Confluentes Mathematici, Vol. 1, No. 2 (2009) 225–247

ON REALITY PROPERTY OF WRONSKI MAPS

E. MUKHIN*, V. TARASOV*,† and A. VARCHENKO‡

*Department of Mathematical Sciences,
Indiana University – Purdue University Indianapolis,
402 North Blackford St, Indianapolis, IN 46202-3216, USA

†St. Petersburg Branch of Steklov Mathematical Institute,
Fontanka 27, St. Petersburg, 191023, Russia

‡Department of Mathematics,
University of North Carolina at Chapel Hill,
Chapel Hill, NC 27599-3250, USA

Received 28 June 2008
Revised 4 September 2008

We prove that if all roots of the discrete Wronskian with step 1 of a set of quasi-exponentials with real bases are real, simple and differ by at least 1, then the complex span of this set of quasi-exponentials has a basis consisting of quasi-exponentials with real coefficients. This theorem generalizes the statement of the B. and M. Shapiro conjecture about spaces of polynomials.

The proof is based on the Bethe ansatz method for the *XXX* model.

Keywords: Discrete Wronski map; B. and M. Shapiro conjecture; Bethe ansatz; *XXX* model.

AMS Subject Classification: 81R12, 14P05

1. Introduction

The B. and M. Shapiro conjecture asserts that if the Wronskian of N polynomials with complex coefficients has real roots only, then the space spanned by these polynomials has a basis consisting of polynomials with real coefficients. This conjecture has many algebro-geometric reformulations and has generated a lot of interest in the past decade, see for example [14, 15].

The B. and M. Shapiro conjecture in the case of two polynomials was proved in [3] by complex-analytic methods. In [9] we proved the general case using a different approach. We showed that a generic space of polynomials V can be constructed by the Bethe ansatz method for the periodic Gaudin model. It turns out that the coefficients of the monic differential operator D of order N annihilating V are eigenvalues of the transfer matrices — linear operators, acting on the space of states of the Gaudin model. If the roots of the Wronskian of V are real, then the transfer matrices are self-adjoint with respect to a positive definite Hermitian form, hence, their eigenvalues are real. This implies that the coefficients of the differential operator D are real, which gives the existence of a basis for V consisting of polynomials with real coefficients.

In this paper we prove a similar statement about spaces of quasi-exponentials by the same method. Namely, we prove that if the Wronskian of N quasi-exponentials $e^{\lambda_i x}p_i(x)$, where λ_i are real numbers and $p_i(x)$ are polynomials with complex coefficients, has real roots only, then the space spanned by these quasi-exponentials has a basis such that all polynomials have real coefficients, see Theorem 4.1. The proof is based on the Bethe ansatz for the quasi-periodic Gaudin model. The case $\lambda_1 = \cdots = \lambda_N = 0$ is the statement of the original B. and M. Shapiro conjecture.

Using the Bethe ansatz for the quasi-periodic XXX model, we obtain a similar statement about spaces of quasi-exponentials with the Wronskian replaced by the discrete Wronskian, see Theorem 2.1. In this case, a new phenomenon occurs: the statement is true only if some additional restrictions are imposed on the roots of the discrete Wronskian. For example, it is sufficient to require that the roots of the discrete Wronskian differ by at least one. The first item of Theorem 2.1 for $N = 2$ and $\lambda_1 = \lambda_2 = 0$ follows from Theorem 1 in [4].

We also consider spaces of quasi-polynomials of the form $x^{z_i}p_i(x, \log x)$, where z_i are real numbers and $p_i(x, y)$ are polynomials with complex coefficients, and their Wronskians. Theorem 5.2 describes sufficient conditions for such a space to have a basis consisting of polynomials with real coefficients. Theorem 5.2 is a statement bispectral dual to Theorem 2.1 in the sense of [13, 1].

Theorems 2.1, 4.1 and 5.2 have reformulations in terms of explicit matrices depending on two groups of complex parameters, see Theorems 6.2, 6.4 and 6.6. For example, if a matrix

$$\begin{pmatrix} a_1 & \dfrac{1}{\lambda_2 - \lambda_1} & \dfrac{1}{\lambda_3 - \lambda_1} & \cdots & \dfrac{1}{\lambda_N - \lambda_1} \\ \dfrac{1}{\lambda_1 - \lambda_2} & a_2 & \dfrac{1}{\lambda_3 - \lambda_2} & \cdots & \dfrac{1}{\lambda_N - \lambda_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{1}{\lambda_1 - \lambda_N} & \dfrac{1}{\lambda_2 - \lambda_N} & \dfrac{1}{\lambda_3 - \lambda_N} & \cdots & a_N \end{pmatrix}$$

has real eigenvalues and the numbers $\lambda_1, \ldots, \lambda_N$ are real, then the numbers $a_1, \ldots, a_N$ are real. Those reformulations, see Corollaries 6.3, 6.5, are related to properties of Calogero–Moser spaces. They also imply a criterion for the reality of irreducible representations of Cherednik algebras, see [7].

The paper is organized as follows. We state the discrete version of B. and M. Shapiro conjecture in Sec. 2 and prove this result in Sec. 3. In Sec. 4, we deduce Theorem 4.1 for spaces of quasi-exponentials from Theorem 2.1. In Sec. 5, we consider spaces of quasi-polynomials, see Theorem 5.2. In Sec. 6, we reformulate our results in terms of matrices.

The results of the paper were presented at the workshop "Contemporary Schubert Calculus and Schubert Geometry" at Banff, Canada, in March 2007 and the conference "ISLAND III: Algebraic Aspects of Integrable Systems" in Scotland in July 2007.

2. Spaces of Quasi-Exponentials and the Discrete Wronski Map

2.1. *Formulation of the statement*

A function of the form $p(x)Q^x$, where Q is a nonzero complex number with the argument fixed, and $p(x) \in \mathbb{C}[x]$, is called *a quasi-exponential function with base* Q.

Fix a natural number $N \geq 2$. Let $\mathbf{Q} = (Q_1, \dots, Q_N)$ be a sequence of nonzero complex numbers with their arguments fixed. We always assume that if $Q_i = Q_j$ for some i, j, then the chosen arguments of Q_i and Q_j are the same.

We call a complex vector space of dimension N spanned by quasi-exponential functions $p_i(x)Q_i^x$, $i = 1, \dots, N$, a *space of quasi-exponentials with bases* $\mathbf{Q}$.

A quasi-exponential function $p(x)Q^x$ is called *real* if $Q \in \mathbb{R}^\times$ and $p(x) \in \mathbb{R}[x]$. The space of quasi-exponentials is called *real* if it has a basis consisting of real quasi-exponential functions.

The discrete Wronskian of functions $f_1(x), \dots, f_N(x)$ is the determinant

$$\mathrm{Wr}^d(f_1, \dots, f_N) = \det \begin{pmatrix} f_1(x) & f_1(x+1) & \cdots & f_1(x+N-1) \\ f_2(x) & f_2(x+1) & \cdots & f_2(x+N-1) \\ \vdots & \vdots & \vdots & \vdots \\ f_N(x) & f_N(x+1) & \cdots & f_N(x+N-1) \end{pmatrix}. \tag{2.1}$$

The discrete Wronskians of two bases for a vector space of functions differ by multiplication by a nonzero number.

Let V be a space of quasi-exponentials with bases $\mathbf{Q}$. The discrete Wronskian of any basis for V is a quasi-exponential of the form $w(x)\prod_{j=1}^N Q_j^x$, where $w(x) \in \mathbb{C}[x]$. The unique representative with a monic polynomial $w(x)$ is called *the discrete Wronskian of* V and is denoted by $\mathrm{Wr}^d(V)$.

Theorem 2.1. *Let V be a space of quasi-exponentials with real bases $\mathbf{Q} \in (\mathbb{R}^\times)^N$, and let $\mathrm{Wr}^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N Q_j^x$. Assume that $z_1, \dots, z_n$ are real. We have*:

(1) *If $|z_i - z_j| \geq 1$ for all $i \neq j$, then the space V is real.*
(2) *Let $Q_1, \dots, Q_N$ be either all positive or all negative. Assume that there exists a subset $I \subset \{1, \dots, n\}$ such that $|z_i - z_j| \geq 1$ for $i \neq j$ provided either $i, j \in I$ or $i, j \notin I$. Then the space V is real.*

Theorem 2.1 is proved in Sec. 3.

Part (1) of Theorem 2.1 for $N = 2$ and $\lambda_1 = \lambda_2 = 0$ follows from Theorem 1 in [4].

2.2. *Examples*

For $\mathbf{Q} \in (\mathbb{R}^\times)^N$ let $\mathcal{L}_n(\mathbf{Q})$ be the set of points $\mathbf{z} = (z_1, \dots, z_n)$ of $\mathbb{R}^n$ such that all spaces of quasi-exponentials with bases $\mathbf{Q}$ and the discrete Wronskian $\mathrm{Wr}^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N Q_j^x$ are real.

The inequalities on $z_1, \dots, z_n$ described in Theorem 2.1 give an n-dimensional subset of $\mathcal{L}_n(\mathbf{Q})$ which does not depend on $\mathbf{Q}$. In examples, these inequalities are sharp in the sense that the corresponding hyperplanes are tangent to the boundary of the set $\bigcup_{\mathbf{Q}} \mathcal{L}_n(\mathbf{Q})$.

A larger subset of $\mathcal{L}_n(\mathbf{Q})$ which depends on $\mathbf{Q}$ is described in Proposition 3.10. In examples, this subset coincides with $\mathcal{L}_n(\mathbf{Q})$, however, its description is rather ineffective.

Example 2.2. Consider the case $N = 2$, $\mathbf{Q} = (1, Q)$, $\deg p_1 = 1$, $\deg p_2 = 1$. Then the discrete Wronskian has two zeros, which we assume to be at 0 and A. This case corresponds to the equation on a, b,

$$\mathrm{Wr}^d(x + a, Q^x(x + b)) = Q^x(Q - 1)x(x - A),$$

which has two solutions:

$$a = \frac{-QA + A - 2Q \pm \sqrt{(Q-1)^2A^2 + 4Q}}{2(Q-1)},$$

$$b = -1 - A + \frac{QA - A + 2Q \mp \sqrt{(Q-1)^2A^2 + 4Q}}{2(Q-1)}.$$

The solutions are real for real Q, A if and only if $A^2 \geq -4Q/(Q-1)^2$. Theorem 2.1 claims that the solutions are real if $A^2 \geq 1$, which gives a sufficient condition because $1 \geq -4Q/(Q-1)^2$ for real Q. The condition is sharp because $1 = -4Q/(Q-1)^2$ for $Q = -1$.

Example 2.3. Consider the case $N = 2$, $\mathbf{Q} = (1, 1)$, $\deg p_1 = 1$, $\deg p_2 = 3$. Then the discrete Wronskian has three zeros, which we assume to be at $0, A$ and B. This case corresponds to the equation on a, b, c,

$$\mathrm{Wr}^d(x + a, x^3 + bx^2 + c) = 2x(x - A)(x - B),$$

which has two solutions:

$$a = -1/2 - (A+B)/3 \pm 1/3\sqrt{-AB - 3/4 + A^2 + B^2},$$

$$b = -3/2 - A - B \pm \sqrt{-4AB - 3 + 4A^2 + 4B^2},$$

$$c = 1/2 + 2(A+B)/3 + 1/3\sqrt{-AB - 3 + 4A^2 + 4B^2} + AB.$$

The solutions are real for real A, B if and only if $A^2 + B^2 - AB - 3/4 \geq 0$.

The set $A^2 + B^2 - AB - 3/4 < 0$ in the real plane with coordinates A, B is the interior of an ellipse centered at the origin. This ellipse is inscribed in the hexagon formed by the lines $|A| = 1$, $|B| = 1$ and $|A - B| = 1$, and is tangent to the sides of the hexagon at the points $(1,1/2)$, $(-1,-1/2)$, $(1/2,-1/2)$, $(1/2,1)$, $(-1/2,-1)$, $(-1/2,1/2)$. Theorem 2.1 claims that the numbers a,b,c are real if the point (A,B) is not inside the hexagon.

3. Proof of Theorem 2.1

3.1. *The discrete Wronskian is a finite algebraic map*

Fix a natural number $N \geq 2$, natural numbers $n_1,\dots,n_k$ such that $\sum_{i=1}^{k} n_i = N$, and a natural number l. Fix $\mathbf{Q} = (Q_1,\dots,Q_1,\dots,Q_k,\dots,Q_k) \in \mathbb{C}^N$, where $Q_i \neq 0$, $Q_i \neq Q_j$ if $i \neq j$ and Q_i is repeated n_i times.

For a natural number d, let $\mathbb{C}_d[x] \subset \mathbb{C}[x]$ be the space of all polynomials of degree less than d. For $m \leq d$, let $Gr(m,d)$ be the Grassmannian of all m-dimensional subspaces in $\mathbb{C}_d[x]$. It is an irreducible projective complex variety of dimension $m(d-m)$.

Let

$$n = lN. \tag{3.1}$$

Define the *discrete Wronski map*:

$$\mathrm{Wr}^d_{\mathbf{Q}}\colon Gr(n_1,l+n_1) \times Gr(n_2,l+n_2) \times \cdots \times Gr(n_k,l+n_k) \to Gr(1,n+1),$$

as follows. For $i=1,\dots,k$, let $V_i \in Gr(n_i,l+n_i)$ and let $p_{i,1}(x),\dots,p_{i,n_i}(x) \in \mathbb{C}[x]$ be a basis for V_i. Then the map $\mathrm{Wr}^d_{\mathbf{Q}}$ sends the point $V_1 \times \cdots \times V_k$ to the line spanned by the polynomial

$$\mathrm{Wr}^d(p_{1,1}(x)Q_1^x,\dots,p_{1,n_1}(x)Q_1^x,\dots,p_{k,1}(x)Q_k^x,\dots,p_{k,n_k}(x)Q_k^x)\prod_{i=1}^{k} Q_i^{-n_i x}.$$

Let $V \in Gr(m,d)$. Then V has a unique basis consisting of monic polynomials $p_j(x) \in \mathbb{C}[x]$, $j=1,\dots,m$, of the form

$$p_j(x) = x^{d_j} + \sum_{s=1}^{d_j} a_{j,s} x^{d_j - s}$$

such that $d > d_r > d_s$ whenever $r > s$ and $a_{j,s} = 0$ whenever $d_j - s = d_r$ for some r. We call this basis the *standard basis for V*.

Proposition 3.1. *The discrete Wronski map is a finite algebraic map.*

Proof. The sets $Gr(n_1,l+n_1) \times Gr(n_2,l+n_2) \times \cdots \times Gr(n_k,l+n_k)$ and $Gr(1,n+1)$ are projective algebraic varieties of dimension n. The discrete Wronski map is a well-defined algebraic map. We only need to show that every point of $Gr(1,n+1)$ has a finite number of preimages.

Fix a monic polynomial $w(x) \in \mathbb{C}_{n+1}[x]$.

Let $V_1 \in Gr(n_1,l+n_1),\dots,V_k \in Gr(n_k,l+n_k)$ be such that $\mathrm{Wr}^d_{\mathbf{Q}}(V_1 \times \cdots \times V_k) = \mathbb{C}w(x)$, and let $p_{i,j}(x) = x^{d_{i,j}} + \sum_{s=1}^{d_{i,j}} a_{i,j,s} x^{d_{i,j}-s}$, $i=1,\dots,k$, $j=1,\dots,n_i$, be the standard basis for V_i. Here $d_{i,j}$ are non-negative integers such that $d_{i,j} < l + n_i$ and $d_{i,r} > d_{i,s}$ if $r > s$.

We have

$$\begin{aligned}&\mathrm{Wr}^d(p_{1,1}(x)Q_1^x, \ldots, p_{k,n_k}(x)Q_k^x)\\&\quad = w(x) \prod_{1\le i<j\le k} (Q_j - Q_i) \prod_{i=1}^{k} \left(Q_i^{n_i x} \prod_{1\le j<s\le n_i} (d_{i,s} - d_{i,j}) \right).\end{aligned} \tag{3.2}$$

Consider Eq. (3.2) as a system of algebraic equations on the nontrivial coefficients $a_{i,j,s}$ of the polynomials $p_{i,j}(x)$. The number of equations equals the number of variables. We claim that this system has finitely many solutions.

Assume that there exist infinitely many solutions. Then there exists a curve of solutions $a^t_{i,j,s}$, $t \in \mathbb{R}_+$, such that some of the coefficients $a^t_{i,j,s}$ tend to infinity in the limit $t \to \infty$.

Consider the limit $t\to\infty$. There exist $\alpha^t_{1,2,1} \in \mathbb{C}$ such that the main terms of polynomials $p^t_{1,1}$ and $p^t_{1,2} - \alpha^t_{1,2,1} p^t_{1,1}$ are linearly independent. Indeed, let $t^{b_{1,1}} q_{1,1}(x)$ be the main term of $p^t_{1,1}(x)$ and let $\deg q_{1,1} = c_{1,1}$. Let $t^{b_{1,2}} q_{1,2}(x)$ be the main term of $p^t_{1,2}(x)$. We set $\alpha^t_{1,2,1} = \beta a^t_{1,2,d_{1,2}-c_{1,1}}/a^t_{1,1,d_{1,1}-c_{1,1}}$ if $q_{1,2}(x) = \beta q_{1,1}(x)$ for some $\beta \in \mathbb{C}$ and $\alpha^t_{1,2,1} = 0$ otherwise.

Similarly, we find $\alpha^t_{i,j,r} \in \mathbb{C}$ such that for $i = 1, \ldots, k$, the main terms as $t \to \infty$ of polynomials

$$\tilde p^t_{i,j}(x) = p^t_{i,j}(x) + \sum_{r=1}^{j-1} \alpha^t_{i,j,r} p^t_{i,r}(x),$$

$j = 1, \ldots, n_i$, are linearly independent.

Let $a^t_{i,j,s}$ tend to infinity and all $a^t_{i,r,l}$ with $r < j$ remain bounded. Then $\tilde p_{i,r}(x) = p_{i,r}(x)$ for $r = 1, \ldots, j$. Hence, the coefficient of $x^{d_{i,j}-s}$ of $\tilde p^t_{i,j}(x)$ tends to infinity.

To obtain the main term of $\mathrm{Wr}^d(\tilde p^t_{1,1}(x)Q_1^x, \ldots, \tilde p^t_{k,n_k}(x)Q_k^x)$ we can replace the polynomials $\tilde p^t_{i,j}(x)$ with their main terms. It follows that $\mathrm{Wr}^d(\tilde p^t_{1,1}(x)Q_1^x, \ldots, \tilde p^t_{k,n_k}(x)Q_k^x)$ has unbounded coefficients as $t \to \infty$. But it is equal to $\mathrm{Wr}^d(p^t_{1,1}(x)Q_1^x, \ldots, p^t_{k,n_k}(x)Q_k^x)$, which does not depend on t. It is a contradiction.

Therefore, the number of tuples $V_1 \in Gr(n_1, l + n_1), \ldots, V_k \in Gr(n_k, l + n_k)$ such that $\mathrm{Wr}^d_{\mathbf{Q}}(V_1 \times \cdots \times V_k) = \mathbb{C}w(x)$, and such that $\deg p_{i,j}(x) = d_{i,j}$, where $p_{i,j}$, $j = 1, \ldots, n_i$, is the standard basis for V_i, is finite for each choice of $d_{i,j}$. The proposition follows. $\square$

For a non-negative integer l, we call a space V of quasi-exponentials with bases $\mathbf{Q} = (Q_1, \ldots, Q_1, \ldots, Q_k, \ldots, Q_k)$, where Q_i is repeated n_i times, $Q_i \neq 0$ and $Q_i \neq Q_j$ for $i \neq j$, a *weight zero space of quasi-exponentials of type* l if V has a basis of the form $\{p_{i,j}(x)Q_i^x, i = 1, \ldots, k,\ j = 1, \ldots, n_i\}$, where $p_{i,j}(x)$ are polynomials of degree $l + j - 1$. We call a space V of quasi-exponentials a *weight zero space* if there exists an $l \in \mathbb{Z}_{\ge 0}$ such that V is a weight zero space of type l.

Corollary 3.2. *If Theorem* 2.1 *holds for weight zero spaces of quasi-exponentials, then it holds for all spaces of quasi-exponentials.*

Proof. Let V be a space of quasi-exponentials with real bases $\mathbf{Q}$. We can choose l such that V has a basis of the form $\{p_{i,j}(x)Q_i^x, i = 1, \ldots, k, j = 1, \ldots, n_i\}$, where $p_{i,j}(x)$ are polynomials of degree at most $l + j - 1$. Then V defines an element $\tilde{V} \in Gr(n_1, l+n_1) \times Gr(n_2, l+n_2) \times \cdots \times Gr(n_k, l+n_k)$. Let $w(t)$, $t \in \mathbb{R}_{\geq 0}$, be a continuous curve in $Gr(1, n+1)$ such that $w(0) = \mathrm{Wr}^d_{\mathbf{Q}}(\tilde{V})$. Then by Proposition 3.1, there exists a continuous curve $\tilde{V}_t \in Gr(n_1, l+n_1) \times Gr(n_2, l+n_2) \times \cdots \times Gr(n_k, l+n_k)$ such that $\tilde{V}_0 = \tilde{V}$ and $\mathrm{Wr}^d_{\mathbf{Q}}(\tilde{V}_t) = w(t)$. If the corresponding spaces of quasi-exponentials V_t are real for $t > 0$, then V is real.

The set of points $V \in Gr(n, l+n)$ with the standard basis $p_j(x)$, and $\deg p_j(x) = l + j - 1, j = 1, \ldots, n$, is dense in $Gr(n, l+n)$. Therefore, the set of points in $Gr(n_1, l+n_1) \times \cdots \times Gr(n_k, l+n_k)$ corresponding to weight zero spaces of quasi-exponentials is dense. By Proposition 3.1, the image of this set under the discrete Wronski map is dense in $Gr(1, n+1)$. The corollary follows. □

3.2. *Reduction to the case of generic* $\mathbf{Q}$

In this section we show that it is sufficient to prove Theorem 2.1 for the case of generic $\mathbf{Q} = (Q_1, \ldots, Q_N)$.

Fix some natural number l. For $i = 1, \ldots, N$, let $q_i(x) = \sum_{j=0}^{l-1} q_{i,j}x^j$. Set

$$p(x, Q, \mathbf{Q}) = x^l + \sum_{j=1}^{N} \prod_{r=1}^{j-1} (Q - Q_r) q_j(x). \tag{3.3}$$

Fix natural numbers $n_1, \ldots, n_k$, such that $\sum_{i=1}^{k} n_i = N$.

For $\sum_{j=1}^{s-1} n_j < i \leq \sum_{j=1}^{s} n_j$ we set $m(i) = s$, $r(i) = i - \sum_{j=1}^{s-1} n_j - 1$. Let $\mathbf{Q}^0 = (Q_1^0, \ldots, Q_1^0, \ldots, Q_k^0, \ldots, Q_k^0)$, where Q_i^0 repeats n_i times. The ith coordinate of $\mathbf{Q}^0$ is $Q^0_{m(i)}$.

We show that $q_{i,j}$ can be considered as coordinates on the affine space of weight zero spaces of quasi-exponentials of type l. For $i = 1, \ldots, N$, let

$$p_i^0(x) = (Q^{-r(i)}(x + Q\partial_Q)^{r(i)} p(x, Q, \mathbf{Q}^0))|_{Q=Q^0_{m(i)}}. \tag{3.4}$$

Clearly $p_i^0(x)$ is a polynomial in x of degree $l + r(i)$.

Lemma 3.3. *Let V be a weight zero space of quasi-exponentials of type l with bases $\mathbf{Q}^0 \in (\mathbb{R}^\times)^N$. Then there exist unique $q_i(x) = \sum_{j=0}^{l-1} q_{i,j}x^j \in \mathbb{C}[x], i = 1, \ldots, N$, such that $\{p_i^0(x)(Q^0_{m(i)})^x\}$ is a basis for V. Moreover, V is real if and only if $q_{i,j}$ are real for all i, j.*

Proof. For $i = 1, \ldots, N$, we have

$$p_i^0(x) = C_i q_i(x) + \tilde{p}_i^0(x),$$

where $\tilde{p}_i^0(x)$ is a polynomial in x and $q_{s,j}$, $s < i$, with real coefficients, and

$$C_i = r(i)! \prod_{j<i} (Q_i^0 - Q_j^0)^{r(i)n_j}.$$

Note that C_i is real and nonzero. The lemma follows. □

Next, we study the dependence of the discrete Wronskian of the weight zero spaces of quasi-exponentials on their exponents $\mathbf{Q}$ in the coordinates $q_{i,j}$.

Lemma 3.4. *The function*

$$W(x, \mathbf{Q}) = \frac{\prod_{i=1}^N Q_i^{-x}}{\prod_{i<j}(Q_j - Q_i)} \operatorname{Wr}^d(p(x, Q_1, \mathbf{Q})Q_1^x, \dots, p(x, Q_N, \mathbf{Q})Q_N^x) \tag{3.5}$$

is a polynomial in variables $x, Q_1, \dots, Q_N$.

Proof. If $Q_i = Q_j$, then $\operatorname{Wr}^d(p(x, Q_1, \mathbf{Q})Q_1^x, \dots, p(x, Q_N, \mathbf{Q})Q_N^x) = 0$. Therefore, all denominators cancel. □

Lemma 3.5. *We have*

$$\begin{aligned} W(x, \mathbf{Q}^0) &= c(\mathbf{Q}^0) \operatorname{Wr}^d(p_1^0(x)(Q_{m(1)}^0)^x, \dots, p_N^0(x)(Q_{m(N)}^0)^x), \\ c(\mathbf{Q}^0) &= \frac{\prod_{i=1}^k (Q_i^0)^{-n_i x}}{\prod_{1\le i<j\le k}(Q_j^0 - Q_i^0)^{n_i n_j} \prod_{i=1}^k \prod_{j=1}^{n_i-1}(n_i - j)^j}. \end{aligned} \tag{3.6}$$

Proof. For a function $f(Q)$, we call

$$\tau_{Q,h}^{(1)} f(Q) = \frac{f(Q+h) - f(Q)}{h}$$

the *discrete derivative of* f. The *nth discrete derivative of function* $f(Q)$, is defined recursively $\tau_{Q,h}^{(n)} f(Q) = \tau_{Q,h}^{(1)} \tau_{Q,h}^{(n-1)} f(Q)$. If $f(Q)$ is a smooth function, then $\lim_{h\to 0} \tau_{Q,h}^{(n)} f(Q) = f^{(n)}(Q)$, where $f^{(n)}(Q)$ is the nth derivative of $f(Q)$ with respect to Q.

Let

$$\mathbf{Q}_h^0 = (Q_1^0, Q_1^0 + h, \dots, Q_1^0 + (n_i - 1)h, \dots, Q_k^0, Q_k^0 + h, \dots, Q_k^0 + (n_k - 1)h),$$

where h is small, and we assume that the argument of $Q_i^0 + jh$ continuously depends on h. Since the function $W(x, \mathbf{Q})$ is a polynomial, we can compute $W(x, \mathbf{Q}^0)$ as the limit $\lim_{h\to 0} W(x, \mathbf{Q}_h^0)$.

Taking suitable linear combinations of rows of the matrix used to compute the discrete Wronskian in formula (3.5) for $W(x, \mathbf{Q}_h^0)$, we obtain the matrix whose (i, j) entry equals

$$(\tau_{Q,h}^{(r(i))} (p(x + j - 1, Q, \mathbf{Q}_h^0)(Q)^{x+j-1}))|_{Q=Q_{m(i)}^0 + r(i)h} \tag{3.7}$$

and whose determinant equals

$$\prod_{i=1}^k h^{n_i(n_i-1)/2} \operatorname{Wr}^d(p(x, Q_1, \mathbf{Q})Q_1^x, \dots, p(x, Q_N, \mathbf{Q})Q_N^x)|_{\mathbf{Q}=\mathbf{Q}_h^0}.$$

Comparing expression (3.7) with the right-hand side of (3.4), we get formula (3.6) from formula (3.5) in the limit $h \to 0$. □

Proposition 3.6. *Assume that Theorem* 2.1 *holds for generic values of* $\mathbf{Q}$. *Then Theorem* 2.1 *holds for all* $\mathbf{Q} \in (\mathbb{R}^\times)^N$.

Proof. Let $z_1, \dots, z_n$ be real, satisfying one of the conditions in Theorem 2.1. Let V be a weight zero space of quasi-exponentials with exponents $\mathbf{Q}^0 \in (\mathbb{R}^\times)^N$ such that $\mathrm{Wr}^d(V) = \prod_{s=1}^n (x - z_s)$.

Consider the equation $W(x, \mathbf{Q}) = \prod_{s=1}^n (x - z_s)$ as a system of n equations on n variables $q_{i,j}$ depending on parameters $\mathbf{Q} = (Q_1, \dots, Q_N)$. It is a system of algebraic equations with polynomial dependence on parameters. Moreover, the number of solutions for any $\mathbf{Q}$ is finite by Proposition 3.1 and Lemma 3.3.

By Lemma 3.3, the space V corresponds to a solution $\{q_{i,j}^0\}$ of this system with parameters $\mathbf{Q}^0$. Then there exist smooth functions $q_{i,j}(\mathbf{Q})$ defined in the neighborhood of $\mathbf{Q}^0$ such that $q_{i,j}(\mathbf{Q}^0) = q_{i,j}^0$ and $\{q_{i,j}(\mathbf{Q})\}$ is a solution of the system with parameters $\mathbf{Q}$.

By the assumption of the proposition, all $q_{i,j}(\mathbf{Q})$ are real if $\mathbf{Q}$ is real and generic. It follows that all $q_{i,j}^0$ are real.

This proves Theorem 2.1 for the weight zero spaces of quasi-exponentials. Then the proposition follows from Corollary 3.2. □

3.3. *Bethe algebra*

In this section we recall some results of [10, 11].

Let $W = \mathbb{C}^N$ with a chosen basis $v_1, \dots, v_N$.

For an operator $M \in \operatorname{End} W$, we denote $M^{(i)} = 1^{\otimes(i-1)} \otimes M \otimes 1^{\otimes(n-i)}$. Similarly, for an operator $M \in \operatorname{End}(W^{\otimes 2})$, we denote by $M^{(ij)} \in \operatorname{End}(W^{\otimes n})$ the operator acting as M on the ith and jth factors of $W^{\otimes n}$.

Let $R(x) = x + P \in \operatorname{End}(W^{\otimes 2})$ be the *rational R-matrix*. Here $P \in \operatorname{End}(W^{\otimes 2})$ is the flip map: $P(x \otimes y) = y \otimes x$ for all $x, y \in W$. Let $E_{ab} \in \operatorname{End} W$ be the linear operator with the matrix $(\delta_{ia}\delta_{jb})_{i,j=1}^N$.

Let the *Yangian* $Y(\mathfrak{gl}_N)$ be the complex unital associative algebra with generators $T_{ab}^{\{s\}}$, $a, b = 1, \dots, N$, $s \in \mathbb{Z}_{\geq 1}$, and relations

$$R^{(12)}(x-y)T^{(13)}(x)T^{(23)}(y) = T^{(23)}(y)T^{(13)}(x)R^{(12)}(x-y), \tag{3.8}$$

where $T(x) = \sum_{a,b=1}^N E_{ab} \otimes T_{ab}(x)$ and $T_{ab}(x) = \delta_{ab} + \sum_{s=1}^\infty T_{ab}^{\{s\}} x^{-s}$.

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra, and the coproduct is given by

$$\Delta(T_{ab}(x)) = \sum_{i=1}^N T_{ib}(x) \otimes T_{ai}(x).$$

The Yangian $Y(\mathfrak{gl}_N)$ is a flat deformation of $U\mathfrak{gl}_N[t]$, the universal enveloping algebra of the current algebra $\mathfrak{gl}_N[t]$.

Given $z \in \mathbb{C}$, define the $Y(\mathfrak{gl}_N)$-module structure on the space W by letting $T_{ab}(x)$ act as $E_{ba}/(x-z)$. We denote this module $W(z)$ and call it the *evaluation module.*

For a matrix $M = (M_{ij})$ with possibly noncommuting entries, we define the row determinant by $\operatorname{rdet}(M) = \sum_{\sigma \in S_N} (-1)^\sigma M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{N\sigma(N)}$.

Let $\mathbf{Q} = (Q_1, \dots, Q_N) \in (\mathbb{C}^\times)^N$. Let $Q = \operatorname{diag}(Q_1, \dots, Q_N)$ be the diagonal matrix with diagonal entries Q_i . Let $\partial = \partial/\partial x$. Define the universal difference operator by

$$\mathcal{D}_{\mathbf{Q}} = \operatorname{rdet}(1 - QT(x)e^{-\partial}).$$

Write

$$\mathcal{D}_{\mathbf{Q}} = 1 - B_{1,\mathbf{Q}}(x)e^{-\partial} + B_{2,\mathbf{Q}}(x)e^{-2\partial} - \cdots + (-1)^N B_{N,\mathbf{Q}}(x)e^{-N\partial}.$$

Then $B_{i,\mathbf{Q}}(x)$ are series in x^{-1} with coefficients in $Y(\mathfrak{gl}_N)$. The series $B_i(x)$ coincides with the higher transfer-matrices, see [2, 10].

We call the unital subalgebra of $Y(\mathfrak{gl}_N)$ generated by the coefficients of the series $B_{i,\mathbf{Q}}(x)$, $i = 1, \dots, N$, the *Bethe algebra* and denote it by $\mathcal{B}_{\mathbf{Q}}$. It is known that the Bethe algebra is commutative, see [8].

Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. Let $\mathbf{W}(\mathbf{z}) = W(z_1) \otimes \cdots \otimes W(z_n)$ be the tensor product of the evaluation modules.

Let $\bar{B}_{i,\mathbf{Q}}(x), i = 1, \dots, N$, be the image of $B_{i,\mathbf{Q}}(x)$ in $(\operatorname{End} \mathbf{W}(\mathbf{z}))[[x^{-1}]]$. The series $\bar{B}_{i,\mathbf{Q}}(x)$ is summed up to a rational function in x.

Let $K_i, i = 1, \dots, n$, be the *qKZ Hamiltonians* in $\mathbf{W}(\mathbf{z})$:

$$K_i = R^{(i,i-1)}(z_i - z_{i-1}) \cdots R^{(i,1)}(z_i - z_1) Q^{(i)} R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}). \tag{3.9}$$

Lemma 3.7. *For $i = 1, \dots, n$, we have*

$$K_i = \prod_{j, j \neq i} (z_i - z_j) \operatorname{Res}_{x = z_i} \bar{B}_{1,\mathbf{Q}}(x).$$

In particular, K_i belongs to the image of $\mathcal{B}_{\mathbf{Q}}$ in $\operatorname{End} \mathbf{W}(\mathbf{z})$.

Proof. The formula is proved by a direct computation. □

The space $W^{\otimes n}$ has the standard *tensor Shapovalov form*:

$$\langle v_{a_1} \otimes \cdots \otimes v_{a_n}, v_{b_1} \otimes \cdots \otimes v_{b_n} \rangle = \prod_{i=1}^n \delta_{a_i b_i}.$$

Recall that the module $\mathbf{W}(\mathbf{z})$ as a vector space is identified with $W^{\otimes n}$. Let $\langle , \rangle_{\mathbf{R}}$ be the form on $\mathbf{W}(\mathbf{z})$ defined by

$$\begin{aligned}
\langle v, w \rangle_{\mathbf{R}} &= \langle v, \mathbf{R} w \rangle, \\
\mathbf{R} &= R^{(n-1,n)}(z_{n-1} - z_n) \cdots R^{(2,n)}(z_2 - z_n) \cdots R^{(2,3)}(z_2 - z_3) \\
&\quad \times R^{(1,n)}(z_1 - z_n) \cdots R^{(1,3)}(z_1 - z_3) R^{(1,2)}(z_1 - z_2).
\end{aligned}$$

We call this form the *Yangian form.*

Lemma 3.8. ([10]) *For any $b \in \mathcal{B}_{\mathbf{Q}}, v, w \in \mathbf{W}(\mathbf{z})$ we have*

$$\langle bv, w\rangle_{\mathbf{R}} = \langle v, bw\rangle_{\mathbf{R}}.$$

Let $v \in \mathbf{W}(\mathbf{z})$ be an eigenvector of the Bethe algebra $\mathcal{B}_{\mathbf{Q}}$. For $i = 1, \dots, N$, let $B_{i,\mathbf{Q},v}(x)$ be the rational function in x with complex coefficients such that

$$B_{i,\mathbf{Q}}(x)v = B_{i,\mathbf{Q},v}(x)v.$$

We denote by $\mathcal{D}_{\mathbf{Q},v}$ the scalar difference operator

$$\mathcal{D}_{\mathbf{Q},v} = 1 - B_{1,\mathbf{Q},v}(x)e^{-\partial} + B_{2,\mathbf{Q},v}(x)e^{-2\partial} - \cdots + (-1)^N B_{N,\mathbf{Q},v}(x)e^{-N\partial}.$$

Given a scalar difference operator $\mathcal{D}$, we call the space of solutions $f(x)$ of the equation $\mathcal{D}f(x) = 0$ such that $f(x)$ is a linear combination of quasi-exponential functions the *quasi-exponential kernel of the operator* $\mathcal{D}$.

Let $\mathcal{U}$ be the complex span of 1-periodic quasi-exponentials $e^{2\pi\sqrt{-1}kx}, k \in \mathbb{Z}$.

Lemma 3.9. ([11]) *Let $v \in \mathbf{W}(\mathbf{z})$ be an eigenvector of the Bethe algebra $\mathcal{B}_{\mathbf{Q}}$. Then the quasi-exponential kernel of the operator $\mathcal{D}_{\mathbf{Q},v}$ has the form $V_v \otimes \mathcal{U}$, where V_v is an N-dimensional complex space of quasi-exponentials with bases $\mathbf{Q}$, and the discrete Wronskian* $\mathrm{Wr}^d(V_v) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N Q_j^x$.

Moreover, for generic $\mathbf{z}, \mathbf{Q}$, and every N-dimensional complex space V of quasi-exponentials with bases $\mathbf{Q}$ and $\mathrm{Wr}^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N Q_j^x$, *there exists an eigenvector $v \in \mathbf{W}(\mathbf{z})$ of the Bethe algebra $\mathcal{B}_{\mathbf{Q}}$ such that the quasi-exponential kernel of the operator $\mathcal{D}_{\mathbf{Q},v}$ has the form $V \otimes \mathcal{U}$.*

3.4. *Proof of Theorem 2.1 for the case of generic $Q_1, \dots, Q_N$*

Let all $z_1, \dots, z_n$ be real. Let all $Q_1, \dots, Q_N$ also be real and nonzero.

Let $W^{\mathbb{R}}$ be the real part of W generated by the chosen basis $v_1, \dots, v_N$, and let $\mathbf{W}^{\mathbb{R}}(\mathbf{z}) = W^{\mathbb{R}}(z_1) \otimes \cdots \otimes W^{\mathbb{R}}(z_n)$ be the real part of $\mathbf{W}(\mathbf{z})$. Let $Y^{\mathbb{R}}(\mathfrak{gl}_N)$ be the real unital algebra generated by $T_{a,b}^{\{s\}}, a, b = 1, \dots, N$, $s \in \mathbb{Z}_{\geq 1}$, and relations (3.8). Let $\mathcal{B}_{\mathbf{Q}}^{\mathbb{R}} \subset Y^{\mathbb{R}}(\mathfrak{gl}_N)$ be the real subalgebra generated by the coefficients of the series $B_{i,\mathbf{Q}}(x), i = 1, \dots, N$. Clearly, $\mathcal{B}_{\mathbf{Q}}^{\mathbb{R}}$ acts in the space $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$.

For $g \in \mathcal{B}_{\mathbf{Q}}^{\mathbb{R}}$, define the form $\langle , \rangle_{\mathbf{R}g}$ on $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$ by the formula

$$\langle v, w\rangle_{\mathbf{R}g} = \langle v, gw\rangle_{\mathbf{R}} = \langle v, \mathbf{R}gw\rangle.$$

The form $\langle , \rangle_{\mathbf{R}g}$ is a real bilinear symmetric form.

Proposition 3.10. *Let $z_1, \dots, z_n$ be real numbers. Let $g \in \mathcal{B}_{\mathbf{Q}}^{\mathbb{R}}$ be such that the form $\langle , \rangle_{\mathbf{R}g}$ is positive definite on $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$. Let V be a space of quasi-exponentials with bases $\mathbf{Q}$ and* $\mathrm{Wr}^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N Q_j^x$. *Then V is real.*

Proof. Since the condition of a form being positive definite is open, we can assume that $\mathbf{z}, \mathbf{Q}$ are generic. Then by Lemma 3.9, there exists a vector $v \in \mathbf{W}(\mathbf{z})$, such that v is an eigenvector of the Bethe algebra $\mathcal{B}_{\mathbf{Q}}$, and $V \otimes \mathcal{U}$ is the quasi-exponential kernel of the operator $\mathcal{D}_{\mathbf{Q},v}$. By Lemma 3.8, the coefficients of the operator $\mathcal{D}_{\mathbf{Q}}$ are rational functions in x which are symmetric operators with respect to the form $\langle , \rangle_{\mathbf{R}g}$. Since this form is positive definite on $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$, the coefficients of the operator $\mathcal{D}_{\mathbf{Q},v}$ are rational functions with real coefficients.

Let Q be a real number. Consider the equation $\mathcal{D}_{\mathbf{Q},v}(p(x)Q^x) = 0$ as a system of equations for the coefficients of the polynomial $p(x) = \sum_{i=0}^{n} a_{n-i}x^i$. This is a system of linear equations with real coefficients. Therefore, the space of solutions has a real basis. The proposition follows. □

In Example 2.3, the converse to Proposition 3.10 is also true. Namely, let $Q_1, \ldots, Q_N$ be real. Let $z_1, \ldots, z_n$ be real numbers such that every space of quasi-exponentials V with bases $\mathbf{Q}$ and $\mathrm{Wr}^d(V) = \prod_{i=1}^{n}(x - z_i)\prod_{j=1}^{N} Q_j^x$, is real. Then there exists $g \in \mathcal{B}_{\mathbf{Q}}$ such that the form $\langle , \rangle_{\mathbf{R}g}$ is positive definite on $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$. However, the existence of such $g \in \mathcal{B}_{\mathbf{Q}}$ is usually difficult to check.

We deduce Theorem 2.1 from Proposition 3.10.

Lemma 3.11. *Assume that $z_i - z_j > 1$ if $i > j$. Then the restriction of the Yangian form to $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$ is a positive definite bilinear form.*

Proof. The restriction of the tensor Shapovalov form to $(W^{\mathbb{R}})^{\otimes n}$ is a positive definite bilinear form. In the limit $z_1 \gg z_2 \gg \cdots \gg z_n$, the Yangian form on $\mathbf{W}(\mathbf{z})$ tends to the tensor Shapovalov form. Moreover, the Yangian form is nondegenerate if $z_i - z_j > 1$ for all $i > j$. The lemma follows, since the dependence of the Yangian form on $\mathbf{z}$ is continuous. □

The first part of Theorem 2.1 with the additional condition $z_i - z_j \neq 1$ for all i, j follows from Lemma 3.11 and Proposition 3.10 with $g = 1$. Then the condition that $z_i - z_j \neq 1$ for all i, j can be dropped by the continuity with respect to z_i, see Proposition 3.1.

Assume that $Q_1, \ldots, Q_N$ are all positive. Assume there exists $0 \le s \le n$ such that $z_i - z_j > 1$ if either $s \ge i > j \ge 1$ or $n \ge i > j > s$. Consider $G_s = (K_1 K_2 \cdots K_s)^{-1}$, where K_i are given by (3.9).

If $\mathbf{z}$ is generic, then there exists an element $g_s \in \mathcal{B}_{\mathbf{Q}}$ which acts on $\mathbf{W}(\mathbf{z})$ by G_s. Indeed, $K_i \in \mathrm{End}(\mathbf{W}(\mathbf{z}))$ are in the image of the Bethe algebra by Lemma 3.7 and the inverse of a nondegenerate operator in a finite-dimensional space can be written as a polynomial of the operator itself.

Lemma 3.12. *Assume that $Q_1, \ldots, Q_N$ are all positive. Assume there exists $0 \le s \le n$ such that $z_i - z_j > 1$ if either $s \ge i > j \ge 1$ or $n \ge i > j > s$. Then the form $\langle , \rangle_{\mathbf{R}G_s}$ is a positive definite bilinear form on $\mathbf{W}^{\mathbb{R}}(\mathbf{z})$.*

Proof. We have

$$\begin{aligned}\mathbf{R}G_s &= (R^{(s-1,s)} \cdots R^{(2,s)} \cdots R^{(2,3)} R^{(1s)} \cdots R^{(1,3)} R^{(1,2)}) \\ &\quad \times (R^{(n-1,n)} \cdots R^{(s+2,n)} \cdots R^{(s+2,s+3)} R^{(s+1,n)} \cdots R^{(s+1,s+3)} R^{(s+1,s+2)}) \\ &\quad \times (Q^{(1)} \cdots Q^{(s)})^{-1},\end{aligned}$$

where $R^{(i,j)} = R^{(i,j)}(z_i - z_j)$. In the limit $z_1 \gg z_2 \gg \cdots \gg z_n$, the form $\langle , \rangle_{\mathbf{R}G_s}$ tends to the positive definite form $\langle , \rangle_s$ given by

$$\langle v, w\rangle_s = \langle v, (Q^{(1)} \cdots Q^{(s)})^{-1} W\rangle.$$

Moreover, the form $\langle , \rangle_{\mathbf{R}G_s}$ is clearly nondegenerate if $z_i - z_j > 1$ for all $i > j$ such that either $i \leq s$ or $j > s$. The lemma follows since the dependence of the form $\langle , \rangle_{\mathbf{R}G_s}$ on $\mathbf{z}$ is continuous. □

The second part of Theorem 2.1 with positive $Q_1, \ldots, Q_N$, and the additional condition that $z_i - z_j \neq 1$ for all i, j, follows from Lemma 3.12 and Proposition 3.10 with $g = G_s$. Then the condition that $z_i - z_j \neq 1$ for all i, j can be dropped by the continuity with respect to z_i, see Proposition 3.1.

The second part of Theorem 2.1 with negative $Q_1, \ldots, Q_N$ follows from the case of positive $Q_1, \ldots, Q_N$.

4. Spaces of Quasi-Exponentials and the Differential Wronski Map

4.1. *Formulation of statement*

A function of the form $p(x)e^{\lambda x}$, where $\lambda \in \mathbb{C}$ and $p(x) \in \mathbb{C}[x]$, is called *a quasi-exponential function with exponent* λ.

Fix a natural number $N \geq 2$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$. We call a complex vector space of dimension N spanned by N quasi-exponential functions $p_i(x)e^{\lambda_i x}$, $i = 1, \ldots, N$, a *space of quasi-exponentials with exponents* $\boldsymbol{\lambda}$.

A quasi-exponential function $p(x)e^{\lambda x}$ is called *real* if $\lambda \in \mathbb{R}$ and $p(x) \in \mathbb{R}[x]$. The space of quasi-exponentials V is called *real* if it has a basis consisting of real quasi-exponential functions.

The Wronskian of functions $f_1(x), \ldots, f_N(x)$ is the determinant

$$\mathrm{Wr}(f_1, \ldots, f_N) = \det \begin{pmatrix} f_1 & f_1' & \cdots & f_1^{(N-1)} \\ f_2 & f_2' & \cdots & f_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ f_N & f_N' & \cdots & f_N^{(N-1)} \end{pmatrix}. \tag{4.1}$$

The Wronskians of two bases for a vector space of functions differ by multiplication by a nonzero number.

Let V be a space of quasi-exponentials with exponents $\boldsymbol{\lambda}$. The Wronskian of any basis for V is a quasi-exponential of the form $w(x)e^{\sum_{i=1}^{N} \lambda_i x}$, where $w(x) \in \mathbb{C}[x]$.

The unique representative with a monic polynomial $w(x)$ is called *the Wronskian of* V and is denoted by $\mathrm{Wr}(V)$.

Theorem 4.1. *Let* V *be a space of quasi-exponentials with real exponents* $\boldsymbol{\lambda} \in \mathbb{R}^N$. *If zeros of the Wronskian* $\mathrm{Wr}(V)$ *are real, then the space* V *is real.*

Theorem 4.1 is proved in Sec. 4.2.

Theorem 4.1 in the case $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$ is the statement of the B. and M. Shapiro conjecture proved in [3] for $N = 2$ and in [9] for all N.

4.2. *Proof of Theorem* 4.1

Theorem 4.1 can be proved similarly to Theorem 2.1. However, it is not difficult to deduce Theorem 4.1 from Theorem 2.1; we do that in this section.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k)$, where λ_i is repeated n_i times. Consider the *Wronski map*:

$$\mathrm{Wr}_{\boldsymbol{\lambda}}\colon Gr(n_1, l+n_1) \times Gr(n_2, l+n_2) \times \cdots \times Gr(n_k, l+n_k) \to Gr(1, n+1),$$

which maps $V_1 \times \cdots \times V_k$ to the line spanned by

$$\mathrm{Wr}(p_{1,1}(x)e^{\lambda_1 x}, \ldots, p_{1,n_1}(x)e^{\lambda_1 x}, \ldots, p_{k,1}(x)e^{\lambda_k x}, \ldots, p_{k,n_k}(x)e^{\lambda_k x}) \prod_{i=1}^{k} e^{-n_i\lambda_i x}.$$

Here n is given by (3.1), and we used the notation of Sec. 3.1 for the bases for V_i.

Proposition 4.2. *The Wronski map is a finite algebraic map.*

Proof. The proof of Proposition 4.2 is similar to the proof of Proposition 3.1. $\square$

For $h \in \mathbb{C}, h \neq 0$, *the discrete Wronskian with step* h of functions $f_1(x), \ldots, f_N(x)$ is the determinant

$$\mathrm{Wr}_h^d(f_1, \ldots, f_N) = \det \begin{pmatrix} f_1(x) & f_1(x+h) & \cdots & f_1(x+h(N-1)) \\ f_2(x) & f_2(x+h) & \cdots & f_2(x+h(N-1)) \\ \vdots & \vdots & \vdots & \vdots \\ f_N(x) & f_N(x+h) & \ldots & f_N(x+h(N-1)) \end{pmatrix}. \tag{4.2}$$

The discrete Wronskians with step h of any two bases for a vector space of functions differ by multiplication by a nonzero number.

Let V be a space of quasi-exponentials with exponents $\boldsymbol{\lambda}$. The discrete Wronskian with step h of any basis for V is a quasi-exponential of the form $w(x)\prod_{j=1}^{N} e^{\lambda_j x}$, where $w(x) \in \mathbb{C}[x]$. The unique representative with a monic polynomial $w(x)$ is called *the discrete Wronskian of* V *with step* h and is denoted by $\mathrm{Wr}_h^d(V)$.

Theorem 2.1 implies the following statement.

Corollary 4.3. *Let h be real. Let V be a space of quasi-exponentials with real exponents $\boldsymbol{\lambda} \in (\mathbb{R}^\times)^N$, and let $\mathrm{Wr}_h^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N e^{\lambda_j x}$. Assume that $z_1, \dots, z_n$ are real and $|z_i - z_j| \geq |h|$ for all $i \neq j$. Then the space V is real.*

Proof. Let V be a space of quasi-exponentials with real exponents $\boldsymbol{\lambda} \in (\mathbb{R}^\times)^N$, and let $\mathrm{Wr}_h^d(V) = \prod_{i=1}^n (x - z_i) \prod_{j=1}^N e^{\lambda_j x}$. Then

$$\bar{V} = \{f(xh) \mid f(x) \in V\}$$

is a space of quasi-exponentials with real bases $(e^{h\lambda_1}, \dots, e^{h\lambda_N})$, and

$$\mathrm{Wr}^d(\bar{V}) = \prod_{i=1}^n (x - z_i/h) \prod_{j=1}^N e^{h\lambda_j x}\,.$$

Therefore, the corollary follows from Theorem 2.1. □

Define the *discrete Wronski map with step h*:

$$\mathrm{Wr}_{\boldsymbol{\lambda},h}^d \colon Gr(n_1, l + n_1) \times Gr(n_2, l + n_2) \times \cdots \times Gr(n_k, l + n_k) \to Gr(1, n + 1),$$

which maps $V_1 \times \cdots \times V_k$ to the line spanned by

$$\mathrm{Wr}_h^d(p_{1,1} e^{\lambda_1 x}, \dots, p_{1,n_1}(x) e^{\lambda_1 x}, \dots, p_{k,1}(x) e^{\lambda_k x}, \dots, p_{k,n_k}(x) e^{\lambda_k x}) \prod_{i=1}^k e^{-n_i \lambda_i x},$$

where we used the notation of Sec. 3.1 for the bases for V_i. Let

$$\bar{\mathrm{Wr}}_{\boldsymbol{\lambda}}^d \colon \mathbb{C} \times Gr(n_1, l + n_1) \times Gr(n_2, l + n_2) \times \cdots \times Gr(n_k, l + n_k) \to \mathbb{C} \times Gr(1, n + 1)$$

be the map which equals

$$\begin{aligned} \mathrm{id} \times \mathrm{Wr}_{\boldsymbol{\lambda},h}^d \quad &\text{on } \{h\} \times Gr(n_1, l + n_1) \times \cdots \times Gr(n_k, l + n_k), \quad h \in \mathbb{C}^\times, \\ \mathrm{id} \times \mathrm{Wr}_{\boldsymbol{\lambda}} \quad &\text{on } \{0\} \times Gr(n_1, l + n_1) \times \cdots \times Gr(n_k, l + n_k). \end{aligned}$$

Lemma 4.4. *The map $\bar{\mathrm{Wr}}_{\boldsymbol{\lambda}}^d$ is a continuous map of smooth varieties.*

Proof. Taking linear combinations of the columns in the matrix used to compute the determinant (4.2), we obtain the matrix of the discrete derivatives which tend to the usual derivatives as $h \to 0$. In particular, let $p_1(x), \dots, p_N(x) \in \mathbb{C}[x]$ and $\boldsymbol{\lambda} \in \mathbb{C}^N$. Then the function $\mathrm{Wr}_h^d((p_1(x) e^{\lambda_1 x}, \dots, p_N(x) e^{\lambda_N x})$ is a smooth function of h and

$$\begin{aligned} &\mathrm{Wr}_h^d(e^{\lambda_1 x} p_1(x), \dots, e^{\lambda_N x} p_N(x)) \\ &\quad = h^{N(N-1)/2} \mathrm{Wr}(p_1(x) e^{\lambda_1 x}, \dots, p_N(x) e^{\lambda_N x}) + o(h^{N(N-1)/2}) \end{aligned}$$

as $h \to 0$, see (4.1), (4.2). The lemma follows. □

From Proposition 4.2, we obtain that it is sufficient to prove Theorem 4.1 for the case of distinct zeros of the Wronskian $\mathrm{Wr}(V)$.

Let $w(x) = \prod_{i=1}^n (x - z_i)$ and $z_i \neq z_j, i \neq j$. Let V be a space of quasi-exponentials with exponents $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ such that $\mathrm{Wr}(V) = w(x)e^{\sum_{i=1}^N \lambda_i x}$. Then by Lemma 4.4, there exists a family V_h of spaces of quasi-exponentials with exponents $\boldsymbol{\lambda}$ such that $\mathrm{Wr}_h^d(V_h) = \mathrm{Wr}(V)$ and $V_h \to V$ as $h \to 0$. Then by Corollary 4.3, for real h such that $|h| \leq \min_{1\leq i<j\leq n} |z_i - z_j|$, the spaces V_h are real. It follows that the space V is real.

Theorem 4.1 is proved.

5. Unramified Spaces of Quasi-Polynomials

5.1. *Formulation of the statement*

A function of the form $p(x, \log x)x^z$, where z is a complex number, and $p(x, y) \in \mathbb{C}[x, y]$, is called *a quasi-polynomial function with exponent* z.

The quasi-polynomials are multi-valued functions and the exponents are defined modulo integers. This does not present any difficulty in this paper since we use only algebraic properties of the quasi-polynomial functions.

Fix a natural number $n \geq 2$. Let $\mathbf{z} = (z_1, \dots, z_n)$ be a sequence of complex numbers. We call a complex vector space of dimension n spanned by quasi-polynomial functions $p_i(x, \log x)x^{z_i}$, $i = 1, \dots, n$, a *space of quasi-polynomials with exponents* $\mathbf{z}$.

A quasi-polynomial function $p(x, \log x)x^z$ is called *real* if $z \in \mathbb{R}$ and $p(x, y) \in \mathbb{R}[x, y]$. The space of quasi-polynomials is called *real* if it has a basis consisting of real quasi-polynomial functions.

The space of quasi-polynomials V is called *non-degenerate* if it does not contain monomials of the form x^z.

Given a space of quasi-polynomials V, let $\mathcal{D}_V = x^n\partial^n + \cdots$ be the unique differential operator of order n with kernel V and top coefficient x^n. The space of quasi-polynomials V is called *unramified* if coefficients of $\mathcal{D}_V$ are rational functions of x.

Let V be an n-dimensional unramified space of quasi-polynomials with exponents $\mathbf{z}$.

The operator $\mathcal{D}_V$ is Fuchsian. Let $\chi_V^{(\infty)}(\alpha)$ and $\chi_V^{(0)}(\alpha)$ be the indicial polynomials of $\mathcal{D}_V$ at $x = \infty$ and $x = 0$ respectively. The polynomials $\chi_V^{(\infty)}(\alpha), \chi_V^{(0)}(\alpha)$ are polynomials in α of degree n. For a natural number k and $c \in \mathbb{C}$, the polynomial $\chi_V^{(\infty)}(\alpha)$ is divisible by $(\alpha - c)^k$ if and only if there exists a polynomial $p(x, y) \in \mathbb{C}[x, y]$ such that $p(0, y) = y^{k-1}$ and $x^c p(1/x, \log x) \in V$.

For a natural number k and $c \in \mathbb{C}$, the polynomial $\chi_V^{(0)}(\alpha)$ is divisible by $(\alpha - c)^k$ if and only if there exists a polynomial $p(x, y) \in \mathbb{C}[x, y]$ such that $p(0, y) = y^{k-1}$ and $x^c p(x, \log x) \in V$.

Lemma 5.1. *There exists a unique monic polynomial* $Y_V(x)$ *such that*

$$\frac{\chi_V^{(0)}(\alpha)}{\chi_V^{(\infty)}(\alpha)} = \frac{Y_V(\alpha - 1)}{Y_V(\alpha)}.$$

Proof. Note that if k is a natural number and c is a complex number, then

$$\frac{\alpha - c - s}{\alpha - c} = \frac{Y(\alpha - 1)}{Y(\alpha)},$$

where $Y(\alpha) = \prod_{i=0}^{s-1}(\alpha - c - i)$. Clearly, for every root c of $\chi_V^{(\infty)}(\alpha)$ of order k, we have the corresponding roots of $\chi_V^{(0)}(\alpha)$ of the form $c - s_i$ such that s_i are natural numbers and the sum of orders of the roots $c - s_i$ is k. The lemma follows. □

The Wronskian of any basis for V has the form $w(x)x^r$, where $r \in \mathbb{C}$ and $w(x) \in \mathbb{C}[x]$, $w(0) \neq 0$. The unique representative with a monic polynomial $w(x)$ is called the *Wronskian of* V and is denoted by $\mathrm{Wr}(V)$.

Theorem 5.2. *Let V be an unramified space of quasi-polynomials with real exponents $\mathbf{z} \in \mathbb{R}^n$, $Y_V = \prod_{i=1}^m(x-\tilde{z}_i)$ and $\mathrm{Wr}(V) = x^r \prod_{i=1}^N(x-Q_i)$, where $\prod_{i=1}^N Q_i \neq 0$. Assume that $Q_1, \ldots, Q_N$ are real. We have*:

(1) *If $|\tilde{z}_i - \tilde{z}_j| \geq 1$ for all $i \neq j$, then the space V is real.*
(2) *Let $Q_1, \ldots, Q_N$ be either all positive or all negative. Assume that there exists a subset $I \subset \{1, \ldots, n\}$ such that $|\tilde{z}_i - \tilde{z}_j| \geq 1$ for $i \neq j$ provided either $i, j \in I$ or $i, j \notin I$. Then the space V is real.*

Theorem 5.2 is proved in Sec. 5.2.

5.2. *Proof of Theorem* 5.2

If $x^z \in V$ for some $z \in \mathbb{R}$, then $\tilde{V} = (x\partial - z)V$ is an unramified space of quasi-polynomials of dimension $n-1$, with the same exponents (except maybe for z). We have $\mathrm{Wr}(V) = \mathrm{Wr}(\tilde{V})$ and $Y_{\tilde{V}} = Y_V$. Moreover, if $\tilde{V}$ is real then V is real. Therefore, without loss of generality we can assume that V is non-degenerate.

Let V be an unramified non-degenerate space of quasi-polynomials with real exponents $\mathbf{z} \in \mathbb{R}^n$, $Y_V = \prod_{i=1}^m(x - \tilde{z}_i)$ and $\mathrm{Wr}(V) = x^r \prod_{i=1}^N(x - Q_i)$, where $r \in \mathbb{C}$, $Q_i \neq 0$. Let

$$\mathcal{D}_V = (x\partial)^n + A_1(x)(x\partial)^{n-1} + \cdots + A_n(x)$$

be the unique differential operator of order n with kernel V and the top coefficient x^n. The coefficients $A_i(x)$ are rational functions in x.

Let $\bar{A}_0(x) \in \mathbb{C}[x]$ be a monic polynomial such that $\bar{A}_i(x) = A_i(x)\bar{A}_0(x)$ is a polynomial for $i = 1, \ldots, n$, and polynomials $\bar{A}_0(x), \ldots, \bar{A}_n(x)$ are relatively prime. Write

$$A_0(x)\mathcal{D}_V = \sum_{i=1}^{s}\sum_{j=1}^{n} \bar{A}_{ij} x^i (x\partial)^j,$$

where $\bar{A}_{ij} \in \mathbb{C}$ and $s = \max_i(\deg \bar{A}_i(x))$. It is sufficient to prove that $\bar{A}_{ij}$ are real numbers.

Define a difference operator with polynomial coefficients $\mathcal{D}_V^*$ by the formula

$$\mathcal{D}_V^* = \sum_{i=1}^{s} \sum_{j=1}^{n} \bar{A}_{ij} x^j e^{-i\partial}.$$

Proposition 5.3. *The quasi-exponential kernel of the operator $\mathcal{D}_V^*$ has the form $V^* \otimes \mathcal{U}$, where V^* is a space of quasi-exponentials with bases $(\bar{Q}_1, \dots, \bar{Q}_s)$ and $\bar{Q}_i \in \{Q_1, \dots, Q_N\}$ for $i = 1, \dots, s$. Moreover,* $\mathrm{Wr}^d(V^*) = Y_V$.

Proof. Proposition 5.3 is proved using a suitable integral transform in the same way as Theorem 4.1 in [13]. An alternative proof can be found in [1]. □

By Theorem 2.1, the space V^* is real, and therefore, all $\bar{A}_{ij}$ are real numbers.

6. Reformulations

6.1. *The discrete case*

In this section we give a reformulation of Theorem 2.1.

Fix $\mathbf{Q} = (Q_1, \dots, Q_N) \in (\mathbb{C}^\times)^N$, such that $Q_i \neq Q_j$ if $i \neq j$.

Let S be the $N \times N$ Vandermonde matrix with the (i, j) entry

$$s_{ij} = Q_i^{j-1}.$$

We have $\det S = \prod_{i<j} (Q_j - Q_i)$.

Let $\bar{S}$ be the $N \times N$ matrix with the (i, j) entry

$$\bar{s}_{ij} = (j-1) Q_i^{j-1}.$$

Let $A = \mathrm{diag}(a_1, \dots, a_N)$ be the diagonal matrix with diagonal entries $a_1, \dots, a_N$.

Clearly, the discrete Wronskian of N quasi-exponentials with linear polynomial part is given by

$$\mathrm{Wr}_1^d((x-a_1)Q_1^x, \dots, (x-a_N)Q_N^x) = \left(\prod_{i=1}^{N} Q_i^x \right) \det((x-A)S + \bar{S}). \qquad (6.1)$$

Denote $\bar{S} S^{-1} = M$. Let m_{ij} denote the (i, j) entry of M.

Lemma 6.1. *We have*

$$m_{ij} = Q_i \frac{\prod_{s \neq i,j} (Q_i - Q_s)}{\prod_{s \neq j} (Q_j - Q_s)} \quad (i \neq j),$$

$$m_{ii} = Q_i \sum_{s \neq i} \frac{1}{Q_i - Q_s}.$$

Proof. Let s_{ij}^* denote the i, j entry of W^{-1}. Define the polynomials $s_i^*(u) = \sum_{s=1}^{N} s_{js}^* u^{s-1}$. We have

$$\deg s_j^*(u) = N - 1, \quad s_j^*(Q_i) = \delta_{ij}.$$

Therefore,

$$s_j^*(u) = \frac{\prod_{s\neq j}(u - Q_s)}{\prod_{s\neq j}(Q_j - Q_s)}.$$

Furthermore, we have

$$m_{ij} = Q_i \left(\frac{d}{du}\ s_j^*\right)\bigg|_{u=Q_i}.$$

The lemma follows. ☐

Define the matrix $\mathcal{Z}^d$ by

$$\mathcal{Z}^d = \begin{pmatrix} a_1 & \dfrac{Q_1}{Q_2 - Q_1} & \dfrac{Q_1}{Q_3 - Q_1} & \cdots & \dfrac{Q_1}{Q_N - Q_1} \\ \dfrac{Q_2}{Q_1 - Q_2} & a_2 & \dfrac{Q_2}{Q_3 - Q_2} & \cdots & \dfrac{Q_2}{Q_N - Q_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{Q_N}{Q_1 - Q_N} & \dfrac{Q_N}{Q_2 - Q_N} & \dfrac{Q_N}{Q_3 - Q_N} & \cdots & a_N \end{pmatrix}. \tag{6.2}$$

Let $D = \mathrm{diag}(d_1, \ldots, d_N)$ be the diagonal matrix with diagonal entries

$$d_i = \prod_{s\neq i}(Q_i - Q_s).$$

Let $B = \mathrm{diag}(m_{11}, \ldots, m_{NN})$ be the diagonal matrix with diagonal entries m_{ii}.

Then we have the equality of matrices

$$A + B - D^{-1}\bar{S}S^{-1}D = \mathcal{Z}^d. \tag{6.3}$$

Theorem 6.2. *Let $Q_1, \ldots, Q_N$ be distinct real numbers and $a_1, \ldots, a_N$ complex numbers. Let $z_1, \ldots, z_N$ be eigenvalues of the matrix $\mathcal{Z}^d$. Assume that $z_1, \ldots, z_N$ are real. Then we have*:

(1) *If $|z_i - z_j| \geq 1$ for all $i \neq j$, then the numbers $a_1, \ldots, a_N$ are real.*
(2) *Let Q_i be either all positive or all negative. Assume that there exists a subset $I \subset \{1, \ldots, N\}$ such that $|z_i - z_j| \geq 1$ for $i \neq j$ provided either $i, j \in I$ or $i, j \notin I$. Then the numbers $a_1, \ldots, a_N$ are real.*

Proof. By formulas (6.1), (6.3) the eigenvalues of the matrix $\mathcal{Z}^d$ are zeros of the discrete Wronskian $\mathrm{Wr}^d((x - \bar{a}_1)Q_1^x, \ldots, (x - \bar{a}_N)Q_N^x)$, where $\bar{a}_i = a_i + m_{ii}$. Since m_{ii} are real, Theorem 6.2 follows from Theorem 2.1. ☐

Remark. We are not aware of a direct proof of Theorem 6.2.

Corollary 6.3. *Let Q and Z be complex $N \times N$-matrices such that Q is invertible and*

$$Z - Q^{-1}ZQ = 1 - K,$$

where K is a rank-one matrix. Assume that all eigenvalues of Q and Z are real. Let $z_1, \dots, z_N$ be the eigenvalues of Z. Then we have:

(1) *If $|z_i - z_j| \geq 1$ for $i \neq j$, then there exists an invertible matrix C such that $C^{-1}QC$ and $C^{-1}ZC$ are real matrices.*
(2) *Let eigenvalues of Q be either all positive or all negative. Assume that there exists a subset $I \subset \{1, \dots, N\}$ such that $|z_i - z_j| \geq 1$ for $i \neq j$ provided either $i, j \in I$ or $i, j \notin I$. Then there exists an invertible matrix C such that $C^{-1}QC$ and $C^{-1}ZC$ are real matrices.*

Proof. Let $\tilde{\mathcal{M}}_N$ be the set of pairs of complex $N \times N$ matrices (Z, Q) such that Q is invertible and such that the rank of the matrix $Z - Q^{-1}ZQ - 1$ is one. We call $(Z_1, Q_1), (Z_2, Q_2) \in \tilde{\mathcal{M}}_N$ equivalent if there exists an invertible $N \times N$ matrix C such that $Z_2 = C^{-1}Z_1C$ and $Q_2 = C^{-1}Q_1C$. Let $\mathcal{M}_N$ be the set of equivalence classes.

Define a map:

$$\tau \colon \mathcal{M}_N \to \mathbb{C}^N/S_N \times \mathbb{C}^N/S_N,$$

which sends the class of (Z, Q) to $(\operatorname{Spec} Z,\ \operatorname{Spec} Q)$.

Then similarly to [6], one can show that $\mathcal{M}_N$ is a smooth variety and the map τ is a finite map of degree $N!$, [5].

Let $\mathcal{M}_N^{\mathbb{R}} \subset \mathcal{M}_N$ be the subset of classes of pairs $(Z, Q) \in \mathcal{M}_N$ with real matrices Z, Q. By Proposition 3.1 of [7], the subset $\mathcal{M}_N^{\mathbb{R}}$ is closed.

If Q is a diagonalizable matrix with eigenvalues Q_i, then there exists a matrix C such that $C^{-1}QC$ is diagonal and then it is easy to see that Z is given by (6.2).

Therefore, the corollary follows from Theorem 6.2 by continuity. □

6.2. *The smooth case*

In this section we give a reformulation of Theorem 4.1.

Fix $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, such that $\lambda_i \neq \lambda_j$ if $i \neq j$. Define the matrix $\mathcal{Z}$ by

$$\mathcal{Z} = \begin{pmatrix} a_1 & \frac{1}{\lambda_2 - \lambda_1} & \frac{1}{\lambda_3 - \lambda_1} & \cdots & \frac{1}{\lambda_N - \lambda_1} \\ \frac{1}{\lambda_1 - \lambda_2} & a_2 & \frac{1}{\lambda_3 - \lambda_2} & \cdots & \frac{1}{\lambda_N - \lambda_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_1 - \lambda_N} & \frac{1}{\lambda_2 - \lambda_N} & \frac{1}{\lambda_3 - \lambda_N} & \cdots & a_N \end{pmatrix}. \tag{6.4}$$

Theorem 6.4. *Let $\lambda_1, \dots, \lambda_N$ be distinct real numbers. If all eigenvalues of the matrix $\mathcal{Z}$ are real, then the numbers $a_1, \dots, a_N$ are real.*

Proof. By a computation similar to the one in Sec. 6.1, we obtain that the eigenvalues of the matrix $\mathcal{Z}$ are zeros of the Wronskian $\mathrm{Wr}((x-\tilde{a}_1)e^{\lambda_1 x}, \dots, (x-\tilde{a}_N)e^{\lambda_N x})$, where

$$\tilde{a}_i = a_i + \sum_{s \neq i} \frac{1}{\lambda_i - \lambda_s}.$$

Therefore, Theorem 6.4 follows from Theorem 4.1. $\square$

The matrix $\mathcal{Z}$ in relation to the Wronskian of quasi-exponentials appeared in [16].

Corollary 6.5. ([7]) *Let Q and Z be complex $N \times N$-matrices such that*

$$[Q, Z] = 1 - K,$$

where K is a rank-one matrix. If all eigenvalues of Q and Z are real, then there exists an invertible matrix C such that $C^{-1}QC$ and $C^{-1}ZC$ are real matrices.

Proof. The proof is similar to the proof of Corollary 6.3. $\square$

We are not aware of a direct proof of Theorem 6.4.

Using the duality studied in [12], one can show that the case of quasi-exponentials with linear polynomials is generic, so Theorem 4.1 can be deduced from Theorem 6.4.

Moreover, a proof of the B. and M. Shapiro conjecture can be obtained from Theorem 6.4 for the case of a nilpotent matrix $\mathcal{Z}$.

6.3. *Trigonometric case*

In this section we give a dual version of Theorem 6.2.

Fix complex numbers $z_1, \dots, z_N$ such that $z_i - z_j \neq 1$.

Define the matrix $\mathcal{Q}^d$ by

$$\mathcal{Q}^d = \begin{pmatrix} b_1 & \dfrac{b_2}{z_1 - z_2 + 1} & \dfrac{b_3}{z_1 - z_3 + 1} & \cdots & \dfrac{b_N}{z_1 - z_N + 1} \\ \dfrac{b_1}{z_2 - z_1 + 1} & b_2 & \dfrac{b_3}{z_2 - z_3 + 1} & \cdots & \dfrac{b_N}{z_2 - z_N + 1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{b_1}{z_N - z_1 + 1} & \dfrac{b_2}{z_N - z_2 + 1} & \dfrac{b_3}{z_N - z_3 + 1} & \cdots & b_N \end{pmatrix}.$$

Theorem 6.6. *Let $z_1, \dots, z_N$ be real numbers such that $z_i - z_j \neq 1$ and let $b_1, \dots, b_N$ be complex numbers. Let $Q_1, \dots, Q_N$ be eigenvalues of the matrix $\mathcal{Q}^d$. Assume that $Q_1, \dots, Q_N$ are nonzero distinct real numbers. Then we have*:

(1) *If $|z_i - z_j| > 1$ for all $i \neq j$, then the numbers $b_1, \dots, b_N$ are real.*
(2) *Let Q_i be either all positive or all negative. Assume that there exists a subset $I \subset \{1, \dots, N\}$ such that $|z_i - z_j| \geq 1$ for $i \neq j$ provided either $i, j \in I$ or $i, j \notin I$. Then the numbers $b_1, \dots, b_N$ are real.*

Proof. Let $Z = \mathrm{diag}(z_1, \ldots, z_N)$ be the diagonal matrix with diagonal entries z_i. Then $\mathcal{Q}^d Z - Z\mathcal{Q}^d - \mathcal{Q}^d$ is a rank 1 matrix. Theorem 6.6 follows from Corollary 6.3.

Alternatively, since $\mathcal{Q}^d$ is invertible, $z_1, \ldots, z_N$ are all distinct. Then one can show that the eigenvalues of the matrix $\mathcal{Q}^d$ are zeros of the Wronskian $\mathrm{Wr}(x^{z_1}(x - \tilde{b}_1), \ldots, x^{z_N}(x - \tilde{b}_N))$, where

$$\tilde{b}_i = b_i \prod_{s \neq i} \frac{z_i - z_s}{z_i - z_s - 1}$$

and deduce Theorem 6.6 from Theorem 5.2. $\square$

Acknowledgments

E.M. is supported in part by NSF grant DMS-0601005. V.T. is supported in part by RFFI grant 08-01-00638. A.V. is supported in part by NSF grant DMS-0555327.

References

1. Yu. Berest and O. Chalykh, Calogero–Moser correspondence: Trigonometric case, preprint (2007) 1–15.
2. A. Chervov and D. Talalaev, Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence, arXiv:hep-th/0604128.
3. A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, *Ann. Math.* **155** (2002) 105–129.
4. A. Eremenko, A. Gabrielov, M. Shapiro and Vainshtein, Rational functions and real Schubert calculus, *Proc. Amer. Math. Soc.* **134** (2006) 949–957.
5. P. Etingof, private communication.
6. P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism, *Invent. Math.* **147** (2002) 243–348.
7. E. Horozov and M. Yakimov, The real loci of Calogero–Moser spaces and the Shapiro–Shapiro conjecture, arXiv:0710.5291.
8. P. P. Kulish and E. K. Sklyanin, *Quantum Spectral Transform Method. Recent Developments*, Lect. Notes in Phys., Vol. 151 (Springer, 1982), pp. 61–119.
9. E. Mukhin, V. Tarasov and A. Varchenko, The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz, math.AG/0512299, to appear in *Ann. Math.*
10. E. Mukhin, V. Tarasov and A. Varchenko, Bethe eigenvectors of higher transfer matrices, *J. Stat. Mech.* (2006), no. 8, P08002, 1–44.
11. E. Mukhin, V. Tarasov and A. Varchenko, Generating operator of *XXX* or Gaudin transfer matrices has quasi-exponential kernel, in *SIGMA Symmetry Integrability Geom. Methods Appl.* Vol. 3 (2007), Paper 060 (electronic), pp. 1–31.
12. E. Mukhin, V. Tarasov and A. Varchenko, Bispectral and (gl_N, gl_M) dualities, *Funct. Anal. Math.* **1** (2006) 55–80.
13. E. Mukhin, V. Tarasov and A. Varchenko, Bispectral and (gl_N, gl_M) dualities, discrete versus differential, *Adv. Math.* **218** (2008) 216–265.
14. J. Ruffo, Y. Sivan, E. Soprunova and F. Sottile, Experimentation and conjectures in the real Schubert calculus for flag manifolds, math.AG/0507377.

15. F. Sottile, Enumerative real algebraic geometry, in *Algorithmic and Quantitative Real Algebraic Geometry* (Piscataway, NJ, 2001), pp. 139–179. DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Vol. 60 (Amer. Math. Soc., 2003).
16. G. Wilson, Collisions of Calogero–Moser particles and an adelic Grassmannian, *Invent. Math.* **133** (1998) 1–41.