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Stability analysis of sliding-mode feedback control

by

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Abstract: This paper provides new analytic tools leading to the first rigorous stability and robustness analysis of sliding-mode feedback controllers. Unrestrictive conditions are given under which these controllers are stabilizing in the presence of large disturbances, as well as modeling, actuator and observation errors. The stability conditions invoke the existence of two Lyapunov-type functions, the first associated with passage to the sliding set in finite time, and the second with convergence to the desired state. In this approach, account is taken, from the outset, of implementational constraints. We provide a framework for establishing stability and robustness of the closed-loop system, for a variety of implementation schemes. We illustrate our results by means of two examples of the type frequently encountered in the sliding-mode literature.

Keywords: sliding-mode control, discontinuous control, Lyapunov functions, stabilization, robustness.

1. Introduction

Sliding-mode control is a well established technique for dynamic system stabilization which has generated a large literature: we refer to the monographs of Utkin (1992) and Edwards and Spurgeon (1998) and overviews of the field by Slotine and Li (1991) and Young et al. (1999). Under sliding-mode control, the state is first driven towards a subset Σ of the state space, the *sliding set*. Subsequently, the state trajectory remains near Σ and moves asymptotically to a value consistent with the desired equilibrium. A major advantage of sliding-mode controls is their stabilizing properties for dynamic systems subject to *large* disturbances. On the negative side, implementation of sliding-mode controllers typically involves the use of large, rapidly switching control signals, leading in some applications to excessive energy expenditure or reduced lifetime of the control actuators.

Sliding-mode controllers, relating the control u to the current state x, commonly take the form

$$u(x) \;=\; g(x) + \chi(x) \tag{1}$$

in which $g(x)$ and $\chi(x)$ are respectively smooth and discontinuous terms. The purpose of the discontinuous term $\chi(x)$ is to force the state to approach Σ at a uniformly positive rate (controllers containing such terms are said to satisfy the 'sliding condition'). The continuous term $g(x)$ can be thought of as preliminary feedback, configuring the system so that the sliding condition can be achieved. Since, if ever the state trajectory departs from Σ, the controller drives it back towards Σ, we expect that the state trajectory attains Σ and then remains in it in some sense. If sliding-mode control is implemented digitally, with a high sample rate, the control values generated by the control law are typically observed to switch rapidly, after the state trajectory first crosses Σ, in such a manner that the state trajectory remains close to Σ, and lies in Σ, in the limit as the sample period tends to zero.

The classical approach to analysing the closed-loop response of a system under sliding-mode control is to seek a smooth control law that approximates the effects of the rapidly switching control observed under digital implementation. Since rapid switching keeps the state close to Σ, we may decide to view the state trajectory as attaining the sliding set and then being generated by an 'equivalent' control law,

$$u_{\text{equiv}}(x) \;=\; g(x) + \chi_{\text{equiv}}(x)$$

in which $\chi_{\text{equiv}}(x)$ is a smooth function, chosen to ensure that the time derivative of the evolving state vector is tangent to Σ. In terms of this heuristic, traditional techniques (Lyapunov theory and eigenvalue analysis in the case that the sliding set is a linear subspace) have then been used, for example in Utkin (1992), to study the stabilizing properties of the equivalent control on the sliding surface.

The purpose of this paper is to provide analytic tools for a truly rigorous stability analysis of sliding-mode controllers. We give broad, unrestrictive conditions under which these controllers are stabilizing, in the presence not only of large disturbances, but also of modeling, actuator and observation errors. These conditions invoke the existence of two Lyapunov-type functions V_1 and V_2, the first associated with approach to the sliding set in finite time, and the second with convergence to the desired final state.

A distinctive feature of our approach is that it takes account, from the outset, of implementational constraints. (The question of how the action of sliding-mode control is affected by implementational effects such as time delays and regularization of the switching function has also been addressed in Utkin's book.) Besides such effects, we assume from the start that the feedback is implemented in the 'sample-and-hold' sense, which allows one to dispense with the heuristic argument in which the state is assumed to lie in the sliding set after a certain time.

The main theorem concerns the stabilizing properties of a sample-and-hold implementation of the feedback controller (1), in the presence of modeling, actuator and measurement errors, when the bounds on these errors and the sample period are suitably small. The principal advantages of our approach over the classical one are

(a): By examining the behaviour of the sample-and-hold implementation of the controller directly, we circumvent the additional hypotheses and the analytical apparatus associated with defining generalized solutions of Filippov type (via equivalent controls), and also the difficulties of choosing the 'correct' equivalent control from among the candidate equivalent controls, when they are not unique. (See Filippov (1988), Utkin (1992), and the references therein.)

(b): Our approach provides the first rigorous basis for the conclusion that the controller retains its stabilizing properties when we take account of both the way in which it is implemented, and the presence of small modeling, actuator and measurement errors.

Theorem 1 below gives conditions under which the controller is stabilizing, for a 'delay-free, zero-order hold (ZOH)' digital control implementation scheme; that is, one in which the state is measured and the corresponding control value is calculated instantaneously at each sample time, and this control value is applied until the next sample time. The focus on a particular implementation scheme, and an idealised one at that – in practice, time will be required to capture the measurement and to calculate the new control value – might seem restrictive. But this is not the case. Indeed we show in Section 5 that many practical implementation schemes (ZOH digital control with time delay, and various schemes involving the regularization of the discontinuous controller on a boundary layer about the sliding set, for example) can be interpreted as a delay-free ZOH scheme with measurement and/or actuator error, and are therefore covered by this paper. *Our delay-free ZOH controller implementation description should therefore be regarded not as a restriction, but rather as a convenient vehicle, for studying the robustness properties of sliding-mode control systems for a wide range of controller implementation schemes.*

While the use of a pair of Lyapunov functions to establish stability properties of digitally implemented sliding-mode control systems, in the presence of modeling and measurement errors, is apparently new, this paper includes several ingredients from earlier research. The sliding-mode controller is discontinuous; the use of a single control Lyapunov function to establish the stabilizing properties of sample-and-hold implementations of discontinuous controllers has been systematically studied, and is the subject of numerous papers (see for example

Clarke et al. (1997, 2000) and Sontag (1999)). The link between the existence of a single smooth control Lyapunov function and the robustness of the corresponding control system to measurement errors is also well understood (see Ledyaev and Sontag (1999)).

In Euclidean space, the length of a vector x is denoted by $|x|$, and the closed unit ball $\{x : |x-a| \le R\}$ by $B(a,R)$. $d_\Sigma(x)$ denotes the Euclidean distance of the point x from the set Σ, namely $\min\{|x-x'| : x' \in \Sigma\}$.

2. System Description

Let $\{t_0 = 0, t_1, t_2, \ldots,\}$ be a strictly increasing sequence of numbers such that $t_i \to \infty$; we refer to such a sequence as a *partition* (of $[0,\infty)$). Consider the dynamic system which relates the state trajectory $x(\cdot)$ to the control signal $u(\cdot)$ as follows:

$$\left.\begin{array}{ll} x(0) = x_0 & \\ \dot{x}(t) \ = \ f(x(t),u(t),d(t)) & \text{a.e. } t \in [0,\infty) \\ u(t) \ = \ a_i + g(x(t_i)+m_i) + v_i & \text{a.e. } t \in [t_i,t_{i+1}),\ i = 0,1,2,\ldots \\ v_i \ \in \ \chi(x(t_i)+m_i) & i = 0,1,2,\ldots \\ d(t) \in D & \text{a.e. } t \in [0,\infty)\,, \end{array}\right\} \qquad (2)$$

in which $f : R^n \times R^m \times R^k \to R^n$ and $g : R^n \to R^m$ are given functions and $\chi : R^n \rightsquigarrow R^m$ is a given set-valued function satisfying

$$\chi(x) \ \subset \ V \quad \text{for all } x \in R^m\,,$$

and $D \subset R^k$ and $V \subset R^m$ are given sets.

Notice that we have replaced the function χ in the control law (1) by a set-valued function χ; this introduces a useful extra degree of flexibility into the ensuing theory, which we will exploit in Section 5. In the preceding equations, $d(\cdot) : [0,\infty) \to R^k$ is a measurable function describing the disturbance signal. The sequences $\{a_i\}$ and $\{m_i\}$ describe the n-vector actuator errors and m-vector measurement errors at successive sample instants, respectively. Because the feedback law is applied in a sample-and-hold manner (constant control on partition subintervals), a physically meaningful state $x(\cdot)$ is generated by the scheme,depending of course on the initial state x_0 and the partition, the control values v_i, the errors m_i and a_i, and the disturbance $d(\cdot)$.

The following hypotheses will be imposed:

(H1) f and g are continuous and of linear growth: there exist $c_f, c_g > 0$ such that

$$\begin{aligned} |f(x,u,d)| \ &\le \ c_f(1+|x|+|u|) \quad \text{for all } (x,u,d) \in R^n \times R^m \times D \\ |g(x)| \ &\le \ c_g(1+|x|) \quad \text{for all } x \in R^n. \end{aligned}$$

(H2) For any bounded sets $X \subset R^n$ and $U \subset R^m$, there exists $K = K(X, U) > 0$ such that the following Lipschitz condition holds:

$$|f(x, u, d) - f(x', u, d)| \; \leq \; K|x - x'| \quad \text{for all } x, x' \in X,\ (u, d) \in U \times D\ .$$

(H3) V and D are closed bounded sets.

3. Lyapunov Functions for Sliding-Mode Control

We assume that the control feedback design

$$u \; = \; g(x) + v \quad \text{for some } v \in \chi(x)$$

has been carried out on the basis of a Lyapunov stability analysis, in ignorance of the measurement and actuator errors, and on the basis of a possibly inaccurate nominal dynamic model:

$$\dot{x} \; = \; f_0(x, u, d)$$

in which the function f_0 differs from the true dynamic function f. The only hypothesis imposed on f_0 is that it be continuous. To study the effects of sliding-mode control via Lyapunov stability analysis, it is helpful to introduce not one, but two Lyapunov-like functions $V_1 : R^n \to [0, \infty)$ and $V_2 : R^n \to [0, \infty)$, a decrease function $W : R^n \to [0, \infty)$ associated with V_2, and a subset $\Sigma \subset R^n$ of the state space (the *sliding set*). Σ is assumed to be a closed set containing the origin (the desired equilibrium). V_1 will be used to capture the property that the sliding-mode control drives the state arbitrarily close to Σ, in finite time. V_2 is associated with the subsequent motion of the state to a neighbourhood of the origin. V_1, V_2 and W will be required to satisfy the following conditions.

(LF1) V_1 is a continuous nonnegative function, and $V_1(x) = 0$ if and only if $x \in \Sigma$. Furthermore, the restriction of V_1 to $R^n \backslash \Sigma$ is continuously differentiable, and there exists $\omega_1 > 0$ such that

$$\langle \nabla V_1(x), f_0(x, g(x) + v, d)) \rangle \leq -\omega_1 \quad \text{for all } v \in \chi(x),\ x \in R^n \backslash \Sigma,\ d \in D.$$

Note that V_1 is not assumed to be differentiable at points in Σ. Now write

$$F_0(x) \; := \; \left\{ \lim_{i \to \infty} f_0(x, g(x) + v_i, d) \; : \; v_i \in \chi(x_i),\ x_i \to x,\ d \in D \right\}\ ,$$

that is, the set of all possible limits of sequences of the form $f_0(x, g(x) + v_i, d)$, where $v_i \in \chi(x_i)$, where x_i is any sequence converging to x, and where d is a point in D. We may think of $F_0(x)$ as consisting of all possible velocity values $\dot{x}$ (in limiting terms, and for the nominal dynamics given by f_0) when the state is at x. Note that $F_0(x)$ reduces to $f_0(x, g(x) + \chi(x), D)$ if χ is single-valued and continuous at x. Another case of special interest is that in which $\chi(x)$ takes a given value χ_+ everywhere on or to one side of a given sliding hypersurface Σ of dimension $n-1$, and a value χ_- on the opposite side. Then, at any point x of Σ, we have

$$F_0(x) = \{f_0(x, g(x) + \chi_+, d) : d \in D\} \cup \{f_0(x, g(x) + \chi_-, d) : d \in D\}.$$

The set $F_0(x)$ is used to express the decrease condition satisfied by V_2.

(LF2) V_2 and W are continuous nonnegative functions such that $V_2(0) = W(0) = 0$ and

$$V_2(x) > 0 \quad \text{and} \quad W(x) > 0 \quad \text{for } x \in \Sigma\backslash\{0\}.$$

Furthermore, the restriction of V_2 to $R^n\backslash\{0\}$ is continuously differentiable, and

$$\sup_{w \in F_0(x)} \langle \nabla V_2(x), w\rangle \;\le -W(x) \quad \text{for all } x \in \Sigma\backslash\{0\}.$$

Observe that (LF1) and (LF2) incorporate variants of the usual 'infinitesimal decrease' condition of Lyapunov functions. That of (LF2) is stated with the help of F_0 because it would not make sense to simply require, for example, that the inner product $\langle \nabla V_2(x), f(x, g(x) + \chi(x), d)\rangle$ be negative when x lies in Σ. The reason for this is that the set Σ may be 'thin', and χ may be discontinuous; an implementation might never actually evaluate χ at any points in Σ, so that the values of the inner product on merely the sliding set cannot in themselves assure the required stabilization.

We require one more property of the Lyapunov pair:

(LF3) $V_1 + V_2$ is a proper function; that is, for any $\alpha \ge 0$ the level set $\{x : V_1(x) + V_2(x) \le \alpha\}$ is bounded.

4. Sufficient Conditions for Stabilization

This section provides sufficient conditions for stabilization via sliding-mode control. The desired equilibrium state is taken to be $x = 0$. Since the control system description we have adopted takes account of digital implementation and allows for disturbances, we cannot expect state trajectories to converge to the zero state as t tends to ∞. Instead we give conditions for *practical (or approximate) semiglobal stabilization*; that is, conditions under which, for any two balls $B(0, R)$ (the set of initial states) and $B(0, r)$ (the target set) in R^n, $R > r > 0$, any state trajectory that issues from the set of initial states is driven to the target set.

The theorem below asserts that if the actuator and measurement errors are sufficiently small (the proof gives explicit bounds), if the modeling error between f and f_0 (matched to the gradients of V_1 and V_2) is sufficiently small, and if the partition size is small enough (or equivalently, the sampling rate high enough), then stabilization takes place in a prescribed uniform manner.

We set

$$M(x) \;:=\; \left\{ \lim_{i\to\infty} (f - f_0)(x, g(x) + v_i, d) \;:\; v_i \in \chi(x_i),\; x_i \to x,\; d \in D \right\}.$$

We may think of $M(x)$ as consisting of the relevant modeling error (in limiting terms) at the state x.

THEOREM 1 **Conditions for Practical Semiglobal Stabilization**. *Assume (H1)–(H3). Suppose there exist functions V_1, V_2 and W, and a set Σ satisfying hypotheses (LF1)–(LF3). Choose any numbers $R > 0$, $r > 0$ ($R > r$), $\bar{\omega} \in (0, \omega_1)$ and $\bar{\epsilon} > 0$. Then there exist positive numbers $R_* > R$ (R_* does not depend on r), $e_m, e_a, \delta, e_1, e_2$ and $T > 0$, with the following properties:*
Take any sequences $\{m_i\}$ and $\{a_i\}$ in R^m and R^n respectively, partition $\{t_i\}$, measurable function $d : [0, \infty) \to D$ and $x_0 \in B(0, R)$ satisfying

$$|m_i| \le e_m, \quad |a_i| \le e_a \quad \textit{and} \quad |t_{i+1} - t_i| \le \delta \quad \textit{for all } i. \tag{3}$$

Suppose in addition that we have

$$|\,\langle \nabla V_1(x), f(x, g(x) + v, d) - f_0(x, g(x) + v, d)\rangle\,| \;\le\; e_1 \qquad \textit{for all } v \in \chi(x),\, x \in B(0, R_*)\backslash\Sigma,\, d \in D \tag{4}$$

and

$$|\,\langle \nabla V_2(x), w\rangle\,| \;\le\; e_2 \;\; \textit{for all } w \in M(x),\, x \in [\,B(0, R_*) \cap \Sigma\,] \backslash\{0\}. \tag{5}$$

Let $x(\cdot) : [0, \infty) \to R^n$ be any solution to eqns.(2); one such solution exists. Then

$$x(t) \in B(0, R_*) \quad \textit{for all } t \ge 0 \quad \textit{and} \quad x(t) \in B(0, r) \quad \textit{for all } t \ge T.$$

Furthermore,

$$d_\Sigma(x(t)) \;\le\; \bar{\epsilon} \quad \textit{for all } t \in [V_1(x(0)/\bar{\omega}, \infty).$$

A proof of the theorem is given in the Appendix.

5. Alternative Digital Implementation Schemes

Our earlier analysis of closed-loop system response corresponding to a discontinuous feedback law

$$u \;\in\; g(x) + \chi(x) \tag{6}$$

is based on a model for implementation in which the constant value of u on $[t_i; t_{i+1})$ satisfies

$$u(t) = u \;=\; a_i + g(x(t_i) + m_i) + v_i,\, v_i \in \chi(x(t_i) + m_i) \quad t \in [t_i, t_{i+1}),\; i = 0, 1, \ldots$$

(7)

in which $\{a_i\}$ and $\{m_i\}$ are unmeasured error variables of sufficiently small magnitude.

At first sight, it would appear that the model (7) is relevant to only one implementation scheme (and a highly idealised one at that): instantaneous measurement and zero-order hold digital control. But the model (7) has, in fact, a universal quality and covers a wide range of implementation schemes, as we now illustrate.

Let us suppose that (H1)–(H3) are satisfied and that there exist functions V_1, V_2 and W satisfying (LF1)–(LF3), for the feedback law (6). Fix $R > 0$ and $r > 0$ ($R > r$). Then Theorem 1 establishes that the controller implementation scheme (7) has the semiglobal practical stabilization property (with respect to these parameters); that is, there exist e_m, e_a, δ, e_1, e_2, R_* ($R_* > R$) and $T > 0$ such that each state trajectory starting at time $t = 0$ in $B(0, R)$ and corresponding to (7), remains in $B(0, R_*)$ for all $t \geq 0$ and is confined to $B(0, r)$ for all times $t \geq T$, provided the modeling error is limited by (4) and (5), and the sampling period, acutator and measurement errors satisfy:

$$|m_i| \leq e_m, \quad |a_i| \leq e_a \quad \text{and} \quad |t_{i+1} - t_i| \leq \delta \quad \text{for all } i\,.$$

Let M be a uniform bound on $(g + \chi)(x + m)$ for $|x| \leq R_*$ and $|m| \leq e_m$, let K be a uniform bound on $|f(x, u, d) + a|$ for $|x| \leq R_*$, $|u| \leq M$, $|a| \leq e_a$ and $d \in D$. Finally, let k_g be a Lipschitz constant for g on $B(0, R_* + e_m)$.

1. Digital Control with Time Delay. A more realistic model for digital implementation of the control law is one in which the constant value of u on $[t_i; t_{i+1})$ must satisfy

$$u(t) = u \; \in \; (g + \chi)(x(t_{i-1}) + m_i^{'}) + a_i^{'} \quad t \in [t_i, t_{i+1}), \quad \text{for all } i, \tag{8}$$

in which $m_i^{'}$ and $a_i^{'}$ are error variables assumed to satisfy

$$|m_i^{'}| \leq e_m^{'}, \quad |a_i^{'}| \leq e_a^{'} \quad i = 0, 1, \ldots,$$

for some positive constants $e_m^{'}$ and $e_a^{'}$.

Note that the right side of this relation depends on the *delayed* state. Here, the processor uses the time period $[t_{i-1}, t_i]$ to capture (approximately) the value of the signal x at time t_{i-1} and to calculate (approximately) the constant control value to be implemented over the subsequent time interval. Observe that we can rewrite (8) in the form

$$u(t) \; \in \; (g + \chi)(x(t_i) + m_i) + a_i^{'}$$

by simply defining

$$m_i \; := \; x(t_{i-1}) - x(t_i) + m_i^{'}\,,$$

and that we have $|m_i| \leq K\delta + e_i^{'}$. Theorem 1 therefore applies, provided δ is replaced by $\delta' \in (0, \delta]$ and $e_m^{'}$, δ' and $a_m^{'}$ are chosen to satisfy

$$e_m^{'} + K\delta' \leq e_m \text{ and } e_a^{'} \leq e_a \ .$$

Clearly, the preceding analysis extends to situations in which $x(t_{i-1})$ in (8) is replaced by $x(t_{i-N})$, for some integer $N > 1$.

A number of implementation schemes are aimed at reducing the rapid switching of control values near the switching surface Σ. We now describe how these too are covered by the analytic framework of this paper.

2. Filtering out High Frequency Components of the Control Signal. One approach to smoothing the control signal generated by the discontinuous control law is to pass it through a low-pass filter. Let us consider the case when sampling is uniform (write $\delta' = |t_{i+1} - t_i|$) and the control is scalar. Then the procedure might take the following form:

$$\left.\begin{aligned} u(t) \ &= \ z_i + a_i^{'} \,, && \text{for } t \in [t_i, t_{i+1}) \\ z_i \ &= \ \alpha z_{i-1} + \beta v_i && \\ v_i \ &\in \ (g + \chi)(x(t_i) + m_i^{'}) \,, && \end{aligned}\right\} \tag{9}$$

$i = 0, 1, 2, \ldots$. Here, $a_i^{'}$ and $m_i^{'}$ are error terms that satisfy $|a_i^{'}| \leq e_a^{'}$ and $|m_i^{'}| \leq e_m^{'}$.

The second equation in (9) describes a simple low-pass digital filter (involving the positive parameters α and β). For concreteness, let us assume that the digital filter is obtained by digitizing a first-order low-pass analogue filter with time constant the positive number τ (that is, with transfer function $(1+\tau s)^{-1}$). Now, the digital filter parameters are

$$\alpha \ = \ e^{-\delta'/\tau} \quad \text{and} \quad \beta \ = \ 1 - e^{-\delta'/\tau} \ .$$

Assuming that the digital filter is initialized to zero ($z_0{=}0$), we deduce by means of a simple calculation that $|z_{i-1}| \leq (1-\alpha)^{-1}\beta M = M$ and so

$$|z_i - v_i| \leq 2e^{-\delta'/\tau} M \ .$$

It follows that

$$u(t) \ \in \ (g + \chi)(x(t_i) + m_i) + a_i$$

where $m_i := m_i^{'}$ and $a_i := a_i^{'} + z_i - v_i$.

We see, once again, that Theorem 1 applies, provided the sampling period δ' and the parameters τ, $e_m^{'}$ and $e_a^{'}$ are chosen to satisfy:

$$e_m^{'} \leq e_m \quad \text{and} \quad e_a^{'} + 2e^{-\delta'/\tau} M \leq e_a \ .$$

Clearly, the analysis relating to the two preceding implementation schemes can be combined. For example, if the last equation in (9) is replaced by $v_i \in (g+\chi)(x(t_{i-1})+m_i^{'})$ (that is, the previous measurement of the state is employed to allow for processing time), Theorem 1 applies if the first inequality in the preceding condition is changed to $e_m^{'} + K\delta' \le e_m$.

For the remaining examples of implementation schemes discussed in this paper, we restrict attention for simplicity to the common case when $f_0(x,u,d)$ is affine with respect to the u variable, when the input is scalar and when the control law is continuous feedback + simple switching control:

$$u(t_i) \;=\; (g+\chi)(x(t_i)) \tag{10}$$

with

$$\chi(x) \;=\; \begin{cases} K & \text{if } s(x) > 0 \\ -K & \text{if } s(x) \le 0\,. \end{cases} \tag{11}$$

Here K is a positive constant and $s(x)$ is the linear function

$$s(x) \;=\; \lambda^T x\,,$$

in which λ is a given nonzero n-vector. The sliding set Σ is the set $\{x : s(x) = 0\}$. In this setting, the decrease condition in (LF2) is easily seen to be equivalent to:

$$\langle \nabla V_2(x), f_0(x, g(x)+v, d)\rangle \;<\; -W(x) \quad \text{for all } x \in \Sigma\backslash\{0\},\ v \in [-K,+K],\ d \in D\,. \tag{12}$$

An important implication of the choice of the function χ given by (11) is the following: under the hypotheses of Theorem 1, the assertions of this theorem (with the same constants R_*, etc.) remain valid when χ is replaced by the multifunction

$$\chi^*(x) \;:=\; \begin{cases} \{+K\} & \text{if } s(x) > 0 \\ [-K,K] & \text{if } s(x) = 0 \\ \{-K\} & \text{if } s(x) < 0\,. \end{cases} \tag{13}$$

We note in particular that the hypothesis (LF1) governing decrease of V_1 continues to be satisfied (with the same decrease parameter ω_1), because χ has been modified only on Σ, and decrease of V_1 is not required on this set. On the other hand, the hypothesis (LF2) governing decrease of V_2 continues to be satisfied (with the same decrease function W). This special case covers the example treated in Section 7 below.

3. Regularization/Hysteresis. Fix $\gamma > 0$. Another approach to implementation is to introduce hysteresis. Here the control signal is taken to be

$$u(t) \;=\; g(x(t_i)+m_i^{'}) + \chi(x(t_j)+m_j^{'}) + a_i^{'} \tag{14}$$

where j is the maximum value of the index k such that

$$k \le i \quad \text{and} \quad |s(x(t_k) + m_k^{'})| > \gamma$$

(if no such k exists, we may take j equal to any index between 0 and i). In other words, the value of the discontinuous term in the control law is updated only at times t_j for which the magnitude of the switching function at the measured state, namely $|s(x(t_j) + m_j^{'})|$, exceeds a threshold γ.

An alternative approach ('regularization') is to replace the switching function χ by a continuous function, a piecewise linear function for example:

$$\left.\begin{array}{rcl} u(t) & \in & (g+\chi)(x(t_i)+m_i^{'}) + a_i^{'} \quad t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots \\ \chi_{\text{reg}}(x) & = & \begin{cases} +K & \text{if } s(x) > \gamma \\ (K/\gamma)s(x) & \text{if } -\gamma \le s(x) \le \gamma \\ -K & \text{if } s(x) < -\gamma \end{cases} \end{array}\right\} \tag{15}$$

Other continuous functions, taking values in the interval $[-K, +K]$ at points in the state space close to Σ, may be used: spline functions or sigmoidal functions, for example.

In both these implementation schemes, $\{a_i^{'}\}$ and $\{m_i^{'}\}$ are error sequences satisfying $|a_i^{'}| \le e_a^{'}$, $|m_i^{'}| \le e_m^{'}$ for all i. We now show that, for both these schemes, it is possible to derive closed-loop stability properties from Theorem 1. We make use of the following property of the switching function $s(x) = \lambda^T x$: for any $x \in R^n$

$$|s(x)| \le \gamma \quad \text{if and only if} \quad d_\Sigma(x) \le |\lambda|^{-1}\gamma\ . \tag{16}$$

For either implementation scheme, two situations may possibly arise:

(i) : $|s(x(t_i) + m_i^{'})| > \gamma$. Because, in this case, $j = i$ in (14) and $\chi(x)$, $\chi^*(x)$ and $\chi_{\text{reg}}(x)$ all coincide when $|s(x)| > \gamma$, we have

$$u(t) \ = \ (g+\chi)(x(t_i)+m_i)+a_i \subset (g+\chi^*)(x(t_i)+m_i)+a_i \quad \text{for } t \in [t_i, t_{i+1})$$

where $a_i = a_i^{'}$ and $m_i = m_i^{'}$. So $|a_i| \le e_a'$ and $|m_i| \le e_m'$ for all such i.

(ii) : $|s(x(t_i) + m_i^{'})| \le \gamma$. In this case we deduce from (16) that there exists an n-vector m_i such that $s(x(t_i) + m_i) = 0$ and

$$|m_i^{'} - m_i| \le |\lambda|^{-1}\gamma. \tag{17}$$

Using the fact that the range of both $\chi(\cdot)$ and $\chi_{\text{reg}}(\cdot)$ are contained in $[-K, +K] = \chi^*(0)$, we deduce that (for both control laws (14) and (15))

$$\begin{aligned} u(t) \ \in \ & g(x(t_i) + m_i^{'}) + [-K, +K] + a_i^{'} \\ & \subset \ (g+\chi^*)(x(t_i) + m_i) + a_i,\ t \in [t_i, t_{i+1}) \end{aligned}$$

where $a_i := a_i' + g(x(t_i) + m_i') - g(x(t_i) + m_i)$. In both situations then, in view of (17) the error terms satisfy:

$$|a_i| \le e_a' + k_g|\lambda|^{-1}\gamma \quad \text{and} \quad |m_i| \le e_m' + |\lambda|^{-1}\gamma \quad \text{for all } i\ .$$

Bearing in mind our earlier observation that the assertions of Theorem 1 remain valid when χ^* replaces χ, we have confirmed that both implementations (14) and (15) have the semiglobal practical stabilization property provided that e_a', e_m' , γ and λ are chosen to satisfy the conditions:

$$e_m' + |\lambda|^{-1}\gamma \le e_m \quad \text{and} \quad e_a' + k_g|\lambda|^{-1}\gamma \le e_a\ .$$

6. A Second-Order Example

We now illustrate in a simple example how sliding modes can yield robust feedback stabilization in the presence of arbitrarily large modeling error (at the price of large and active control laws); we also interpret in our context the known issue of 'matching' the errors. The setting is a familiar one in texts on sliding-mode control (see for example Slotine and Li (1991)). We take $n = 2$ and denote points in state space by (x, y). The dynamics are given by

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = h(x(t)) + u(t),$$

The goal is to stabilize the state to the origin by means of continuous + switching state feedback, where the switching term is bounded in magnitude by some constant $L > 0$.

The choice of sliding set is $\Sigma := \{(x, y) : x + y = 0\}$, a choice motivated by the fact that if the (x, y) could be restricted to a neighborhood of Σ (by some discontinuous feedback strategy), the dynamics would then imply $\dot{x} \approx -x$, which in turn seems to imply the stabilization of x to 0. As for the component y of the state, note that the corresponding differential equation

$$\dot{y}(t) = h(x(t)) + u(t)$$

leaves the fate of y somewhat in doubt; of course, this differential equation is irrelevant on the sliding set itself, except (possibly) as a limiting idealization. On the other hand, the relation $y(t) \approx -x(t)$ tends to confirm that y should converge to 0. Given that in practice the state (x, y) will *not* be exactly in Σ, a rigorous analysis requires a different approach; Theorem 1 provides this.

We take a nominal dynamic system having the same structure:

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = h_0(x(t)) + u(t),$$

where the modeling error $h - h_0$ may be large. (However, we suppress the disturbance signal, for ease of exposition.) It is assumed that h and h_0 are continuous and have linear growth, and that $h - h_0$ is globally bounded; we also assume that h is Lipschitz on bounded sets.

We wish to place ourselves in the general framework considered by Theorem 1, for $n = 2, m = 1$ and

$$f(x, y, u) = [y, h(x) + u]^T, \; V = [-L, L].$$

Thus we seek a feedback $u(x, y) = g(x, y) + \chi(x, y)$, where $\chi(x, y) \subset [-L, L]$. The nominal function f_0 is simply the same as f, but with h replaced by h_0.

We must choose Lyapunov functions in accordance with (LF1)–(LF3). A natural choice for V_1 is $V_1(x, y) := |x+y|$, which is continuous, zero precisely on Σ, and continuously differentiable on $R^2\backslash\Sigma$. The decrease condition required in hypothesis (LF1) of the Theorem becomes

$$\frac{x + y}{|x + y|} \{h_0(x) + y + g(x, y) + \chi(x, y)\} \leq -\omega_1.$$

This suggests taking $g(x, y) = -h_0(x) - y$ and $\chi(x, y) = -L \operatorname{sgn}(x + y)$. (Here, $\operatorname{sgn}(x)$ equals $+1$ when x is positive, -1 when x is negative; the precise definition of χ when $x + y = 0$ is not important, but let us set it equal to the interval $[-L, L]$ for definiteness.) With these choices, we see that ω_1 can be taken to be L. Note that hypotheses (H1)–(H3) are satisfied.

There are many possible choices for V_2, but a function depending only upon x suggests itself, for the reason that ∇V_2 then has a zero inner product with $f - f_0$: the Lyapunov function is 'matched' to the modeling error. This automatically assures that the bound (5) on modeling error will be satisfied. We take $V_2(x, y) = x^2$. With these choices, we see that (LF2) and (LF3) are satisfied, for $W(x, y) = x^2$.

As mentioned in the Remark following the proof of the Theorem, the only constraint on modeling error is that given for e_1 by (33)(43). This leads to an explicit estimate for how large L must be:

Proposition 1 *Suppose that L is taken larger than*

$$\|h - h_0\|_\infty := \sup_{x \in R^n} |(h - h_0)(x)|.$$

Then, for any $0 < r < R$, for all sufficiently small levels of actuator and measurement error, and for all sufficiently fine partitions, the feedback given above stabilizes initial points in $B(0, R)$ to $B(0, r)$.

In order to apply Theorem 1, we set $\omega_1 = L$ and then take any $\bar{\omega}$ in the open interval $(0, L - \|h - h_0\|_\infty)$. The left side of (4) is bounded by $\|h - h_0\|_\infty$, which is seen to provide a suitable choice of e_1 in (33)(43) (for all sufficiently small values of e_a, e_m and δ). As mentioned, (5) holds automatically, and so the Theorem applies.

7. Linear Scalar Control

Consider the linear system in control canonical form, relating the n-vector state $x(t)$ to the scalar control $u(t)$:

$$\frac{d}{dt}x(t) \;=\; Ax(t) + b(d(t) + u(t)) \tag{18}$$

in which

$$A \;=\; \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & \\ 0 & 0 & 0 & . & . & 0 & 1 \\ -a_0 & -a_1 & . & . & . & . & -a_{n-1} \end{bmatrix} \quad \text{and} \quad b \;=\; \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}.$$

Here $a_0, \ldots, a_{n-1}$ are known parameters, while $d(t)$ is an unmeasured scalar disturbance signal, assumed to satisfy, for some given $d_{\max} > 0$, the condition

$$|d(t)| \;\leq\; d_{\max} \text{ for all } t\,.$$

We seek state feedback

$$u \;=\; \phi(x), \tag{19}$$

to achieve closed-loop asymptotic stabilization, for arbitrary disturbance signals $d(\cdot)$.

This design problem is addressed in the sliding-mode literature as follows (see, e.g., Edwards and Spurgeon (1998)). Fix coefficients $\lambda_0, \ldots, \lambda_{n-1}$ of a Hurwitz polynomial, of degree $n-1$,

$$\lambda(\sigma) \;=\; \lambda_0 + \lambda_1\sigma + \ldots + \lambda_{n-2}\sigma^{n-2} + \sigma^{n-1}$$

and define the scalar-valued function of the state

$$s(x) \;=\; \lambda_0 x_1 + \lambda_1 x_2 + \ldots + \lambda_{n-2}x_{n-1} + x_n\,. \tag{20}$$

Define also the k-vector

$$k \;=\; \mathrm{col}\{a_0, a_1 - \lambda_0, \ldots, a_{n-1} - \lambda_{n-2}\}\,.$$

Fix $K > d_{\max}$. Consider now the control law (19) in which

$$\phi(x) \;=\; k^T x - K\,\mathrm{sgn}\{s(x)\}\,. \tag{21}$$

Here, $\mathrm{sgn}(\cdot)$ is the 'signum function'

$$\mathrm{sgn}(s) \;:=\; \begin{cases} +1 & \text{if } s > 0 \\ -1 & \text{if } s \leq 0\,. \end{cases}$$

The rationale here is that, if we substitute control law (19) into (18) and take account of the fact that $\frac{d}{dt}x_i = x_{i+1}$, for $i = 1, \ldots, n-1$, there results (in the case $s(x(t)) \neq 0$)

$$\begin{aligned}
\frac{d}{dt}\,|s(x(t))| \quad &= \quad \mathrm{sgn}\{s(x(t))\}[d/dt\, x_n + \lambda_{n-2}x_n + \ldots + \lambda_0 x_2] \\
&= \quad \mathrm{sgn}\{s(x(t))\}[-a_0x_1 + \ldots - a_{n-1}x_n + a_0x_1 \\
&\qquad\quad +(a_1 - \lambda_0)x_2 + \ldots + (a_{n-1} - \lambda_{n-2})x_n \\
&\qquad\quad -K\,\mathrm{sgn}\{s(x(t)\} + d(t) + \lambda_{n-2}x_n + \ldots + \lambda_0 x_2] \\
&= \quad \mathrm{sgn}\{s(x(t))\}[-K\mathrm{sgn}\{s(x(t))\} + d(t)] \ \leq \ -\omega\,,
\end{aligned}$$

where $\omega = (K - d_{\max})\,(>\ 0\,)$. These calculations suggest that, if $s(x(0)) \neq 0$, then $x(t)$ arrives at the set

$$\Sigma \ := \ \{x\,:\,s(x) = 0\} \tag{22}$$

at a positive time $\bar{T} \leq |s(x(0))|/(K - d_{\max})$, where it remains thereafter. Moreover, for $t > \bar{T}$, we have that

$$s(x(t)) \ = \ \frac{d^{n-1}}{dt^{n-1}}x_1(t) + \lambda_{n-2}\frac{d^{n-2}}{dt^{n-2}}x_1(t) + \ldots + \lambda_0 x_1(t) \ = \ 0\,. \tag{23}$$

Since $\lambda(\sigma)$ is a Hurwitz polynomial, we can surmise that $x_1(t)$ (and hence also $x_2(t), \ldots, , x_n(t)$) converge to zero, as $t \to \infty$.

We now use the methods of this paper to justify these conclusions, taking account of implementational effects and the presence of modeling and measurement errors. We show:

PROPOSITION 2 *Take $R > 0$ and $r > 0$ ($R > r$). Take also $d_{max} > 0$ and $K > d_{max}$. Then there exist positive numbers e_a, e_m, δ, R_* and T with the following properties:*

Take any partition $\{t_i\}$, measurable function $d(\cdot) : [0,\infty) \to R$, and sequences $\{a_i\}$ and $\{m_i\}$ satisfying

$$|t_{i+1} - t_i| < \delta,\ |a_i| \leq e_a,\ |m_i| \leq e_m \textit{ for all } i,\ |d(t)| \leq d_{max} \textit{ for all } t\,.$$

Then, for any $x_0 \in B(0,R)$, the solution $x : [0,\infty) \to R^n$ to

$$\left.\begin{array}{rcll}
\dot{x}(t) & = & Ax(t) + b(d(t) + u(t))) & \textit{a.e. } t \in [0,\infty) \\
u(t) & = & a_i + \phi(x(t_i) + m_i) & \textit{a.e. } t \in [t_i, t_{i+1}) \\
x(0) & = & x_0 &
\end{array}\right\} \tag{24}$$

where $\phi(\cdot)$ is the mapping (21), satisfies $|x(t)| \leq R_*$ for all $t \geq 0$ and

$$x(t) \ \in \ B(0,r) \quad \textit{for all } t \geq T\,.$$

Proof. To prove the proposition, we apply Theorem 1, making the following identifications:

$$\begin{aligned} f(x,u,d) &= f_0(x,u,d) = Ax + b(u+d) \\ g(x) &= k^T x,\ \chi(x) = -K\,\mathrm{sgn}\{s(x)\} \\ D &= [-d_{\max}, +d_{\max}] \text{ and } V = [-K, +K]\,. \end{aligned}$$

(Thus, in contrast to the example of the previous section, we now suppress modeling error; equivalently, we view it as being subsumed by the disturbance term.) The hypotheses (H1)–(H3) on the dynamics are satisfied for this choice of data. The assertions of the proposition will have been proved then, if we can construct functions V_1, V_2 and W satisfying the conditions (LF1)–(LF3).

We take $V_1 : R^n \to [0, \infty)$ to be the function

$$V_1(x) = |s(x)|,$$

where $s(\cdot)$ is the function (20). The sliding set is

$$\Sigma = \{x : V_1(x) = 0\} = \{x : s(x) = 0\}.$$

We show that V_1 satisfies (LF1). V_1 is continuous, and continuously differentiable on $R^n \backslash \Sigma$. Write

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & \\ 0 & 0 & 0 & . & . & 0 & 1 \\ 0 & -\lambda_0 & . & . & . & . & -\lambda_{n-2} \end{bmatrix}$$

For any $x \in R^n \backslash \Sigma$ and $d \in D$ we calculate:

$$\begin{aligned} &\langle \nabla V_1(x), f(x, g(x) + \chi(x), d)\rangle \\ &\quad = \langle \nabla V_1(x), A_0 x + b(-K\mathrm{sgn}\{s(x)\} + d)\rangle \\ &\quad = \mathrm{sgn}\,\{s(x)\}[\lambda_0, \ldots, \lambda_{n-2}, 1] \\ &\qquad\qquad [x_2, \ldots, x_n, (-\lambda_0 x_2 \ldots - \lambda_{n-2} x_n - K\mathrm{sgn}\{s(x)\} + d)]^T \\ &\quad = -K + \mathrm{sgn}\,\{s(x)\}\, d \le -\omega\,, \end{aligned}$$

where ω is the positive number $\omega = K - d_{\max}$. We have confirmed (LF1).

With a view to constructing V_2 and W, we introduce the matrices

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & . & . & . & \\ 0 & . & . & . & . & 1 \\ -\lambda_0 & . & . & . & . & -\lambda_{n-2} \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} I_{(n-1)\times(n-1)} \\ \\ -\underline{\lambda}^T \end{bmatrix}$$

in which $\underline{\lambda}$ is the vector

$$\underline{\lambda} \;=\; \mathrm{col}\{\lambda_0, \ldots, \lambda_{n-2}\}\;.$$

Note that J has full column rank and that $\Sigma = \mathrm{range}\{J\}$. It follows that $J^T J$ is invertible and, given any $x \in R^n$, there exists a unique $\xi \in R^{n-1}$ (it is $\xi = (J^T J)^{-1} J^T x$) such that $x = J\xi$. Since $\bar{A}$ is a 'stable' matrix, there exists a symmetric matrix $\bar{P}$ and $\gamma > 0$ and $c > 0$ such that

$$\xi^T \bar{P}\bar{A}\xi \;\leq\; -\gamma|\xi|^2 \text{ and } \xi^T \bar{P}\xi \;\geq c|\xi|^2 \quad \text{for all } \xi \in R^{n-1}.$$

(See Willems (1970).) Define

$$P \;=\; \begin{bmatrix} \bar{P} & \underline{0} \\ \underline{0}^T & 0 \end{bmatrix}$$

in which $\underline{0}$ is the $(n-1)$-vector $\mathrm{col}\{0, \ldots, 0\}$. Finally we set

$$V_2(x) \;=\; \frac{1}{2} x^T P x \quad \text{for all } x \in R^n\;.$$

The function W is taken to be

$$W(x) \;=\; \frac{1}{2}\gamma|[J^T J]^{-1} J^T x|^2.$$

We observe that V_2 and W are continuously differentiable, nonnegative, and vanish at the origin. Take any $x \in \Sigma\backslash\{0\}$. Then $x = J\xi$, where ξ is the nonzero vector $\xi = (J^T J)^{-1} J^T x$.
Note the following identities

$$J^T P J \;=\; \bar{P}, \quad J^T P \;=\; [\bar{P}\;\underline{0}], \;\; A_0 J \;=\; \begin{bmatrix} \bar{A} \\ e^T \end{bmatrix}\,, \tag{25}$$

in which e is some (possibly nonzero) $(n-1)$-vector. With the help of the first identity, we deduce that

$$W(x) \;=\; \frac{1}{2}|[J^T J]^{-1} J^T J\xi|^2 = \frac{\gamma}{2}|\xi|^2 \;>\; 0,$$
$$V_2(x) = \frac{1}{2} x^T P x = \frac{1}{2}\xi^T J^T P J \xi = \frac{1}{2}\xi^T \bar{P}\xi \;>\; 0.$$

We have shown that V_2 and W are positive on $\Sigma\backslash\{0\}$. To complete the verification of (LF2), it remains to check its decrease property.
Let x be any point in $\Sigma\backslash\{0\}$. Note that any element w of $F_0(x)$ is of the form $f(x, g(x) + v, d)$ for some $v \in V$ and $d \in D$. We calculate

$$\begin{aligned}
&\langle \nabla V_2(x), f(x, g(x) + v, d)\rangle \\
&\;=\; \langle \nabla V_2(x), A_0 x + b(v + d)\rangle \\
&\;=\; x^T P A_0 x + 0 \\
&\;=\; \xi^T J^T P A_0 J \xi \;=\; \xi^T [\bar{P}\;\underline{0}] \begin{bmatrix} \bar{A} \\ e^T \end{bmatrix} \xi \\
&\;\leq\; -\gamma|\xi|^2 \;\;=\; -\gamma|[J^T J]^{-1} J^T x|^2 \;<\; -W(x)\,.
\end{aligned}$$

(To derive these relations we have used (25) and the fact that $Pb = 0$.) The decrease property of (LF2) is confirmed.
Finally we must show that $V_1 + V_2$ is proper (condition (LF3)). Fix a number α and take any n-vector x satisfying

$$V_1(x) + V_2(x) \ \leq \ \alpha \ .$$

Define the $n \times n$ nonsingular matrix $M \ := \ [J, \ b]$ and write $\zeta = M^{-1}x$. A simple calculation gives $V_1(x) = |\zeta_n|$ and

$$V_2(x) = \frac{1}{2}\,[\zeta_1, \ldots, \zeta_{n-1}]\,\bar{P}\,[\zeta_1, \ldots, \zeta_{n-1}]^T \ \geq \ \frac{1}{2}\sigma|[\zeta_1, \ldots, \zeta_{n-1}]^T|^2 \ ,$$

where $\sigma > 0$ is the minimum eigenvalue of $\bar{P}$. It follows that

$$\frac{1}{2}\sigma(|\zeta_1|^2 + \ldots |\zeta_{n-1}|^2) + |\zeta_n| \ \leq \ \alpha \ .$$

It follows that ζ, and hence x, is confined to a bounded set depending only upon α. Thus $V_1 + V_2$ is proper. The functions V_1, V_2 and W have been shown to satisfy (LF1)–(LF3). The proof of the proposition is complete.

Appendix: Proof of Theorem 1.

We shall be considering state trajectories generated by eqns.(2) for partitions and errors which satisfy bounds of the type (3)-(5). We begin with the following result, which addresses the issue of stabilization *near* the sliding set; that is, for small values of V_1.

Lemma 1. For any $0 < \beta < b$ there exists $\alpha > 0$ arbitrarily small such that the set

$$S := \{x : V_1(x) \leq \alpha, \ V_2(x) \leq b\}$$

is invariant in the following sense. Let $S^+ := \{x : V_1(x) \leq 2\alpha, V_2(x) \leq 2b\}$. Then, for all sufficiently small values of the positive parameters $e_m, e_a, \delta, e_1, e_2$, for any trajectory $x(\cdot)$ generated by (2) with $x(0) \in S$ and for data satisfying

$$|m_i| \leq e_m, \quad |a_i| \leq e_a \quad \text{and} \quad |t_{i+1} - t_i| \leq \delta \quad \text{for all } i \tag{26}$$

$$|\,\langle \nabla V_1(x), f(x, g(x) + v, d) - f_0(x, g(x) + v, d)\rangle\,| \ \leq \ e_1$$
$$\forall v \in \chi(x),\ x \in S^+ \backslash \Sigma,\ d \in D \tag{27}$$

$$|\,\langle \nabla V_2(x), f(x, g(x) + v, d) - f_0(x, g(x) + v, d)\rangle\,| \ \leq \ e_2$$
$$\forall v \in \chi(x),\ x \in S^+ \backslash \{0\},\ d \in D \tag{28}$$

we have $x(t) \in S \quad \forall t \geq 0$. Further, there exists $T_\alpha > 0$ such that we have $V_2(x(t)) \leq \beta \ \ \forall t \geq T_\alpha$ for any such trajectory x.

Proof. A simple proof by contradiction (based on the hypotheses (LF2) and (LF3)) shows that if the positive values of α, e_a, e_m are sufficiently small, then there exists $\omega_\alpha > 0$ such that

$$V_1(x) \leq 2\alpha,\ \beta/2 \leq V_2(x) \leq 2b,\ v \in \chi(x),\ d \in D,\ |a| \leq e_a,\ |m| \leq e_m \Longrightarrow \langle \nabla V_2(x), f_0(x, a + g(x+m) + v, d)\rangle < -\omega_\alpha. \quad (29)$$

We fix such values. We next specify in what sense the remaining parameters δ, e_1 and e_2 must be small.

A *modulus of continuity* for a function h on a set X refers to a continuous nondecreasing function $\mu : [0,\infty) \to [0,\infty)$ with $\mu(0) = 0$ such that

$$|h(x) - h(y)| \leq \mu(|x-y|)\ \forall x, y \in X.$$

(Such a function always exists if h is continuous and X compact.) We define the following compact sets:

$$\begin{aligned} S_1 &:= \{x : \alpha/2 \leq V_1(x) \leq 2\alpha,\ \ V_2(x) \leq 2b\} \\ S_2 &:= \{x : V_1(x) \leq 2\alpha,\ \ \beta/4 \leq V_2(x) \leq 2b\}. \end{aligned}$$

We introduce the following moduli of continuity: μ_1 for V_1 on S^+, μ_2 for V_2 on S^+, ν_1 for ∇V_1 on S_1, ν_2 for ∇V_2 on S_2. We set

$$\phi := \sup\{|f(x, a + g(x+m) + v, d)| : x \in S^+, |a| \leq e_a, |m| \leq e_m, v \in V, d \in D\}$$

$$\begin{aligned} G_1 &:= \max\{|\nabla V_1(x)| : x \in S_1\} \\ G_2 &:= \max\{|\nabla V_2(x)| : x \in S_2\}. \end{aligned}$$

We also require a modulus ρ of uniform continuity such that

$$x \in S^+, v \in V, d \in D \Longrightarrow |f(x, a + g(x+m) + v, d) - f(x, g(x) + v, d)| \leq \rho(|(a,m)|). \quad (30)$$

Hypotheses (H2)–(H3) imply the existence of a Lipschitz constant K for f in the following sense:

$$x, x' \in S^+, |a| \leq e_a, |m| \leq e_m, v \in V, d \in D \Longrightarrow |f(x, a + g(x+m) + v, d) - f(x', a + g(x+m) + v, d)| \leq K|x - x'|. \quad (31)$$

We now fix any $\bar\omega \in (0, \omega_1)$ and we choose positive numbers δ, e_1, e_2 small enough so that

$$\mu_1(\delta\phi) \ < \ \alpha/2,\ \ \mu_2(\delta\phi) < \beta/4, \quad (32)$$

$$e_1 + G_1\rho(e_a + e_m) + \delta G_1 K\phi + \phi\nu_1(\delta\phi) < \omega_1 - \bar\omega, \quad (33)$$

$$e_2 + G_2\rho(e_a + e_m) + \delta G_2 K\phi + \phi\nu_2(\delta\phi) < \omega_\alpha/2 \quad (34)$$

(if necessary we reduce e_a and e_m and consequently ϕ in order to obtain (33) and (34)). With these choices, we now proceed to verify the assertions of the Lemma. Accordingly, let $x(\cdot)$ be any solution of eqns.(2), for data satisfying (26)-(28), and with $x(0) \in S$. We prove that $x(t) \in S$ for $0 = t_0 < t \leq t_1$ (the next sampling time), which will confirm the stated invariance. While $x(t)$ remains in S^+, the estimate $|\dot{x}(t)| \leq \phi$ applies; in conjunction with (32), this implies that $V_1(x(t)) < 2\alpha$ and $V_2(x(t)) < 2b$ for $0 < t \leq t_1$, whence $x(t)$ lies in S^+ throughout the interval $[0, t_1]$. We now wish to prove that we have $V_1(x(t)) \leq \alpha$ on the interval. Observe that if there is a point $\tau \in [0, t_1]$ for which $V_1(x(\tau)) < \alpha/2$, then this follows from (32), so we may limit ourselves to the case in which $x(t) \in S_1$ for all $t \in [0, t_1]$. Fix any $t \in (0, t_1)$. Then, for some $t' \in (0, t)$ and $d \in D$, we have

$$\begin{aligned}
&V_1(x(t)) - V_1(x(0)) \\
&\leq t \,\langle \nabla V_1(x(t')), f(x(t'), a_0 + g(x(0) + m_0) + v_0, d)\rangle \text{ (mean value)} \\
&\leq t \,\langle \nabla V_1(x(0)), f(x(t'), a_0 + g(x(0) + m_0) + v_0, d)\rangle + t\phi\nu_1(\delta\phi)
\end{aligned}$$

(by definition of ν_1, and since $|x(t') - x(0)| \leq t\phi \leq \delta\phi$)

$$\begin{aligned}
&\leq t \,\langle \nabla V_1(x(0)), f(x(0), a_0 + g(x(0) + m_0) + v_0, d)\rangle \\
&\qquad + t^2 G_1 K\phi + t\phi\nu_1(\delta\phi) \text{ (by (31))} \\
&\leq t \,\langle \nabla V_1(x(0)), f(x(0), g(x(0)) + v_0, d)\rangle + tG_1\rho(e_a + e_m) \\
&\qquad + t^2 G_1 K\phi + t\phi\nu_1(\delta\phi) \text{ (by (30))} \\
&\leq t \,\langle \nabla V_1(x(0)), f_0(x(0), g(x(0)) + v_0, d)\rangle + te_1 + tG_1\rho(e_a + e_m) \\
&\qquad + t^2 G_1 K\phi + t\phi\nu_1(\delta\phi) \text{(in view of (27))} \\
&\leq t\{-\omega_1 + e_1 + G_1\rho(e_a + e_m) + \delta G_1 K\phi + \phi\nu_1(\delta\phi)\} \;\; \text{(by (LH1))} \\
&\leq -t\bar{\omega} \;\; \text{(in view of (33))}.
\end{aligned}$$

It follows that $V_1(x(t)) \leq \alpha$ as claimed. In fact, the argument implies that for any two successive nodes $x(t_i)$ and $x(t_{i+1})$, either there exists $\tau \in (t_i, t_{i+1})$ for which $V_1(x(\tau)) < \alpha/2$, or else we have a type of decrease condition:

$$V_1(x(t)) - V_1(x(t_i)) \leq -\bar{\omega}(t - t_i), \quad t \in [t_i, t_{i+1}].$$

The same argument as above applied to V_2 (invoking (29) rather than (LH1) at the appropriate point) establishes that for any two successive nodes $x(t_i)$ and $x(t_{i+1})$, either there exists $\tau \in (t_i, t_{i+1})$ for which $V_2(x(\tau)) < \beta/4$, or else we have

$$V_2(x(t)) - V_2(x(t_i)) \leq -(t - t_i)\omega_\alpha/2, \quad t \in [t_i, t_{i+1}].$$

It follows that $V_2(x(t)) \leq b \;\; \forall t \geq 0$, which confirms that S has the stated invariance property. Furthermore, the decrease property implies that some node

$x(t_i)$ must satisfy $V_2(x(t_i)) \leq \beta/4$ for a value of t_i no greater than $T_\alpha := 2b/\omega_\alpha$ (since V_2 is nonnegative). It follows from (32) and the decrease property that all subsequent nodes $x(t_j)$ must then satisfy $V_2(x_j)) \leq \beta/2$, whence $V_2(x(t)) \leq \beta \; \forall t \geq T_\alpha$. This completes the proof of Lemma 1.

We next address the issue of stabilization for points *distant* from the sliding set; that is, for which V_1 is not necessarily small. We are given $R > 0$ and we consider initial conditions $x(0) \in B(0, R)$. If the measurement errors m_i are bounded *a priori* by a parameter E_m, and the actuator errors by E_a (which we may suppose without loss of generality), then hypotheses (H1)–(H3) imply the existence of constants c_1, c_2 such that, for any state trajectory x,

$$|\dot{x}(t)| \leq c_1 |x(t)| + c_2(E_m + E_a).$$

With $|x(0)| \leq R$, this yields via Gronwall's Lemma the following bound:

$$|x(t)| \leq e^{c_1 t}\{R + c_2(E_a + E_m)/c_1\} =: h(t), \quad t \geq 0. \tag{35}$$

We set

$$T_R := \max_{|x| \leq R} V_1(x)/\bar{\omega}, \tag{36}$$

where, as before, $\bar{\omega}$ is a given number in $(0, \omega_1)$, and we take R' to be the right side of (35) when $t = T_R$:

$$R' = h(T_R), \tag{37}$$

a choice which clearly depends only upon V_1, R, $\omega_1, \bar{\omega}$, and of course the parameters appearing in the hypotheses (H1)–(H3).

Lemma 2. For any $\alpha > 0$, for sufficiently small values of the positive parameters e_m, e_a, δ, e_1, for any trajectory $x(\cdot)$ generated by (2) with $x(0) \in B(0, R)$ and for data satisfying

$$|m_i| \leq e_m, \quad |a_i| \leq e_a \quad \text{and} \quad |t_{i+1} - t_i| \leq \delta \quad \text{for all } i \tag{38}$$

$$\begin{aligned} &|\langle \nabla V_1(x), f(x, g(x) + v, d) - f_0(x, g(x) + v, d)\rangle| \leq e_1 \\ &\qquad \forall v \in \chi(x),\ x \in B(0, R')\backslash\Sigma,\ d \in D, \end{aligned} \tag{39}$$

there exists $\bar{t} \in [0, V_1(x(0))/\bar{\omega}]$ such that $V_1(x(\bar{t})) \leq \alpha$, and such that

$$x(t) \in B(0, R'),\ t \in [0, \bar{t}].$$

Proof. We set

$$\begin{aligned} &V_1(R') := \max\{V_1(y) : |y| \leq R'\} \\ &S_1 := \{x : \alpha/2 \ \leq \ V_1(x) \ \leq \ V_1(R')\} \\ &\phi := \sup\{|f(x, a + g(x+m) + v, d)| : \\ &\qquad x \in B(0, R'), |a| \leq E_a, |m| \leq E_m, v \in V, d \in D\} \\ &G := \max\{|\nabla V_1(x)| : x \in S_1\}. \end{aligned}$$

Let μ be a modulus of continuity for V_1 on $B(0,R')$, ν a modulus of continuity for ∇V_1 on S_1, and ρ a modulus of uniform continuity such that

$$x \in B(0,R'), v \in V, d \in D \Longrightarrow \\ |f(x, a+g(x+m)+v,d) - f(x,g(x)+v,d)| \le \rho(|(a,m)|). \quad (40)$$

Hypotheses (H2)–(H3) imply the existence of a Lipschitz constant K for f in the following sense:

$$x, x' \in B(0,R'), |a| \le e_a, |m| \le e_m, v \in V, d \in D \Longrightarrow \\ |f(x, a+g(x+m)+v,d) - f(x', a+g(x+m)+v,d)| \le K|x-x'|. \quad (41)$$

We choose positive numbers δ, e_1, e_2 and $e_a < E_a$, $e_m < E_m$ small enough so that

$$\mu(\delta\phi) < \ \alpha/4, \ \ \delta\bar{\omega} < \alpha/2 \quad (42)$$

$$e_1 + G\rho(e_a+e_m) + \delta GK\phi + \phi\nu(\delta\phi) < \omega_1 - \bar{\omega}\,. \quad (43)$$

With these choices, we now proceed to verify the assertions of the Lemma. Accordingly, let $x(\cdot)$ be any solution of eqns.(2), for data satisfying (38)-(39), and with $x(0) \in B(0,R)$.

We may suppose that $V(x(0)) > \alpha$, for otherwise there is nothing to prove. Consider the subintervals $[t_i, t_{i+1}]$ $(i \ge 0)$ for which $t_{i+1} \le V_1(x(0))/\bar{\omega}$ (at least the first subinterval is of this type, by the second part of (42)). It follows from (35) and the definition of R' that for any such subinterval, we have $x(t) \in B(0,R')$ for $t \in [t_i, t_{i+1}]$.

Case 1. For some such subinterval, there is a point $\tau \in [t_i, t_{i+1}]$ for which $V_1(x(\tau)) < \alpha/2$. In this case it is clear that the assertion of the Lemma holds.

Case 2. The remaining case is that in which $x(t)$ lies in S_1 throughout the subinterval, for all the subintervals $[t_i, t_{i+1}]$ in question. Then, for any $t \in [t_i, t_{i+1}]$, for some $t' \in (t_i, t_{i+1})$ and $d \in D$, we have (by the mean value theorem)

$$V_1(x(t)) - V_1(x(t_i)) = (t-t_i)\,\langle \nabla V_1(x(t')), f(x(t'), a_i + g(x(t_i)+m_i) + v_i, d)\rangle\,.$$

The right side is in turn bounded above by

$$\begin{aligned}
&(t-t_i)\,\langle \nabla V_1(x(t_i)), f(x(t'), a_i + g(x(t_i)+m_i)+v_i, d)\rangle \\
&\quad + (t-t_i)\phi\nu(\delta\phi) \quad \text{(by definition of } \nu\text{, and since } |x(t')-x(t_i)| \le \delta\phi) \\
&\le (t-t_i)\,\langle \nabla V_1(x(t_i)), f(x(t_i), a_i + g(x(t_i)+m_i)+v_i, d)\rangle \\
&\quad + (t-t_i)\{\delta GK\phi + \phi\nu(\delta\phi)\} \quad \text{(by the Lipschitz condition (41))} \\
&\le (t-t_i)\,\langle \nabla V_1(x(t_i)), f(x(t_i), g(x(t_i))+v_i, d)\rangle \\
&\quad + (t-t_i)\{G\rho(e_a+e_m) + \delta GK\phi + \phi\nu(\delta\phi)\} \quad \text{(see (40))}.
\end{aligned}$$

In view of (39), this last expression is in turn bounded above by

$$\begin{aligned}
&(t-t_i)\{\langle \nabla V_1(x(t_i)), f_0(x(t_i), g(x(t_i))+v_i, d)\rangle + e_1 + G\rho(e_a+e_m)\\
&\quad + \delta GK\phi + \phi\nu(\delta\phi)\}\\
&\le (t-t_i)\{-\omega_1 + e_1 + G\rho(e_a+e_m) + \delta GK\phi + \phi\nu(\delta\phi)\} \text{ (by (LH1))}\\
&\le -(t-t_i)\bar{\omega} \text{ (in view of (43)).}
\end{aligned}$$

If Case 1 never holds, then this decrease conclusion is valid up to the last partition point t_{j+1} no greater than $V_1(x(0))/\bar{\omega}$. For that last one, we have $V_1(x(t_{j+1}) \ge \alpha/2$ (since otherwise Case 1 would apply) and $t_{j+1} > V_1(x(0))/\bar{\omega} - \delta$. Summing in the decrease estimate yields

$$V_1(x(t_{j+1})) - V_1(x(0)) \le -t_{j+1}\bar{\omega},$$

which, combined with the preceding inequality, gives

$$V_1(x(t_{j+1})) \le \delta\bar{\omega} < \alpha/2 \quad \text{(in light of (42)).}$$

This contradiction shows that Case 2 does not occur, and the Lemma is proved.

We now proceed to prove the Theorem by combining the 'near Σ' and the 'distant from Σ' analyses given above. Given $r, R, \bar{\omega}$ and $\bar{\epsilon}$ as in its statement, we define R' as above, via (37). Now set

$$b := \max\{V_2(x) : x \in B(0,R')\},$$

and choose $\bar{\alpha} \in (0, T_R/2)$ and $\beta \in (0,b)$ small enough so that

$$\begin{aligned}
&\{x : V_1(x) \le \bar{\alpha},\ V_2(x) \le \beta\} \subset B(0,r)\\
&\{x : V_1(x) \le \bar{\alpha},\ V_2(x) \le b\} \subset \{x : d_\Sigma(x) < \bar{\epsilon}\}.
\end{aligned}$$

Apply Lemma 1 to find $\alpha \in (0,\bar{\alpha}]$ so that the conclusions of that Lemma hold, for certain bounds on the errors and the mesh size. Suppose now that these bounds are further reduced (if necessary) to make Lemma 2 operative. Then any state trajectory $x(\cdot)$ emanating from $B(0,R)$, in the presence of data satisfying the bounds in question, satisfies $V_1(x(\bar{t})) \le \alpha$ at a certain time $\bar{t} \le V_1(x(0))/\bar{\omega}$ for which $x(\bar{t}) \in B(0,R')$. Thus we have

$$x(\bar{t}) \in S := \{x : V_1(x) \le \alpha,\ V_2(x) \le b\} \subset \{x : d_\Sigma(x) < \bar{\epsilon}\}.$$

The set S is invariant (in the sense of Lemma 1) and for some R'' (in view of (LF3)) we have

$$S^+ := \{x : V_1(x) \le 2\alpha, V_2(x) \le 2b\} \subset \{x : V_1(x) \le T_R, V_2(x) \le 2b\} \subset B(0,R'').$$

Set $R_* := \max\{R', R''\}$. Then R_* depends upon R, ω_1, $\bar{\omega}$, the functions V_1 and V_2, the parameters in the hypotheses (H1)–(H3), but not upon r. Since $B(0,R_*)$ contains both S^+ and $B(0,R')$, the bounds on modeling error prescribed by the lemmas (that is, (27),(28) and (39)) are all subsumed by the corresponding bounds relative to the set $B(0,R_*)$ (that is, the relations (4) and (5) in the statement of the Theorem). All the conclusions of the Theorem now follow.

Remark (error estimates). The statement of the Theorem does not make explicit how small the error parameters must be, but some information does emerge in the proof. For example, let us suppose that we are concerned primarily with modeling error. Observe that if actuator and measurement error are small enough, a suitably small value of the mesh size δ will allow in the inequalities (33)(34)(43) values of e_1 and e_2 as close as desired to $\omega_1 - \bar{\omega}$ and $\omega_\alpha/2$ respectively. (Thus the maximum allowable modeling error is determined by the available decrease rate.) Note that a trade-off exists: taking $\bar{\omega}$ smaller allows for greater error, but this may make the convergence slower (see (36)). These observations can yield estimates on how large the modeling error can be (to still be able to guarantee the conclusions of the Theorem). See the example of Section 6 in this connection, where the corresponding bound e_2 is rendered irrelevant because the Lyapunov function V_2 and the modeling error are 'matched': the inner product of ∇V_2 and $f - f_0$ vanishes.

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