}\right\|_{L^\infty(0,T;H^1)} \le M_3 \Big\}.
\end{aligned}
$$

We consider the application

$$
\begin{array}{rccc}
\Theta : & R_T(M_1,M_2,M_3) & \longrightarrow & X_T\\
& (\widetilde{u}+h,\widetilde{\varphi},\widetilde{\sigma}) & \longmapsto & (u+h,\varphi,\sigma)
\end{array}
$$

where $u$, $\varphi$ and $\sigma$ are the unique solutions of (3.1), (3.7) and (3.18) respectively, with

$$
\begin{aligned}
&\widetilde{\eta} = \eta(\widetilde{\varphi}),\\
&G_1 = -\alpha^2 \Delta\widetilde{\varphi}\nabla\widetilde{\varphi} + \operatorname{div}\widetilde{\sigma} - (\widetilde{u}+h)\cdot\nabla(\widetilde{u}+h) + 2\operatorname{div}(\eta(\widetilde{\varphi})D(h)),\\
&G_2 = -\widetilde{\varphi}\,(\widetilde{u}+h),\\
&G_3 = \nu(\widetilde{\varphi})D(\widetilde{u}+h).
\end{aligned}
$$

A fixed point $(v=u+h,\varphi,\sigma)$ of $\Theta$ is clearly a solution of the problem (1.3). We will apply the Shauder fixed point theorem. The three following points thus should be verified:

- **(i)** $R_T(M_1,M_2,M_3)$ is non empty for $M_1$, $M_2$, $M_3$ large enough and for any $T$,
- **(ii)** the existence of $T^*>0$ small enough with $\Theta(R_{T^*}(M_1,M_2,M_3)) \subset R_{T^*}(M_1,M_2,M_3)$,
- **(iii)** $R_{T^*}$ is a convex compact subset of $X_{T^*}$ and $\Theta$ is continuous.

**Step (i)** Let $u^*$ and $\varphi^*$ be the unique solutions on $\mathbb{R}^+$ of the following problems

$$
\begin{aligned}
&\frac{\partial u^*}{\partial t} - \Delta u^* + \nabla p^* = 0, \quad \operatorname{div} u^* = 0, \quad u^*(0)=u_0, \quad u^*|_\Gamma = 0,\\
&\frac{\partial \varphi^*}{\partial t} + \Delta^2\varphi^* = 0, \quad \varphi^*(0)=\varphi_0, \quad \left.\frac{\partial \varphi^*}{\partial n}\right|_\Gamma = \left.\frac{\partial \Delta\varphi^*}{\partial n}\right|_\Gamma = 0.
\end{aligned}
$$

We easily show that there is a constant $J$ such that for any time $T > 0$ we have

$$\|u^*\|^2_{L^\infty(0,T;V_1\cap H^2)} + \|u^*\|^2_{L^2(0,T;V_1\cap H^3)} + \left\|\frac{\partial u^*}{\partial t}\right\|^2_{L^\infty(0,T;V_0)} + \left\|\frac{\partial u^*}{\partial t}\right\|^2_{L^2(0,T;V_1)} \le J|\Delta v_0|_2^2,$$
$$\|\varphi^*\|^2_{L^\infty(0,T;\Phi_4)} + \|\varphi^*\|^2_{L^2(0,T;\Phi_6)} + \left\|\frac{\partial \varphi^*}{\partial t}\right\|^2_{L^\infty(0,T;\Phi_0)} + \left\|\frac{\partial \varphi^*}{\partial t}\right\|^2_{L^2(0,T;\Phi_2)} \le J\|\varphi_0\|_4^2.$$

If we choose $M_1 \ge J|\Delta u_0|_2^2$, $M_2 \ge J\|\varphi_0\|_4^2$ and $M_3 \ge \|\sigma_0\|_2$ then $(u^* + h, \varphi^*, \sigma_0) \in R_T(M_1, M_2, M_3)$. So, if $M_1$, $M_2$ and $M_3$ are sufficiently large then $R_T(M_1, M_2, M_3) \ne \emptyset$ for all $T > 0$.

**Step (ii)** This step breaks up into two parts. In the first one, we obtain estimates on the source terms $G_1$, $G_2$ and $G_3$ so that we can apply the lemmas 3.1, 3.2 and 3.3. In fact, we will show that the sources terms are a little more regular in time than wath we require in each lemma. This will make it possible to set up the second part of the step which consists in finding $M_1$, $M_2$, $M_3$ and $T^*$ such as $R_{T^*}(M_1, M_2, M_3)$ is stable by the application $\Theta$.

**Estimate of $G_1$:** The regularity which is required on $G_1$ in the lemma 3.1 is:

$$G_1 \in L^2(0,T;H^1), \quad \frac{\partial G_1}{\partial t} \in L^2(0,T;H^{-1}),$$
$$G_1 \cdot \frac{\partial G_1}{\partial t} \in L^1(0,T;L^1) \quad \text{et} \quad G_1(0) \in L^2.$$

Recall that $G_1$ is given by

$$G_1 = -\alpha^2 \Delta\widetilde{\varphi}\nabla\widetilde{\varphi} + \operatorname{div}\widetilde{\sigma} - (\widetilde{u} + h)\cdot\nabla(\widetilde{u} + h) + 2\operatorname{div}(\eta(\widetilde{\varphi})D(h)).$$

We assume that $(\widetilde{u} + h, \widetilde{\varphi}, \widetilde{\sigma}) \in R_T$. Using lemma 2.1, the product $\Delta\widetilde{\varphi}\nabla\widetilde{\varphi}$ verifies

$$\|\Delta\widetilde{\varphi}\nabla\widetilde{\varphi}\|^2_{L^2(0,T;H^1)} \le CT\ \|\Delta\widetilde{\varphi}\|^2_{L^\infty(0,T;H^1)}\|\nabla\widetilde{\varphi}\|^2_{L^\infty(0,T;H^2)} \le CTM_2^2.$$

The same method allows us to verify that the other terms are regular too:

$$\|\operatorname{div}\widetilde{\sigma}\|^2_{L^2(0,T;H^1)} \le CT\ M_3,$$
$$\|(\widetilde{u}+h)\cdot\nabla(\widetilde{u}+h)\|^2_{L^2(0,T;H^1)} \le CT\ \|\widetilde{u}+h\|^4_{L^\infty(0,T;H^2)} \le CT\ (M_1^2 + \|h\|_2^4),$$
$$\|\operatorname{div}(\eta(\widetilde{\varphi})D(h))\|^2_{L^2(0,T;H^1)} \le CT\ \|\eta(\widetilde{\varphi})\|^2_{L^\infty(0,T;H^2)}\|h\|_3^2 \le CT\ (1 + M_2^2)\ \|h\|_3^2.$$

Remark that for the last inequality, we use the following relation

$$\begin{aligned}\|\eta(\widetilde{\varphi})\|_2^2 &= |\eta(\widetilde{\varphi})|_2^2 + |\eta'(\widetilde{\varphi})\nabla\widetilde{\varphi}|_2^2 + |\eta''(\widetilde{\varphi})\nabla\widetilde{\varphi}\nabla\widetilde{\varphi}|_2^2 + |\eta'(\widetilde{\varphi})\nabla\nabla\widetilde{\varphi}|_2^2 \\ &\le |\eta|_\infty^2 + |\eta'|_\infty^2|\nabla\widetilde{\varphi}|_2^2 + |\eta''|_\infty^2|\nabla\widetilde{\varphi}|_4^4 + |\eta'|_\infty^2|\nabla\nabla\widetilde{\varphi}|_2^2 \\ &\le C(1 + M_2^2).\end{aligned} \tag{3.21}$$

We deduce finally $G_1 \in L^2(0,T;H^1)$ with

$$\|G_1\|^2_{L^2(0,T;H^1)} \le CT(M_1^2 + M_2^2 + M_3 + \|h\|_2^4 + \|h\|_3^2 + M_2^2\|h\|_3^2). \tag{3.22}$$

For the time derivative, we have by definition

$$\begin{aligned}\frac{\partial G_1}{\partial t} = -\alpha^2\Delta\frac{\partial\widetilde{\varphi}}{\partial t}\ \nabla\widetilde{\varphi} - \alpha^2\Delta\widetilde{\varphi}\ \nabla\frac{\partial\widetilde{\varphi}}{\partial t} + \operatorname{div}\frac{\partial\widetilde{\sigma}}{\partial t} + 2\operatorname{div}\left(\eta'(\widetilde{\varphi})\frac{\partial\widetilde{\varphi}}{\partial t}D(h)\right) \\ - \frac{\partial\widetilde{u}}{\partial t}.\nabla(\widetilde{u}+h) - (\widetilde{u}+h).\nabla\frac{\partial\widetilde{u}}{\partial t}.\end{aligned} \tag{3.23}$$

Many terms are much more regular than what is required. If we write $\partial_t G_1 = f + g$ where $f$ corresponds to the terms of the first line of (3.23) and $g$ corresponds to the terms of the second line:

$$f = -\alpha^2 \Delta \frac{\partial \widetilde{\varphi}}{\partial t} \nabla \widetilde{\varphi} - \alpha^2 \Delta \widetilde{\varphi} \, \nabla \frac{\partial \widetilde{\varphi}}{\partial t} + \operatorname{div} \frac{\partial \widetilde{\sigma}}{\partial t} + 2 \operatorname{div} \left( \eta'(\widetilde{\varphi}) \frac{\partial \widetilde{\varphi}}{\partial t} D(h) \right),$$
$$g = -\frac{\partial \widetilde{u}}{\partial t} . \nabla (\widetilde{u} + h) - (\widetilde{u} + h) . \nabla \frac{\partial \widetilde{u}}{\partial t}.$$

The terms of $f$ will be not only $H^{-1}$ but also $L^2$, what will facilitate the estimate of the product $G_1 \cdot f$. Start with the estimate of $f$ in $L^2$. We use like previously results of interpolation and products of Sobolev spaces which make it possible to affirm that

$$\left\| \Delta \frac{\partial \widetilde{\varphi}}{\partial t} \nabla \widetilde{\varphi} \right\|^2_{L^2(0,T;L^2)} \leq C \left\| \Delta \frac{\partial \widetilde{\varphi}}{\partial t} \right\|^2_{L^2(0,T;L^2)} \| \nabla \widetilde{\varphi} \|^2_{L^\infty(0,T;H^2)} \leq C M_2^2,$$
$$\left\| \Delta \widetilde{\varphi} \, \nabla \frac{\partial \widetilde{\varphi}}{\partial t} \right\|^2_{L^2(0,T;L^2)} \leq C \sqrt{T} \| \Delta \widetilde{\varphi} \|^2_{L^\infty(0,T;H^2)} \left\| \nabla \frac{\partial \widetilde{\varphi}}{\partial t} \right\|^2_{L^4(0,T;L^2)} \leq C \sqrt{T} M_2^2,$$
$$\left\| \operatorname{div} \frac{\partial \widetilde{\sigma}}{\partial t} \right\|^2_{L^2(0,T;L^2)} \leq C T \left\| \frac{\partial \widetilde{\sigma}}{\partial t} \right\|^2_{L^\infty(0,T;H^1)} \leq C T M_3.$$

Developping the term $2 \operatorname{div} (\eta'(\widetilde{\varphi}) \partial_t \widetilde{\varphi} D(h))$, we find

$$\left\| \operatorname{div} \left( \eta'(\widetilde{\varphi}) \frac{\partial \widetilde{\varphi}}{\partial t} D(h) \right) \right\|^2_{L^2(0,T;L^2)} \leq C T M_2^2 \|h\|_3^2 + C \sqrt{T} M_2 \|h\|_3^2.$$

For $f$, an estimate of the following type was obtained (we indicate here the dependence of each constant because it is essential to observe that the constant in front of $M_2^2$ depends neither on time $T$, nor of constants $M_1$, $M_2$, $M_3$.):

$$\|f\|^2_{L^2(0,T;L^2)} \leq C(\Omega) M_2^2 + (\sqrt{T} + T) \; C(\Omega, M_1, M_2, M_3, \|h\|_3).$$

The terms of $g$ are estimated on the following way:

$$\left\| \frac{\partial \widetilde{u}}{\partial t} . \nabla (\widetilde{u} + h) \right\|^2_{L^2(0,T;H^{-1})} \leq C T \left\| \frac{\partial \widetilde{u}}{\partial t} \right\|^2_{L^\infty(0,T;L^2)} \left\| \widetilde{u} + h \right\|^2_{L^\infty(0,T;H^2)} \leq C T M_1 (M_1 + \|h\|_2^2),$$
$$\left\| (\widetilde{u} + h) . \nabla \frac{\partial \widetilde{u}}{\partial t} \right\|^2_{L^2(0,T;H^{-1})} \leq C T \left\| \widetilde{u} + h \right\|^2_{L^\infty(0,T;H^2)} \left\| \frac{\partial \widetilde{u}}{\partial t} \right\|^2_{L^\infty(0,T;H^{-1})} \leq C T (M_1 + \|h\|_2^2) M_1.$$

We find:

$$\|g\|^2_{L^2(0,T;H^{-1})} \leq C T M_1 (M_1 + \|h\|_2^2).$$

Adding the estimates concerning $f$ and the estimates concerning $g$, we obtain $\partial_t G_1 \in L^2(0,T;H^{-1})$ with the inegality:

$$\left\| \frac{\partial G_1}{\partial t} \right\|^2_{L^2(0,T;H^{-1})} \leq C(\Omega) M_2^2 + (\sqrt{T} + T) \; C(\Omega, M_1, M_2, M_3, \|h\|_3). \tag{3.24}$$

The product $G_1 \cdot \partial_t G_1$ is written $G_1 \cdot f + G_1 \cdot g$. Since $G_1 \in L^2(0,T;H^1)$ and $f \in L^2(0,T;L^2)$, it is clear that $G_1 \cdot f \in L^1(0,T;L^1)$ with the estimate:

$$\|G_1 \cdot f\|^2_{L^1(0,T;L^1)} \leq C \|G_1\|_{L^2(0,T;H^1)} \|f\|_{L^2(0,T;L^2)}.$$

Concerning the product $G_1 \cdot g$, we remark that

$$G_1 \cdot g = b\left( \frac{\partial \widetilde{u}}{\partial t}, \widetilde{u} + h, G_1 \right) + b\left( \widetilde{u} + h, \frac{\partial \widetilde{u}}{\partial t}, G_1 \right).$$

where $b(u,v,w)=(u.\nabla v,w)$. If $u\in v_0$ then this trilinear application verifies $b(u,v,w)=-b(u,w,v)$

$$G_1\cdot g=-b\Big(\frac{\partial\,\widetilde{u}}{\partial t},G_1,\widetilde{u}+h\Big)-b\Big(\widetilde{u}+h,G_1,\frac{\partial\,\widetilde{u}}{\partial t}\Big)\in L^1(0,T;L^1).$$

We deduce the following estimate

$$\begin{aligned}\|G_1\cdot g\|_{L^1(0,T;L^1)} &\le C\sqrt{T}\Big\|\frac{\partial\,\widetilde{u}}{\partial t}\Big\|_{L^\infty(0,T;V_0)}|\nabla G_1|_{L^2(0,T;L^2)}|\widetilde{u}+h|_{L^\infty(0,T;V_0\cap H^2)}\\ &\le C\sqrt{T}M_2\|G_1\|_{L^2(0,T;H^1)}.\end{aligned}$$

Adding the estimate of $G_1\cdot f$ like that of $G_1\cdot g$, we find:

$$\Big\|G_1\cdot\frac{\partial G_1}{\partial t}\Big\|_{L^1(0,T;L^1)}\le C\|G_1\|_{L^2(0,T;H^1)}(\sqrt{T}M_2+\|f\|_{L^2(0,T;L^2)}). \tag{3.25}$$

Concerning the estimate of $|G_1(0)|_2^2$, we could write it like a direct consequence of the estimates of $G_1$ in $L^2(0,T;H^1)$ and of $\partial_t G_1$ in $L^2(0,T;H^{-1})$. These estimate will be dependent on $T$, which we do not wish here (see the remark 3.1). We prefer to write:

$$|G_1(0)|_2^2\le C(\|\varphi_0\|_3^4+\|\sigma_0\|_1^2+\|u_0\|_2^4+\|h\|_2^4+\|h\|_2^2\|u_0\|_1^2+\|h\|_2^2+\|\varphi_0\|_3^2\|h\|_1^2). \tag{3.26}$$

**Estimate of $G_2$:** For the source term of the Cahn-Hilliard equation, the lemma 3.2 require:

$$\begin{aligned}&G_2\in L^2(0,T;L^2),\quad \operatorname{div} G_2\in L^2(0,T;H^2),\quad G_2.n=0\quad\text{on }\Gamma,\\ &\operatorname{div}\frac{\partial G_2}{\partial t}\in L^2(0,T;L^2)\quad\text{and}\quad \operatorname{div} G_2(0)\in L^2.\end{aligned}$$

Recall that $G_2=-\widetilde{\varphi}\,(\widetilde{u}+h)$, so $\operatorname{div}\,(G_2)=-(\widetilde{u}+h)\cdot\nabla\widetilde{\varphi}$. It is clear that if $(\widetilde{u}+h,\widetilde{\varphi},\widetilde{\sigma})\in R_T$ then all these conditions are satisfied. More precisely, we have

$$\begin{aligned}&\|G_2\|^2_{L^2(0,T;L^2)}\le CT\|v\|^2_{L^\infty(0,T;L^2)}\|\varphi\|^2_{L^\infty(0,T;H^2)}\le CTM_1M_2,\\ &\|\operatorname{div}\,(G_2)\|^2_{L^2(0,T;H^2)}\le CT\|v\|^2_{L^\infty(0,T;H^2)}\|\nabla\varphi\|^2_{L^\infty(0,T;H^2)}\le CTM_1M_2,\\ &\Big\|\frac{\partial v}{\partial t}.\nabla\varphi\Big\|^2_{L^2(0,T;L^2)}\le CT\Big\|\frac{\partial v}{\partial t}\Big\|^2_{L^\infty(0,T;L^2)}\|\nabla\varphi\|^2_{L^\infty(0,T;H^2)}\le CTM_1M_2,\\ &\Big\|v\cdot\nabla\frac{\partial\varphi}{\partial t}\Big\|^2_{L^2(0,T;L^2)}\le C\sqrt{T}\|v\|^2_{L^\infty(0,T;H^2)}\Big\|\frac{\partial\varphi}{\partial t}\Big\|^2_{L^4(0,T;H^1)}\le C\sqrt{T}M_1M_2,\\ &|\operatorname{div}\,G_2(0)|_2^2\le C(|u_0|_2^2+|h|_2^2)\|\varphi_0\|_3^2.\end{aligned} \tag{3.27}$$

**Estimate of $G_3$:** Finaly, for the third source term appearing in the lemma 3.3:

$$G_3=\nu(\widetilde{\varphi})D(\widetilde{u}+h),$$

we have, for $(r,s)=(\infty,1)$ or $(r,s)=(2,2)$, the inequality:

$$\begin{aligned}\|G_3\|^2_{L^r(0,T;H^s)}&\le\ \|\nu(\widetilde{\varphi})\|^2_{L^\infty(0,T;H^2)}\|D(\widetilde{u})\|^2_{L^r(0,T;H^s)}+C\ \|\nu(\widetilde{\varphi})\|^2_{L^\infty(0,T;H^2)}\|D(h)\|^2_s\\ &\le\ \|\nu(\widetilde{\varphi})\|^2_{L^\infty(0,T;H^2)}(M_1+\|h\|_3^2).\end{aligned}$$

To control the norm of $\nu(\widetilde{\varphi})$ in $H^2$ we use the same method that for $\eta(\varphi)$ (see (3.21)), we obtain

$$\|\nu(\widetilde{\varphi})\|_2^2 \leq C(1+M_2^2).$$

We deduce that if $(\widetilde{u}+h, \widetilde{\varphi}, \widetilde{\sigma}) \in R_T$ then $G_3 \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ with

$$\|G_3\|^2_{L^\infty(0,T;H^1)} + \|G_3\|^2_{L^2(0,T;H^2)} \leq C\ (1+M_2^2)(M_1 + \|h\|_3^2). \tag{3.28}$$

To apply the three lemmas, two other remarks are necessary. First of all, we must check that if $(\widetilde{u}+h, \widetilde{\varphi}, \widetilde{\sigma}) \in R_T$ then $\nabla\widetilde{\varphi} \in L^\infty(0,T;H^2)$ and $\partial_t\widetilde{\varphi} \in L^2(0,T;L^\infty)$ (these assumptions are necessary in the lemma 3.1 to show that $\widetilde{\eta} = \eta(\widetilde{\varphi})$ is regular enough). We have clearly

$$\|\nabla\widetilde{\varphi}\|^2_{L^\infty(0,T;H^2)} \leq CM_2,$$

and for the second estimate, we use an interpolation result:

$$[L^\infty(0,T;\Phi_0), L^2(0,T;\Phi_2)]_{1/4} = L^{8/3}(0,T;L^\infty),$$

which implies $\partial_t\widetilde{\varphi} \in L^2(0,T;L^\infty)$ with

$$\left\|\frac{\partial\widetilde{\varphi}}{\partial t}\right\|^2_{L^2(0,T;L^\infty)} \leq T^{1/4}\left\|\frac{\partial\widetilde{\varphi}}{\partial t}\right\|^2_{L^{8/3}(0,T;L^\infty)} \leq CT^{1/4}\left\|\frac{\partial\widetilde{\varphi}}{\partial t}\right\|^{1/2}_{L^\infty(0,T;\Phi_0)}\left\|\frac{\partial\widetilde{\varphi}}{\partial t}\right\|^{3/2}_{L^2(0,T;\Phi_2)} \leq CT^{1/4}M_2.$$

The second remark comes from lemma 3.2 in which we saw that if the function $G_2$ is defined on $[0,T]$ then the solution undoubtedly only exists until a time $T_1 \leq T$. To avoid this problem, it suffices to choose $T$ small enough. Indeed, if $T$ verifies:

$$T \leq \frac{1}{k_{20}(\|\varphi_0\|_4^2, CTM_1M_2)} \tag{3.29}$$

then using the inequality $\|\operatorname{div} G_2\|^2_{L^2(0,T;H^2)} \leq CTM_1M_2$ and the growth of $k_{20}$ in each variable, we have

$$T \leq \frac{1}{k_{20}(\|\varphi_0\|_4^2, \|\operatorname{div} G_2\|^2_{L^2(0,T;H^2)})} = \frac{1}{\Lambda}.$$

Henceforth, we will take $T$ small enough to verify this condition (3.29) and we will be able to thus apply the results of the lemma 3.2 on $[0,T]$ as well in dimension 2 and in dimension 3.

Applying the lemmas 3.1, 3.2 and 3.3 and using the estimates for $G_1$, $G_2$ and $G_3$ (all the functions $k_i$ being non-decreasing in each variable), we can affirm that $(v,\varphi,\sigma) \in R_T$ as soon as

$$\begin{cases} k_1\Big(\|u_0\|_2^2,\ T\widetilde{C},\ M_2^2\widehat{C} + (\sqrt{T}+T)\widetilde{C},\ (\sqrt{T}+T)\widetilde{C},\ |G_1(0)|_2^2,\ M_2\widehat{C},\ T^{1/4}\widetilde{C}\Big) \leq M_1, \\ k_2\Big(\|\varphi_0\|_4^2,\ T\widetilde{C},\ T\widetilde{C},\ (\sqrt{T}+T)\widetilde{C},\ |\operatorname{div}(G_2(0))|_2^2\Big) \leq M_2, \\ k_3\Big(\|\sigma_0\|_2^2,\ C\ (1+M_2^2)(M_1+\|h\|_3^2),\ C\ (1+M_2^2)(M_1+\|h\|_3^2),\ M_1,\ \sqrt{T}M_1,\ \|h\|_3^2,\ M_2\Big) \leq M_3. \end{cases}$$

In theses estimates, wo notations were introduced. The noted constants $\widehat{C}$ depend neither on $T$, nor of $M_1$, $M_2$ or $M_3$ whereas the constants $\widetilde{C}$ do not depend on time $T$: $\widetilde{C} = \widetilde{C}(\Omega,\ M_1,\ M_2,\ M_3,\ \|h\|_3)$.

By using once again the growth of the functions $k_i$ in each variable and the fact that theses functions $k_i$ doesn't depend on $T$, it is clear that it is enough to choose three positive constants $M_1$, $M_2$ and $M_3$ verifying the strict inequalities for $T = 0$:

$$\begin{cases} k_1\Big(\|u_0\|_2^2,\ 0,\ M_2^2\widehat{C},\ 0,\ |G_1(0)|_2^2,\ M_2\widehat{C},\ 0\Big) < M_1, \\ k_2\Big(\|\varphi_0\|_4^2,\ 0,\ 0,\ 0,\ |\operatorname{div} G_2(0)|_2^2\Big) < M_2, \\ k_3\Big(\|\sigma_0\|_2^2,\ C\ (1+M_2^2)(M_1+\|h\|_3^2),\ C\ (1+M_2^2)(M_1+\|h\|_3^2),\ M_1,\ 0,\ \|h\|_3^2,\ M_2\Big) < M_3. \end{cases}$$

Thus, we choose successively $M_2$, $M_1$ and $M_3$ satisfying these estimates, and finally we deduce the existence of a time $T^* > 0$ such as $\Theta(R_{T^*}) \subset R_{T^*}$.

**Step (iii)** Clearly $R_T$ is convex and it is easily seen that it is closed in $X_T$. Moreover, from Ascoli theorem $R_T$ is relatively compact in $X_T$, see [25].

Let us show the sequential continuity of the application $\Theta$ from $R_T$ to $X_T$ : assume that $(\widetilde{u}_n + h, \widetilde{\varphi}_n, \widetilde{\sigma}_n)$ is a sequence in $R_T$ converging to $(\widetilde{u} + h, \widetilde{\varphi}, \widetilde{\sigma})$ in $X_T$. If we note $(u_n + h, \varphi_n, \sigma_n)$ and $(u + h, \varphi, \sigma)$ the images by $\Theta$ of $(\widetilde{u}_n + h, \widetilde{\varphi}_n, \widetilde{\sigma}_n)$ and $(\widetilde{u} + h, \widetilde{\varphi}, \widetilde{\sigma})$ respectively then it is sufficient to prove that $(u_n + h, \varphi_n, \sigma_n)$ tends to $(u + h, \varphi, \sigma)$ in $X_T$.

In a first step, we show that $u_n$ tends to $u$ in $V_0$. The functions $u_n$ and $u$ being defined by

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - 2\operatorname{div}\,(\eta(\widetilde{\varphi}_n)D(u_n)) + \nabla p_n = G_1(\widetilde{u}_n, \widetilde{\varphi}_n, \widetilde{\sigma}_n), \quad \operatorname{div} u_n = 0, \\ \dfrac{\partial u}{\partial t} - 2\operatorname{div}\,(\eta(\widetilde{\varphi})D(u)) + \nabla p = G_1(\widetilde{u}, \widetilde{\varphi}, \widetilde{\sigma}), \quad \operatorname{div} u = 0. \end{cases}$$

Making the difference, posing $\bar{u} = u_n - u$ and multiplying the equation obtained by $\bar{u}$, we deduce

$$\frac{d}{dt}\left(\frac{|\bar{u}|_2^2}{2}\right) + \eta_1|\nabla\bar{u}|_2^2 \le 2|\eta'|_\infty\|\widetilde{\varphi}_n - \widetilde{\varphi}\|_1\|\nabla u_n\|_1|\nabla\bar{u}|_2 + \int_\Omega (G_1(\widetilde{u}_n, \widetilde{\varphi}_n, \widetilde{\sigma}_n) - G_1(\widetilde{u}, \widetilde{\varphi}, \widetilde{\sigma})).\bar{u},$$
$$\frac{d}{dt}\left(\frac{|\bar{u}|_2^2}{2}\right) + \frac{\eta_1}{2}|\nabla\bar{u}|_2^2 \le C\|\widetilde{\varphi}_n - \widetilde{\varphi}\|_1^2 + \|G_1(\widetilde{u}_n, \widetilde{\varphi}_n, \widetilde{\sigma}_n) - G_1(\widetilde{u}, \widetilde{\varphi}, \widetilde{\sigma})\|_{-1}^2.$$

It is then enough to prove that $G_1(\widetilde{u}_n, \widetilde{\varphi}_n, \widetilde{\sigma}_n)$ tends to $G_1(\widetilde{u}, \widetilde{\varphi}, \widetilde{\sigma})$ in $H^{-1}$. For this, we recall that $G_1$ is defined in our case by

$$G_1(u, \varphi, \sigma) = -\alpha^2\Delta\varphi\nabla\varphi + \operatorname{div}\sigma - (u + h)\cdot\nabla(u + h) + 2\operatorname{div}\,(\eta(\varphi)D(h)).$$

We know that the sequences $\Delta\widetilde{\varphi}_n$ and $\frac{\partial\Delta}{\partial t}\widetilde{\varphi}_n$ are bounded in $L^\infty(0, T; H^2)$ and $L^\infty(0, T; H^{-2})$ respectively. According to the results due to J. Simon [25], we deduce from it that $\Delta\widetilde{\varphi}_n$ admits a limit in $H^2$, which is equal to $\Delta\widetilde{\varphi}$. In the same way, $\widetilde{\varphi}_n$ tends to $\widetilde{\varphi}$ in $\Phi_4$, and thus the product $\nabla\widetilde{\varphi}_n\ \Delta\widetilde{\varphi}_n$ tends to $\nabla\widetilde{\varphi}\ \Delta\widetilde{\varphi}$ in $H^2$. The other terms treat same manner, by noticing in particular that $\widetilde{u}_n$ tends to $\widetilde{u}$ in $V_2$ and that $\operatorname{div}\widetilde{\sigma}_n$ tends to $\operatorname{div}\widetilde{\sigma}$ in $L^2$.

We identically prove the convergences of $\varphi_n$ to $\varphi$ in $\Phi_0$ and of $\sigma_n$ to $\sigma$ in $L^2$.

# 4 Uniqueness

In this section, we show that the solution obtained in theorem 2.1 is the only one in the class of regular solutions. Precisely, the result reads as follow:

**Theorem 4.1**
*Let $T > 0$. Problem (1.3) admits at most one solution $(u+h, \varphi, \sigma)$ in the class*

$$u \in L^\infty(0,T;V_1 \cap H^2) \cap L^2(0,T;V_1 \cap H^3), \quad \frac{\partial u}{\partial t} \in L^\infty(0,T;V_0) \cap L^2(0,T;V_1),$$
$$\varphi \in L^\infty(0,T;\Phi_4) \cap L^2(0,T;\Phi_6), \quad \frac{\partial \varphi}{\partial t} \in L^\infty(0,T;\Phi_0) \cap L^2(0,T;\Phi_2),$$
$$\sigma \in L^\infty(0,T;H^2), \quad \frac{\partial \sigma}{\partial t} \in L^\infty(0,T;H^1).$$

**Proof :** To prove this result, as usual we take the difference of two solutions $(v_1, \varphi_1, \sigma_1)$ and $(v_2, \varphi_2, \sigma_2)$ belonging to the class specified in the statement of the theorem. The scalar $\varphi = \varphi_1 - \varphi_2$, the vector $v = v_1 - v_2$ and the tensor $\sigma = \sigma_1 - \sigma_2$ satisfy the following: for any $w \in V_1$, any $\psi \in \Phi_3$ and any $\tau \in H^1$,

$$\begin{aligned}\int_\Omega \frac{\partial v}{\partial t}.w + \int_\Omega (v \cdot \nabla v_1).w + \int_\Omega (v_2 \cdot \nabla v).w - \int_\Omega \operatorname{div} \sigma.w \\ + 2\int_\Omega (\eta(\varphi_1) - \eta(\varphi_2)) D(v_1) : D(w) + 2\int_\Omega \eta(\varphi_2) D(v) : D(w) \\ = -\alpha^2 \int_\Omega (w \cdot \nabla \varphi_1)\Delta\varphi - \alpha^2 \int_\Omega (w \cdot \nabla\varphi)\Delta\varphi_2,\end{aligned}$$

$$\begin{aligned}\int_\Omega \frac{\partial \varphi}{\partial t}\psi + \int_\Omega (v \cdot \nabla\varphi_1)\psi + \int_\Omega (v_2 \cdot \nabla\varphi)\psi \\ - \alpha^2 \int_\Omega (B(\varphi_1) - B(\varphi_2))\nabla\Delta\varphi_1 \cdot \nabla\psi - \alpha^2 \int_\Omega B(\varphi_2)\nabla\Delta\varphi \cdot \nabla\psi \\ + \int_\Omega (BF''(\varphi_1) - BF''(\varphi_2))\nabla\varphi_1 \cdot \nabla\psi + \int_\Omega BF''(\varphi_2)\nabla\varphi \cdot \nabla\psi = 0,\end{aligned}$$

$$\begin{aligned}\int_\Omega \frac{\partial \sigma}{\partial t} : \tau + \int_\Omega (v \cdot \nabla\sigma_1) : \tau + \int_\Omega (v_2 \cdot \nabla\sigma) : \tau + \int_\Omega l(\varphi_2)\sigma : \tau \\ + \int_\Omega (l(\varphi_1) - l(\varphi_2))\sigma_1 : \tau + \int_\Omega (g(\sigma, \nabla v_1) + g(\sigma_2, \nabla v)) : \tau \\ = \int_\Omega (\nu(\varphi_1) - \nu(\varphi_2)) D(v_1) : \tau + \int_\Omega \nu(\varphi_2) D(v) : \tau,\end{aligned}$$

Taking $(w, \psi, \tau) = (v, \Delta\varphi, \sigma)$ as function test we deduce after some computations (see [9] for similar estimates):

$$\frac{d}{dt}|v|_2^2 + \eta_1|\nabla v|_2^2 \le C(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2) + \frac{\alpha^4 B_1}{2}|\nabla\Delta\varphi|_2^2.$$

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2\right) + \frac{\alpha^4 B_1}{2}|\nabla\Delta\varphi|_2^2 \le C(|v|_2^2 + |\nabla\varphi|_2^2) + C|\Delta^2\varphi_1|_2^{4/3}|\nabla\varphi|_2^2.$$

$$\frac{d}{dt}|\sigma|_2^2 + l_1|\sigma|_2^2 \le C(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2) + C|\sigma|_2^2(\|\sigma_1\|_2^2 + \|\sigma_2\|_2^2 + \|v_1\|_3 + \|v_1\|_2^2) + \frac{\eta_1}{2}|\nabla v|_2^2.$$

Combining these three equations, we obtain the following energy inequality

$$\frac{d}{dt}\left(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2\right) \le C\beta_4(t)\left(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2\right)$$

with $\beta_4 = 1 + |\Delta^2\varphi_1|_2^{4/3} + \|\sigma_1\|_2^2 + \|\sigma_2\|_2^2 + \|v_1\|_3 + \|v_1\|_2^2$ belonging to $L^1(0,T)$. Since the initial values are zero, we deduce from Gronwall lemma that $(v, \varphi, \sigma) = 0$. ■

# 5 Global existence

Concerning the global existence, we cannot hope to obtain strong solutions in the three dimensional case taking into account the fact that the problem is still open for the only Navier-Stokes problem! In the two dimensional case, the Navier-Stokes equation is not an obstacle (see for example [26, 28]) but no result of global existence (with unspecified data) is know for monophasic models. We thus will restrict here on the case where the data are small. In [15], C. Guillopé and J.C. Saut prove the global existence of strong solution with small data supposing moreover that the fluid is not very viscoelastic (what amounts for us imposing $\nu$ small). In [13], E.F. Cara, F. Guillen and R.R. Ortega free themselves from this hypothesis but prove only an existence on $[0, T]$ for all $T$ (for smaller data than $\beta(T)$, where $\beta$ tends to 0 when $T$ increase...). In this part, we will prove that this assumption of slightly viscoelastic fluid can be removed. In addition, we always take place in the diphasic case and thus it should be supposed that the data in $\varphi$ are "small". Precisely, when we speak about small data, we will suppose that $\varphi_0$ is close to a constant $\omega$ such that $F''(\omega) > 0$ (this assumption is usual, we say that $\omega$ is a metastable state for potential $F$). We prove then (see theorem 2.2):

**Theorem 5.1**
*Assume $u_0 \in V_1 \cap H^2$, $\varphi_0 \in \Phi_4$, $\sigma_0 \in H^2$, $\eta$, $B$, $l$ and $\nu$ satisfy* (2.1)*, $F$ satisfies* (2.2), (2.3), (2.4) *and $h$ satisfies* (2.5)*. Let $w$ lying in a metastable region of $F$.*

*If $h$, $u_0$, $\sigma_0$ and $\varphi_0 - \omega$ are small enough in their respective space then the results of the theorem 2.1 hold on $\mathbb{R}^+$. Moreover, $\|u\|_2$, $\|\sigma\|_2$ and $\|\varphi - \omega\|_3$ remain small.*

**Proof** : By definition of the metastability of $\omega$ we assume that $F''(\omega) > 0$ and $B(\omega) > 0$. The functions $F$ and $B$ being regular, we know that there exists a neighborhood $\mathcal{V}$ of $\omega$ in $\mathbb{R}$ on which $F'' \geq 0$ and $B > 0$. We can so consider two functions $F_\omega$, $B_\omega$ verifying

$$
\begin{aligned}
& F''_\omega = F'' \quad \text{and} \quad B_\omega = B \quad \text{on } \mathcal{V}, \\
& F \in \mathcal{C}^4(\mathbb{R}, \mathbb{R}^+), \quad B \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^+), \\
& F_\omega(\omega) = 0, \ F'_\omega(\omega) = 0, \ F_\omega \text{ is convex and all its derivatives are bounded on } \mathbb{R}.
\end{aligned}
$$

It is clear that $B_\omega$ verifies (2.1) and that $F_\omega$ verifies the hypotheses (2.2), (2.3) and (2.4). Moreover, the convexity of $F_\omega$ implies that we can choose (in the assumption (2.4))

$$\forall x \in \mathbb{R}, \quad F_1(x) = 1 \quad \text{and} \quad F_2(x) = 0.$$

In the sequel of the proof, we work with the modified problem, obtained by remplacing $B$ and $F$ by $B_\omega$ and $F_\omega$. Since we will show that $\varphi$ remains close to $\omega$ (and precisely in the neighborhood $\mathcal{V}$), the solution obtained for the modified problem will be also solution of the initial problem (see [4]).

We know that to show an estimate with small data, the quadratic terms are easily controled. For this reason we write the system (1.3) in the following form

$$
\begin{aligned}
& \frac{\partial u}{\partial t} - 2 \operatorname{div} (\eta(\varphi) D(u)) + \nabla p = \operatorname{div} \sigma + H_1, \\
& \operatorname{div} u = 0,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
& \frac{\partial \sigma}{\partial t} + v \cdot \nabla \sigma + l(\varphi)\sigma = \nu(\varphi) D(u) + H_3, \\
& v = u + h,
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
& \frac{\partial \varphi}{\partial t} - \operatorname{div} (B(\varphi) \nabla \mu) = \operatorname{div} (H_2), \\
& \mu = -\alpha^2 \Delta \varphi + F'(\varphi),
\end{aligned}
\tag{5.3}
$$

with the conditions

$$u(0) = u_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad u|_\Gamma = 0, \quad \left.\frac{\partial \varphi}{\partial n}\right|_\Gamma = \left.\frac{\partial \mu}{\partial n}\right|_\Gamma = 0.$$

and we define

$$H_1 = -\alpha^2\Delta\varphi\nabla\varphi + v\cdot\nabla v + 2\,\mathrm{div}\,(\eta(\varphi)D(h)),$$
$$H_2 = -(u+h)\varphi,$$
$$H_3 = g(\sigma,\nabla v) + \nu(\varphi)D(h).$$

Let us prove energy estimates on the solutions of (5.1), (5.2) and (5.3):

**Step 1: Estimate on the velocity.** Let us take again the estimate of $u$ in $L^\infty(0,T;V_1)\cap L^2(0,T;V_1\cap H^2)$ (section 3.1.1) obtained by carrying out the scalar product of (5.1) by $Au$. It is written in our case

$$\frac{d}{dt}\left(\frac{|\nabla u|_2^2}{2}\right) + \frac{\eta_1}{2}\|u\|_2^2 \le C(|H_1|_2^2 + |\nabla\varphi|_\infty^2|\nabla u|_2^2) + C_1|\,\mathrm{div}\,\sigma|_2^2. \tag{5.4}$$

**Step 2: On the monophasic model.** In fact we can see that using the coupling terms div $\sigma$ and $\nu(\varphi)D(u)$, we can obtain more interesting estimate if we add $(\nu(\omega)\times(5.1), u)$ and $((5.2),\sigma)$:

$$\frac{d}{dt}\left(\nu(\omega)|u|_2^2 + |\sigma|_2^2\right) + \left(\nu(\omega)\eta_1|\nabla u|_2^2 + l_1|\sigma|_2^2\right) \le \int_\Omega \nu(\omega)H_1.u + \int_\Omega H_3.\sigma$$
$$+ \int_\Omega \nu(\omega)\,\mathrm{div}\,(\sigma).u + \int_\Omega \nu(\varphi)D(u).\sigma$$

With an integration by parts, the coupling terms write

$$\int_\Omega \nu(\omega)\,\mathrm{div}\,(\sigma).u + \int_\Omega \nu(\varphi)D(u).\sigma = \int_\Omega (\nu(\varphi)-\nu(\omega))D(u).\sigma.$$

Using the fact that (see [12, 27] for the laplacian regularity)

$$\begin{aligned}|\varphi-\omega|_\infty &\le |\varphi - m(\varphi)|_\infty + |m(\varphi_0)-\omega|\\ &\le C\|\varphi-m(\varphi)\|_2 + |m(\varphi_0)-\omega|\\ &\le C(|\Delta\varphi|_2 + |\varphi_0-\omega|_\infty),\end{aligned}$$

we obtain

$$\frac{d}{dt}\left(\nu(\omega)|u|_2^2 + |\sigma|_2^2\right) + C_2\left(|\nabla u|_2^2 + |\sigma|_2^2\right) \le C(\|H_1\|_{-1}^2 + |H_3|_2^2) + C(|\Delta\varphi|_2^2 + |\varphi_0-\omega|_\infty^2)|\sigma|_2^2. \tag{5.5}$$

By taking the derivative in time of (5.1) and (5.2), we get the same estimates for $\partial_t u$ and $\partial_t\sigma$, the only difference being principaly in the behavior of the terms

$$2\,\mathrm{div}\left(\eta'(\varphi)\frac{\partial\varphi}{\partial t}D(u)\right) \quad \text{and} \quad \nu'(\varphi)\frac{\partial\varphi}{\partial t}D(u)$$

respectively appearing when we derive the Stokes equation and the Oldroyd law. Multiplying the first term by $\partial_t u$ and integrating on $\Omega$, we get

$$\begin{aligned}\left|\int_\Omega \mathrm{div}\left(\eta'(\varphi)\frac{\partial\varphi}{\partial t}D(u)\right).\frac{\partial u}{\partial t}\right| &\le \left|\int_\Omega \eta''(\varphi)\frac{\partial\varphi}{\partial t}(\nabla\varphi.D(u)).\frac{\partial u}{\partial t}\right|\\ &\quad + \left|\int_\Omega \eta'(\varphi)\left(\frac{\partial\nabla\varphi}{\partial t}.D(u)\right).\frac{\partial u}{\partial t}\right| + \frac{1}{2}\left|\int_\Omega \eta'(\varphi)\frac{\partial\varphi}{\partial t}\Delta u.\frac{\partial u}{\partial t}\right|\\ &\le C\left(\left\|\frac{\partial\varphi}{\partial t}\right\|_2\|\nabla\varphi\|_2|D(u)|_2\left|\frac{\partial u}{\partial t}\right|_2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1\|D(u)\|_2\left|\frac{\partial u}{\partial t}\right|_2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1\|\Delta u\|_1\left|\frac{\partial u}{\partial t}\right|_2\right)\\ &\le C\left(\left\|\frac{\partial\varphi}{\partial t}\right\|_2^2\|\nabla\varphi\|_2^2 + \|u\|_1^2\left|\frac{\partial u}{\partial t}\right|_2^2 + \|u\|_3^2\left|\frac{\partial u}{\partial t}\right|_2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\left|\frac{\partial u}{\partial t}\right|_2\right).\end{aligned}$$

Concerning the second term, we multiply by $\partial_t\sigma$ and integrate on $\Omega$:

$$\left|\int_\Omega \nu'(\varphi)\frac{\partial\varphi}{\partial t}D(u):\frac{\partial\sigma}{\partial t}\right| \le C\left|\frac{\partial\varphi}{\partial t}\right|_2\|u\|_3\left|\frac{\partial\sigma}{\partial t}\right|_2 \le C\left(\left|\frac{\partial\varphi}{\partial t}\right|_2^2\left|\frac{\partial\sigma}{\partial t}\right|_2 + \|u\|_3^2\left|\frac{\partial\sigma}{\partial t}\right|_2\right).$$

We deduce:

$$\begin{aligned}\frac{d}{dt}\left(\nu(\omega)\left|\frac{\partial u}{\partial t}\right|_2^2 + \left|\frac{\partial\sigma}{\partial t}\right|_2^2\right) + C_3\left(\left\|\frac{\partial u}{\partial t}\right\|_1^2 + \left|\frac{\partial\sigma}{\partial t}\right|_2^2\right) \le C\Bigg(&\left\|\frac{\partial H_1}{\partial t}\right\|_{-1}^2 + \left|\frac{\partial H_3}{\partial t}\right|_2^2 + \left|\frac{\partial\varphi}{\partial t}\right|_2^2\left|\frac{\partial\sigma}{\partial t}\right|_2 + \|u\|_3^2\left|\frac{\partial\sigma}{\partial t}\right|_2\\
&+\left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\|\sigma\|_1^2 + \left\|\frac{\partial u}{\partial t}\right\|_1^2\|\sigma\|_2^2 + (|\Delta\varphi|_2^2 + |\varphi_0-\omega|_\infty^2)\left|\frac{\partial\sigma}{\partial t}\right|_2^2\\
&+\left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\|\nabla\varphi\|_2^2\|u\|_1^2 + \|u\|_3^2\left|\frac{\partial u}{\partial t}\right|_2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\left|\frac{\partial u}{\partial t}\right|_2\Bigg).\end{aligned} \tag{5.6}$$

**Step 3.1: Estimate for $\mathbb{P}\,\mathrm{div}\,\sigma$.** We recall that $\mathbb{P}$ is the orthogonal projection from $L^2$ on the subspace of divergence-free vector fields with vanishing normal component on the boundary. Our next task is to get an estimate for $\mathbb{P}\,\mathrm{div}\,\sigma$. For this purpose, we start out by applying the divergence operator to (5.2). We next apply the Leray projection and we deduce

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{P}\,\mathrm{div}\,\sigma + \mathbb{P}(v\cdot\nabla\,\mathrm{div}\,\sigma) + l(\omega)\mathbb{P}\,\mathrm{div}\,\sigma =& \mathbb{P}((l(\omega)-l(\varphi))\,\mathrm{div}\,\sigma) - \mathbb{P}(\nabla u:\nabla\sigma)\\
&-\mathbb{P}(l'(\varphi)\nabla\varphi.\sigma) + \mathbb{P}\,\mathrm{div}\,H_3 + \mathbb{P}(\nu'(\varphi)\nabla\varphi.D(u))\\
&+\mathbb{P}((\nu(\varphi)-\nu(\omega))\Delta u) + \nu(\omega)\mathbb{P}(\Delta u).\end{aligned}$$

Using (5.1), we obtain an estimate of $\mathbb{P}(\Delta u) = -Au$:

$$\nu(\omega)\mathbb{P}(\Delta u) = \frac{\nu(\omega)}{\eta(\omega)}\left(\frac{\partial u}{\partial t} - \mathbb{P}((\eta(\varphi)-\eta(\omega))\Delta u) - \mathbb{P}\,\mathrm{div}\,\sigma - \mathbb{P}H_1 + 2\mathbb{P}(\eta'(\varphi)\nabla\varphi.D(u))\right).$$

The equation for $\mathbb{P}\,\mathrm{div}\,\sigma$ becomes

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{P}\,\mathrm{div}\,\sigma + \mathbb{P}(v\cdot\nabla\,\mathrm{div}\,\sigma) + \left(l(\omega)+\frac{\nu(\omega)}{\eta(\omega)}\right)\mathbb{P}\,\mathrm{div}\,\sigma = \mathbb{P}((l(\omega)-l(\varphi))\,\mathrm{div}\,\sigma) + \mathbb{P}(\nabla u:\nabla\sigma)&\\
-\mathbb{P}(l'(\varphi)\nabla\varphi.\sigma) + \mathbb{P}\,\mathrm{div}\,H_3 + \mathbb{P}(\nu'(\varphi)\nabla\varphi.D(u)) + \mathbb{P}((\nu(\varphi)-\nu(\omega))\Delta u)&\\
+\frac{\nu(\omega)}{\eta(\omega)}\left(\frac{\partial u}{\partial t} - \mathbb{P}((\eta(\varphi)-\eta(\omega))\Delta u) - \mathbb{P}H_1 + 2\mathbb{P}(\eta'(\varphi)\nabla\varphi.D(u))\right).&\end{aligned} \tag{5.7}$$

**Lemma 5.1**
*We have $\mathbb{P}(v\cdot\nabla\,\mathrm{div}\,\sigma) = (v\cdot\nabla)\mathbb{P}\,\mathrm{div}\,\sigma + R$ with (see [24]):*

$$|R|_2^2 \le C\|\sigma\|_1^2|\nabla v|_\infty^2.$$

**Proof of the lemma:** We split $\mathrm{div}\,\sigma = T + \nabla q$ where $T$ is a divergence-free vector field with vanishing normal component on the boundary ($\mathbb{P}\,\mathrm{div}\,\sigma = T$). Hence

$$\mathbb{P}((v\cdot\nabla)\,\mathrm{div}\,\sigma) = \mathbb{P}((v\cdot\nabla)T) + \mathbb{P}((v\cdot\nabla)\nabla q) = (v\cdot\nabla)T - (Id-\mathbb{P})((v\cdot\nabla)T) + \mathbb{P}((v\cdot\nabla)\nabla q)$$

with the relations

$$(v\cdot\nabla)\nabla q = \nabla((v\cdot\nabla)q) - \nabla v\cdot\nabla q \quad \text{so} \quad \mathbb{P}(v\cdot\nabla)\nabla q) = -\mathbb{P}(\nabla v\cdot\nabla q),$$

$$(v\cdot\nabla)T - (T\cdot\nabla)v \in V_0 \quad \text{so} \quad (Id-\mathbb{P})((v\cdot\nabla)T) = (Id-\mathbb{P})((T\cdot\nabla)v).$$

We deduce:

$$\begin{aligned}\mathbb{P}((v\cdot\nabla)\,\mathrm{div}\,\sigma) &= (v\cdot\nabla)T - (Id-\mathbb{P})((T\cdot\nabla)v) - \mathbb{P}(\nabla v\cdot\nabla q)\\
&= (v\cdot\nabla)T - (Id-\mathbb{P})((\mathbb{P}\,\mathrm{div}\,\sigma\cdot\nabla)v) - \mathbb{P}(\nabla v.(\,\mathrm{div}\,\sigma - \mathbb{P}\,\mathrm{div}\,\sigma))\end{aligned}$$

The bound on the rest $R$ is now obvious. ■

We multipy (5.7) by $\mathbb{P}\,\mathrm{div}\,\sigma$ and we integrate. We find, using the lemma 5.1:

$$\frac{d}{dt}\Big(|\mathbb{P}\,\mathrm{div}\,\sigma|_2^2\Big) + \widetilde{C}|\mathbb{P}\,\mathrm{div}\,\sigma|_2^2 \le C\Big((\|\nabla\varphi\|_2^2 + |\varphi_0-\omega|_\infty^2)(\|\sigma\|_1^2+\|u\|_2^2) + \|v\|_3^2\|\sigma\|_1^2 + |\,\mathrm{div}\,H_3|_2^2 + \left|\frac{\partial u}{\partial t}\right|_2^2 + |H_1|_2^2\Big). \tag{5.8}$$

**Step 3.2: Control of $Au$.** Since

$$Au = \frac{1}{\eta(\omega)}\Big(-\frac{\partial u}{\partial t} + \mathbb{P}((\eta(\varphi)-\eta(\omega))\Delta u) + \mathbb{P}\,\mathrm{div}\,\sigma + \mathbb{P}H_1 - 2\mathbb{P}(\eta'(\varphi)\nabla\varphi.D(u))\Big), \tag{5.9}$$

an estimate on $|\mathbb{P}\,\mathrm{div}\,\sigma|_2$ allows us to deduce a bound on $|Au|_2$ and by Stokes regularity, a bound on $\|u\|_2$ :

$$\|u\|_2^2 \le C\Big(\left|\frac{\partial u}{\partial t}\right|_2^2 + (|\Delta\varphi|_2^2 + |\varphi_0-\omega|_\infty^2)\|u\|_2^2 + |\mathbb{P}\,\mathrm{div}\,\sigma|_2^2 + |H_1|_2^2 + \|\nabla\varphi\|_2^2\|u\|_1^2\Big). \tag{5.10}$$

**Step 3.3: Estimate on $\sigma$.** Taking the inner product $(\nabla(5.2), \nabla\sigma)$ we find

$$\frac{d}{dt}\Big(\frac{|\nabla\sigma|_2^2}{2}\Big) + l_1|\nabla\sigma|_2^2 \le C(|\nabla v|_\infty^2|\nabla\sigma|_2^2 + |\nabla\varphi|_\infty^2|\sigma|_2^2 + |\nabla\varphi|_\infty^2|\nabla u|_2^2 + |\nabla H_3|_2^2) + C\|u\|_2^2.$$

Using the estimate (5.10), we obtain

$$\frac{d}{dt}\Big(|\nabla\sigma|_2^2\Big) + 2l_1|\nabla\sigma|_2^2 \le C(|\nabla v|_\infty^2|\nabla\sigma|_2^2 + |\nabla\varphi|_\infty^2|\sigma|_2^2 + |\nabla\varphi|_\infty^2|\nabla u|_2^2 + |\nabla H_3|_2^2) + C\Big(\left|\frac{\partial u}{\partial t}\right|_2^2 + (|\Delta\varphi|_2^2 + |\varphi_0-\omega|_\infty^2)\|u\|_2^2 + |H_1|_2^2 + \|\nabla\varphi\|_2^2\|u\|_1^2\Big) + \widehat{C}|\mathbb{P}\,\mathrm{div}\,\sigma|_2^2. \tag{5.11}$$

We can combine estimates to cancel the term $\widehat{C}|\mathbb{P}\,\mathrm{div}\,\sigma|_2^2$ in the right-hand side : the sum $2\widehat{C}(5.8) + \widetilde{C}(5.11)$ gives

$$\frac{d}{dt}\Big(|\nabla\sigma|_2^2 + |\mathbb{P}\,\mathrm{div}\,\sigma|_2^2\Big) + C\Big(|\nabla\sigma|_2^2 + |\mathbb{P}\,\mathrm{div}\,\sigma|_2^2\Big) \le C\Big(\|v\|_3^2\|\sigma\|_1^2 + \|H_3\|_1^2 + |H_1|_2^2 + (\|\nabla\varphi\|_2^2 + |\varphi_0-\omega|_\infty^2)(\|\sigma\|_1^2 + \|u\|_2^2)\Big) + C\left|\frac{\partial u}{\partial t}\right|_2^2. \tag{5.12}$$

**Step 4: More estimates...** Using arguments similar to those presented in the previous step, we first apply $\nabla\mathbb{P}\,\mathrm{div}$ to (5.2), using $\nabla\mathbb{P}(5.1)$ we deduce

$$\frac{d}{dt}\Big(\frac{|\nabla\mathbb{P}\,\mathrm{div}\,\sigma|_2^2}{2}\Big) + \Big(l(\omega) + \frac{\nu(\omega)}{\eta(\omega)}\Big)|\nabla\mathbb{P}\,\mathrm{div}\,\sigma|_2^2 \le C\Big((\|\nabla\varphi\|_2^2 + |\varphi_0-\omega|_\infty^2)(\|\sigma\|_2^2 + \|u\|_3^2) + \|\nabla\varphi\|_2^4(|\sigma|_2^2 + \|u\|_1^2) + \|v\|_3^2\|\sigma\|_2^2 + |\nabla\,\mathrm{div}\,H_3|_2^2 + \left|\frac{\partial\nabla u}{\partial t}\right|_2^2 + |\nabla H_1|_2^2\Big).$$

**Remark 5.1**
*During this estimate, we only know that $\sigma \in L^\infty(H^2)$. It is thus not allowe a priori to make the computation $\nabla\mathbb{P}\,div\,(u\cdot\nabla\sigma)$. We can justify this by regularisation just like we do it during the proof of the Oldroyd lemma, section 3.3.1.*

Next, by the Stokes operator regularity properties we obtain a bound for $\|u\|_3$ (using (5.9)):

$$\|u\|_3^2 \le C_0\Big(\left\|\frac{\partial u}{\partial t}\right\|_1^2 + |\nabla\mathbb{P}\,\mathrm{div}\,\sigma|_2^2\Big) + C((\|\nabla\varphi\|_2^2 + |\varphi_0-\omega|_\infty^2)\|u\|_3^2 + |\nabla H_1|_2^2 + \|\nabla\varphi\|_2^4\|u\|_1^2). \tag{5.13}$$

Computing $(\nabla\nabla(5.2), \nabla\nabla\sigma)$, we have

$$\frac{d}{dt}\left(|\nabla\nabla\sigma|_2^2\right) + 2l_1|\nabla\nabla\sigma|_2^2 \leq C(|\nabla v|_\infty^2\|\nabla\sigma\|_1^2 + |\nabla\varphi|_\infty^2|\nabla\sigma|_2^2 + |\nabla\varphi|_\infty^4|\sigma|_2^2 + |\nabla\varphi|_\infty^2\|u\|_3^2 + |\nabla H_3|_2^2)$$
$$+ C\left(\left\|\frac{\partial u}{\partial t}\right\|_1^2 + \|\nabla\varphi\|_2^2\|u\|_3^2 + |\nabla H_1|_2^2 + |\nabla\varphi|_\infty^4\|u\|_1^2\right) + C|\mathbb{P}\nabla \operatorname{div}\sigma|_2^2,$$

that is combined with the estimate for $|\nabla\mathbb{P}\operatorname{div}\sigma|_2$ gives

$$\frac{d}{dt}\left(|\nabla\nabla\sigma|_2^2 + |\nabla\mathbb{P}\operatorname{div}\sigma|_2^2\right) + C\left(|\nabla\nabla\sigma|_2^2 + |\nabla\mathbb{P}\operatorname{div}\sigma|_2^2\right) \leq C\Big(\|H_3\|_2^2 + \|H_1\|_1^2 + \|v\|_3^2\|\sigma\|_2^2$$
$$+ (\|\nabla\varphi\|_2^2 + |\varphi_0 - \omega|_\infty^2)(\|\sigma\|_2^2 + \|u\|_3^2) + \|\nabla\varphi\|_2^4(|\sigma|_2^2 + \|u\|_1^2)\Big) + C\left\|\frac{\partial u}{\partial t}\right\|_1^2. \tag{5.14}$$

Adding (5.12) and (5.14) we find

$$\frac{d}{dt}\left(\|\nabla\sigma\|_1^2 + \|\mathbb{P}\operatorname{div}\sigma\|_1^2\right) + C_4\left(\|\nabla\sigma\|_1^2 + \|\mathbb{P}\operatorname{div}\sigma\|_1^2\right) \leq C\Big(\|H_3\|_2^2 + \|H_1\|_1^2 + \|v\|_3^2\|\sigma\|_2^2$$
$$+ (\|\nabla\varphi\|_2^2 + |\varphi_0 - \omega|_\infty^2)(\|\sigma\|_2^2 + \|u\|_3^2) + \|\nabla\varphi\|_2^4(|\sigma|_2^2 + \|u\|_1^2)\Big) + C_5\left\|\frac{\partial u}{\partial t}\right\|_1^2. \tag{5.15}$$

**Step 5: Estimates on the Cahn-Hilliard equation.** The equation (5.3) was already studied in the lemma 3.2. In the present case, we can use the fact that the potential we consider is convex (that is $F_2(m(\varphi_0)) = 0$). Moreover, the potential and all its derivate being bounded, the estimate (3.16) write

$$\frac{d}{dt}\left(\|\nabla\varphi\|_3^2 + \frac{2}{\alpha^2}\int_\Omega F(\varphi)\right) + C\left(\|\nabla\varphi\|_5^2 + \frac{2}{\alpha^2}\int_\Omega F(\varphi) + \left\|\frac{\partial\varphi}{\partial t}\right\|_2^2\right) \leq C(\|\nabla\varphi\|_3^4 + \|\nabla\varphi\|_3^8 + \|\operatorname{div} H_2\|_2^2). \tag{5.16}$$

**Step 6: Estimates on source terms.** Now it is sufficient to estimate the norm of $H_1$, $H_2$ and $H_3$. We have easily

$$\|H_1\|_1^2 \leq C(\|u\|_1^2\|u\|_3^2 + \|\nabla\varphi\|_2^4 + \|u\|_2^2\|h\|_2^2 + \|h\|_2^4 + (1 + \|\nabla\varphi\|_1^4)\|h\|_3^2),$$
$$\|\operatorname{div} H_2\|_2^2 \leq C(\|u\|_2^2\|\nabla\varphi\|_2^2 + \|h\|_2^2\|\nabla\varphi\|_2^2),$$
$$\|H_3\|_2^2 \leq C(\|\sigma\|_2^2\|u\|_3^2 + \|\sigma\|_2^2\|h\|_3^2 + (1 + \|\nabla\varphi\|_1^4)\|h\|_3^2),$$

and for the time derivatives

$$\left\|\frac{\partial H_1}{\partial t}\right\|_{-1}^2 \leq C\left(\left|\frac{\partial u}{\partial t}\right|_2^2\|u\|_2^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\|\nabla\varphi\|_2^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\|h\|_2^2 + \left\|\frac{\partial u}{\partial t}\right\|_1^2\|h\|_1^2\right),$$
$$\left|\frac{\partial H_3}{\partial t}\right|_2^2 \leq C\left(\left|\frac{\partial\sigma}{\partial t}\right|_2^2\|u\|_3^2 + \left|\frac{\partial\sigma}{\partial t}\right|_2^2\|h\|_3^2 + \left\|\frac{\partial u}{\partial t}\right\|_1^2\|\sigma\|_2^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|_1^2\|h\|_2^2\right).$$

**Step 7: Conclusion.** From the computation

$$\frac{C_4}{2C_1}\times(5.4) + (5.5) + \frac{C_4 + C_5}{C_3}\times(5.6) + \frac{C_4}{2C_0}\times(5.13) + (5.15) + (5.16)$$

and from the bounds on the terms $H_i$, we can write

$$y' + C\ z \leq\ z\ f(y) + C_{11}\ z + C_{12}, \quad z \geq y \geq 0$$

with

$$y = \|u\|_1^2 + \|\sigma\|_2^2 + |\partial_t u|_2^2 + |\partial_t\sigma|_2^2 + \|\mathbb{P}\operatorname{div}\sigma\|_1^2 + \|\nabla\varphi\|_3^2 + \int_\Omega F(\varphi),$$
$$z = \|u\|_3^2 + \|\sigma\|_2^2 + \|\partial_t u\|_1^2 + |\partial_t\sigma|_2^2 + \|\mathbb{P}\operatorname{div}\sigma\|_1^2 + \|\nabla\varphi\|_5^2 + \|\partial_t\varphi\|_2^2 + \int_\Omega F(\varphi),$$
$$C_{11} = \|h\|_3^2 + |\varphi_0 - \omega|_\infty^2 \quad \text{and} \quad C_{12} = \|h\|_3^2 + \|h\|_2^4.$$

The function $f$ is continuous and vanishes in 0 (in fact the product $zf(y)$ contains all the nonlinear terms).

The following lemma prove that if $y(0)$, $C_{11}$ and $C_{12}$ are small enough then $y$ remains small on the interval $[0, T^*]$. Consequently $y(T^*)$ is small and we can apply again the existence theorem 2.1: we find a solution in $[T^*, 2T^*]$. We can repeat this argument in each interval $[0, nT^*]$, $n \in \mathbb{N}$. We proved that

$$u \in L^\infty(\mathbb{R}^+; V_1) \cap L^2_{loc}(\mathbb{R}^+; V_1 \cap H^3), \quad \frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; V_0) \cap L^2_{loc}(\mathbb{R}^+; V_1),$$
$$\varphi \in L^\infty(\mathbb{R}^+; \Phi_4) \cap L^2_{loc}(\mathbb{R}^+; \Phi_6), \quad \frac{\partial \varphi}{\partial t} \in L^2_{loc}(\mathbb{R}^+; \Phi_2),$$
$$\sigma \in L^\infty(\mathbb{R}^+; H^2), \quad \frac{\partial \sigma}{\partial t} \in L^\infty(\mathbb{R}^+; L^2).$$

It remains to be shown that

$$u \in L^\infty(\mathbb{R}^+; V_1 \cap H^2), \quad \frac{\partial \varphi}{\partial t} \in L^\infty(\mathbb{R}^+; \Phi_0), \quad \text{and} \quad \frac{\partial \sigma}{\partial t} \in L^\infty(\mathbb{R}^+; H^1).$$

To prove that $u \in L^\infty(\mathbb{R}^+; V_1 \cap H^2)$, we proceed like in the proof of the lemma 3.1: We first write

$$\nabla p = -\alpha^2 \Delta\widetilde{\varphi} \nabla\widetilde{\varphi} + \operatorname{div} \widetilde{\sigma} - \widetilde{v} \cdot \nabla\widetilde{v} + 2 \operatorname{div} (\widetilde{\eta} D(v)) - \frac{\partial u}{\partial t}$$

which implies $p \in L^\infty(\mathbb{R}^+; L^2/\mathbb{R})$. Next, we rewrite the same equation to make appear the Stokes operator $Au$, and we obtain the desired result (see the last part of 3.1).

Using the estimate (3.17) with $\operatorname{div} G_2 = v \cdot \nabla\varphi$ it's easy to conclude for the quantity $\partial_t \varphi$. In the same way, the estimate (3.20) with $G_3 = \nu(\varphi) D(v)$ allows to obtain the result for $\partial_t \sigma$. ■

**Lemma 5.2**
*Let $y$ and $z$ be two $\mathcal{C}^1$ functions defined on $[0, T]$ and satisfying*

$$y' + \alpha\, z \leq \; z\, f(y) + \beta, \quad 0 \leq y \leq z.$$

*Assume that $f$ is continuous and $f(0) = 0$, $\alpha$ and $\beta$ being positive constants. Let $M$ be a positive real such that $2f(M) < \alpha$. If $\beta < M\alpha/2$ then*

$$y(0) < M \quad \Longrightarrow \quad y < M \text{ on } [0, T]. \tag{5.17}$$

**Proof** : Suppose that (5.17) is not true, and set $\overline{T} = \inf\{t \in [0, T],\ y(t) > M\}$. Clearly we have $y(\overline{T}) = M$ and $y'(\overline{T}) \geq 0$. Moreover, using the differential inequation evaluated in $\overline{T}$:

$$y'(\overline{T}) \leq z(\overline{T})(f(M) - \alpha) + \beta.$$

From $z(\overline{T}) \geq y(\overline{T}) = M$ and from the choice for $M$ we have $y'(\overline{T}) < 0$. This gives a contradiction. ■

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