A Maximum Principle for Hybrid Optimal Control Problems with Pathwise State Constraints

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***Abstract*— This paper provides necessary conditions of optimality, in the form of a maximum principle , for a broad class of hybrid optimal control problems, in which the dynamics of the constituent processes take the form of differential equations with control terms, and restrictions on the transitions or switches between operating modes are described by collections of functional equality and inequality constraints. Different choices of the constraint functionals capture a wide range of possible autonomous and controlled switching strategies. A notable feature of our formulation is the provision it makes for pathwise state constraints on the continuous variables.**

***Index Terms*— Hybrid control, Maximum Principle, State Constraints**

I. INTRODUCTION

Many complex control tasks, for example those associated with controlling an autonomous vehicle to carry out a sequence of manoeuvres or with controlling a collection of interacting process units, involve two levels of decision making. At the higher level, it is necessary to set the values of 'discrete' decision variables determining, possibly, the sequencing of operations or the selection of way stations. At the lower level, we are concerned with providing 'continuous' input signals that control the constituent devices, consistent with the upper level decision making. Hybrid control addresses the problem of integrating discrete and continuous decision making in this context.

Hybrid optimal control is an approach to hybrid control, in which we seek strategies to minimize a cost function, or criterion of best performance. One approach to seeking optimal strategies, followed by Shaikh and Caines [7] is to carry out an iterated two-step procedure, according to which we alternate between minimizing the cost function over the discrete and continuous variables.

'Hybrid maximum principle' is the name given to some set of first order necessary conditions of optimality, akin to the traditional maximum principle, relating to a selection of continuous variables in a hybrid optimal control problem, that optimizes the cost function for a fixed choice of the discrete variables. For some simple cases of hybrid optimal control problems, it can be used to carry out the minimization over the continuous variables analytically. For other cases, such conditions inspire numerical schemes for minimizing the cost over the continuous time variables and are of assistence in establishing properties of limit points of sequences of continuous variables generated by such schemes ('convergence analysis'). The role of the hybrid maximum principle, then, is to attend to the 'minimization over continuous variables' step of two-step interative schemes for solving optimal (and related numerical optimization schemes) for solving hybrid optimal control problems.

A characteristic of hybrid systems is the occurence of switches in the structure of the dynamics, constraints sets, etc., governing the evolution of the continuous variables. In a hybrid optimal control problem a particular choice of the discrete variables typically fixes a sequence of dynamical systems for the continuous variables and imposes certain constraints on transition times and on the values of continuous variables just before, and just after, a transition occurs.

A very general formulation of a dynamic optimization problem that arises from fixing the discrete variables in a hybrid optimal control problem, takes the form of a finite collection of 'continuous' control systems and a collection of constraints on the times transitions between these control systems occur, as well as on the values of the continuous variables before and after each transition time. Such a formulation, referred to as optimal control of 'multiprocesses', was proposed by Clarke and Vinter, and was aimed at unifying earlier work on 'multi-stage' dynamic optimization reported in, for example, the aeronautical control and resource economics literature. However with the growth interest hybrid control, we can expect that the principal forum for future application of such conditions will be to hybrid control. The papers [3], [4] provide a Maximum Principle for optimal multiprocesses of a very general nature, in which the constraints take the form of set inclusions and the dynamics of the consituent processes are governed by (possibly nonsmooth) differential inclusions.

The philosophy of [3] was to derive optimality conditions in the absence of any presumed structure on the constraints governing transitions times and end-values of constituent processes (apart from their being expressible in terms of closed set inclusions). In such conditions, degeneracy manifests itself as lack of information in the optimality conditions. (If, for example, the conditions are satisfied with the cost multiplier taking the value zero, they convey no useful properties of minimizers). The conditions are of significance then, only when additional hypotheses (typically of a 'controllability' nature) are imposed which exclude satisfaction of the optimality conditions in some trivial sense. This approach is followed also by Sussmann [9], but in the narrower context of autonomous problems (dynamical behaviour invariant under time translations), for which the time intervals associated with the individual 'subprocesses' are contiguous.

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An alternative philosophy, followed by Shaikh and Caines [7] and Garavello and Piccoli [5] is to focus at the outset on specific classes of constraints relevant to hybrid control and to introduce hypotheses on the dynamics and transition constraints ensuring the validity of the hybrid maximum principle in a non-degenerate form. We add that the work in [6], [7], and that in [8], on 'non-degenerate' hybrid maximum principles is part of a programme aimed at the development of numerical methods and schemes for optimizing over continuous and discrete variables.

The contribution of this paper is to present a hybrid maximum principle, for a generalization of the dynamic optimization problem considered in [3], in which we introduce pathwise constraints on the continuous 'state' variables. For simplicity of presentation, we have adopted a 'controlled differential equation' description of the dynamics and expressed 'transition' constraints as functional equality/inequality constraints. All data is assumed to be differentiable. These restrictions can be relaxed, at the cost of a much higher analytical overhead.

For our purposes, a hybrid system incorporates a number of 'continuous' control systems described as follows: for $i = 1, \ldots M$:

$$\begin{aligned} \dot{x}_i(t) &= f_i(t, x_i(t), u_i(t)) \quad \text{a.e. } t \in [s_i, t_i] && \text{(I.1)} \\ u_i(t) &\in U_i \quad \text{a.e. } t \in [s_i, t_i] && \text{(I.2)} \\ x_i(t) &\in A_i \quad \text{a.e. } t \in [s_i, t_i] \,. && \text{(I.3)} \end{aligned}$$

Here, for each i, n_i, m_i are positive integers, $f_i : R \times R^{n_i} \times R^{m_i} \to R^{n_i}$ is a given function and $U_i \subset R^{m_i}$ and $A_i \subset R^{n_i}$ are given sets. It is assumed that (for each i) the state constraint set A_i has the representation

$$A_i \;=\; \{x \,|\, \phi_i(x) \le 0\}$$

in which $\phi_i(x) : R^{n_i} \to R^{k_i}$ is a given function. (The inequality is interpreted in a componentwise sense.) In what follows, the k_i scalar valued functions associated with ϕ_i will be denoted ϕ_i^j, $j = 1, \ldots, k_i$.

A *process* for the hybrid system is an M-tuple $\{(s_i, t_i, x_i(.), u_i(.))\}_{i=1}^M$, whose i'th element comprises numbers s_i, t_i ($s_i < t_i$), an absolutely continuous function $x_i : [s_i, t_i] \to R^{n_i}$ and a measurable function $u_i : [s_i, t_i] \to R^{m_i}$ satisfying (I.1)– (I.3).

Suppose we are given scalar valued functions

$$\begin{aligned} \psi^j : \Pi_{i=1}^M \left(R \times R^{n_i} \times R \times R^{n_i}\right) \to R, \\ j = 0, \ldots, d_1, d_1+1, \ldots, d_1+d_2 \end{aligned}$$

for some integers $d_1 \ge 0$, $d_2 \ge 0$, and functions

$$L_i : R \times R^{n_i} \times R \times R^{n_i}, \quad i = 1, \ldots, M.$$

Consider now the optimization problem:

$$(P) \left\{ \begin{array}{l} \text{Minimize } \psi^0\left(\{(s_i, x_i(s_i), t_i, x_i(t_i))\}_{i=1}^M\right) \\ \qquad + \sum_i \int_{s_i}^{t_i} L_i(t, x_i(t), u_i(t))dt \\ \text{over processes} \{(s_i, t_i, x_i(.), u_i(.))\}_{i=1}^M \\ \qquad \text{satisfying the constraints} \\ \psi^j\left(\{(s_i, x_i(s_i), t_i, x_i(t_i))\}_{i=1}^M\right) \;\le\; 0 \\ \quad \text{for } j = 1, \ldots, d_1 \\ \psi^j\left(\{(s_i, x_i(s_i), t_i, x_i(t_i))\}_{i=1}^M\right) \;=\; 0 \\ \quad \text{for } j = d_1+1, \ldots, d_1+d_2 \,. \end{array} \right.$$

The cases '$d_1 = 0$' and $d_2 = 0$ correpond to 'no inequality constraints' and 'no equality constraints', respectively.

A common situation is that in which the time intervals $[s_i, t_i]$ are contiguous and $n_1 = \ldots = n_M = n$ and $m_1 = \ldots = m_M = m$ (for some n and m). But the general theory we develop will not impose these requirements, i.e. our framework allows for a wide variety of 'linked' processes, each of which may have control and state vectors of different dimensions.

In the literature, transitions between contiguous time intervals $[s_i, t_i]$ and $[s_{i+1}, t_{i+1}]$ are classified according to whether they are 'autonomous' or 'controlled'. In [1] autonomous transitions are those for which the value of the state $x_{i+1}(s_{i+1})$ after a transition is uniquely determined by the value of the state $x_i(t_i)$ before the transition time $t_i (= s_{i+1})$, while for controlled transitions, the value of $x_{i+1}(s_{i+1})$ is a choice variable. In the context of hybrid system necessary conditions in, for instance, [6], [7], [8], autonomous transitions are those for which the discrete state variable must make a jump to a predetermined value when the continuous state component enters a manifold of co-dimension 1 (sometimes termed a guard), while a controlled transition denotes a controlled jump of the discrete state component at an arbitrary time instant and to an arbitrary continuous state component value (outside a guard).The controlled jump may be further constrained by discrete dynamics for the discrete state component and the number of jumps may be constrained by a bound on the number of allowable jumps. Clearly the constraint functionals in (P) can be chosen to accommodate a wide variety of autonomous and controlled transitions.

A process which achieves the minimum in (P) over all processes satisfying the constraints is called a minimizer for (P).

Let $\{\bar{s}_i, \bar{t}_i, \bar{x}_i(.), \bar{u}_i(.)\}_{i=1}^M$ be a minimizing process of interest. Define $\tilde{f}_i = (L_i, f_i)$ for $i = 1, \ldots, M$. We shall impose the following hypotheses on the data.

(H1): $\psi^j, j = 0, \ldots, d_1 + d_2$ are continuously differentiable functions.

(H2): There exists a Borel measurable function $k_i : [\bar{s}_i, \bar{t}_i] \times R^{m_i} \to R$ and $\epsilon > 0$ such that $t \to k_i(t, \bar{u}_i(t))$ is integrable and

$$|\tilde{f}_i(t'', x'', u) - \tilde{f}_i(t', x', u)| \;\le\; k_i(t, u)|(t'', x'') - (t', x')|$$

for all $(t'', x''), (t', x') \in (t, \bar{x}_i(t)) + \epsilon B$, a.e. $t \in [\bar{s}_i, \bar{t}_i]$.

(H3): $\tilde{f}_i(t, x, .)$ is Borel measurable for each (t, x).

(H4): U_i is a Borel set.

(H5): ϕ_i^j is continuously differentiable for $j = 1, \dots, k_i$.

II. A Maximum Principle for Hybrid Optimal Control Problems with State Constraints.

In this section we state necessary conditions for a process $\{(\bar{s}_i, \bar{t}_i, \bar{x}_i(.), \bar{u}_i(.))\}_{i=1}^M$ to be a minimizer, in the form of a maximum principle. The optimality conditions take the form of maximum principle like conditions on each element $(\bar{s}_i, \bar{t}_i, \bar{x}_i(.), \bar{u}_i(.))$, $i = 1, \dots, M$ in the process, with common cost multiplier λ_0, coupled through relationships involving the boundary values of the adjoint variables. Note that the number M of 'locations' or 'discrete state regimes' is fixed and also the sequence in which they occur; this reflects the fact that the maximum principle addresses the problem merely of minimizing the cost over the continuous variables, for fixed values of the discrete variables.

It is well known in optimal control that the 'Lagrange multiplier' asssociated with a unilateral state constraint takes the form of a function of bounded variation defining the 'integrator' in a Stieltjes integral. Such multipliers arise also in optimal hybrid control, when state constraints are present. $NBV^+([\bar{s}, \bar{t}]; R^k)$ is the space of R^n-valued functions of bounded variation on the interval $[\bar{s}, \bar{t}]$, which are (component-wise) non-decreasing and right continuous on the half open interval $(\bar{s}, \bar{t}]$. Given a continuous function $\phi : [\bar{s}, \bar{t}] \to R^k$ and an element $\mu \in NBV^+([\bar{s}, \bar{t}]; R^k$, we denote by $\int_{[\bar{s}, \bar{t}]} \phi(t) \mu(dt)$ the Stieltjes integral of $\phi(.)$ with respect to the 'integrator' μ. An R^k-valued function μ in $NBV^+([\bar{s}, \bar{t}]; R^k)$ can also be thought of as a collection of k scalar valued functions in $NBV^+([\bar{s}, \bar{t}]; R)$. We write these scalar valued functions μ^j, $j = 1 \dots, k$.

Define $H_i^\lambda : R \times R^{n_i} \times R^{n_i} \times R^{m_i} \to R$:

$$H_i^\lambda(t, x, p, u) \;=\; p^T f_i(t, x, u) - \lambda L_i(t, x, u)$$

for $i = 1, \dots, M$ and $\lambda \ge 0$.

Theorem 2.1: Let $\{(\bar{s}_i, \bar{t}_i, \bar{x}_i(.), \bar{u}_i(.))\}_{i=1}^M$ be a minimizer for (P). Assume (H1)-(H5) are satisfied. Then there exist

- non-negative numbers λ_j, $j = 0, \dots, d_1$
- numbers λ_j, $j = d_1 + 1, \dots, d_1 + d_2$

and, for $i = 1 \dots, M$,

- an element $\mu_i \in NBV^+([\bar{s}_i, \bar{t}_i]; R^{k_i})$,
- an abs. continuous function $p_i : [\bar{s}_i, \bar{t}_i] \to R^{n_i}$,
- a continuous function $h_i : [\bar{s}_i, \bar{t}_i] \to R$,

with the properties (i)–(vii) below.
(In these relationships, $q_i(t)$ is taken to be

$$q_i(t) \;=\; p_i(t) + \int_{[\bar{s}_i, t]} \nabla_x \phi_i(\bar{x}_i(s)) \mu_i(ds)$$

if $t \in (\bar{s}_i, \bar{t}_i]$ and $q_i(\bar{s}_i) = p_i(\bar{s}_i)$.)

(i): (non-triviality of Lagrange Multipliers)
$\left(\{\lambda_i\}_{i=0}^{d_1+d_2}, \{p_i\}_{i=1}^M, \{\mu_i\}_{i=1}^M\right) \;\neq\; 0$

(ii): (Transversality Conditions)
$$\{(-h_i(\bar{s}_i), q_i(\bar{s}_i), h_i(\bar{t}_i), -q_i(\bar{t}_i))\}_{i=1}^M \;=\; \nabla\left(\sum_{j=0}^{d_1+d_2} \lambda_j \psi^j \{(\bar{s}_i, \bar{x}_i(\bar{s}_i), \bar{t}_i, \bar{x}_i(\bar{t}_i))\}_{i=1}^M\right)$$

(iii): (Comp. Slackness of Endpoint Multipliers)
If, for some $j \in \{1, \dots, d_1\}$,
$\psi^j\left(\{(\bar{s}_i, \bar{x}_i(\bar{s}_i), \bar{t}_i, \bar{x}_i(\bar{t}_i))\}_{i=1}^M\right) \;<\; 0$, then $\lambda_j = 0$.

and, for $i = 1, \dots, M$,

(iv): (Continuity of the Maximized Hamiltonian)
$h_i(t) = \sup_{u \in U_i} H_i(t, \bar{x}_i(t), q_i(t), u)$ a.e. $t \in [\bar{s}_i, \bar{t}_i]$,

(v): (Adjoint Equations)
$-\dot{p}_i(t) = \;(H_i)_x\,(t, \bar{x}_i(t), q_i(t), \bar{u}_i(t))$ a.e. $t \in [\bar{s}_i, \bar{t}_i]$

(vi): (Maximization of the Hamiltonian).
$h_i(t) = H_i(t, \bar{x}_i(t), q_i(t), \bar{u}_i(t))$ a.e. $t \in [\bar{s}_i, \bar{t}_i]$

(vii): (Comp. Slackness of State Constraint Multipliers)
If, for some $j \in \{1, \dots, k_i\}$, $\phi_i^j(\bar{x}_i(t)) < 0$, then $\mu_j(.)$ is constant on a relative neighborhood of t in $[\bar{s}_i, \bar{t}_i]$.

The idea behind the derivation of the optimality conditions is to associate a minimizer for the hybrid optimal control problem with a minimizer for a conventional optimal control problem, apply a known state constrained Maximum Principle to this latter problem, and to interpret the conditions in terms of the data of the original problem. The reformulation involves the introduction of additional control variables. The new control variables induce a change of independent variable, the result of which is to replace each sub-interval $[s_i, t_i]$ associated with a process $(s_i, t_i, x_i(.), u_i(.))$ by a common sub-interval of the real line. A similar technique has been used in other contexts, to derive first order optimality conditions for conventional optimal control problems, for example, or to establish constancy of the maximized Hamiltonian along an optimal process for autonomous optimal control problems. (See [2], [10].) Details of the derivation will be given elsewhere.

It is of interest to explore the detailed implications of the above optimality conditions for a number of significant cases of problem (P). We consider one such case, in which the time intervals are contiguous and the state variable is continuous across a transition between time intervals. Fix $\psi : R^n \to R$, $x_0 \in R^n$, $[\bar{s}, \bar{t}] \subset R$. Assume that, for some integers n and k,

$$n_1, \dots, n_M = n \quad \text{and} \quad k_1, \dots, k_M = k\;.$$

Consider the hybrid optimal control problem

$$(P)\left\{\begin{array}{l}\text{Minimize } \psi(x_N(\bar{t}))\\ \text{over processes}\{(s_i,t_i,x_i(.),u_i(.))\}_{i=1}^M\\ \qquad\quad\text{satisfying the constraints}\\ s_1=\bar{s}, t_1=s_2,\ldots,t_{M-1}=s_M\\ x_1(s_1)=x_0, x_1(t_1)=x_2(s_1),\\ \qquad \ldots,x_{M-1}(t_{M-1})=x_M(s_M)\end{array}\right.$$

By considering the special case of (P), in which $d_1=0$, $d_2=2M+1$ and
$(\psi^1,\ldots,\psi^{2M+1}=$
$(s_1-\bar{s},t_M-\bar{t},t_1-s_2,\ldots,t_{M-1}=s_M),x_1(s_1)=x_0,x_1(t_1)=x_2(t_2),\ldots,x_{M-1}(t_{M-1})=x_M(s_M))$

we may deduce the following optimality conditions from Thm. 2.1:

Let $\{(\bar{s}_i,\bar{t}_i,\bar{x}_i(.),\bar{u}_i(.))\}_{i=1}^M$ be a minimizer. Assume that the data for the above optimization problem, regarded as a special case of (P), satisfies (H1)-(H5). Assume furthermore that, for $i=1,\ldots,M$,

$$\phi_1(\bar{x}_1(\bar{s}_1))<0,\phi_1(\bar{x}_1(\bar{t}_1))<0,\ldots,$$
$$\phi_N(\bar{x}_N(\bar{s}_N))<0,\phi_N(\bar{x}_N(\bar{t}_N))<0\ .$$

Then there exists a non-negative number λ_0 and, for $i=1,\ldots,M$,

- *an element $\mu_i\in NBV^+([\bar{s}_i,\bar{t}_i];R^{k_i})$,*
- *an abs. continuous function $p_i:[\bar{s}_i,\bar{t}_i]\to R^{n_i}$,*
- *a continuous function $h_i:[\bar{s}_i,\bar{t}_i]\to R$,*

with the properties (a)–(e) below (in which

$$q_i(t)\ =\ p_i(t)+\int_{[\bar{s}_i,t]}\nabla_x\phi_i(\bar{x}_i(s))\mu_i(ds)$$

if $t\in(\bar{s}_i,\bar{t}_i]$ and $q_i(\bar{s}_i)=p_i(\bar{s}_i)$.)

(a): *(non-triviality of Lagrange Multipliers)*
$\left(\lambda_0,\{\mu_i\}_{i=1}^M\right)\ \neq\ 0$

(b): *(Transversality Conditions)*
$h_1(\bar{t}_1)=h_2(\bar{s}_2),\ldots,h_{M-1}(\bar{t}_{M-1})=h_M(\bar{s}_M)$

$q_1(\bar{t}_1)=q_2(\bar{s}_2),\ldots,q_{M-1}(\bar{t}_{M-1})=q_M(\bar{s}_M)$

$-q_N=\lambda_0\nabla\psi(\bar{x}_M(\bar{t}_M))$

and, for $i=1,\ldots,M$,

(c): *(Continuity of the Maximized Hamiltonian)*
$h_i(t)=\sup_{u\in U_i}H_i(t,\bar{x}_i(t),q_i(t),u)$ *a.e.* $t\in[\bar{s}_i,\bar{t}_i]$,

(d): *(Adjoint Equations)*
$-\dot{p}_i(t)=\ (H_i)_x\,(t,\bar{x}_i(t),q_i(t),\bar{u}_i(t))$ *a.e.* $t\in[\bar{s}_i,\bar{t}_i]$

(e): *(Maximization of the Hamiltonian).*
$h_i(t)=H_i(t,\bar{x}_i(t),q_i(t),\bar{u}_i(t))$ *a.e.* $t\in[\bar{s}_i,\bar{t}_i]$

(f): *(Comp. Slackness of the State Constraint Multipliers)*
For $j=1,\ldots,k_i$
'$\phi_i^j(\bar{x}_i(t))<0$' *implies* '$\mu_j^j(.)$ *is constant on a relative neighbourhood of t in $[\bar{s}_i,\bar{t}_i]$'.*

Notice that, in this case, the transversality conditions translate into a number of conditions, which include the assertion that the maximized Hamiltonian is continuous across a 'free' transition time and that, if the continuous state does not jump at a transition time, then neither does the adjoint variable.

III. A Simple Example

We consider a simple example, which exemplifies multi-stage optimal control problems with state constraints. A robot arm of unit mass, initially at rest, moves along a line from the origin. It picks up a load of unit mass when it has travelled a distance L from its starting point, and returns to its starting point. At the end of the manoeuvre the velocity of the robot arm must be zero. A double integrator model is used to describe the effect of a force u, which is regarded as the control, on the robot arm displacement y. Denote by T and τ, the overall time of the manoeuvre and of the pickup time respectively. It is assumed that both the velocity $\dot{y}$ of the robot arm and the control are subject to pathwise constraints:

$$|\dot{y}(t)|\ \leq 1 \quad\text{and}\quad |u(t)|\ \leq 1 \quad \text{a.e. } t\in[0,T]\ .$$

Finally, we assume that the load is stationary just before pickup. It is struck by the robot arm at the pickup time τ, and the subsequent velocity of robot arm and load is governed by the law of conservation of momentum:

$$2\times v(\tau^+)\ =\ 1\times v(\tau^-)\ .$$

Regarding y and $\dot{y}$ and as the x_1 and x_2 components respectively of a 2-vector state variable x, we can formulate the problem of determining a 'minimum time' strategy as a 'multi-stage' optimal problem:

$$\left\{\begin{array}{l}\text{Minimize } T\\ \text{over numbers } T>0,\ \tau\in[0,T] \text{ and } ((x(.),u(.)) \text{ s.t}\\ \dot{x}\ =\ \left\{\begin{array}{ll}Ax(t)+b_1u(t), & \text{a.e. } t\in[0,\tau)\\ Ax(t)+b_2u(t), & \text{a.e. } t\in(\tau,T]\end{array}\right.\\ u(t)\in[-1,+1] \quad \text{a.e. } t\in[0,T]\\ |x_2(t)|\leq 1 \quad \text{a.e. } t\in[0,T]\\ x(0)=x(T)=0\\ x_1(\tau^+)=x_1(\tau^-)\\ x_2(\tau^+)=(1/2)x_2(\tau^-)\ .\end{array}\right.$$

Here,

$$A=\left[\begin{array}{cc}0&1\\0&0\end{array}\right],\ b_1=\left[\begin{array}{c}0\\1\end{array}\right],\ b_2=\left[\begin{array}{c}0\\1/2\end{array}\right].$$

This problem can be alternatively expressed as a special case of (P), in which the hybrid system has two 'modes', those corresponding to the unloaded and the loaded weight of the robot arm respectively. Assume that

$$L>2\ .$$

It can be shown that a minimizer exists for this problem, and that the hypotheses of the Thm. 2.1 are satisfied with respect to every minimizer. Furthermore, there is a unique process satisfies the conditions of Thm. 2.1. Since this is a necessary condition, it follows that there is a unique minimizer and it satisfies the conditions of the hybrid maximum principle. Write T^*, τ^* and $u^*(.)$ for the optimal time horizon, pickup time and control respectively. It can be deduced from the hybrid maximum principle that:

$$T^* = 3 + 2L,\ \tau^* = 1 + L$$

and writing

$$t_1 = 1,\ t_2 = L,\ t_3 = 3 + L,\ t_4 = 2L + 1$$

we have

$$u^*(t) = \begin{cases} +1 & 0 < t < t_1 \\ 0 & t_1 < t < t_2 \\ -1 & t_2 < t < t_3 \\ 0 & t_3 < t < t_4 \\ +1 & t_4 < t < T^* \end{cases}$$

Figure 1 shows a phase plane portrait of the optimal state trajectory. Full details will be reported elsewhere. Notice an interesting feature of the optimal strategy: while policies are permitted for which the velocity of the robot arm at the pickup time is non-zero, the optimal policy is to pick up the load when the robot arm is stationary. This phenomenon is connected with the fact that the change of velocity at pickup is modelled by an appeal to 'conservation of momentum'. The optimal pickup may occur at a time when the robot arm velocity is non-zero, if another law is invoked to determine the robot arm velocity after pickup.

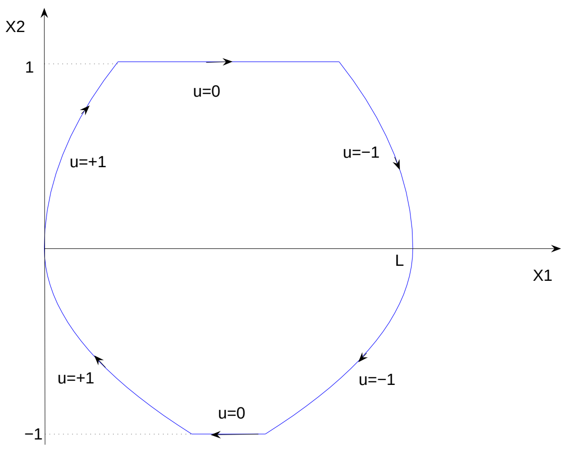

Fig. 1. **Phase plane portrait of the optimal state trajectory**

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