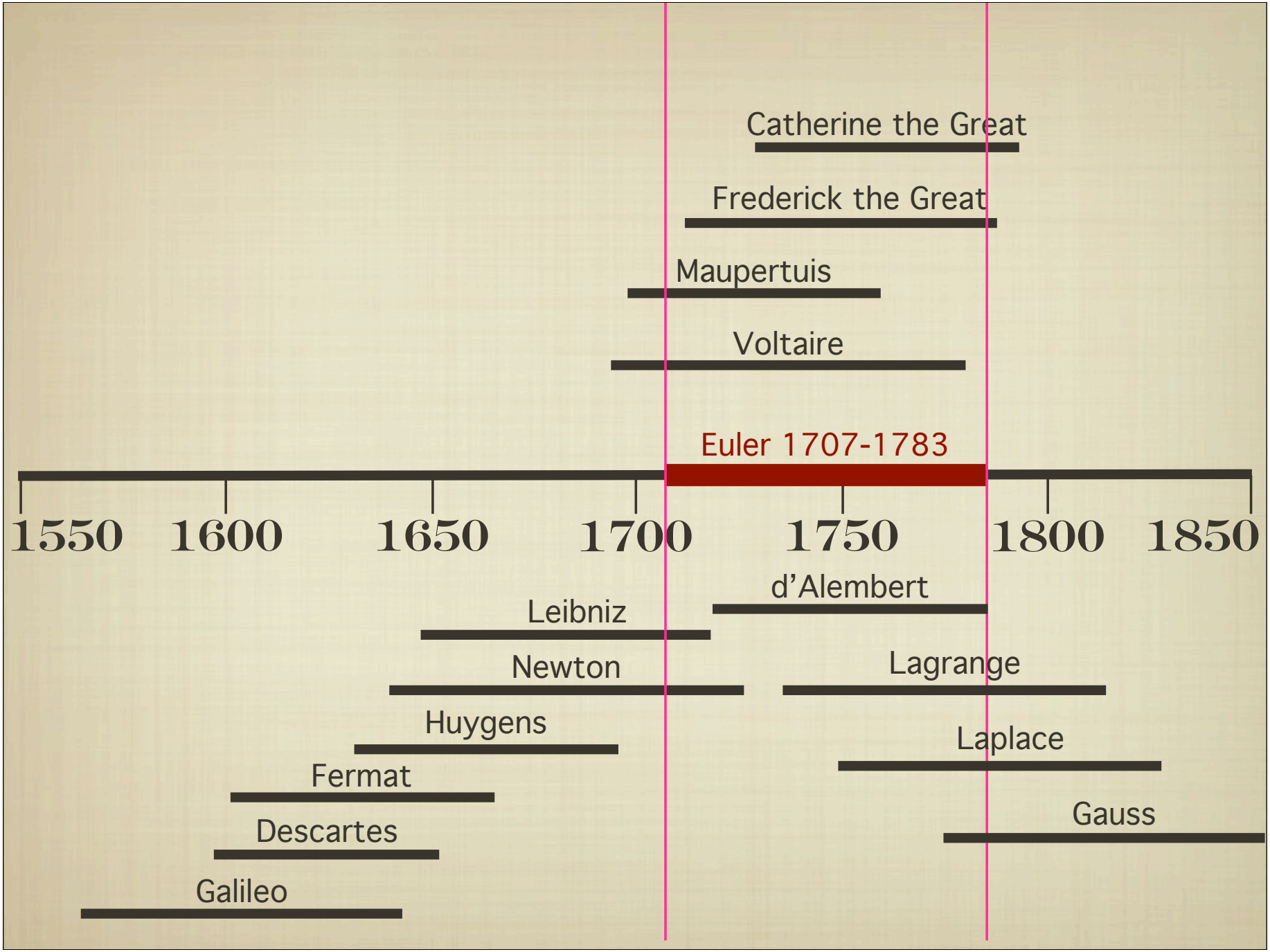
Euler : Life, the universe and optimization

Francis Clarke
Institut universitaire de France
et Université de Lyon


Catherine the Great
Frederick the Great
Maupertuis
Voltaire
Euler 1707-1783
1550
1600
1650
1700
1750
1800
1850
d'Alembert
Leibniz
Newton
Lagrange
Huygens
Laplace
Fermat
Gauss
Descartes
Galileo

Leonhard Euler

1707-1783

Born in Bâle (Switzerland)
in 1707

- *prodigy*
- *university at 14*
- *Johann Bernoulli*
- ~~*theology*~~ → math

- **Academy of Saint-Petersburg (19 to 34)**

  Catherine I

- **Academy of Berlin (34 to 59)**

  Frederick the Great, Maupertuis, Voltaire

- **Academy of Saint-Petersburg (59 to 76)**

  Catherine the Great, Russian school

Publications

pure mathematics:
number theory, algebra, combinatorics, series, differential and integral calculus, geometry, topology, complex variables, probability, calculus of variations…

applied mathematics:
naval science, mechanics, optics, hydrodynamics, elasticity, electricity, numerical analysis, acoustics, music, astronomy, optimization…

- 800 publications (+ correspondence)
- Omnia Opéra : 80 volumes
- between 1725 and 1800, Euler = $\frac{1}{3}\left(\text{math} + \text{math phys} + \text{mech engin}\right)$
- blind as of 1771
- one publication per week in 1775

Publications : some highlights

- ***Introductio***
  1748
- ***Differential and Integral Calculus***
  1755-1770
- ***Lettres à une princesse d'Allemagne***
  1768
- ***Algebra*** **(Russian, German, French, Latin, English...)**
  1770

2587 pages

bestseller

INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS VSU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

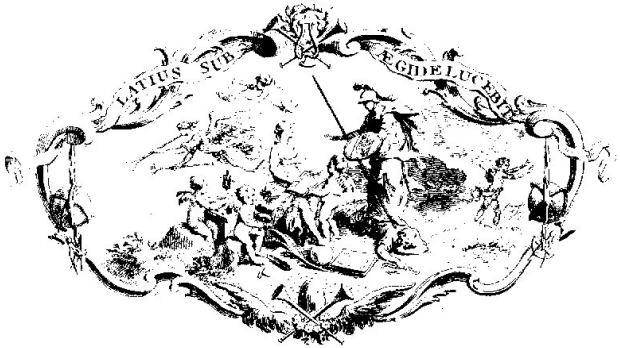
INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI
Impensis Academiae Imperialis Scientiarum
1768.

INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ
Socio.

TOMUS PRIMUS.



LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

Three equations in Euler's career

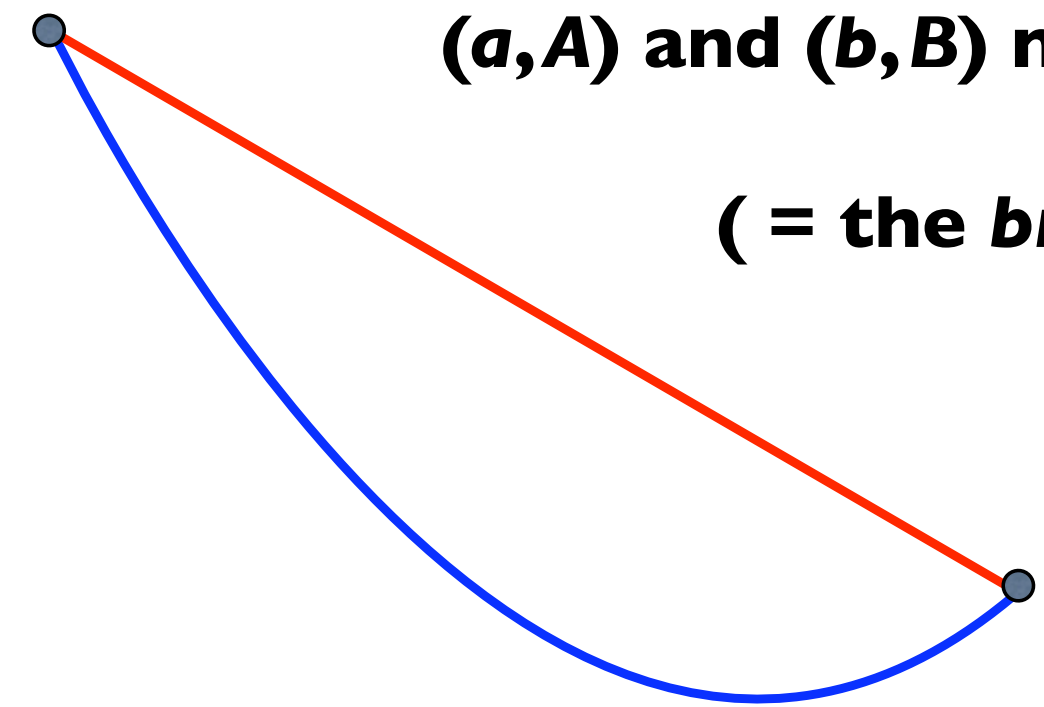
A) $1 + 1/4 + 1/9 + 1/16 \ldots + 1/k^2 + \ldots = \pi^2/6$

B) $e^{\pi i} = -1$

C) Euler's equation in

optimization over curves

= calculus of variations



Which curve y(x) joining the points (a,A) and (b,B) minimizes the time of descent?

(= the *brachistochrone*)

We seek to minimize

$$\int_a^b \frac{\sqrt{1+y'(x)^2}}{\sqrt{y(x)-A}}\,dx$$

relative to the curves y(x) satisfying

y(a) = A, y(b) = B

Euler's monograph of 1744 :

Méthodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes sive Solutio Problematis Isoperimetrici Latissimo Sensu

We find :

- **Statement of the general problem**
- **Euler's equation**
- **Principle of least action**
- **Method of multipliers for constraints**
- **100 examples**

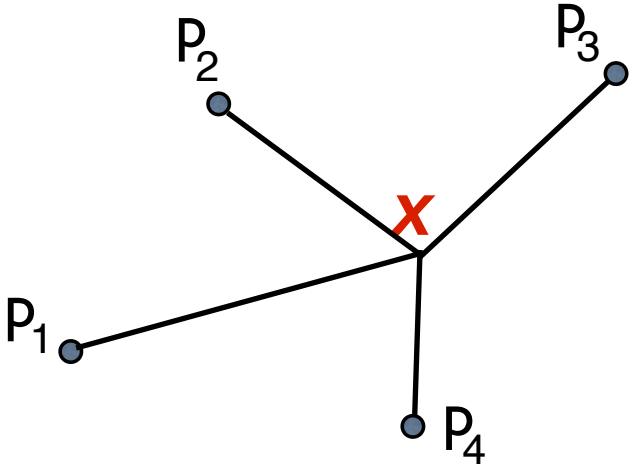
'Nothing at all takes place in the universe in which some rule of maximum or minimum does not appear'

Axiom :

The behavior of a physical system corresponds to a minimum.

Recall : the case of a static equilibrium…

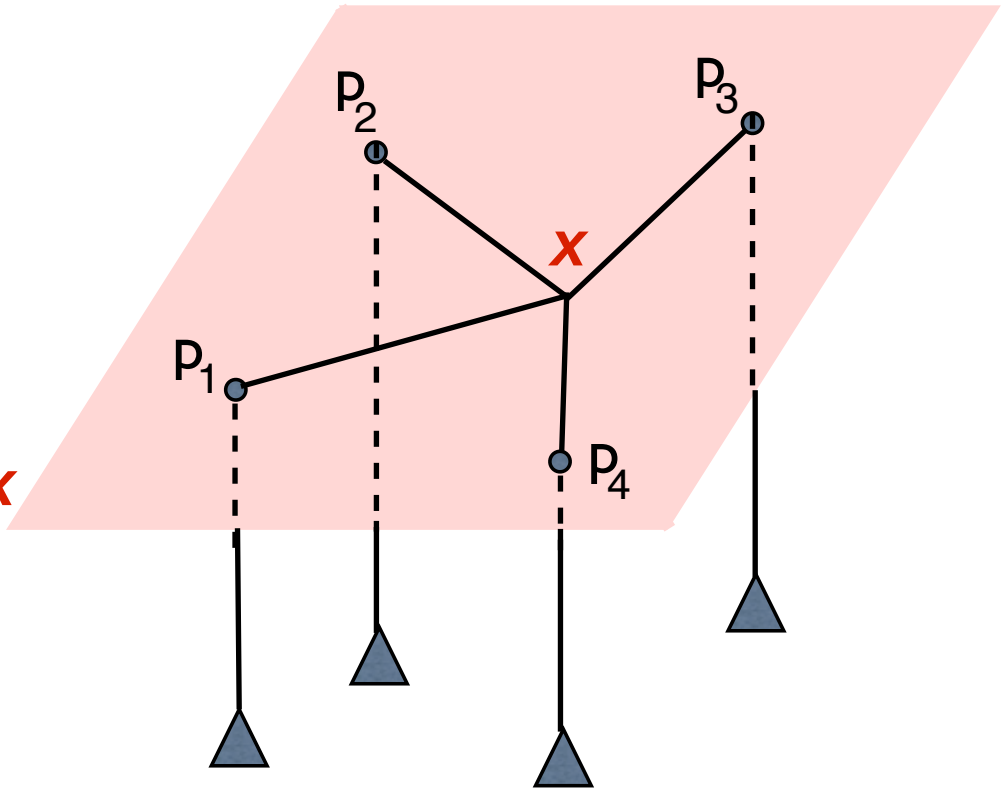
Problem of Torricelli / Steiner



We seek a point *x* which is *central* relative to P_1, P_2, P_3, P_4 : that is, which minimizes

$$|P_1 - x| + |P_2 - x| + |P_3 - x| + |P_4 - x|$$

Solution by furniture :



At equilibrium, the point *x* minimizes the potential energy of the system (d'Alembert / Maupertuis)

It is surprising that Euler, to whom you have been so generous, has obtained so easily your permission to leave after twenty-six years.

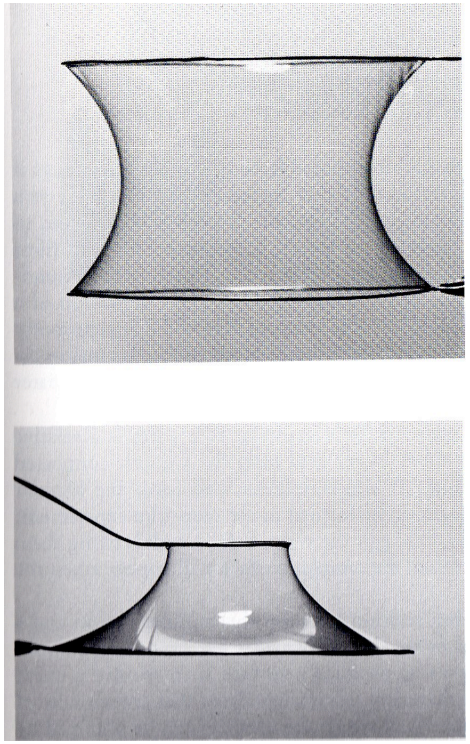
Jean le Rond
d'Alembert

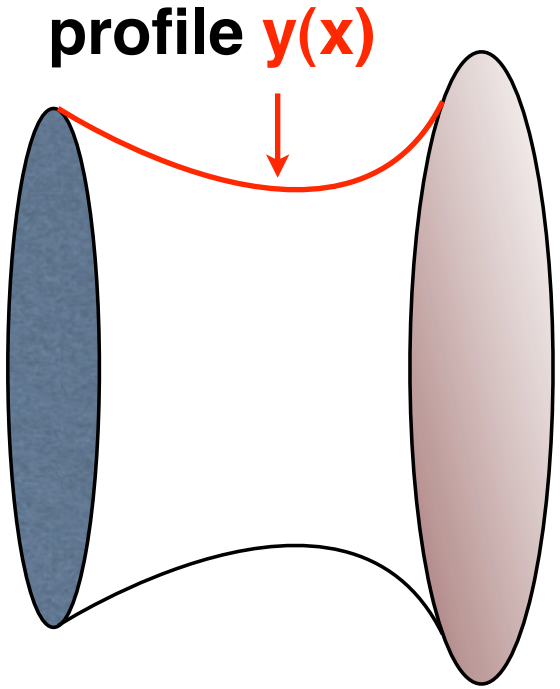
Paris 1717 – Paris 1783

Il est bien singulier que M. Euler, comblé de biens par Votre Majesté, lui et sa famille, ait obtenu son congé si aisément après vingt-six ans de séjour.

Lettre au roi de Prusse
26 mai 1766

One of Euler's examples





The observed surface is of minimal area

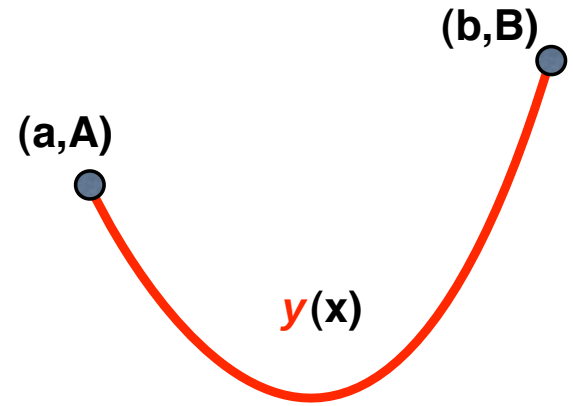
$$\min_{y(\cdot)} \int_a^b y(x)\sqrt{1+y'(x)^2}\,dx \quad \Rightarrow \quad \frac{d}{dx}\left\{\frac{y'(x)\,y(x)}{\sqrt{1+y'(x)^2}}\right\} = \sqrt{1+y'(x)^2}$$

Euler's Principle of Least Action is also a *dynamic* extension of the static case.

Examples : trajectory of a ball, orbit of a planet, oscillation of a pendulum

The method of *multipliers* allows one to treat constraints.

Exemple



The curve *y*, a *catenary*, minimizes the potential energy relative to curves of prescribed length.

Two personalities closely linked to Euler, and in particular to his monograph :

Maupertuis* and *Lagrange

Pierre-Louis Moreau de Maupertuis

Born Saint-Malo 1698

- soldier
- explorer
- *héros de salon*
- *causeur*
- philosopher of science

- named president of his Academy by Frederick the Great
- recruits Euler in 1741

Voltaire, on Maupertuis, *before* knowing him :

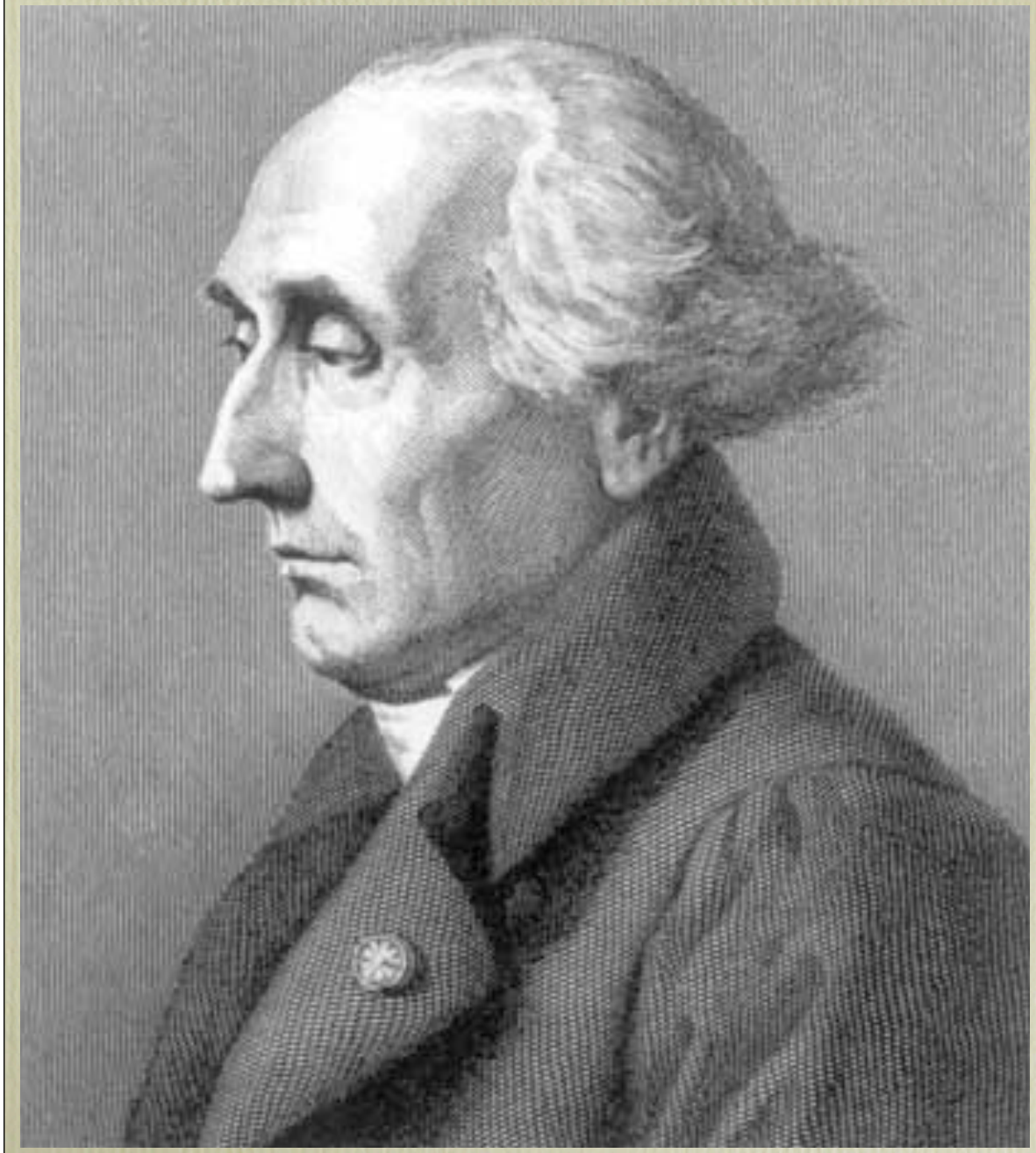
Héros de la physique, Argonautes nouveaux
Qui franchissez les monts, qui traversez les eaux,
Dont le travail immense et l'exact mesure
De la terre étonnée ont fixé la figure.

and *after* :

Courrier *de la physique, Argonautes nouveaux*
Qui franchissez les monts, qui traversez les eaux,
Ramenez des climats, soumis aux trois couronnes
Vos perches, vos secteurs, et surtout deux Lapones!
Vous avez confirmé dans ces lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui.

great scandal

In 1744, one year after receiving Euler's manuscript, Maupertuis publishes the principle of least action as his own... Euler refuses to condemn him



Joseph Louis Lagrange

Born Turin 1736

- **Writes to Euler in 1755, describes the method of *variations***
- **Euler names the subject in his honor : *calculus of variations***
- **Euler is his mentor until his death**

Lagrange

'Euler-Lagrange equation'
'Lagrange multipliers' !

- ***1788 : Mécanique Analytique***
  = calculus of variations + multipliers + least action
- **He replaces Euler in Berlin in 1766 ; Frederick writes :**

  Europe's greatest king desires to have at his court Europe's greatest mathematician

- **Euler and Lagrange : the two great mathematicians of 18th**
- **Later, Lagrange emphasizes the regularity of functions :**

  Théorie des Fonctions analytiques, contenant les Principes du Calcul différentiel, dégagés de toute Considération d'infiniment Petits, d'Evanouissans, de Limites et de Fluxions, et réduits à l'Analyse algébrique des Quantités finies (1797)

Three phenomena related to non regularity, and studied neither by Euler nor by Lagrange

- **Abrupt transitions**

  19th century : Jacobi, conjugate points
  20th : catastrophes

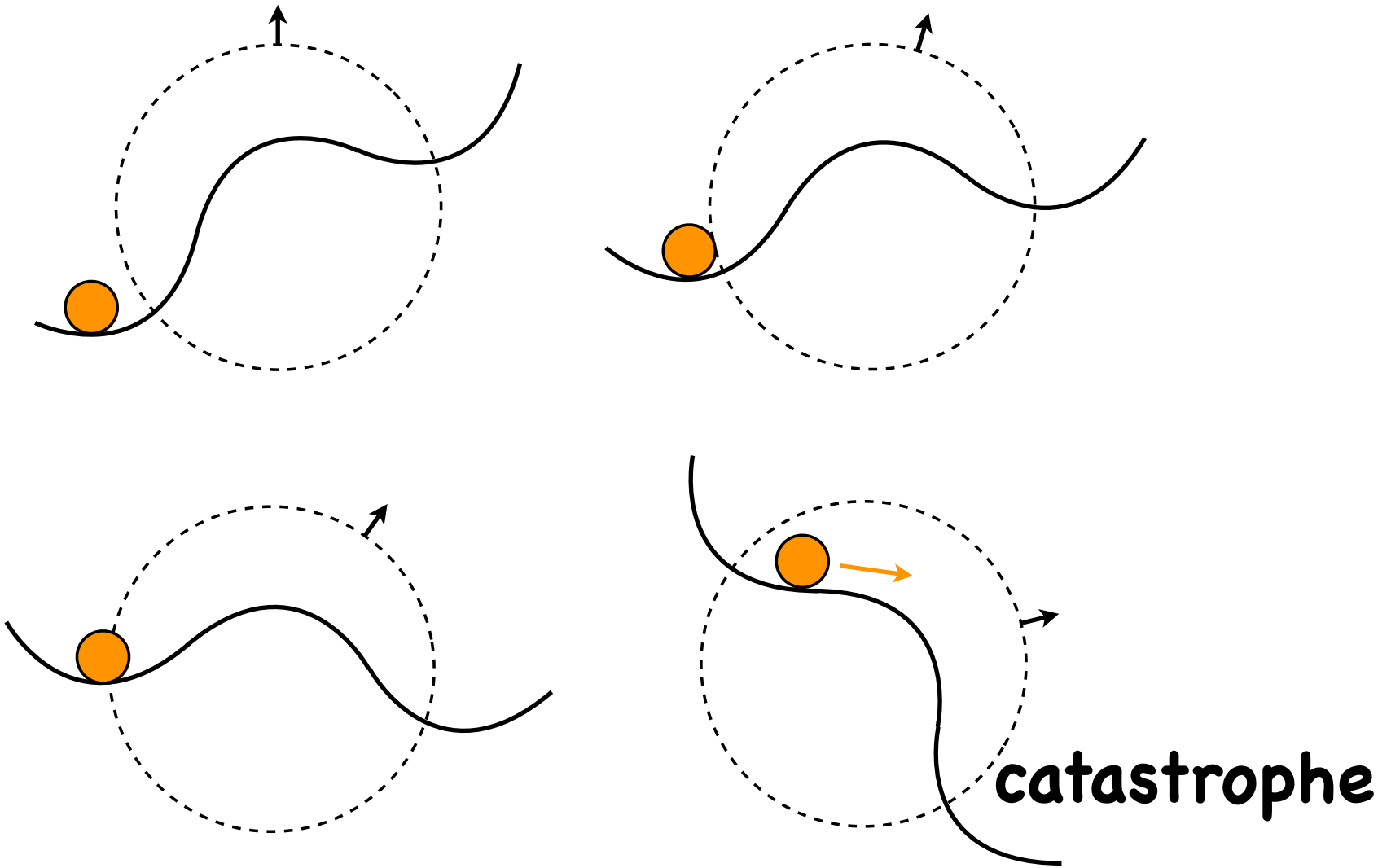
- **Nonsmooth solutions (with corners)**

  19th : duBois-Reymond, Weierstrass
  20th : Hilbert, Sobolev, De Giorgi…

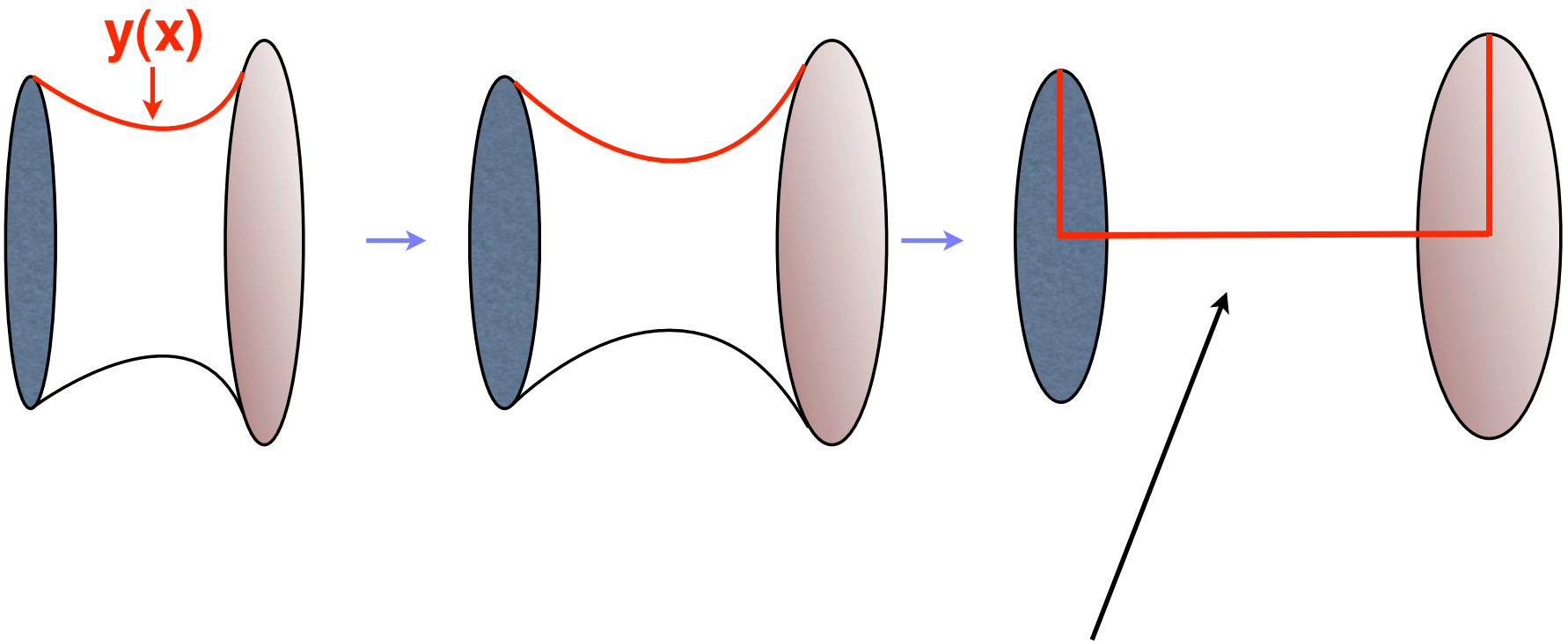
- **Nonsmooth behavior (non differentiable)**

  20th : NonsmoothAnalysis

Equilibrium shifts (first irregularity)



Solutions with corners (second irregularity)



The Goldschmidt solution (1831)

A design problem: the optimal column

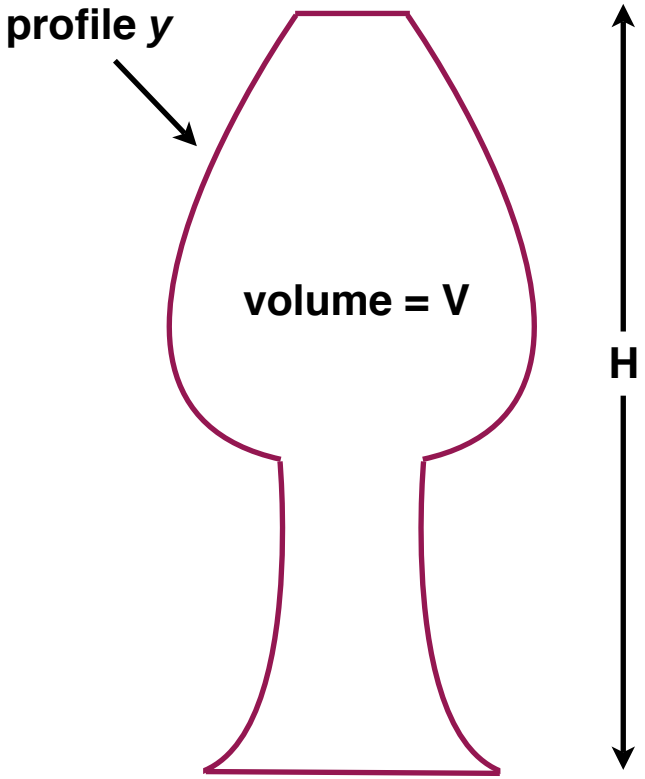
Ainsi c'est un problème de maximis et minimis de déterminer la courbe qui, par sa rotation autour de son axe formera une colonne capable de supporter la plus grande charge possible, la hauteur et la masse de la colonne étant données.

Lagrange (1770) *Sur la figure des colonnes*

To find the curve which by its revolution determines the column of greatest efficiency.

Truesdell

Designing a column of revolution



1. Choose a profile *y*

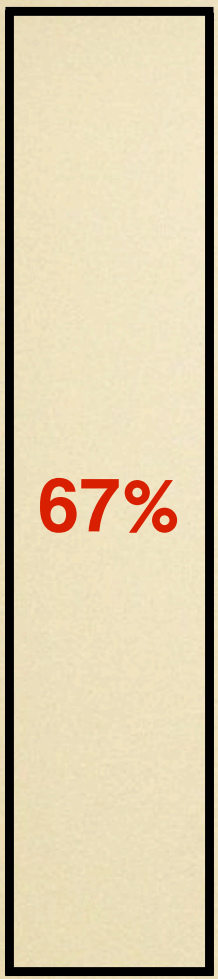
2. Effect a rotation to generate a column C(*y*)

3. Respect the constraints on volume and height

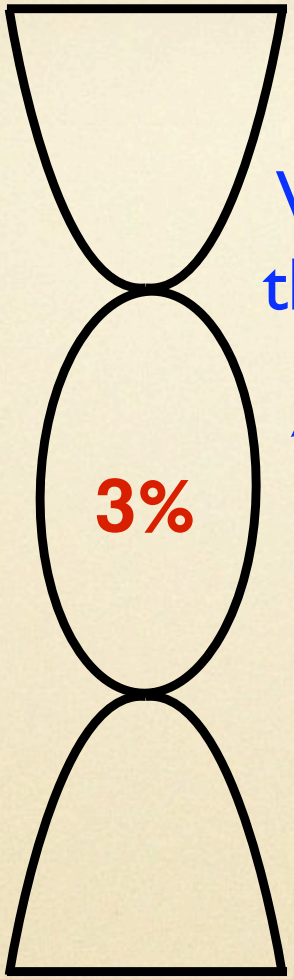
4. Calculate (as per Euler) the strength (maximal load) *f*(*y*) of the column C(*y*)

5. Maximize f(*y*) over *y*

Three solutions

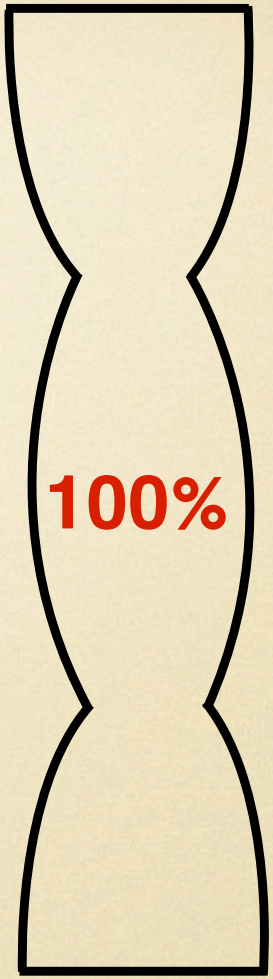


Lagrange
1770



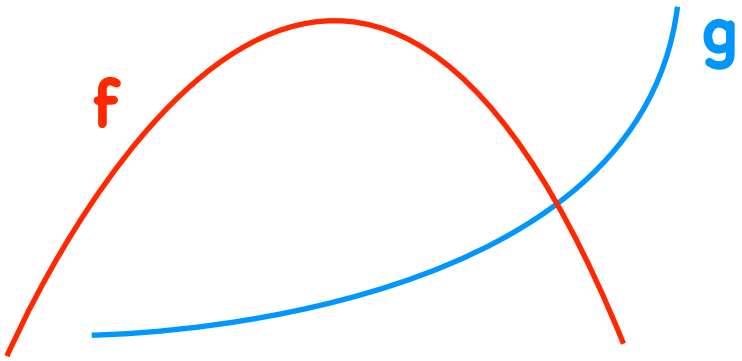
Where is the error?

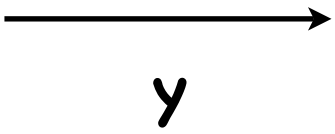
Keller & Tadjbaksh
1960

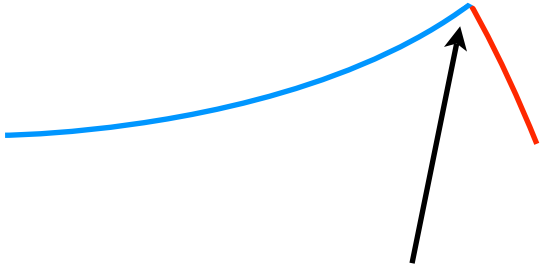


Cox & Overton
1992
(generalized gradients)

Nonsmooth Functions (third irregularity)







the function min(f,g) has a corner here

The function "maximal load supported by a column of profile y" is a nonsmooth function of y ... which is where the error was made

Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivées.

Hermite

Optimal control is an extension of the calculus of variations: there is a function u(t) (the **control**) corresponding to certain parameters of the system that can be varied in order to influence its evolution (= the curve y(t))

We now take charge of the optimization, rather than letting nature do it

Canonical example : aeronautics ; control of a rocket by its engines and guidance system
(Russian school, > 1950, Pontryagin, multipliers…)

Numerous applications : transport, communication, energy production, finance, management, chemical production, medicine, renewable resources…

An example

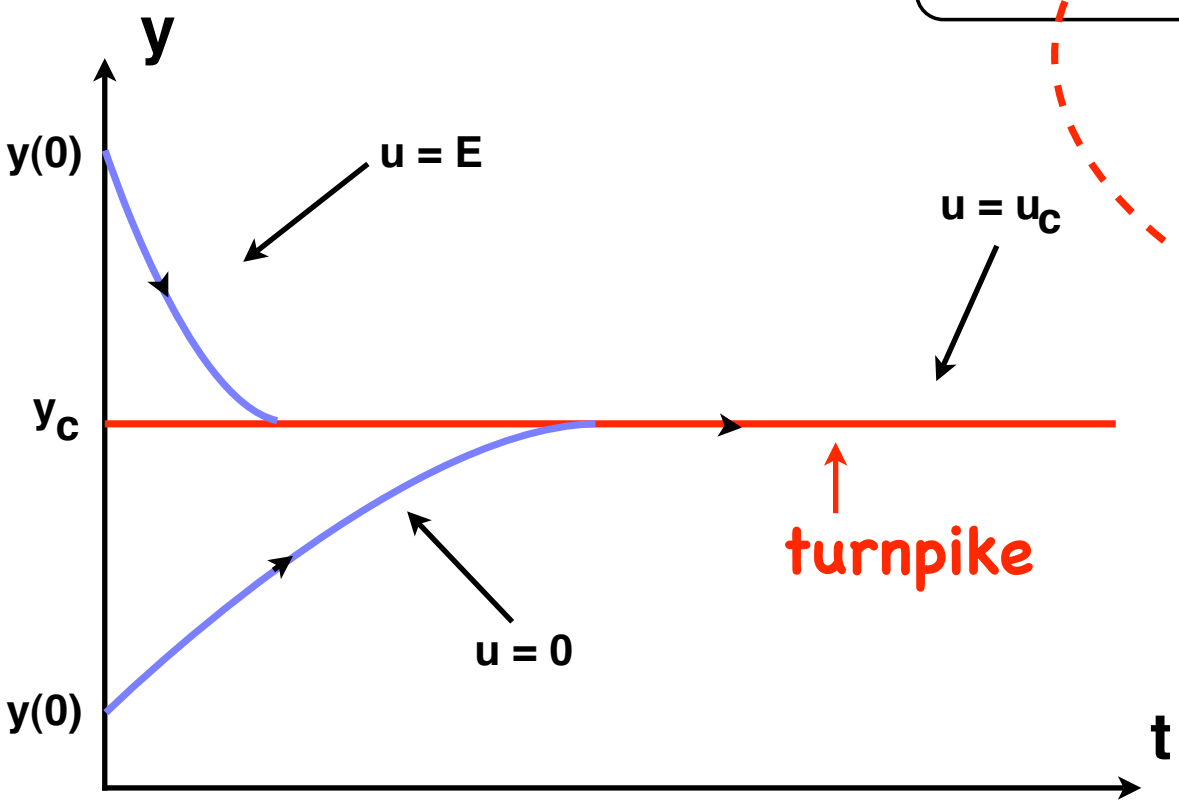
A model in renewable resources

(Clark, Clarke, Munro / Econometrica)

$$y'(t) = G(y(t)) - u(t)\,y(t)$$

$$\max \int_0^\infty e^{-\delta t}\{\pi\, y(t) - k\}\, u(t)\, dt$$

$$0 \le u(t) \le E$$

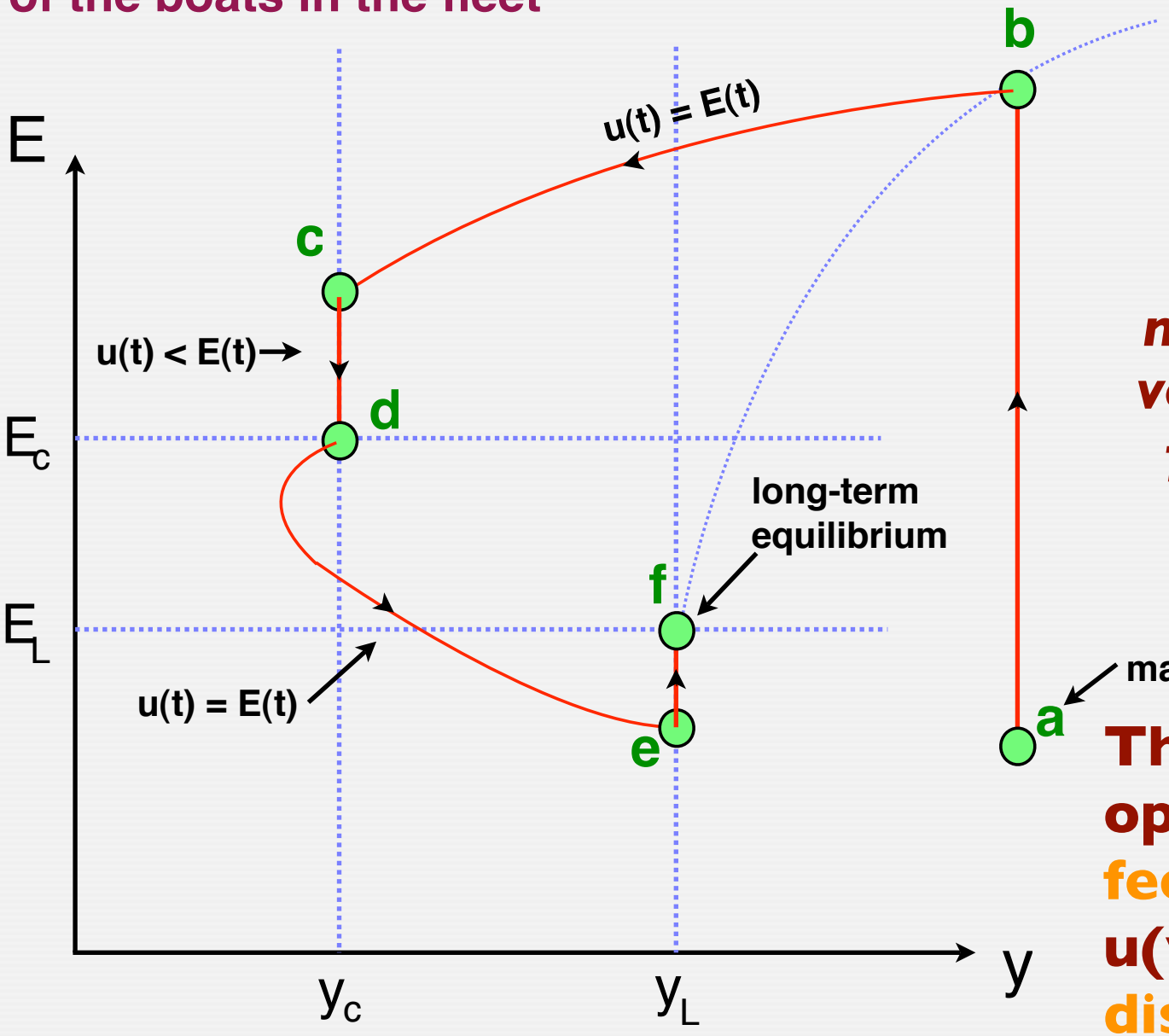


y = biomass
u = fishing effort
G = natural growth law
E = effort bound
δ = discount rate
π = resource price
k = effort cost

Simple, but...

If δ is sufficiently large, then $y_c = 0$ (extinction)

Optimal strategy in the presence of investment AND depreciation of the boats in the fleet



Proof:
nonsmooth verification functions

There is an optimal feedback u(y,E)... discontinuous!

Feedback Control

Feedbacks are controls depending on the current state y (rather than t) :

$$u(y), \text{ not } u(t)$$

Classic Example : a thermostat

It turns out that discontinuous feedbacks are essential :

- hybrid systems (robotics)
- large unknown perturbations (landing)
- pursuit/evasion (the boy and the crocodile)

Discontinuous Feedback : a current research area, and another 'irregularity'.

Scientific epilogue

The contributions of Euler discussed today :

- **Calculus of variations**

  still fundamental to physics, engineering + control (the modern face of the subject)

- **The method of multipliers**

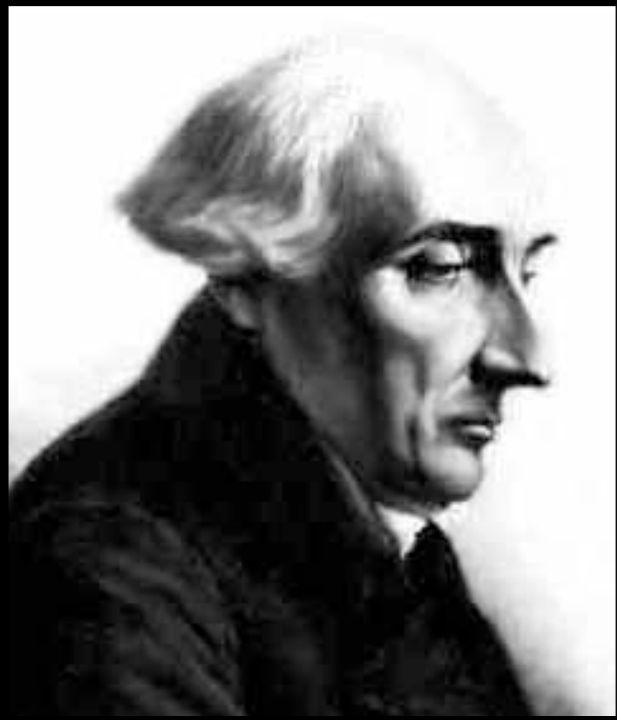
  central to optimization of many kinds, notably in optimal control

- **Principle of least action**

  relativity, quantum mechanics, string theory

These ideas still live...
Euler could contribute today

Biographical epilogue



Euler *Maupertuis* *Lagrange*

Maupertuis

- Resigns in disgrace in 1753 after the

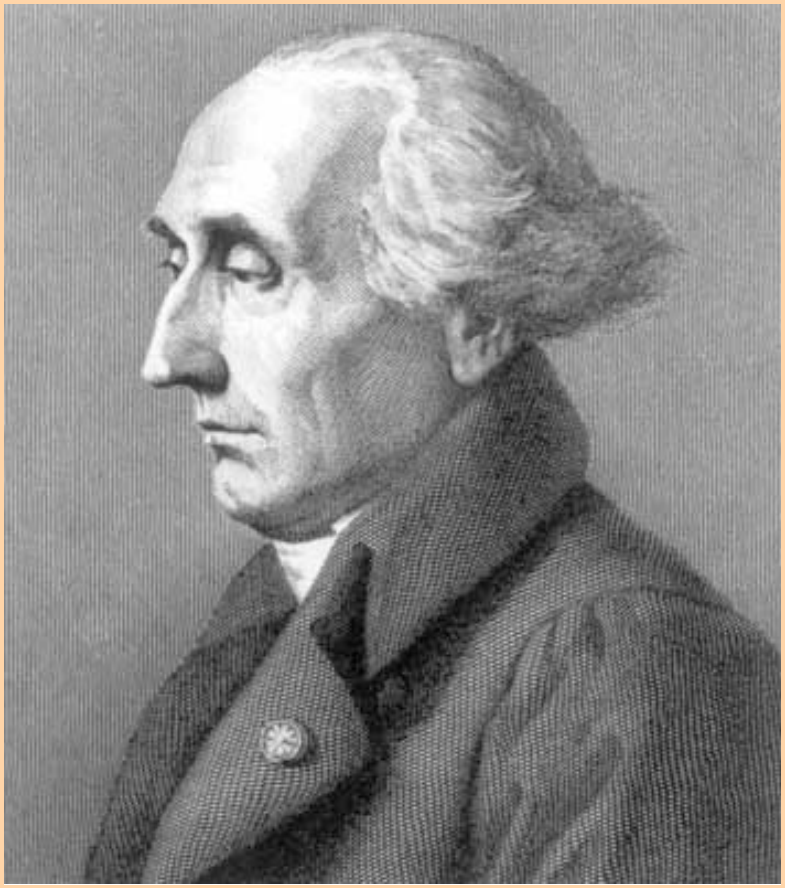
great scandal

in which he is opposed to Voltaire and the Paris academy

- Contracts tuberculosis
- Dies in 1759, at the age of 61, in Bâle, in Bernoulli's house

No street in Paris, no lunar crater, But :

Some people still speak of Maupertuis' principle of least action (!)



Lagrange

- After 20 years in Berlin, he joins the Paris Academy in 1786
- During the revolution : metric system, Ecole Normale and Polytechnique
- Under Napoléon : senator, count of the Empire, grand officer of the Légion d'honneur
- His 'greatest treasure' : his young wife, whom he marries at the age of 56
- Dies in Paris in 1813 at the age of 77

- **a great man, a genius, generous and modest**
- ***Euler was the first to cite the work of others fairly and positively*** **(Truesdell)**

Euler

Dies in September 1783 in Saint-Petersburg at the age of 76

R. I. P.

Last day :

- **morning : modeling montgolfières (a recent invention)**
- **afternoon : calculations on the orbit of Uranus (recently discovered)**
- **evening : a stroke, sudden death**

Last words : *I die*

Euler : Life, the universe and optimization

THE END