Equilibrium analysis for a mass-conserving model in presence of cavitation

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Abstract

We study the existence of equilibrium positions for the load problem in Lubrication Theory. The problem consists of two surfaces in relative motion separated by a small distance filled by a lubricant. The system is described by the modified Reynolds equation (Elrod-Adams model) which describes the behavior of the lubricant and an extra integral equation given the balance of forces. The balance of forces allows to obtain the unknown position of the surfaces, defined with one degree of freedom.

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1 Introduction and problem setting

Lubrication is the process used in mechanical systems to carry the load between two surfaces in relative motion and close proximity. The narrow space between the surfaces is filled by the lubricant, its characteristics allow to avoid the direct contact between the surfaces and reduce the wear. In that case, when distance is strictly positive we say that the system is in Hydrodynamic regime. The force induced by the pressure of the fluid is developed by the relative motion of the surfaces and it depends on the geometry of the space filled by the lubricant.

For simplicity, we assume that the bottom surface is planar and moves with a constant horizontal translation velocity. The lubricant is assumed incompressible and the distance between the surfaces belongs to the range of admissible distances satisfying the thin-film hypothesis, and therefore the pressure fluid does not depend on the vertical coordinate.

Let us denote by Ω the two-dimensional domain in which the hydrodynamical contact occurs. We suppose, for simplicity, that $\Omega =]0,1[^2$ and the boundary $\partial\Omega$ is split into two parts: Γ_0 (defined by $\{x_1 = 0\}$) and the rest of the boundary, denoted by $\partial\Omega - \Gamma_0$. We also assume that the fluid flux at Γ_0 is a given constant $\mu > 0$. Without loss of generality we may assume that the velocity of the bottom surface is oriented in the direction of the x_1 - axis and its normalized value is equal to 1.

In order to take into account the cavitation in the fluid we introduce the so-called Elrod-Adams model in the stationary case. The problem consists of finding p (the pressure of the lubricant) and θ (the volume fraction occupied by the fluid), solution of the following system (see for example [9] and [1]):

$$\left\{\begin{array}{l} \nabla \cdot [h^3(x)\nabla p] = \frac{\partial(\theta h)}{\partial x_1} \quad x \in \Omega \\ \theta \in H(p), \quad p \geq 0 \\ p = 0 \quad x \in \partial\Omega - \Gamma_0 \\ h\theta - h^3 \frac{\partial p}{\partial x_1} = \mu \quad \text{on } \Gamma_0 \end{array}\right. \tag{1.1}$$

where h is the non-dimensional distance (the gap) between the surfaces and H is the Heaviside multivalued function defined by

$$H(p) = \left\{\begin{array}{cc} 1 & p > 0 \\ [0,1] & p = 0 \\ 0 & p < 0. \end{array}\right.$$

In many lubricated systems, the position of the surface is unknown and h may present some degrees of freedom. We reduce our study to the case where h will be given up to one degree of freedom which is the vertical translation, which results as a equilibrium position between the hydrodynamic force $\int_\Omega p(x)\, dx$ and the known exterior force F (assumed constant) applied upon the upper surface. Then, we assume that h is defined as follows

$$h(x) = h_0(x_1) + a \tag{1.2}$$

where $a > 0$ accounts for the vertical translation and $h_0 : [0,1] \to [0,\infty[$ is a given regular non-negative function which represents the gap corresponding to $a = 0$ and defines the geometry of the space. For simplicity we assume that the surface is rigid (i.e. h_0 is independent of the forces applied), depends only on x_1 and is $C^1([0,1])$ function. We also suppose that $\min_{x_1 \in [0,1]} h_0(x_1) = 0$, which allows to say that a represents the minimum distance between the two surfaces.

We are interested in the equilibrium positions of the system, which are defined as the stationary solutions of the equation defined by the second Newton′s Law. Then, for a given constant force F, the problem consists in finding $a > 0$ such that,

$$\int_\Omega p dx = F \tag{1.3}$$

with h is defined in (1.2) and (p, θ) is a solution of (1.1).

The problem for a general $h_0 \in C^1(\Omega)$ presents a high complexity and non-existence of solutions may occur for particular shapes h_0 and some exterior forces. Therefore we should restrict the study to the case where the contact region in the limit case ($a = 0$) satisfies the following assumptions: There exist $\gamma \in]0, 1[$, $\alpha > 1$ and m_1, m_2 two positive constants such that

$$h_0(\gamma) = 0 \tag{1.4}$$

$$h_0'(x_1) < 0 \quad \text{for } x_1 \in [0, \gamma] \tag{1.5}$$

$$h_0'(x_1) > 0 \quad \text{for } x_1 \in [\gamma, 1] \tag{1.6}$$

$$h_0 \text{ is convex on } [0, \gamma] \tag{1.7}$$

$$m_1 |x_1 - \gamma|^\alpha \leq h_0(x_1) \leq m_2 |x_1 - \gamma|^\alpha \quad \text{for } x_1 \in]0, 1[\tag{1.8}$$

$$m_1 |x_1 - \gamma|^{\alpha-1} \leq |h_0'(x_1)| \leq m_2 |x_1 - \gamma|^{\alpha-1} \quad \text{for } x_1 \in]0, 1[\tag{1.9}$$

The weak formulation of problem (1.1) consists of finding (p, θ) in $V^+ \times L^\infty(\Omega)$ such that

$$\left\{ \begin{array}{l} \int_\Omega h^3 \nabla p \cdot \nabla \varphi dx = \int_\Omega h\theta \frac{\partial \varphi}{\partial x_1} dx + \int_{\Gamma_0} \mu \varphi d\sigma \quad \forall \varphi \in V \\ \theta \in H(p) \end{array} \right. \tag{1.10}$$

where

$$V = \left\{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \partial\Omega - \Gamma_0 \right\} \text{ and } V^+ = \{\varphi \in V : \varphi \geq 0\}.$$

The main result of this paper is the following

Theorem 1.1. Under assumptions (1.4)-(1.9), there exists at least one solution $(p, \theta, a) \in V^+ \times L^\infty(\Omega) \times]0, +\infty[$ to (1.10),(1.2) and (1.3).

Remark. We notice that if $a \geq \mu$, a solution to (1.1) or (1.10) is $p = 0$ and $\theta(x) = \frac{\mu}{h_0(x)+a}$. Then, if the external force F is zero there exist infinity many solutions of the coupled problem (1.10), (1.2), (1.3) defined by $(0, \frac{\mu}{h_0(x)+a}, a)$ for any $a \geq \mu$.
In this paper we focus on the case $F > 0$ which is more relevant from a physical point of view.

Up to our knowledge the equilibrium problem in lubrication with the use of the Elrod-Adams model from a theoretical point of view was not considered before. The existing literature in numerical simulations and applications has been growing in the last forty years, nevertheless a deep mathematical study of the existence of equilibrium positions has not been accomplished. Uniqueness /

multiplicity of solutions are not studied in the paper, the existing techniques to prove uniqueness of inverse problems can not be applied directly and new ideas should be introduced to obtain such results. There exist in the literature other theoretical studies of equilibrium problems in lubrication using different models as Reynolds equation (see for example [11], [4]) or Reynolds inequality (see [5] or [8]).

The content of the paper is the following: in Section 2 we consider a regularization of the problem and we prove the existence of at least a solution of this regularized problem, as well as appropriate estimations. In Section 3 we prove the main result by passing to the limit and in Section 4 we give some numerical simulations.

2 The regularized problem

2.1 Setting of the regularized problem

In this section we study a regularized problem of (1.1) with h given by (1.2), obtained by replacing in (1.1) the Heaviside function H by a regular approximation H_ϵ defined by

$$H_\epsilon(z) = \begin{cases} 1 & z \geq \epsilon \\ \frac{z}{\epsilon} & 0 \leq z \leq \epsilon \\ 0 & z \leq 0. \end{cases}$$

The regularized problem is as follows

$$\begin{cases} \nabla \cdot [h^3(x) \nabla p_{\epsilon,a}] = \frac{\partial [h H_\epsilon(p_{\epsilon,a})]}{\partial x_1} \quad \text{in } \Omega, \\ p_{\epsilon,a} = 0 \quad \text{in } \partial\Omega - \Gamma_0, \\ h H_\epsilon(p_{\epsilon,a}) - h^3 \frac{\partial p_{\epsilon,a}}{\partial x_1} = \mu \quad \text{on } \Gamma_0. \end{cases} \tag{2.1}$$

The variational formulation of the problem consists of finding $p_{\epsilon,a} \in V$ such that

$$\int_\Omega h^3 \nabla p_{\epsilon,a} \cdot \nabla \varphi dx = \int_\Omega h H_\epsilon(p_{\epsilon,a}) \frac{\partial \varphi}{\partial x_1} dx + \int_{\Gamma_0} \mu \varphi d\sigma \quad \forall \varphi \in V. \tag{2.2}$$

For any given $a > 0$ it is well known that the problem (2.2) has a unique solution (see Bayada-Vázquez [2]) where the main ideas of the proofs are summarized for the journal bearing problem (see Bayada-Martin-Vázquez [3] and the references therein for more details). Since the proof for the slide is similar to the journal-bearing system, we omit the details. We also have for $\epsilon \to 0$

$$p_{\epsilon,a} \rightharpoonup p_a \qquad \text{weakly in } V \tag{2.3}$$

$$H_\epsilon(p_{\epsilon,a}) \rightharpoonup \theta_a \qquad \text{weakly star in } L^\infty(\Omega) \tag{2.4}$$

where (p_a, θ_a) is a solution of (1.10).

The weak formulation of the regularized coupled problem (1.10), (1.2), (1.3) consists of finding p_ϵ solution of (2.2) and $a_\epsilon > 0$ such that

$$\int_\Omega h_\epsilon^3 \nabla p_\epsilon \cdot \nabla \varphi dx = \int_\Omega h_\epsilon H_\epsilon(p_\epsilon) \frac{\partial \varphi}{\partial x_1} dx + \int_{\Gamma_0} \mu \varphi d\sigma \quad \forall \varphi \in V \tag{2.5}$$

$$\int_{\Omega} p_{\epsilon} dx = F \tag{2.6}$$

$$h_{\epsilon}(x) = h_0(x_1) + a_{\epsilon}. \tag{2.7}$$

In order to prove the existence of a solution of the coupled problem (2.5), (2.6), (2.7) we re-write the problem as an scalar problem with only one unknown. To reduce it, we introduce the function

$$g_{\epsilon} : a \in]0, +\infty[\to g_{\epsilon}(a) = \int_{\Omega} p_{\epsilon,a}(x) dx$$

where $p_{\epsilon,a}$ is the unique solution of (2.2) with h defined by (1.2).
Then, (2.5), (2.6), (2.7) is described as follows:
Find $a_{\epsilon} > 0$ such that

$$g_{\epsilon}(a_{\epsilon}) = F. \tag{2.8}$$

Since g_{ϵ} is a continuous function, we just need to obtain large and small values of g_{ϵ} and apply the intermediate value theorem to prove the existence of at least one solution of (2.8).
The most difficult part of the proof is to obtain a large value of g_{ϵ} and it can only be found for a small enough. So, the crucial step of the result is to prove that

$$\limsup_{a \to 0} g_{\epsilon}(a) > F. \tag{2.9}$$

The idea is to bound from below the solution p_{ϵ} by a subsolution of the problem which is large enough when a goes to zero.

Definition 2.1. We say that $q_{\epsilon} \in V$ is a subsolution of (2.2) if it satisfies:

$$\int_{\Omega} h^3 \nabla q_{\epsilon} \cdot \nabla \varphi dx \leq \int_{\Omega} h H_{\epsilon}(q_{\epsilon}) \frac{\partial \varphi}{\partial x_1} dx + \int_{\Gamma_0} \mu \varphi d\sigma \quad \forall \varphi \in V^+. \tag{2.10}$$

The next lemma gives a comparison principle for the solutions of (2.2).

Lemma 2.2. Let p_{ϵ} be a solution of (2.2) and q_{ϵ} a subsolution. Then $p_{\epsilon} \geq q_{\epsilon}$ a.e. on Ω.

Proof. We adapt a technique of Gilbarg and Trudinger [10] to non-linear problems (see also [6]).
We set $w = q_{\epsilon} - p_{\epsilon}$ and our goal is to prove that $w^+ = 0$.
Substracting (2.2) from (2.10) we obtain

$$\int_{\Omega} h^3 \nabla w \cdot \nabla \varphi dx \leq \int_{\Omega} h \left(H_{\epsilon}(q_{\epsilon}) - H_{\epsilon}(p_{\epsilon}) \right) \frac{\partial \varphi}{\partial x_1} dx \leq c_{\epsilon} \int_{\Omega} |w| \cdot \left| \frac{\partial \varphi}{\partial x_1} \right| \quad \forall \varphi \in V^+.$$

Now for any $\delta > 0$ we take $\varphi = \frac{w^+}{w^+ + \delta}$ and the end of the proof is like in Gilbarg and Trudinger [10]. $\square$

2.2 Construction of a subsolution

In order to construct a subsolution we split the domain into several subdomains, we consider first the following subdomains:

$$\Omega_\ell =]0,\gamma[\times]0,1[\quad \text{and} \quad \Omega_a = \left]\beta, \gamma - a^{1/\alpha}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[$$

where $\beta \in]0,\gamma[$ is a constant to be fixed later and $a \in]0,(\gamma-\beta)^\alpha[$.
Let us denote by $R_{\epsilon,a} : \Omega_a \to \mathbb{R}$ the solution of the Reynolds equation

$$\begin{cases} \nabla \cdot [h^3 \nabla R_{\epsilon,a}] = \frac{\partial h}{\partial x_1} & \text{in } \Omega_a \\ R_{\epsilon,a} = \epsilon & \text{in } \partial\Omega_a. \end{cases} \tag{2.11}$$

It is clear that $R_{\epsilon,a} = R_a + \epsilon$ where R_a is the unique solution of

$$\begin{cases} \nabla \cdot [h^3 \nabla R_a] = \frac{\partial h}{\partial x_1} & \text{in } \Omega_a \\ R_a = 0 & \text{in } \partial\Omega_a. \end{cases} \tag{2.12}$$

Since $\frac{\partial h}{\partial x_1} \leq 0$ on Ω_a and thanks to maximum principle we have that $R_a \geq 0$, which implies $R_{\epsilon,a} \geq \epsilon$ on Ω_a.

In the following lemma we construct a subsolution to (2.2).

Lemma 2.3. Let $\xi_\epsilon \in H^2(\Omega_\ell - \Omega_a)$ be such that

$$\xi_\epsilon = R_{\epsilon,a} = \epsilon \qquad \text{on } \partial\Omega_a \tag{2.13}$$

$$\xi_\epsilon = 0 \qquad \text{on } \partial\Omega_\ell - \Gamma_0 \tag{2.14}$$

$$\frac{\partial}{\partial x_1}\left(hH_\epsilon(\xi_\epsilon)\right) - \nabla \cdot (h^3 \nabla \xi_\epsilon) \leq 0 \qquad \text{on } \Omega_\ell - \Omega_a \tag{2.15}$$

$$hH_\epsilon(\xi_\epsilon) - h^3 \frac{\partial \xi_\epsilon}{\partial x_1} \leq \mu \qquad \text{on } \Gamma_0 \tag{2.16}$$

$$h^3 \frac{\partial R_{\epsilon,a}}{\partial \nu} \leq h^3 \frac{\partial \xi_\epsilon}{\partial \nu} \qquad \text{on } \partial\Omega_a \tag{2.17}$$

$$h^3 \frac{\partial \xi_\epsilon}{\partial x_1} \leq 0 \qquad \text{on } \{x_1 = \gamma\} \tag{2.18}$$

where ν denotes the unitary exterior normal to $\partial\Omega_a$.
Then, $q_\epsilon : \Omega \to \mathbb{R}$ defined by

$$q_\epsilon = \begin{cases} R_{\epsilon,a} & \text{on } \Omega_a \\ \xi_\epsilon & \text{on } \Omega_\ell - \Omega_a \\ 0 & \text{on } \Omega - \Omega_\ell \end{cases} \tag{2.19}$$

is a subsolution of (2.2).

Proof. It is clear that q_ϵ is an element of V. For any $\varphi \in V^+$ we have

$$\int_\Omega h^3 \nabla q_\epsilon \cdot \nabla \varphi dx - \int_\Omega h H_\epsilon(q_\epsilon) \frac{\partial \varphi}{\partial x_1} dx - \int_{\Gamma_0} \mu \varphi d\sigma = \int_{\Omega_a} h^3 \nabla R_{\epsilon,a} \cdot \nabla \varphi + \int_{\Omega_\ell - \Omega_a} h^3 \nabla \xi_\epsilon \cdot \nabla \varphi - \int_{\Omega_a} h \frac{\partial \varphi}{\partial x_1} - \int_{\Omega_\ell - \Omega_a} h H_\epsilon(q_\epsilon) \frac{\partial \varphi}{\partial x_1} - \int_{\Gamma_0} \mu \varphi d\sigma$$

since $H_\epsilon(R_{\epsilon,a}) = 1$.
Now we have

$$\int_{\Omega_a} h^3 \nabla R_{\epsilon,a} \cdot \nabla \varphi - \int_{\Omega_a} h \frac{\partial \varphi}{\partial x_1} = \int_{\partial \Omega_a} h^3 \frac{\partial R_{\epsilon,a}}{\partial \nu} \varphi - \int_{\partial \Omega_a} h \nu_1 \varphi \tag{2.20}$$

and also

$$\begin{aligned} &\int_{\Omega_\ell - \Omega_a} h^3 \nabla \xi_\epsilon \cdot \nabla \varphi - \int_{\Omega_\ell - \Omega_a} h H_\epsilon(\xi_\epsilon) \frac{\partial \varphi}{\partial x_1} \\ &= \int_{\Omega_\ell - \Omega_a} \left\{ \frac{\partial}{\partial x_1} \left[h H_\epsilon(\xi_\epsilon) \right] - \nabla \cdot \left[h^3 \nabla \xi_\epsilon \right] \right\} \varphi - \int_{\partial \Omega_a} h^3 \frac{\partial \xi_\epsilon}{\partial \nu} \varphi \\ &+ \int_{\partial \Omega_a} h H_\epsilon(\xi_\epsilon) \nu_1 \varphi - \int_{\Gamma_0} h^3 \frac{\partial \xi_\epsilon}{\partial x_1} \varphi + \int_{\Gamma_0} h H_\epsilon(\xi_\epsilon) \varphi \\ &+ \int_{\{x_1 = \gamma\}} h^3 \frac{\partial \xi_\epsilon}{\partial x_1} \varphi - \int_{\{x_1 = \gamma\}} h H_\epsilon(\xi_\epsilon) \varphi \end{aligned} \tag{2.21}$$

Adding (2.20) and (2.21) and using the hypothesis of the lemma we obtain the result. □

We precise the choice of β in the definition of Ω_a: we choose $\beta \in]0, \gamma[$ such that

$$h_0(\beta) - \beta h_0'(\beta) < \mu \tag{2.22}$$

Such β clearly exists by continuity of h_0 and h_0' since $h_0(\gamma) = h_0'(\gamma) = 0$.
Then there exists $a_0 > 0$ small enough such that

$$h(\beta) - \beta h'(\beta) < \mu, \quad \text{for any } a \in]0, a_0[. \tag{2.23}$$

Then, we may assume that

$$a_0 < \min \left\{ (\gamma - \beta)^\alpha, \left(\frac{\gamma}{2} \right)^\alpha \right\}. \tag{2.24}$$

Now we introduce the auxiliary function $q_{1a} : [0, \gamma] \to \mathbb{R}$ defined by

$$q_{1a}(x_1) = \begin{cases} \frac{h(\beta) + h'(\beta)(x_1 - \beta)}{h(x_1)} & \text{for } 0 \le x_1 \le \beta \\ 1 & \text{for } \beta \le x_1 \le \gamma - a^{1/\alpha} \\ \sin\left(\frac{\pi}{2a^{1/\alpha}} (\gamma - x_1) \right) & \text{for } \gamma - a^{1/\alpha} \le x_1 \le \gamma. \end{cases}$$

Since $q_{1a}'(\beta) = 0$ then $q_{1a} \in H^2(0, \gamma)$ with

$$q_{1a}(\gamma) = 0 \tag{2.25}$$

It is clear that the function q_{1a} is non-negative and from the convexity of h on $]0,\gamma[$ we also deduce

$$0 \leq q_{1a} \leq 1 \quad \text{on } [0,\gamma]. \tag{2.26}$$

We also consider $q_2 : [0,1] \to \mathbb{R}$ defined as follows

$$q_2(x_2) = \begin{cases} \sin(2\pi x_2) & \text{for } 0 \leq x_2 \leq \frac{1}{4} \\ 1 & \text{for } \frac{1}{4} \leq x_2 \leq \frac{3}{4} \\ \sin(2\pi(1-x_2)) & \text{for } \frac{3}{4} \leq x_2 \leq 1. \end{cases}$$

It is clear that $q_2 \in H^2(0,1) \cap H_0^1(0,1)$ and

$$0 \leq q_2 \leq 1 \quad \text{on } [0,1] \tag{2.27}$$

Let us now introduce the functions

$$\xi_{\epsilon,a} : \Omega_\ell \to \mathbb{R} \text{ defined by } \xi_{\epsilon,a}(x_1,x_2) = \epsilon q_{1a}(x_1) q_2(x_2)$$

and $q_{\epsilon,a} : \Omega_\ell \to \mathbb{R}$ defined by

$$q_{\epsilon,a} = \begin{cases} R_{\epsilon,a} & \text{on } \Omega_a \\ \xi_{\epsilon,a} & \text{on } \Omega_\ell - \Omega_a \\ 0 & \text{on } \Omega - \Omega_\ell \end{cases} \tag{2.28}$$

where $R_{\epsilon,a}$ is the solution of (2.11).
Observe that $\xi_{\epsilon,a} = \epsilon$ on $\partial\Omega_a$ and $\xi_{\epsilon,a} = 0$ on $\partial\Omega_\ell - \Gamma_0$.

Lemma 2.4. Let a_0, β given by (2.22)–(2.24). Then for any $a \in]0,a_0]$ there exists $\epsilon_0 = \epsilon_0(a)$ such that for any $\epsilon \in]0,\epsilon_0[$ the function $q_{\epsilon,a}$ given by (2.28) is a subsolution of the problem (2.2).

Proof. We prove that the hypothesis of Lemma 2.3 are verified with $\xi_\epsilon = \xi_{\epsilon,a}$. It is clear that (2.13),(2.14) and (2.18) are satisfied.
Since $R_{\epsilon,\delta} \geq \epsilon$ on Ω_a then $\frac{\partial R_{\epsilon,\delta}}{\partial \nu} \leq 0$ on $\partial\Omega_a$ which implies (2.17) since $\frac{\partial \xi_{\epsilon,a}}{\partial \nu} = 0$ on $\partial\Omega_a$. On the other hand, from (2.26)-(2.27) we deduce that

$$H_\epsilon(\xi_{\epsilon,a}) = q_{1a}(x_1) q_2(x_2) \quad \text{on } \Omega_\ell - \Omega_a$$

so the inequality (2.16) is equivalent to

$$\left[h(\beta) - \beta h'(\beta) - \epsilon h^3(0) q'_{1a}(0)\right] q_2(x_2) \leq \mu \quad \forall x_2 \in [0,1]$$

Since $q'_{1a}(0)$ is independent of ϵ, from (2.23) and (2.27) we deduce that (2.16) is verified for $\epsilon > 0$ small enough depending on a.
It remains to prove (2.15) which is equivalent to

$$\begin{aligned} &q_2(x_2)\,(hq_{1a})'(x_1) - \epsilon q_2(x_2)\,(h^3 q'_{1a})'(x_1) \\ &-\epsilon\,(h^3 q_{1a})(x_1) q''_2(x_2) \leq 0 \qquad \text{in } \Omega_\ell - \Omega_a. \end{aligned} \tag{2.29}$$

Observe that

$$q_2'' = -A_0 q_2, \quad \text{with } A_0 = 4\pi^2 \mathbb{1}_{]0,1/4[\cup]3/4,1[}. \tag{2.30}$$

Then the inequality (2.29) is reduced to

$$(hq_{1a})'(x_1) - \epsilon\left(h^3 q_{1a}'\right)'(x_1) + \epsilon A_0(x_2)\left(h^3 q_{1a}\right)(x_1) \le 0 \quad \text{in } \Omega_\ell - \Omega_a. \tag{2.31}$$

We consider three cases

- Case 1. $0 < x_1 < \beta$
  In this case (2.31) becomes

  $$h'(\beta) - \epsilon\left(h^3 q_{1a}'\right)'(x_1) + \epsilon A_0(x_2)\left(h^3 q_{1a}\right)(x_1) \le 0.$$

  For ϵ small enough (depending on a) this inequality is satisfied since $h'(\beta) < 0$.

- Case 2. $\beta < x_1 < \gamma - a^{1/\alpha}$
  Then (2.31) becomes

  $$h_0'(x_1) + \epsilon A_0(x_2)\left(h^3 q_{1a}\right)(x_1) \le 0. \tag{2.32}$$

  From the fact that h_0 is decreasing and convex on $]0, \gamma[$ and using also (1.9) we deduce $h_0'(x_1) \le -m_1 a^{(\alpha-1)/\alpha}$ for $\beta < x_1 < \gamma - a^{1/\alpha}$.
  Then (2.32) is verified for ϵ small enough (depending on a).

- Case 3. $\gamma - a^{1/\alpha} < x_1 < \gamma$
  We have in this case

  $$q_{1a}'(x_1) = -\frac{\pi}{2a^{1/\alpha}} \cos\left(\frac{\pi}{2a^{1/\alpha}}(\gamma - x_1)\right)$$

  and

  $$q_{1a}'' = -\frac{\pi^2}{4a^{2/\alpha}} q_{1a}.$$

  Dividing (2.31) by q_{1a} we deduce that (2.31) is equivalent to

  $$\begin{aligned} & h_0'(x_1) - \tfrac{\pi}{2a^{1/\alpha}} h(x_1) \operatorname{cotan}\left(\tfrac{\pi}{2a^{1/\alpha}}(\gamma - x_1)\right) + \\ & \tfrac{3\pi}{2a^{1/\alpha}} \epsilon (h^2 h')(x_1) \operatorname{cotan}\left(\tfrac{\pi}{2a^{1/\alpha}}(\gamma - x_1)\right) + \tfrac{\pi^2}{4a^{2/\alpha}} \epsilon h^3(x_1) \\ & + \epsilon A_0(x_2) h^3(x_1) \le 0 \quad \text{in } \Omega_\ell - \Omega_a. \end{aligned} \tag{2.33}$$

  Observe that the three first terms in the above inequality are negative and the last two terms are non-negative. In the sub-case $\gamma - a^{1/\alpha} < x_1 < \gamma - \frac{1}{2}a^{1/\alpha}$, we have $h_0'(x_1) \le -m_1\left(a^{1/\alpha}/2\right)^{\alpha-1}$ and this gives (2.33) for ϵ small enough. In the other sub-case $\gamma - \frac{1}{2}a^{1/\alpha} < x_1 < \gamma$ we have

  $$-\frac{\pi}{2a^{1/\alpha}} h(x_1) \operatorname{cotan}\left(\frac{\pi}{2a^{1/\alpha}}(\gamma - x_1)\right) \le -\frac{\pi}{2a^{1/\alpha}} a = -\frac{\pi}{2} a^{1-1/\alpha}$$

  and this gives (2.33) for ϵ small enough.

This ends the proof of lemma 2.4. □

We now introduce a lower bound for $\int_{\Omega_a} R_a$ in the following lemma.

Lemma 2.5. There exists a constant $c>0$ independent of a and $a_1=(\frac{\mu}{2})^\alpha \leq 1$ such that

$$\int_{\Omega_a} R_a dx \geq c\, a^{1/\alpha-1}, \quad \forall a \in]0,a_1].$$

Proof. We proceed as in [7] and consider the variational formulation of (2.12)

$$\int_{\Omega_a} (h_0+a)^3 \nabla R_a \cdot \nabla\varphi = -\int_{\Omega_a} h_0' \varphi, \quad \forall \varphi \in H_0^1(\Omega_a) \tag{2.34}$$

where $R_a \in H_0^1(\Omega_a)$. From (2.34) we deduce, thanks to Cauchy-Schwarz inequality

$$\int_{\Omega_a} (h_0+a)^3 |\nabla R_a|^2 \geq \sup_{\varphi\in H_0^1(\Omega_a),\varphi\neq 0} \frac{\left|\int_{\Omega_a} h_0'\varphi\right|^2}{\int_{\Omega_a}(h_0+a)^3|\nabla\varphi|^2}. \tag{2.35}$$

Now consider $\psi_1 \in \mathscr{D}(\mathbb{R}), \psi_1>0$, with support included in $[1,2]$ and $\psi_2 \in \mathscr{D}(\mathbb{R}), \psi_2>0$ with support included in $[1/4,3/4]$.
We take in (2.34), $\varphi(x_1,x_2) = \psi_1\left(\frac{\gamma-x_1}{a^{1/\alpha}}\right)\psi_2(x_2)$ which is an element of $H_0^1(\Omega_a)$ with support included in $[\gamma-2a^{1/\alpha}, \gamma-a^{1/\alpha}]\times[1/4,3/4]$.
Since $-h_0'(x_1) \geq m_1(\gamma-x_1)^{\alpha-1}$ we have

$$\begin{aligned}\left|\int_{\Omega_a} h_0'\varphi\right| &\geq m_1 \int_{\gamma-2a^{1/\alpha}}^{\gamma-a^{1/\alpha}}\int_{1/4}^{3/4} (\gamma-x_1)^{\alpha-1}\psi_1\left(\frac{\gamma-x_1}{a^{1/\alpha}}\right)\psi_2(x_2)dx_1dx_2 \\ &\geq m_1 a^{\frac{\alpha-1}{\alpha}} \int_{1/4}^{3/4}\psi_2(x_2)dx_2 \int_{\gamma-2a^{1/\alpha}}^{\gamma-a^{1/\alpha}} \psi_1\left(\frac{\gamma-x_1}{a^{1/\alpha}}\right)dx_1\end{aligned}$$

and we easily see that there exists a positive constant c_1 such that

$$\left|\int_{\Omega_a} h_0'\varphi\right| \geq c_1 a, \quad \forall a < \left(\frac{\gamma}{2}\right)^\alpha \tag{2.36}$$

Using that $h_0(x) \leq m(\gamma-x_1)^\alpha$ we have that :

$$\begin{aligned}\int_{\Omega_a} (h_0+a)^3|\nabla\varphi|^2 \leq \int_{1/4}^{3/4}\int_{\gamma-2a^{1/\alpha}}^{\gamma-a^{1/\alpha}} (m_2(\gamma-x_1)^\alpha+a)^3\cdot \\ \left[\frac{1}{a^{2/\alpha}}\left|\psi_1'\left(\frac{\gamma-x_1}{a^{1/\alpha}}\right)\right|^2|\psi_2(x_2)|^2 + \left|\psi_1\left(\frac{\gamma-x_1}{a^{1/\alpha}}\right)\right|^2|\psi_2'(x_2)|^2\right]\end{aligned}$$

This gives

$$\int_{\Omega_a}(h_0+a)^3|\nabla\varphi|^2 \leq c_2 a^{3-1/\alpha}, \quad \forall\, a < \left(\frac{\gamma}{2}\right)^\alpha \tag{2.37}$$

We deduce from (2.35), (2.36) and (2.37):

$$\int_{\Omega_a}(h_0+a)^3|\nabla R_a|^2 \geq \frac{c_1^2}{c_2}a^{1/\alpha-1}, \quad \forall\, a < \left(\frac{\gamma}{2}\right)^{\alpha}. \tag{2.38}$$

On the other hand taking $\varphi = R_a$ in (2.34) and using the fact that $-h_0' \leq m_2$ on Ω_a and that $R_a > 0$ on Ω_a we deduce

$$m_2\int_{\Omega_a} R_a \geq -\int_{\Omega_a} h_0' R_a = \int_{\Omega_a}(h_0+a)^3|\nabla R_a|^2$$

so we obtain the result using (2.38), where $a_1 = \left(\frac{\nu}{2}\right)^{\alpha}$. □

The main result of this section is

Theorem 2.6. For any $F > 0$ there exists $\epsilon_0 > 0$, $a_2 > 0$ and $a_3 > a_2$ depending possibly on F but independent of ϵ, such that for any $\epsilon \in]0, \epsilon_0]$ there exists at least a solution $(p_\epsilon, a_\epsilon) \in V \times [a_2, a_3]$ of the coupled problem (2.5)-(2.6)-(2.7) or (2.8).
Moreover we have

$$\|p_\epsilon\|_{H^1(\Omega)} \leq C \tag{2.39}$$

where C is a constant independent of ϵ.

Proof. For any fixed $\epsilon > 0$ the continuity of g_ϵ is obvious as a consequence of the continuity of the solution of (2.2) with respect to a. On the other hand, for any $a > 0$ take $\varphi = p_\epsilon$ in (2.2). We have

$$\begin{aligned}\textstyle\int_\Omega (h_0+a)^3|\nabla p_\epsilon|^2 &\leq \textstyle\int_\Omega (h_0+a)\left|\frac{\partial p_\epsilon}{\partial x_1}\right| + \mu\int_{\Gamma_0} p_\epsilon dx_2\\ &\leq \left(\textstyle\int_\Omega (h_0+a)^3|\nabla p_\epsilon|^2\right)^{1/2}\left(\textstyle\int_\Omega \frac{dx}{h_0+a}\right)^{1/2} + \mu C_1\|p_\epsilon\|_{H^1(\Omega)}\\ &\leq \tfrac{1}{2}\textstyle\int_\Omega (h_0+a)^3|\nabla p_\epsilon|^2 + \frac{1}{2a}|\Omega| + \mu C_1\|p_\epsilon\|_{H^1(\Omega)}\end{aligned}$$

with $C_1 > 0$ a constant. We deduce using also the Poincaré inequality:

$$\begin{aligned}C_2 a^3\|p_\epsilon\|^2_{H^1(\Omega)} &\leq \tfrac{1}{a}|\Omega| + 2\mu C_1\|p_\epsilon\|_{H^1(\Omega)}\\ &\leq \tfrac{1}{a}|\Omega| + \tfrac{C_2}{2}a^3\|p_\epsilon\|^2_{H^1(\Omega)} + \tfrac{2}{C_2 a^3}\mu^2 C^2.\end{aligned}$$

We then deduce the existence of a constant $C_3 > 0$ such that

$$\|p_\epsilon\|_{H^1(\Omega)} \leq C_3\left(\frac{1}{a^2}+\frac{1}{a^3}\right), \quad \text{for any } a > 0 \text{ and } \epsilon > 0. \tag{2.40}$$

By Poincaré inequality there exists $C_4 > 0$ such that

$$g_\epsilon(a) \leq C_4\left(\frac{1}{a^2}+\frac{1}{a^3}\right), \quad \text{for any } a > 0 \text{ and } \epsilon > 0. \tag{2.41}$$

Finally from Lemma 2.5, Lemma 2.4 and Lemma 2.2, we deduce that for any $a \in]0, \min\{a_0, a_1\}]$, there exists $\epsilon_0 = \epsilon_0(a)$ such that for any $\epsilon \in]0, \epsilon_0]$ we have

$$g_\epsilon(a) > c\, a^{1/\alpha-1}. \tag{2.42}$$

We now chose $a_2 > 0$ small enough such that

$$c\, a_2^{1/\alpha-1} \geq F \tag{2.43}$$

and $a_3 > a_2$ large enough such that

$$C_4 \left(\frac{1}{a_3^2} + \frac{1}{a_3^3} \right) \leq F. \tag{2.44}$$

It is clear from (2.41)-(2.42)-(2.43)-(2.44) and the continuity of g_ϵ that there exists $a_\epsilon \in [a_2, a_3]$ such that

$$g_\epsilon(a_\epsilon) = F, \qquad \text{for any } \epsilon \in]0, \epsilon_0(a_2)] \tag{2.45}$$

which ends the proof of the existence of the solution of (2.8).
From (2.40) we deduce also (2.39), with $C = C_3 \left(\frac{1}{a_2^2} + \frac{1}{a_2^3} \right)$. □

3 Proof of the main theorem

Now we can prove Theorem 1.1. We can extract a subsequence of ϵ denoted also by ϵ and we have $a^* \in [a_2, a_3]$ and $p^* \in V$ such that $a_\epsilon \to a^*$ and $p_\epsilon \to p^*$ in V weakly. Passing to the limit $\epsilon \to 0$ in a classical manner in (2.5), (2.6) and (2.7) we obtain the result.

4 Numerical results

The goal of this section is the numerical illustration of the theoretical result of Theorem 1.1 as well as some extensions.
We consider here $h_0(x) = |x_1 - 1/2|^\alpha$ corresponding to $\gamma = \frac{1}{2}$, with different values of $\alpha > 0$ and we search for a numerical solution of the coupled problem (1.10)-(1.2)-(1.3) with $\mu = 0.2$. To do this we represent graphically $\int_\Omega p dx$ as a function of $a > 0$, where p is the numerical solution of (1.10) with $h = h_0 + a$.
The simulation code is based on the Elrod-Adams algorithm [9], with the implementation detailed by Ausas et al. in [1].
In section 4.1 we take $\alpha = 2$ (which is included in the case studied theoretically in this work) and we observe numerically the existence of a unique solution for any given $F > 0$.
An interesting question here is to see what happens in the case $\alpha \leq 1$, which is still an open theoretical question. Recall that the stationary problem with Reynolds variational inequality in the place of (1.10) was studied in [7]; in the case $\alpha = 1$ the authors still proved the existence of a solution for any $F > 0$, while in the case $\alpha < 1$ they proved the existence of a solution for $F < F_0$ only, where $F_0 > 0$ is a threshold value. In Section 4.2 ($\alpha = 1$) and Section 4.3 ($\alpha = \frac{1}{2}$) we observe numerically that the results in [7] remain valid with an Elrod-Adams model (system (1.10)-(1.2)-(1.3)).

4.1 Case 1. $\alpha = 2$

The left plot of Fig. 1 represents the parabolic shape of h_0 with $\alpha = 2$ while in the right plot we can see the profile of $\int_\Omega p dx$ (the load) with respect to a. We can conclude that for any $F > 0$ the studied coupled problem has a unique solution $a > 0$ (remark that the theoretical uniqueness result is still an open question).
In Fig. 2 we represent the 3d profile of the pressure $p(x)$ and the contour plot of the corresponding θ field for, respectively, $a = 0.1$, $a = 0.01$ and $a = 0.001$. One can observe that for $a = 0.001$ we have large values of the pressure which is concentrated around the middle of the domain ($x_1 = 1/2$), as predicted theoretically.

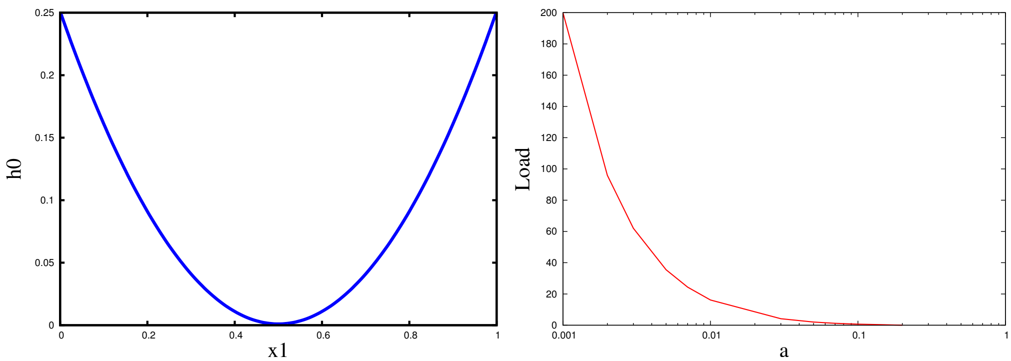
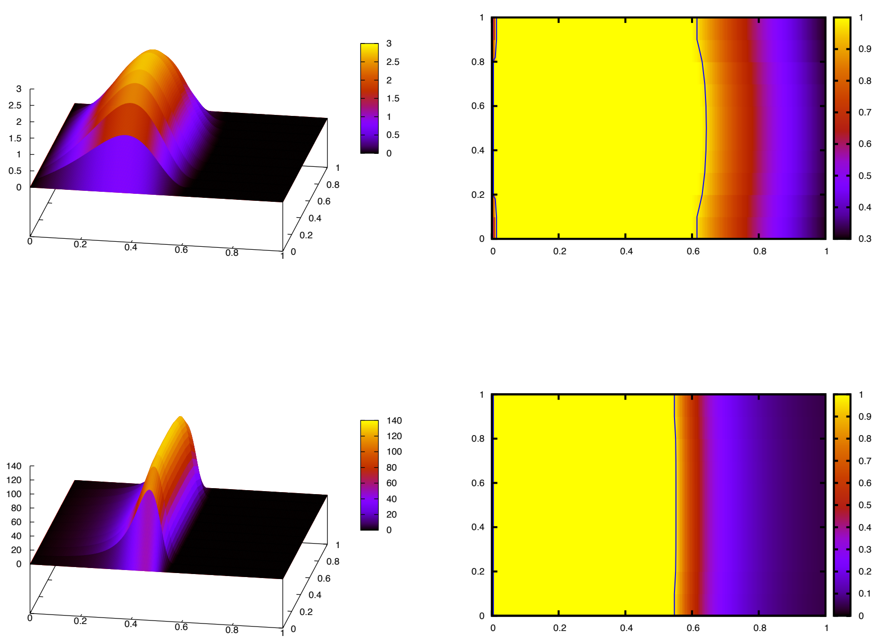


Figure 1: Case $\alpha = 2$. The shape of the upper body h_0 (left plot) and the load versus a (right plot).



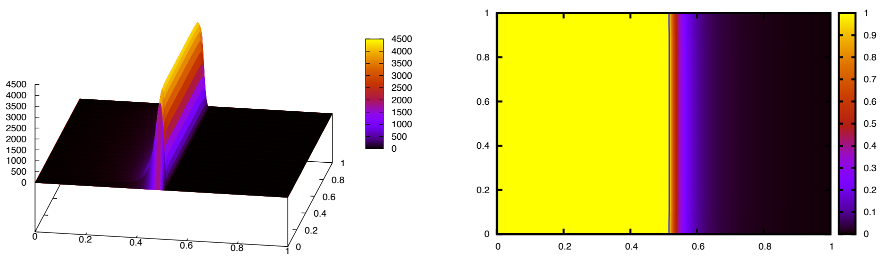


Figure 2: Case $\alpha = 2$. Pressure (left) and corresponding theta-field profiles for different values of a. From top to bottom : $a = 0.1, 0.01, 0.001$.

4.2 Case 2. $\alpha = 1$

In this limit case we can expect that the result of Theorem 1.1 is still valid for any $F > 0$. Nevertheless, we expect (as in [7]) that $\int_\Omega p dx$ goes very slowly to $+\infty$ when $a \to 0$ (as $\log\left(\frac{1}{a}\right)$), which is confirmed by the numerical simulations (see Fig. 3).

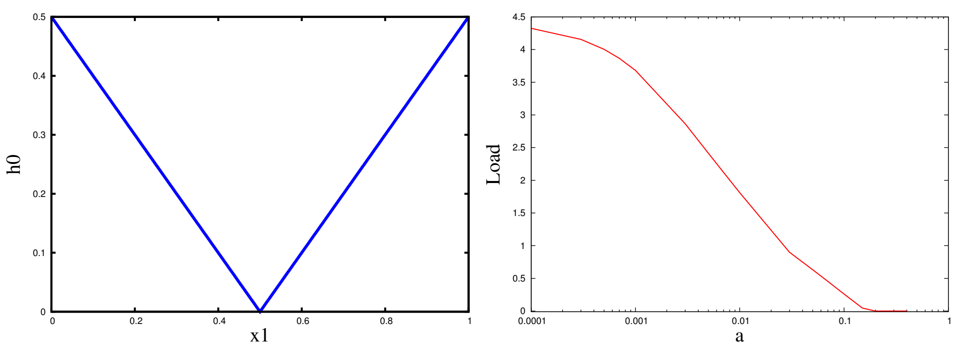
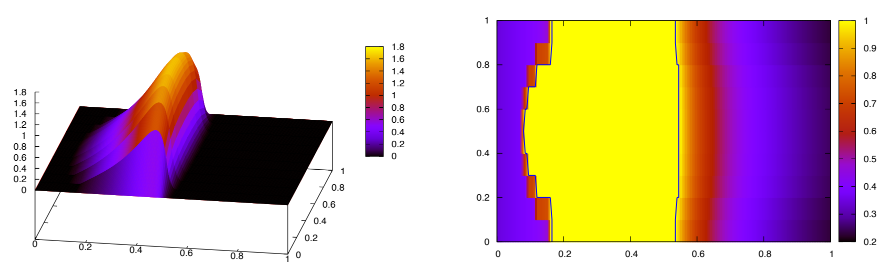


Figure 3: Case $\alpha = 1$. The shape of the upper body h_0 (left plot) and the load versus a (right plot).



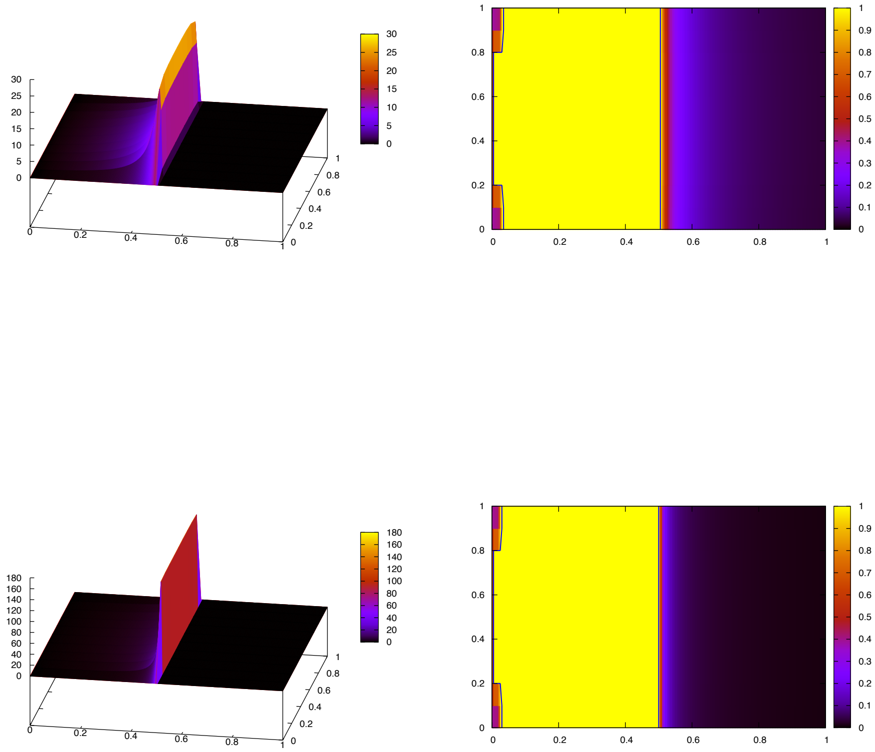


Figure 4: Pressure (left) and corresponding theta field profiles for different values of a. Values of a from top to bottom: $a = 0.1, 0.01, 0.001$.

4.3 Case 3. $\alpha = \frac{1}{2}$

In this case we observe numerically the existence of a threshold $F_0 = 0.235$ such that the result is valid only for $F < F_0$ (see Fig. 5).

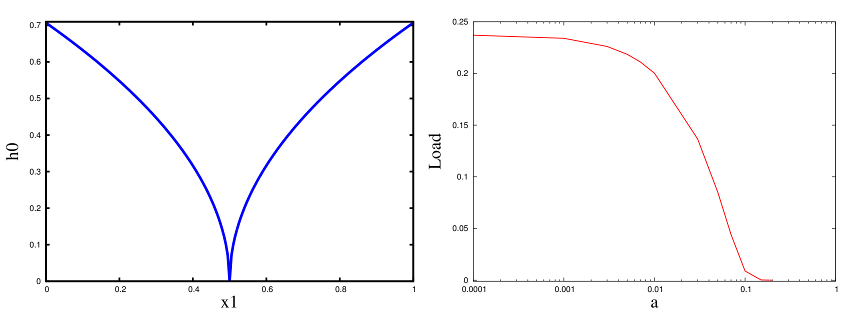


Figure 5: Case $\alpha = 1/2$. The shape of the upper body h_0 (left plot) and the load versus a (right plot).

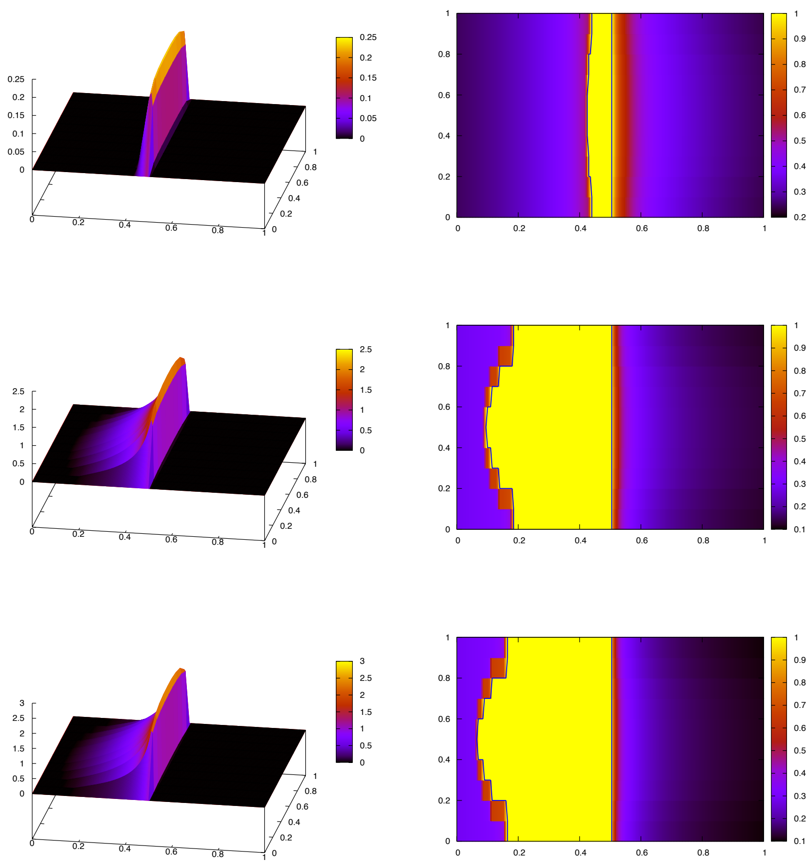


Figure 6: Pressure (left) and corresponding theta-field profiles for different values of a. Values of a from top to bottom: $a = 0.1, 0.01, 0.001$.

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