

# Continuity of solutions to a basic problem in the calculus of variations

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## Abstract

We study the problem of minimizing  $\int_{\Omega} F(Du(x)) dx$  over the functions  $u \in W^{1,1}(\Omega)$  that assume given boundary values  $\phi$  on  $\Gamma := \partial\Omega$ . The Lagrangian  $F$  and the domain  $\Omega$  are assumed convex. A new type of hypothesis on the boundary function  $\phi$  is introduced: the *lower* (or upper) *bounded slope condition*. This condition, which is less restrictive than the familiar bounded slope condition of Hartman, Nirenberg and Stampacchia, allows us to extend the classical Hilbert-Haar regularity theory to the case of semiconvex (or semiconcave) boundary data (instead of  $C^2$ ). We prove in particular that the solution is locally Lipschitz in  $\Omega$ . In certain cases, as when  $\Gamma$  is a polyhedron or else of class  $C^{1,1}$ , we obtain in addition a global Hölder condition on  $\bar{\Omega}$ .

## 1 Introduction

In this article we study the following problem ( $P$ ) in the multiple integral calculus of variations:

$$\min_u \int_{\Omega} F(Du(x)) dx \text{ subject to } u \in W^{1,1}(\Omega), \text{ tr } u = \phi \quad (P)$$

where  $\Omega$  is a domain in  $\mathbb{R}^n$ ,  $u$  is scalar-valued, and  $\text{tr } u$  signifies the trace of  $u$  on  $\Gamma := \partial\Omega$ . The function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *strictly convex* if for any two distinct points  $z_1, z_2$  in  $\mathbb{R}^n$  and  $\lambda \in (0, 1)$  we have

$$F(\lambda z_1 + (1 - \lambda)z_2) < \lambda F(z_1) + (1 - \lambda)F(z_2),$$

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and *coercive* of order  $p > 1$  if for certain constants  $\sigma > 0$  and  $\mu$  we have

$$F(z) \geq \sigma|z|^p + \mu \quad \forall z \in \mathbb{R}^n.$$

Our *basic hypotheses* are as follows: (1)  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) is open, convex, and bounded; (2)  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is strictly convex; (3)  $F$  is coercive of order  $p > 1$ .

Since a convex body has Lipschitz boundary, the trace operator is well-defined. The function  $\phi : \Gamma \rightarrow \mathbb{R}$  which prescribes the boundary values lies in some appropriate class, the choice of which is one of our main considerations. Under basic hypotheses (1)-(3) above, in order to guarantee that the problem is well-posed and admits a solution  $u$ , it is sufficient that  $\phi$  be the restriction to  $\Gamma$  of a Lipschitz function. This follows from well-known lower semicontinuity theorems for the integral functional which go back in spirit to Tonelli's *direct method*; the unique solution then lies in  $W^{1,p}(\Omega)$  necessarily.

The question now becomes the regularity of  $u$ . The principal classical results in this connection are based on two approaches: the theory of De Giorgi (which is also referred to as that of De Giorgi-Nash-Moser), and what has become known as Hilbert-Haar theory. We refer to the books of Giaquinta [8], Giusti [10], and Morrey [17] for detailed discussions.

The theory of De Giorgi can be applied directly to  $(P)$ , but in that case it requires strong assumptions on  $F$ : uniform ellipticity and constrained growth of the same order  $p$  from both above and below (see however Marcellini [13] and the references therein for certain extensions). More relevant to this article are its application to Lagrangians  $F$  that may not satisfy such demanding hypotheses, but are at least *regular*: of class  $C^2$ , with  $\nabla^2 F(z) > 0$  at each  $z$ . Then, provided that one knows *a priori* that  $u$  is Lipschitz, De Giorgi's theorem on elliptic equations can be invoked to deduce higher-order regularity: it follows that  $Du$  is locally Hölder continuous in  $\Omega$ .

The classical Hilbert-Haar approach, which in contrast makes no additional structural assumptions on  $F$ , requires instead that  $\phi$  satisfy the *bounded slope condition* (BSC). This theory cannot be attributed to a single person. Hilbert, Rado, Haar, and von Neumann each made a key contribution to its development; after the BSC was formulated in its present form by Hartman and Nirenberg [12], Stampacchia [18] coined the term BSC and applied it to variational problems.

The bounded slope condition of rank  $K$  is the requirement that given any point  $\gamma$  on the boundary, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma), \quad y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with  $\phi$  at  $\gamma$  whose 'slopes' satisfy  $|\zeta_\gamma^-| \leq K, |\zeta_\gamma^+| \leq K$  and

such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \leq \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

This is a very restrictive requirement on ‘flat parts’ of  $\Gamma$ , since it forces  $\phi$  to be affine. But the BSC becomes more interesting when  $\Omega$  is sufficiently curved.  $\Omega$  is said to be *uniformly convex* if, for some  $\epsilon > 0$ , every point  $\gamma$  on the boundary admits a hyperplane  $H$  through  $\gamma$  such that

$$d_H(\gamma') \geq \epsilon |\gamma' - \gamma|^2 \quad \forall \gamma' \in \Gamma.$$

Miranda’s Theorem [16] states that when  $\Omega$  is uniformly convex, then any  $\phi$  of class  $C^2$  satisfies the BSC. Later, Hartman [11] showed that when  $\Omega$  is uniformly convex and  $\Gamma$  is  $C^{1,1}$ , then  $\phi$  satisfies the BSC if and only if  $\phi$  is itself  $C^{1,1}$ .

The Hilbert-Haar theory (see Chapter 1 of Giusti [10]) affirms that when  $\phi$  satisfies the BSC of rank  $K$ , then the solution  $u$  of problem (P) is globally Lipschitz of rank  $K$ . Thus the two theories work in tandem to obtain the continuous differentiability of the solution.<sup>1</sup>

In this article we introduce a new hypothesis on  $\phi$ , the *lower bounded slope condition* (lower BSC) of rank  $K$ : given any point  $\gamma$  on the boundary, there exists an affine function  $y \mapsto \langle \zeta_\gamma, y - \gamma \rangle + \phi(\gamma)$  with  $|\zeta_\gamma| \leq K$  such that

$$\langle \zeta_\gamma, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \quad \forall \gamma' \in \Gamma.$$

This requirement (which can be viewed as a one-sided BSC) has an alternate characterization:

**Proposition 1.1**

*A function  $\phi : \Gamma \rightarrow \mathbb{R}$  satisfies the lower BSC of rank  $K$  if and only if  $\phi$  is the restriction to  $\Gamma$  of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  which is convex and globally Lipschitz of rank  $K$ .*

**Proof** The sufficiency follows from the fact that a function  $f$  as described admits at each point  $x$  an element  $\zeta$  in its subdifferential  $\partial f(x)$ : then  $|\zeta| \leq K$  and

$$f(x') - f(x) \geq \langle \zeta, x' - x \rangle \quad \forall x' \in \mathbb{R}^n,$$

and the lower BSC follows (since  $f = \phi$  on  $\Gamma$ ). For the necessity, we set

$$f(x) := \sup_{\gamma \in \Gamma} \langle \zeta_\gamma, x - \gamma \rangle + \phi(\gamma),$$

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<sup>1</sup>More precisely, Hilbert-Haar theory affirms the existence of a minimizer in the class of Lipschitz functions. To deduce that this solution coincides with the Sobolev solution  $u$ , one must rule out the Lavrentiev phenomenon. This can be done in several ways: see [6][3][14].

which is easily seen to coincide with  $\phi$  on  $\Gamma$ , and to be convex and Lipschitz of rank  $K$ .  $\square$

Being one-sided, the lower BSC naturally admits a counterpart: an *upper* BSC that is satisfied by  $\phi$  exactly when  $-\phi$  satisfies the lower BSC.

In the context of the lower (or upper) BSC, the property that  $\Omega$  be curved has less importance than before; flat parts of the boundary do not force  $\phi$  to be affine. Nonetheless, curvature can still serve a purpose: Bousquet [1] has shown that when  $\Omega$  is uniformly convex, then  $\phi$  satisfies the lower (upper) BSC if and only if it is the restriction to  $\Gamma$  of a function which is semiconvex (semiconcave), a familiar and useful property in pde's (see for example [4]). In the uniformly convex case, therefore, the results of this article extend Hilbert-Haar theory to boundary data that is semiconvex or semiconcave rather than  $C^2$  (or  $C^{1,1}$ ).

It turns out that the one-sided BSC hypothesis has significant implications for the regularity of the solution  $u$ , although, unsurprisingly, it implies less than the full two-sided BSC. In fact, the one-sided BSC, which goes considerably beyond the full BSC with respect to the domains and boundary conditions which it allows, nonetheless gives the crucial regularity property:  $u$  is locally Lipschitz in  $\Omega$ . This allows us to assert that  $u$  is a weak solution of the Euler equation, in the absence of any 'natural growth conditions' on  $F$ , and also permits the application of the theory of higher-order regularity when the Lagrangian is regular. Another desirable regularity property is continuity at the boundary. In a variety of situations, it turns out that the lower or upper BSC does imply that, and even a global Hölder condition in some cases.

We proceed now to state the main results of the article. In accordance with the usual convention, the continuity assertions on  $u$  refer to the existence of a representative of  $u$  having the stated property.

**Theorem 1.1**

*Under the basic hypotheses, if  $\phi$  satisfies the lower bounded slope condition, then the solution  $u$  of (P) is locally Lipschitz in  $\Omega$  and lower semicontinuous on  $\bar{\Omega}$ . There is a constant  $\bar{K}$  with the property that for any subdomain  $\Omega'$  of distance  $\delta > 0$  from  $\Gamma$ , we have*

$$|u(x) - u(y)| \leq (\bar{K}/\delta)|x - y| \quad \forall x, y \in \Omega'.$$

*In addition,  $u$  is continuous on  $\bar{\Omega}$  if one of the following holds:*

- (a)  $\Gamma$  is a polyhedron, or
- (b)  $\Gamma$  is  $C^{1,1}$  and  $p > (n + 1)/2$ , or
- (c)  $\Omega$  is uniformly convex.

In cases (a) and (b),  $u$  satisfies a Hölder condition on  $\bar{\Omega}$ .

**Remark 1.1**

1. The lower semicontinuity is interpreted with  $u$  taken to be equal to  $\phi$  everywhere on  $\Gamma$ .
2. There is a corresponding version of the theorem in which, in its two occurrences, the word ‘lower’ is replaced by ‘upper’.

When  $F$  has more regularity itself, some further properties of  $u$  follow immediately:

**Corollary 1.2**

Let the basic hypotheses hold, and assume that  $\phi$  satisfies the lower bounded slope condition. Then

- (a) If  $F$  is differentiable,  $u$  is a weak solution of the Euler equation:

$$\int_{\Omega} \nabla F(Du(x)) \cdot D\psi(x) \, dx = 0 \quad \forall \psi \in C_c^1(\Omega).$$

- (b) If in addition we assume that  $F$  is of class  $C^2$  and satisfies

$$\nabla^2 F(z) > 0 \quad \forall z,$$

then  $Du$  is locally Hölder continuous in  $\Omega$ .

Theorem 1.1 is elaborated upon and proved in Section 2. The type of regularity it provides is illustrated by the following example due to Piermarco Cannarsa and Pierre Bousquet:

**Example** Let  $n = 2$  and take  $F$  to be the Dirichlet Lagrangian

$$F(Du) := |Du|^2 = u_x^2 + u_y^2,$$

and  $\Omega$  the unit disc. The complex function

$$f(z) := - \sum_{i \geq 1} \frac{z^i}{i^2}$$

is analytic on  $\Omega$  and continuous on  $\bar{\Omega}$ . We define  $u(r, \theta)$  to be its real part, and  $\phi(\theta)$  on  $\Gamma$  to be  $u(1, \theta)$ :

$$\phi(\theta) := - \sum_{i \geq 1} \frac{\cos i\theta}{i^2}.$$

It can be shown that  $u$  (which is of course harmonic in  $\Omega$ ) belongs to  $W^{1,2}(\Omega)$ , from which it follows [17, Theorem 3.3.1] that  $u$  is the

solution to problem (P). The gradient of  $u$  is unbounded on  $\Omega$ , so that  $u$  is locally but not globally Lipschitz.

The function  $\phi$  satisfies a lower (but not full) bounded slope condition [1], and  $F$  is coercive of order 2. Theorem 1.1 tells us that  $u$  satisfies a Hölder condition on the closed disc, and provides the bound  $\bar{K}/d_\Gamma(x)$  for  $|Du(x)|$ .

**Lipschitz boundary conditions.** In the last section of the article we study the case in which, rather than a one-sided bounded slope condition,  $\phi$  satisfies a weaker requirement, a Lipschitz condition. We prove

**Theorem 1.2**

Let  $\phi$  be Lipschitz, where in addition to the basic hypotheses we assume that either

- (a)  $\Gamma$  is a polyhedron, or
- (b)  $\Gamma$  is  $C^{1,1}$  and  $p > (n + 1)/2$ .

Then for certain constants  $\bar{k}$  and  $\beta > 0$  the solution  $u$  has the following property: if  $\Omega'$  is any subdomain of  $\Omega$  of distance  $\delta$  from  $\Gamma$ , then

$$|u(x) - u(y)| \leq (\bar{k}/\delta^\beta)|x - y|^\beta \quad \forall x, y \in \Omega'.$$

In case (a),  $u$  satisfies a Hölder condition on  $\bar{\Omega}$ . In case (b), if in addition  $\Omega$  is uniformly convex, then  $u$  is continuous on  $\bar{\Omega}$ .

**Remark 1.2**

The proof of the classical Hilbert-Haar theorem is based on the Comparison (or Maximum) Principle on spaces of Lipschitz functions. Our proof of Theorem 1.1 uses an extension of the Comparison Principle (by Mariconda and Treu) to the Sobolev setting. In further contrast to earlier work, we use dilation rather than translation to construct comparison functions.

## 2 Proof of Theorem 1.1

### 2.1 Four theorems

We first establish some notation. For any  $x, y \in \Omega, x \neq y$ , we denote by  $\pi_\Gamma(x|y)$  the ‘projection of  $x$  onto  $\Gamma$  in the direction of  $y$ ’; that is, the unique point  $\gamma \in \Gamma$  of the form  $x + t(y - x), t > 0$ . Then  $d_\Gamma(x|y)$ , the ‘distance from  $x$  to  $\Gamma$  in the direction of  $y$ ’, is given by  $d_\Gamma(x|y) := |x - \pi_\Gamma(x|y)|$ . We have  $d_\Gamma(x|z) \geq d_\Gamma(x)$ , where the latter denotes the usual distance function associated with  $\Gamma$ .

In this section we prove the four following theorems, which clearly imply Theorem 1.1. Besides being steps towards its proof, they make

additional assertions. Note for example that the first two do not require  $F$  to be coercive; we simply suppose instead that a solution  $u$  exists. They also provide explicit estimates for certain constants.

**Theorem 2.1**

We posit the basic hypotheses (1) and (2). Let  $u$  be a solution of (P), where  $\phi$  satisfies the lower bounded slope condition of rank  $K$ . Then  $u$  is continuous in  $\Omega$  and lower semicontinuous on  $\overline{\Omega}$ , and we have

$$u(x) \geq \phi(\gamma) - K|x - \gamma| \quad \forall x \in \Omega, \quad \forall \gamma \in \Gamma,$$

as well as

$$u(x) - u(y) \leq \overline{K} \frac{|x - y|}{d_\Gamma(x|y)} \quad \forall x, y \in \Omega, x \neq y$$

where

$$\overline{K} := 2\|\phi\|_{L^\infty(\Gamma)} + K \operatorname{diam} \Omega.$$

**Theorem 2.2**

In addition to the hypotheses of Theorem 2.1, assume either that  $\Gamma$  is a polyhedron, or else that  $\Omega$  is uniformly convex. Then the solution  $u$  is also continuous on  $\overline{\Omega}$ .

**Theorem 2.3**

In addition to the hypotheses of Theorem 2.1, assume that  $\Gamma$  is a polyhedron and that  $F$  is coercive of order  $p > 1$ . Then for some constant  $k_1$ ,  $u$  satisfies

$$-K|x - \gamma| \leq u(x) - \phi(\gamma) \leq k_1|x - \gamma|^{a_1}, \quad \forall x \in \Omega, \quad \forall \gamma \in \Gamma,$$

where

$$a_1 := (p - 1)/(n + p - 1) \in (0, 1).$$

Furthermore,  $u$  satisfies a Hölder condition on  $\overline{\Omega}$  of order

$$a_2 := (p - 1)/(n + 2p - 2) \in (0, a_1).$$

**Theorem 2.4**

In addition to the hypotheses of Theorem 2.1, assume that  $\Omega$  is  $C^{1,1}$  and that  $F$  is coercive of order  $p > (n + 1)/2$ . Then for some constant  $\kappa_1$ ,  $u$  satisfies

$$-K|x - \gamma| \leq u(x) - \phi(\gamma) \leq \kappa_1|x - \gamma|^{\alpha_1}, \quad \forall x \in \Omega, \quad \forall \gamma \in \Gamma,$$

where

$$\alpha_1 := (2p - n - 1)/(2n + 2p - 2) \in (0, 1).$$

Furthermore,  $u$  satisfies a Hölder condition on  $\overline{\Omega}$  of order

$$\alpha_2 := (2p - n - 1)/(4p + n - 3) \in (0, \alpha_1).$$

**Remark 2.1**

1. The proofs of the Hölder conditions yield constants (such as  $k_1$  and  $\kappa_1$ ) that depend on the data of the problem only via the dimension  $n$ , the coercivity coefficients  $\sigma, \mu$  (as regards  $F$ ), the quantities  $\|\phi\|_\infty, K$  (as regards  $\phi$ ), and the geometry of  $\Omega$  (its diameter, together with the density and tightness of corners in the polyhedral case, or the curvature of the boundary in the smooth case).
2. When  $p \leq n$ , then the fact that  $Du \in L^p(\Omega)$  does not in itself imply that  $u$  is continuous. But for  $p > n$ , a well-known theorem of Morrey asserts that  $u$  satisfies a Hölder condition on  $\bar{\Omega}$  of order  $(p - n)/p$ . Theorems 2.3 and 2.4 evidently provide Hölder continuity in some cases in which Morrey's Theorem is not applicable, and, in some cases where it does apply, they yield a better (that is, greater) Hölder exponent.
3. In the presence of coercivity, all the theorems above can be extended to problems in which a constraint of the form  $Du(x) \in C$  a.e. is present, where  $C$  is a given closed convex set. It suffices to apply the (unconstrained) theorem to the solution  $u_i$  of the penalized problem with Lagrangian  $F(z) + id_C(z)$ , whose solution  $u_i$  converges to  $u$ .
4. If the strict convexity hypothesis on  $F$  is weakened to merely convexity, then the solution  $u$  is not necessarily unique and may not have the stated regularity; however, the conclusions of the theorems hold for *some* solution, as can be shown by considering the Lagrangian  $F(z) + \epsilon|z|^2$  and letting  $\epsilon$  go to 0.
5. The methods of this article can be extended to treat more general Lagrangians (depending on  $x, u$ ) by the barrier function technique, as forthcoming work will show [2].

**2.2 Proof of Theorem 2.1**

We denote

$$I_\Omega(v) := \int_\Omega F(Dv(x)) dx,$$

which is always well-defined for any  $v \in W^{1,1}(\Omega)$ , possibly as  $+\infty$ . An element  $v$  of  $W^{1,1}(\Omega)$  is called a *minimizer* (for  $I_\Omega$ ) if  $I_\Omega(v) < +\infty$  and if for any other  $w \in W^{1,1}(\Omega)$  having the same trace on  $\partial\Omega$ , we have  $I_\Omega(w) \geq I_\Omega(v)$ . An affine function is always a minimizer (see [10]), and the sum of a minimizer and a constant yields a minimizer.

There is a well-known *comparison* or *maximum principle* for a given pair of minimizers  $v, w$  on  $\Omega$  (not necessarily having the same trace on  $\partial\Omega$ ) which asserts that if  $v \leq w$  on  $\partial\Omega$ , then  $v \leq w$  in  $\Omega$ . A proof may be found for example in Chapter 1 of [10], in the classical case (see



also [9]), when  $v$  and  $w$  are restricted to being Lipschitz continuous of given rank  $K$ . The proof can be adapted to certain lattices in a Sobolev space setting, as shown by Mariconda and Treu [15]. We now state a special case of their results for convenience.

**Theorem (Comparison Principle)** *Let  $v$  and  $w$  be minimizers for  $I_\Omega$  such that  $\operatorname{tr} v \leq \operatorname{tr} w$  a.e. Then in  $\Omega$  we have  $v \leq w$  a.e.*

(Of course the two ‘almost everywhere’ in this statement refer to  $(n-1)$ - and  $n$ -dimensional measure respectively.) We give the proof of Theorem 2.1 in the stated case in which  $\phi$  satisfies the lower bounded slope condition of rank  $K$ . (The corresponding ‘upper’ case reduces to this one by considering  $\tilde{u} := -u$ , which is a minimizer for  $\tilde{F}(z) := F(-z)$  with boundary condition  $-\phi$ .) By Proposition 1.1 we may suppose that  $\phi$  is a globally defined convex function of (global) Lipschitz rank  $K$ . The Comparison Principle (applied to  $u$  and appropriate constant functions) implies that  $|u|$  is essentially bounded on  $\bar{\Omega}$  by the constant  $M := \|\phi\|_{L^\infty(\Gamma)}$ . The word ‘essentially’ can be dispensed with here if we take  $u$  to be the precise representative (see [7]) of its class, as we do henceforth.

We fix a point  $z$  in  $\Omega$  and a scalar  $\lambda$  in  $(0, 1)$ . For any  $y \in R^n$ , we denote by  $y_\lambda$  the point  $\lambda(y - z) + z$ , and we define the set  $\Omega_\lambda$  as the image of  $\Omega$  under this mapping:  $\Omega_\lambda := \lambda(\Omega - z) + z$ . Note that  $\Omega_\lambda$  is an open convex subset of  $\Omega$ , and that the mapping  $\gamma \rightarrow \gamma_\lambda$  is one-to-one between points  $\gamma \in \Gamma$  and points  $\gamma_\lambda \in \Gamma_\lambda := \partial\Omega_\lambda$ .

We proceed to define on  $\Omega_\lambda$  the function

$$u_\lambda(x) := \lambda u((x - z)/\lambda + z);$$

note that  $u_\lambda(x_\lambda) = \lambda u(x)$  for any  $x \in \Omega$ .

**Lemma 2.1**

We have

$$u_\lambda(\gamma_\lambda) - u(\gamma_\lambda) \leq K'(1 - \lambda), \quad \gamma \in \Gamma \text{ a.e.},$$

where  $K' := \|\phi\|_{L^\infty(\Gamma)} + K \operatorname{diam} \Omega = M + K \operatorname{diam} \Omega$ .

**Proof** Since  $u$  is approximately continuous<sup>2</sup> at almost every point of  $\Omega$ , it follows from Fubini’s theorem and the use of spherical coordinates that for almost every  $\gamma \in \Gamma$ , the half-open line segment  $(z, \gamma]$  has the property that almost all of its points are points of approximate continuity of  $u$ . Additionally, by known properties of Sobolev functions [19], for almost every such  $\gamma$ , the segment is such that the restriction

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<sup>2</sup>Approximate continuity at  $x$  refers to the existence of a measurable set  $C$  satisfying  $\lim_{r \rightarrow 0} |B(x, r) \cap C|/|B(x, r)| = 1$ , and relative to which  $u$  is continuous.

of  $u$  to it is (absolutely) continuous, with  $u(\gamma) = \phi(\gamma)$ . We fix any  $\gamma$  in  $\Gamma$  for which the associated segment  $(z, \gamma]$  has both these properties.

Let  $\zeta$  be an element of  $\partial\phi(\gamma)$ . Then  $|\zeta| \leq K$ , and for any  $x \in \mathbb{R}^n$  we have

$$\phi(x) \geq \phi(\gamma) + \langle \zeta, x - \gamma \rangle =: v(x).$$

In particular,  $u \geq v$  a.e. on  $\Gamma$ . Since  $v$  is affine and hence a minimizer, the Comparison Principle gives  $u \geq v$  a.e. in  $\Omega$ . This last inequality must hold at all points of approximate continuity of  $u$ , and so on a set of full one-dimensional measure in the line segment  $(z, \gamma]$ . Since  $u$  is continuous along the segment, it follows that  $u \geq v$  on the entire segment, whence

$$\begin{aligned} u_\lambda(\gamma_\lambda) - u(\gamma_\lambda) &\leq \lambda\phi(\gamma) - v(\gamma_\lambda) \\ &= (\lambda - 1)\phi(\gamma) - \langle \zeta, \gamma_\lambda - \gamma \rangle \\ &= (\lambda - 1) \{ \phi(\gamma) - \langle \zeta, z - \gamma \rangle \} \\ &\leq (1 - \lambda) \{ M + K \operatorname{diam} \Omega \}. \end{aligned}$$

This proves the Lemma.  $\square$

**Lemma 2.2**

*There is a subset  $X(\lambda, z)$  of full measure in  $\Omega$  such that for every  $x$  in  $X(\lambda, z)$  we have*

$$\lambda u(x) - u(\lambda x + (1 - \lambda)z) \leq K'(1 - \lambda).$$

**Proof** We note first that  $u$  restricted to  $\Omega_\lambda$  is a minimizer. For if it were not, there would exist  $v$  on  $\Omega_\lambda$  with the same boundary values as  $u$ , yet strictly ‘better’ than  $u$  on that subdomain; then the function which equals  $v$  on  $\Omega_\lambda$  and  $u$  on  $\Omega \setminus \Omega_\lambda$  (which lies in  $W^{1,1}(\Omega)$ ) would be strictly better than  $u$  on  $\Omega$ , while having the same trace on  $\Gamma$ , which is a contradiction.

We claim that  $u_\lambda$  is also a minimizer (relative to the domain  $\Omega_\lambda$ ). If this were not the case, there would exist  $v \in W^{1,1}(\Omega_\lambda)$  with the same values on  $\Gamma_\lambda$  as  $u_\lambda$  such that

$$I_{\Omega_\lambda}(v) := \int_{\Omega_\lambda} F(Dv(x)) dx < \int_{\Omega_\lambda} F(Du_\lambda(x)) dx =: I_{\Omega_\lambda}(u_\lambda).$$

Upon making an evident change of variables, the right-hand integral is seen to equal

$$\lambda^n \int_{\Omega} F(Du(x)) dx = \lambda^n I_{\Omega}(u).$$

Now define  $w \in W^{1,1}(\Omega)$  by  $w(x) := (1/\lambda)v(\lambda(x - z) + z)$ , which has the same boundary values at  $\Gamma$  as  $u$ . A simple calculation then leads to  $I_{\Omega}(w) < I_{\Omega}(u)$ , a contradiction which establishes the claim.

The conclusion of Lemma 2.1, in conjunction with the Comparison Principle, when applied to the domain  $\Omega_\lambda$  and the two functions  $u_\lambda$  and  $u + K'(1 - \lambda)$  (restricted to  $\Omega_\lambda$ ), gives rise to

$$u_\lambda(x) - u(x) \leq K'(1 - \lambda), \quad x \in \Omega_\lambda \text{ a.e.},$$

which may be re-expressed as follows: there is a subset  $X(\lambda, z)$  of  $\Omega$  of full measure such that

$$\lambda u(x) - u(\lambda x + (1 - \lambda)z) \leq K'(1 - \lambda) \quad \forall x \in X(\lambda, z),$$

as required.  $\square$

**Lemma 2.3**

Let  $x$  and  $y$  in  $\Omega$  be distinct Lebesgue points of  $u$ . Then

$$u(x) - u(y) \leq \bar{K} \frac{|x - y|}{d_\Gamma(x|y)},$$

where  $\bar{K} := K' + M = 2\|\phi\|_{L^\infty(\Gamma)} + K \text{diam } \Omega$ .

**Proof** Let  $z$  be a point on the line segment  $(x, \pi_\Gamma(x|y))$  satisfying

$$|x - z| = d_\Gamma(x|y) - \mu,$$

where  $\mu$  is a small positive number. Assume for ease of notation that  $z = 0$ . Then for some  $\lambda \in (0, 1)$  we have  $y = \lambda x$ . Let  $\epsilon > 0$  be such that  $B(x, \epsilon) \subset \Omega$ . According to the preceding lemma, we have, for almost all  $x' \in B(x, \epsilon)$ ,

$$\lambda u(x') \leq u(\lambda x') + K'(1 - \lambda).$$

Integrating this inequality over  $B(x, \epsilon)$ , we obtain, after a change of variables on the right side,

$$\frac{\lambda}{|B(x, \epsilon)|} \int_{B(x, \epsilon)} u(x') dx' \leq \frac{1}{|B(y, \lambda\epsilon)|} \int_{B(y, \lambda\epsilon)} u(y') dy' + K'(1 - \lambda).$$

Passing to the limit as  $\epsilon \rightarrow 0$  gives

$$\lambda u(x) - u(y) \leq K'(1 - \lambda),$$

which implies

$$u(x) - u(y) \leq \bar{K}(1 - \lambda)$$

since  $u$  is bounded by  $M$ . In view of the fact that

$$(1 - \lambda)|x| = (1 - \lambda)(d_\Gamma(x|y) - \mu) = |x - y|,$$

and because  $\mu$  is arbitrarily small, we obtain the conclusion of the Lemma.  $\square$

It follows from Lemma 2.3 that a representative of  $u$  is locally Lipschitz in the manner stated in Theorem 2.1. We now examine the behavior at the boundary; we establish a ‘lower Lipschitz’ semicontinuity:

**Lemma 2.4**

Let  $\gamma$  be any point in  $\Gamma$ . Then

$$u(x) \geq \phi(\gamma) - K|x - \gamma| \quad \forall x \in \Omega.$$

**Proof** By the lower bounded slope condition, there exists  $\zeta$ ,  $|\zeta| \leq K$ , such that

$$\phi(\gamma') \geq \phi(\gamma) + \langle \zeta, \gamma' - \gamma \rangle, \gamma' \in \Gamma.$$

Thus  $u \geq v$  a.e. on  $\Gamma$ , where  $v$  is the affine function  $v(x) := \phi(\gamma) + \langle \zeta, x - \gamma \rangle$ . By the Comparison Principle, we have  $u \geq v$  everywhere in  $\Omega$ , since  $u$  and  $v$  are continuous; the Lemma follows.  $\square$

This completes the proof of Theorem 2.1.

### 2.3 Proof of Theorem 2.2

**The polyhedral case.** We consider first the case in which  $\Gamma$  is a polyhedron, arguing by contradiction. Suppose that  $u$  fails to be continuous on  $\bar{\Omega}$ . Then, in view of the lower Lipschitz property noted in Lemma 2.4, there must be a sequence  $x_i$  of points in  $\Omega$  converging to a point  $\gamma \in \Gamma$  and such that, for some  $\epsilon > 0$ , we have

$$\lim_{i \rightarrow +\infty} u(x_i) \geq \phi(\gamma) + \epsilon.$$

Let  $T_\Gamma(\gamma)$  denote the tangent cone of convex analysis to  $\bar{\Omega}$  at  $\gamma$ . Because  $\bar{\Omega}$  is a convex body, it is known (see [5]) that this tangent cone has nonempty interior, and that for any  $v$  in its interior, the point  $x + rv$  lies in  $\Omega$  whenever  $x \in \bar{\Omega}$  is near  $\gamma$  and  $r > 0$  is sufficiently small. Because  $\Omega$  is a polytope, there is a more uniform version of this: for some  $\delta > 0$ , we have

$$x + T(\delta) \subset \Omega \quad \forall x \in \Omega \cap B(\gamma, \delta),$$

where, for any  $r > 0$ ,  $T(r) := T_\Gamma(\gamma) \cap B(0, r)$ . Now pick  $i$  sufficiently large so that  $|x_i - \gamma| < \delta/2$  and  $u(x_i) \geq \phi(\gamma) + 2\epsilon/3$ , and any  $y$  of the form  $x_i + v$  where  $v \in T(\delta)$ ,  $v \neq 0$ . We observe that (in view of the preceding inclusion)  $d_\Gamma(x_i|y) \geq \delta$ , so that by Lemma 2.3, we have  $u(y) \geq u(x_i) - \bar{K}|x_i - y|/\delta$ . Suppose that  $y$  is further asked to satisfy  $|x_i - y| < \epsilon\delta/(3\bar{K})$ . Then we deduce  $u(y) > \phi(\gamma) + \epsilon/3$ . We have

shown that for all  $i$  sufficiently large,  $u(x_i + T(\alpha)) > \phi(\gamma) + \epsilon/3$ , where  $\alpha := \min[\delta, \epsilon\delta/(3\bar{K})]$ . For any  $v$  in the interior of  $T(\alpha)$ , it is clear that  $\gamma + v$  lies in  $x_i + T(\alpha)$  for all  $i$  sufficiently large. In view of this, the preceding inclusion gives

$$u(\gamma + \text{int } T(\alpha)) > \phi(\gamma) + \epsilon/3.$$

However, the closure of  $\gamma + \text{int } T(\alpha)$  contains a neighborhood in  $\Gamma$  of  $\gamma$  (since  $\Omega$  is a polytope), and the trace of  $u$  on  $\Gamma$  is  $\phi$  (continuous). This contradiction proves the theorem in the case when  $\Gamma$  is a polyhedron.

**The uniformly convex case.** The fact that  $u$  is continuous when  $\Omega$  is uniformly convex is a special case of a result due to Pierre Bousquet [1]; we give his proof here for completeness.

**Proposition 2.1**

*When  $\Omega$  is uniformly convex and  $\phi$  is continuous, then  $u$  is continuous.*

**Proof** To see this, let  $\phi_i$  be a sequence of  $C^2$  functions on a neighborhood of  $\Gamma$  converging uniformly to  $\phi$ ; set  $\epsilon_i := \|\phi - \phi_i\|_\infty$ . When  $\Omega$  is uniformly convex, Miranda's Theorem asserts that each  $\phi_i$  satisfies the (full) bounded slope condition of some rank  $K_i$  (these will be unbounded in general). Then Hilbert-Haar theory yields the existence of a Lipschitz function  $u_i$  which minimizes  $I_\Omega(v)$  relative to all Lipschitz functions having value  $\phi_i$  on  $\Gamma$ .

By a result of Mariconda and Treu [14], it follows that  $u_i$  is also a minimizer relative to  $W^{1,1}(\Omega)$  (that is, no Lavrentiev phenomenon is present). On  $\Gamma$  we have

$$|u(\gamma) - u_i(\gamma)| = |\phi(\gamma) - \phi_i(\gamma)| \leq \epsilon_i \text{ a.e.,}$$

so by the Comparison Principle we must have  $|u - u_i| \leq \epsilon_i$  a.e. in  $\Omega$ . But then  $u$  is the uniform limit of a sequence of continuous functions, and so is continuous itself.  $\square$

## 2.4 Proof of Theorem 2.3

A *hyperwedge* signifies a subset  $W$  of a given hyperplane  $H$  having the form

$$W_H(v, \ell, \omega) = \{tv' : 0 \leq t \leq \ell, v' \in H, |v' - v| \leq \omega\},$$

where the unit vector  $v \in H$  is the direction of  $W$ ,  $\ell$  its length, and  $\omega$  its width. Because  $\Gamma$  is a polyhedron, there exist positive  $\ell$  and  $\omega$  such that, for any point  $\gamma \in \Gamma$ , for some hyperplane  $H = H(\gamma)$  and direction  $v = v(\gamma)$ , the set  $\gamma + W_H(v, \ell, \omega)$  lies in  $\Gamma$ . We may also take  $\ell$  and  $\omega$  small enough so that, for some  $m > 0$ , for any  $\gamma \in \Gamma$ , we have

$$x + W_{H(\gamma)}(v(\gamma), \ell, \omega) \subset \Omega \quad \forall x \in \Omega \cap B(\gamma, m).$$

**Lemma 2.5**

There exists a constant  $k$  with the following property: given any  $x \in \Omega$  lying within  $m$  of  $\Gamma$  and its projection  $\gamma$  onto  $\Gamma$ , one has

$$u(x) - \phi(\gamma) \leq k|x - \gamma|^{a_1},$$

where  $a_1$  is defined in the statement of Theorem 2.3.

**Proof** We assume that  $u(x) = \phi(\gamma) + \delta$  for some  $\delta > 0$ , for otherwise there is nothing to prove. Note that  $\gamma$  (which we do not require to be the unique projection) is necessarily in the interior of some facet, and therefore defines an  $(n-1)$ -dimensional subspace  $H$  whose associated hyperplane  $\gamma + H$  contains that facet.

By the remark above, there is a hyperwedge  $W = W_H(v, \ell, \omega)$  such that  $\gamma + W$  lies in  $\Gamma$  (and in  $\gamma + H$ ), and such that  $x + W \subset \Omega$ . For every point  $x' \in x + W$  we have  $d_\Gamma(x|x') \geq \ell$ . In light of Theorem 2.1 we deduce (for such  $x'$ )

$$u(x') \geq u(x) - \bar{K}|x - x'|/\ell = \phi(\gamma) + \delta - \bar{K}|x - x'|/\ell.$$

We associate with every point  $x' \in x + W$  its projection  $\gamma' \in \gamma + H$ . For almost every such  $\gamma'$  (relative to  $(n-1)$ -dimensional measure) we have

$$u(\gamma') = \phi(\gamma') \leq \phi(\gamma) + K|\gamma' - \gamma|,$$

since  $\phi$  is Lipschitz of rank  $K$ . Since  $|\gamma' - \gamma| = |x' - x|$ , it follows that

$$u(x') - u(\gamma') \geq \delta - c|x' - x|,$$

for  $c = K + \bar{K}/\ell$ . This implies (almost everywhere  $x'$ )

$$\int_0^d |Du(x' + t\nu) \cdot \nu| dt \geq \delta - c|x' - x|,$$

where  $\nu$  is the (unit) normal vector to  $\Gamma$  at  $\gamma$  and  $d := |x - \gamma|$ . We now limit the  $x'$  in question to those satisfying  $|x' - x| < \rho_0\delta$ , where  $\rho_0$  is chosen small enough to guarantee both  $c|x' - x| < \delta/2$  and  $|x - x'| < \ell$ ; note that such a  $\rho_0$  can be chosen as a function of just  $M$  and  $\ell$ , since  $\delta$  is bounded above *a priori* by  $2M$ . Set  $S := (x + W) \cap B(x, \rho_0\delta)$ . Then the last inequality gives

$$\int_0^d |Du(x' + t\nu) \cdot \nu| dt \geq \delta/2 \quad \forall x' \in S.$$

Applying Hölder's inequality to the integral on the left, and then integrating over  $x' \in S$ , we obtain, with the help of the change of variables formula,

$$d^{p-1} \int_S \int_{[0,d]} |Du|^p dt dx' \geq c' \delta^p (\rho_0\delta)^{n-1},$$

for a constant  $c'$  depending on the width  $\omega$  of the hyperwedge  $W$  and the dimension  $n$ . The iterated integral on the left is bounded *a priori* because  $F$  is coercive. We immediately deduce an estimate of the form  $\delta^{n+p-1} \leq c'' d^{p-1}$ , which gives the required result, for a certain  $k$ .  $\square$

This proves a boundary estimate of the type that appears in the statement of the theorem, but only for the case in which  $\gamma$  is the projection of  $x$  onto  $\Gamma$ . We proceed now to establish such an estimate generally, for any  $x \in \Omega$  and  $\gamma \in \Gamma$ . Let  $\gamma'$  be a nearest point in  $\Gamma$  to  $x$ . We consider first the case in which  $|x - \gamma'| < m$ . Then we may argue as follows:

$$\begin{aligned} u(x) - \phi(\gamma) &= u(x) - \phi(\gamma') + \phi(\gamma') - \phi(\gamma) \\ &\leq k|x - \gamma'|^{a_1} + K|\gamma' - \gamma| \end{aligned}$$

(by Lemma 2.5, and since  $\phi$  is Lipschitz)

$$\leq k|x - \gamma|^{a_1} + K|\gamma' - x| + K|x - \gamma|$$

(since  $\gamma'$  is a closest point in  $\Gamma$  to  $x$ )

$$\leq k|x - \gamma|^{a_1} + 2K|x - \gamma| \leq k_1|x - \gamma|^{a_1}$$

for some new constant  $k_1$ . By taking  $k_1$  greater than  $2Mm^{-a_1}$ , the upper bound for  $u(x) - \phi(\gamma)$  just obtained will continue to hold in the remaining case in which  $|x - \gamma'| \geq m$  (since  $|x - \gamma| \geq |x - \gamma'|$ ). The general estimate thereby follows.

We now complete the proof of Theorem 2.3. We set  $a' := 1/(1 + a_1)$ ,  $a_2 := a_1 a'$ . We establish the existence of a certain constant  $\bar{k}$  such that, for any two points  $x, x'$  in  $\Omega$ , the following inequality holds:

$$u(x) - u(x') \leq \bar{k}|x - x'|^{a_2}.$$

Let  $\gamma$  be the projection of  $x$  onto  $\Gamma$ .

**Case 1:**  $|x - x'|^{a'} \geq |x - \gamma|$ . We write

$$\begin{aligned} u(x) - u(x') &= [u(x) - \phi(\gamma)] + [\phi(\gamma) - u(x')] \\ &\leq k|x - \gamma|^{a_1} + K|\gamma - x'| \end{aligned}$$

(by the preceding estimate, and since  $\phi$  is Lipschitz)

$$\begin{aligned} &\leq k|x - x'|^{a_1 a'} + K|\gamma - x| + K|x - x'| \\ &\leq k|x - x'|^{a_2} + K|x' - x|^{a'} + K|x - x'| \\ &\leq c|x - x'|^{a_2}, \end{aligned}$$

for a suitable constant  $c$ .

**Case 2:**  $|x - x'|^{a'} < |x - \gamma|$ . We have

$$d_{\Gamma}(x|x') \geq |x - \gamma| > |x - x'|^{a'},$$

so that by Theorem 2.1 it follows that

$$u(x) - u(x') \leq \bar{K}|x - x'|/|x - x'|^{a'} = \bar{K}|x - x'|^{a_2}.$$

We get the required inequality by taking  $\bar{k} := \max(c, \bar{K})$ , and the theorem follows. (Lemma 2.4 continues to apply here, as regards the lower boundary estimate.)

## 2.5 Proof of Theorem 2.4

In the proof we shall use the fact that when  $\Gamma$  is  $C^{1,1}$  (and compact),  $\Omega$  satisfies a uniform *inner ball condition*: there exists  $R > 0$  with the following property: for every point  $\gamma \in \Gamma$  there is a closed ball of radius  $R$  contained in  $\bar{\Omega}$  which contacts the boundary at  $\gamma$ . (This is a familiar property in regularity theory for pde's; see for example [4]). We note as well that there is a unique exterior unit normal vector  $\nu(\gamma)$  to  $\Gamma$  (or the ball) at each boundary point, and  $\nu(\cdot)$  is Lipschitz continuous. Finally, every point in  $\Omega$  of distance less than  $R$  from  $\Gamma$  admits a unique projection (closest point) in  $\Gamma$ .

We assume that  $F$  is coercive of order  $p > (n + 1)/2$ , and we set

$$\alpha_1 := (2p - n - 1)/(2n + 2p - 2) \in (0, 1).$$

We first prove a boundary-based Hölder condition.

### Lemma 2.6

*There exist positive constants  $\kappa$  and  $m$  such that for any  $\gamma \in \Gamma$ , for any  $x \in \Omega$  of the form  $\gamma - t\nu(\gamma)$ ,  $0 < t < m$ , one has*

$$u(x) - \phi(\gamma) \leq \kappa|x - \gamma|^{\alpha_1}.$$

**Proof** We take  $m < R/4$  to begin; it will be subject to another upper bound presently. Note that  $\gamma$  is the closest point to  $x$  in  $\Gamma$ . We assume for notational convenience, and without loss of generality, that  $\gamma - R\nu(\gamma) = 0$ , and we denote by  $H_0$  the hyperplane through 0 that is perpendicular to  $\nu(\gamma)$ . For points  $y$  lying 'above'  $H$  (that is, for which  $\langle y, \nu(\gamma) \rangle > 0$ ) we consider the coordinate system in which a point  $y$  is expressed in the form  $r\theta$ , for some  $\theta \in \Gamma$  and  $0 < r$ . Then points in  $\Gamma$  are those having coordinates of the form  $(1, \theta)$ .

We assume that  $u(x) > \phi(\gamma)$ , for otherwise there is nothing to prove; let

$$\delta_x := u(x) - \phi(\gamma), \quad m_x := |x - \gamma|.$$



We denote by  $S$  the part of the hyperplane  $x + H_0$  lying in  $B(0, R)$ , and by  $R_x$  its radius: the distance in  $x + H_0$  from  $x$  to the relative boundary of  $S$ . If  $\rho$  lies in  $(0, R_x)$  and  $|\theta - \gamma| < \rho$ , then the ray  $r\theta$  meets  $S$  at a point we label  $r_\theta\theta$ . We consider the following domain  $A$ :

$$A := \{y = r\theta : |\theta - \gamma| < \rho, r_\theta \leq r \leq 1\},$$

the points ‘radially above’  $S$  and ‘below’  $\Gamma$ . For any point  $y$  in  $A$  having the form  $r_\theta\theta$  (a point in  $x + H_0$ , on the ‘lower’ boundary of  $A$ ), we have

$$d_\Gamma(x|y) \geq R_x \geq cm_x^{1/2},$$

where  $c$  depends only on  $R$ . By Theorem 2.1 we deduce

$$u(r_\theta\theta) \geq u(x) - \bar{K}|x - r_\theta\theta|/[cm_x^{1/2}].$$

For almost all points  $y$  in  $A$  of the form  $\theta \in \Gamma$  (the ‘upper’ boundary),  $u$  is absolutely continuous along the ray  $r\theta$ , differentiable at almost all points of the ray, and we have  $u(y) = \phi(y)$ , whence

$$u(\theta) = \phi(\theta) \leq \phi(\gamma) + K|\theta - \gamma| = u(x) + \delta_x + K|\theta - \gamma|.$$

We conclude from the last two inequalities that for such  $\theta$ ,

$$\begin{aligned} \int_{r_\theta}^1 |\langle Du(r\theta), \theta \rangle| dr &\geq \delta_x - K|\theta - \gamma| - \bar{K}|r_\theta\theta - x|/[cm_x^{1/2}] \\ &\geq \delta_x - K|\theta - \gamma| - \bar{K}|\theta - \gamma|/[cm_x^{1/2}] \\ &\geq \delta_x - 2(\bar{K}/c)m_x^{-1/2}|\theta - \gamma| \end{aligned}$$

(since  $m_x \leq m$ , provided we choose  $m < [\bar{K}/(cK)]^2$ )

$$\geq \delta_x - c'm_x^{-1/2}\rho,$$

where  $c'$  depends only on  $\bar{K}$  and  $c$ . Let us now specify  $\rho = \rho_0 m_x^{1/2} \delta_x$ ; then we get

$$\int_{r_\theta}^1 |\langle Du(r\theta), \theta \rangle| dr \geq \delta_x [1 - c'\rho_0];$$

we pick  $\rho_0 < 1/c'$  to make this positive. Further, if we choose  $\rho_0 < c/(2M)$  we also have

$$\rho = \rho_0 m_x^{1/2} \delta_x < [c/(2M)](R_x/c)(2M) = R_x$$

(since  $\delta_x \leq 2M$ ), as required above. The previous inequality now becomes

$$\int_{r_\theta}^1 |\langle Du(r\theta), \theta \rangle| dr \geq c''\delta_x.$$

Because  $\Gamma$  lies below the supporting hyperplane  $\gamma + H_0$  to  $\bar{\Omega}$  at  $\gamma$ , there is a constant  $C$  depending only on  $R$  such that

$$(1 - r_\theta) \leq C|x - \gamma| = Cm_x.$$

Applying Hölder's inequality to the integral above, we deduce

$$C'm_x^{p-1} \int_{r_\theta}^1 |\langle Du(r\theta), \theta \rangle|^p dr \geq \delta_x^p$$

for a certain constant  $C'$ . We proceed to integrate this inequality over  $\theta \in \Gamma \cap B(\gamma, \rho)$ . Since  $|Du|^p$  is summable over  $\Omega$  (as follows from coercivity), it is summable over  $A$ , and we get (after invoking the change of variables formula)

$$C''m_x^{p-1} \geq \delta_x^p \rho^{n-1} = \delta_x^p [\rho_0 m_x^{1/2} \delta_x]^{n-1}.$$

where  $C''$  depends on  $R, K, M$  and the coercivity constants of  $F$ , but not on the specific  $x$  or  $\gamma$ . This gives the estimate in the statement of the Lemma, for a certain constant  $\kappa$ .  $\square$

This proves a boundary estimate of the type that appears in the statement of the theorem, but only for the case in which  $\gamma$  is the projection of  $x$  onto  $\Gamma$ . We proceed now to establish the estimate more generally. Let  $x$  be any point within  $m$  of the boundary and  $\gamma \in \Gamma$ . Let  $\gamma'$  be the nearest point in  $\Gamma$  to  $x$ . Then

$$\begin{aligned} u(x) - \phi(\gamma) &= u(x) - \phi(\gamma') + \phi(\gamma') - \phi(\gamma) \\ &\leq \kappa|x - \gamma'|^{\alpha_1} + K|\gamma' - \gamma| \end{aligned}$$

(by Lemma 2.6, and since  $\phi$  is Lipschitz)

$$\begin{aligned} &\leq \kappa|x - \gamma|^{\alpha_1} + K|\gamma' - x| + K|x - \gamma| \\ &\leq \kappa|x - \gamma|^{\alpha_1} + 2K|x - \gamma| \leq \kappa_1|x - \gamma|^{\alpha_1} \end{aligned}$$

for some new constant  $\kappa_1$ .

We now proceed to derive a global Hölder condition, not just based at boundary points, but valid on all of  $\bar{\Omega}$ . It suffices to show that  $u$  satisfies a Hölder condition in a neighborhood of  $\Gamma$ , since on strictly interior domains  $u$  is actually Lipschitz. We define  $\alpha' := 1/(1+\alpha_1)$ ,  $\alpha_2 := \alpha_1\alpha'$ . We will establish that for a certain constant  $\bar{\kappa}$ , for any two points  $x$  and  $x'$  in  $\bar{\Omega}$  within distance  $m$  of  $\Gamma$ , one has

$$u(x) - u(x') \leq \bar{\kappa}|x - x'|^{\alpha_2}.$$

Let  $\gamma$  be the projection of  $x$  onto  $\Gamma$ . We consider two cases, according to whether  $|x - x'|^{\alpha'}$  is greater than  $|x - \gamma|$  or not. In the first case,

we write

$$\begin{aligned} u(x) - u(x') &= [u(x) - \phi(\gamma)] + [\phi(\gamma) - u(x')] \\ &\leq \kappa|x - \gamma|^{\alpha_1} + K|\gamma - x'| \end{aligned}$$

(by Lemma 2.6 and by Lemma 2.4 respectively)

$$\begin{aligned} &\leq \kappa|x - x'|^{\alpha_1} + K|x - x'| + K|x - \gamma| \\ &\leq \kappa|x - x'|^{\alpha_1} + K|x - x'| + K|x - x'|^{\alpha'} \\ &\leq c'|x - x'|^{\alpha_2}, \end{aligned}$$

for a certain new constant  $c'$ .

There remains the other case to consider, in which

$$|x - x'|^{\alpha'} < |x - \gamma|.$$

We then have

$$d_\Gamma(x|x') \geq |x - \gamma| > |x - x'|^{\alpha'}.$$

Applying Theorem 2.1 gives

$$u(x) - u(x') \leq \bar{K} \frac{|x - x'|}{|x - x'|^{\alpha'}} = c''|x - x'|^{1-\alpha'} = c''|x - x'|^{\alpha_2}.$$

The result follows, for  $\bar{\kappa} = \max(c', c'')$ , and Theorem 2.4 is proved. (The other (lower) boundary estimate in the theorem continues to be provided by Lemma 2.4.)

**Proof of Corollary 1.2.** Given  $\psi \in C_c^1(\Omega)$  and any  $t > 0$ , the optimality of  $u$  implies  $I_\Omega(u + t\psi) \geq I_\Omega(u)$ . Upon dividing by  $t$  and then letting  $t$  tend to 0 we obtain

$$\int_\Omega \nabla F(Du(x)) \cdot D\psi(x) dx = 0,$$

where the integration and the limit commute because  $Du(x)$  is bounded for  $x \in \text{supp } \psi$ . Thus  $u$  is a weak solution of the Euler equation, which is the first assertion of Corollary 1.2. The second assertion holds because under the stated assumptions, and as we know  $Du$  to be locally bounded by the theorem, De Giorgi's regularity theorem applies.

### 3 Proof of Theorem 1.2

We are given that  $\phi$  is Lipschitz of rank  $K$  (say) on  $\Gamma$ . We exploit this by writing

$$\phi(\gamma) = \min_{q \in \Gamma} \{\phi(q) + K|\gamma - q|\}.$$

Since the function  $\phi_q(y) := \phi(q) + K|y - q|$  is convex and Lipschitz of rank  $K$  on  $\mathbb{R}^n$ , this reveals  $\phi$  to be the lower envelope (attained for each point) of a family of functions each satisfying the lower bounded slope condition of rank  $K$ . We write  $\phi = \min_q \phi_q$  to signify this fact; note that  $\|\phi_q\|_\infty$  is bounded by a constant depending just on the data of  $(P)$ .

Since  $F$  is coercive, there is (for any given  $q$ ) a solution  $u_q$  to the version of  $(P)$  in which the boundary condition is given by  $\phi_q$ . In light of Theorems 2.3 and 2.4,  $u_q$  is Hölder continuous on  $\bar{\Omega}$ . Note that on  $\Gamma$  we have  $u = \phi \leq \phi_q = u_q$  a.e. By the Comparison Principle, therefore, we have  $u \leq u_q$  a.e. in  $\Omega$ .

The existence of the minimizers  $u_q$  will allow us to prove an analogue of Lemma 2.1 in the proof of Theorem 2.1. Fixing a point  $z$  in  $\Omega$  and a scalar  $\lambda$  in  $(0, 1)$ , we define  $\Omega_\lambda$  and  $u_\lambda$  exactly as in the proof of Theorem 2.1. We set  $\beta := a_1$  if case (a) of the theorem holds, and  $\beta := \alpha_1$  in case (b) (Note:  $a_1$  and  $\alpha_1$  remain as defined in the statements of Theorems 2.3 and 2.4.) Lemma 2.1 now becomes :

**Lemma 3.1**

*There exists a constant  $K'$  depending only on the data of  $(P)$  such that*

$$u(\gamma_\lambda) - u_\lambda(\gamma_\lambda) \leq K'(1 - \lambda)^\beta, \quad \gamma \in \Gamma \text{ a.e.}$$

**Proof** Let  $\gamma \in \Gamma$  be a point for which  $u(\gamma) = \phi(\gamma)$ , and let  $q$  be an index for which  $\phi(\gamma) = \phi_q(\gamma)$ . Then

$$\begin{aligned} u(\gamma_\lambda) - u_\lambda(\gamma_\lambda) &= u(\lambda\gamma) - \lambda u(\gamma) \\ &\leq u_q(\lambda\gamma) - \lambda\phi(\gamma) \end{aligned}$$

(assuming for now that the point  $\lambda\gamma$  is not a point of exception for the inequality  $u \leq u_q$ )

$$\begin{aligned} &= u_q(\lambda\gamma) - \phi_q(\gamma) + (1 - \lambda)\phi_q(\gamma) \\ &\leq k'|\gamma - \lambda\gamma|^\beta + (1 - \lambda)\|\phi_q\|_\infty \end{aligned}$$

(where  $k'$  is either the  $k_1$  of Theorem 2.3 or the  $\kappa_1$  of Theorem 2.4)

$$\leq \bar{k}(1 - \lambda)^\beta,$$

for an appropriate constant  $\bar{k}$ . This argument can be modified to avoid supposing that  $\lambda\gamma$  is not a point of exception, precisely as was done in the proof of Lemma 2.1. We omit the details, since the fact that  $(1 - \lambda)$  is replaced by  $(1 - \lambda)^\beta$  causes no significant changes.  $\square$

The next step in the proof also follows very closely that of Theorem 2.1. From the Comparison Principle we have  $u(x) \leq u_\lambda(x)$  a.e. on  $\Omega_\lambda$ ,

which leads as before to the conclusion that  $u$  is continuous on  $\Omega$  and satisfies

$$u(y) - u(x) \leq \frac{\bar{k}|x - y|^\beta}{d_\Gamma(x|y)^\beta} \quad \forall x, y \in \Omega, \quad x \neq y.$$

This implies that  $u$  is locally Hölder continuous in  $\Omega$  (of uniform order  $\beta$ ) in the manner stated in Theorem 1.2. The fact that  $u$  is continuous on  $\bar{\Omega}$  when  $\Omega$  is uniformly convex was already noted in connection with the proof of Theorem 2.2.

There remains to show that  $u$  satisfies a Hölder condition in case (a); the key to this is to establish an analogue of Lemma 2.5. The proof of the latter requires only minor changes: the appeal to Theorem 2.1 is replaced by one to Theorem 2.3, which has the effect of replacing  $|x - x'|$  by  $|x - x'|^{\alpha_2}$  in the rest of the argument. This merely reduces the ensuing Hölder exponent.

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