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TIME-HARMONIC MAXWELL EQUATIONS IN BIOLOGICAL CELLS — THE DIFFERENTIAL FORM FORMALISM TO TREAT THE THIN LAYER

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We study the behavior of the electromagnetic field in a biological cell modeled by a medium surrounded by a thin layer and embedded in an ambient medium. We derive approximate transmission conditions in order to replace the membrane by these conditions on the boundary of the interior domain. Our approach is essentially geometric and based on a suitable change of variables in the thin layer. Few notions of differential calculus are given in order to obtain the first-order conditions in a simple way, and numerical simulations validate the theoretical results. Asymptotic transmission conditions at any order are given in the last section of the paper. This paper extends to the time-harmonic Maxwell equations the previous works presented in [30, 33, 31, 6].

Keywords: Asymptotic expansion; time-harmonic Maxwell's equations; differential forms on manifolds; finite element method; edge elements.

AMS Subject Classification: 34E05, 34E10, 58J37

1. Introduction and Motivations

The electromagnetic modeling of biological cells has become extremely important since several years, in particular in the biomedical research area. In the simple models [17, 19], the biological cell is a domain with a thin layer composed of a conducting cytoplasm surrounded by a thin insulating membrane. When exposed to an electric field, a potential difference is induced across the cell membrane. This transmembrane potential (TMP) may be of sufficient magnitude to be biologically significant. In particular, if it overcomes a threshold value, complex phenomena such as electropermeabilization or electroporation may occur [37, 38, 25, 24]. The electrostatic pressure becomes so high that the thin membrane is locally destructured: some exterior molecules might be internalized inside the cell. This process holds great promises particularly in oncology and gene therapy, to deliver drug molecules in cancer treatment. This is the reason why an accurate knowledge of the distribution of the electromagnetic field in the biological cell is necessary. Several papers in the bioelectromagnetic research area deal with numerical electromagnetic modeling of biological cells [26, 36, 34]. Actually the main difficulties of the numerical computations lie in the thinness of the membrane (the relative thickness of the membrane is one thousandth of the cell size) and in the high contrast of the electromagnetic parameters of the different cell constituents. We present here an asymptotic method to replace the thin membrane by appropriate transmission conditions on the boundary of the cytoplasm.

In previous papers [30, 33, 31, 6], an asymptotic analysis is proposed to compute the electric potential in domains with thin layer, using the electroquasistatic approximation.[a] However, it is not clear whether the magnetic effects of the field may be neglected. This is the reason why we present in this paper an asymptotic analysis for the time-harmonic Maxwell equations in a domain with thin layer. Our analysis is close to those performed in [30, 33, 31]. Roughly speaking, it is based on a suitable change of variables in the membrane in order to write the explicit dependence of the studied differential operator in terms of small parameter (the thinness of the membrane). The novelty of the paper lies in the use of differential form formalism, which seems to be the good formalism to treat Maxwell's equations in the time-harmonic regime according to Flanders [18], Warnick *et al.* [39, 40] and Lassas *et al.* [20, 21]. The convenience of this formalism allows us to consider the Helmholtz equation and the Maxwell equations in a similar fashion.

Throughout this paper, we consider a material composed of an interior domain surrounded by a thin membrane. This material, representing a biological cell, is embedded in an ambient medium submitted to an electric current density. We study the asymptotic behavior of the electromagnetic field in the three domains (the ambient medium, the thin layer and the cytoplasm) as the thickness of the membrane tending to zero. We derive appropriate transmission conditions at first order on the boundary of the cytoplasm in order to remove the thin layer from the problem. Actually, the influence of the membrane is approached by these transmission conditions. To justify our asymptotic expansion, we provide piecewise estimates of the error between the exact solution and the approximate solution.

The paper is structured as follows. In Sec. 2, we present the studied problem in the differential calculus formalism and we state the main results of the paper. We then provide in Sec. 3 numerical simulations that validate the theoretical results. In particular, we demonstrate that for biological cells, the membrane behavior dramatically changes with respect to the frequency. More precisely, we show that if the "thin layer" model presented here is valid for quite large frequencies, a "very

[a] The electroquasistatic approximation consists in considering that the electric field comes from a potential: $E = -\nabla V$. In this approximation, the curl part of the electric field vanishes and the magnetic field is neglected.

resistive thin layer" model, as described in [32], has to be studied for low frequencies. Section 4 is devoted to the geometry: we perform our change of variables and write the problem in the so-called local coordinates. In Sec. 5, we derive formally our asymptotic expansion, which is rigorously proved in Sec. 6. In Sec. 7, we give recurrence formulas to obtain the asymptotic expansion at any order. The Appendix is devoted to explicit formulas used to derive the conditions.

2. Maxwell's Equations Using Differential Forms

In the following we present the conventions of differential calculus formalism used throughout this paper. We refer the reader to Schwarz [35] and Flanders [18] for complete surveys of the differential calculus.

Notation 2.1. Let p equal 2 or 3 and let k be an integer smaller than p. For a compact, connected and oriented Riemannian manifold of dimension p, $(M, \mathfrak{g})$, of $\mathbb{R}^3$ we denote by $\Omega^k(M)$ the space of k-forms defined on M.

- The exterior product between two differential forms ω and η is denoted by $\omega \wedge \eta$.
- The inner product on $\Omega^k(M)$ is denoted by $\langle \cdot, \cdot \rangle_{\Omega^k}$.
- The Hodge star operator is denoted by $\star$.
- The interior product of a differential form ω with a smooth vector field Y is written $\mathsf{int}(Y)\omega$.
- The L^2-scalar product of two k-differential forms u and v is defined by

$$(u, v)_{L^2\Omega^k(M)} = \int_M \langle u, \overline{v} \rangle_{\Omega^k} \,\mathrm{dvol}_M$$

  and $\| \cdot \|_{L^2\Omega^k(M)}$ denotes the induced norm.

The exterior differential and codifferential operators are respectively denoted by d, δ. The Laplace–Beltrami operator Δ is defined by $\Delta = -\mathrm{d}\delta - \delta\mathrm{d}$.

$L^2\Omega^k(M)$ is the space of the square integrable k-forms of M while for $s \in \mathbb{R}$, $H^s\Omega^k(M)$ is the usual Sobolev space of k-forms. Let $H\Omega^k(\mathrm{d}, M)$ and $H\Omega^k(\delta, M)$ denote

$$H\Omega^k(\mathrm{d}, M) = \{\omega \in L^2\Omega^k(M) : \mathrm{d}\omega \in L^2\Omega^{k+1}(M)\}, \tag{2.1}$$

$$H\Omega^k(\delta, M) = \{\omega \in L^2\Omega^k(M) : \delta\omega \in L^2\Omega^{k-1}(M)\}, \tag{2.2}$$

that are Hilbert spaces when associated with their respective norms

$$\|\omega\|_{H\Omega^k(\mathrm{d},M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\mathrm{d}\omega\|_{L^2\Omega^{k+1}(M)},$$

$$\|\omega\|_{H\Omega^k(\delta,M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\delta\omega\|_{L^2\Omega^{k-1}(M)}.$$

We also denote by $H\Omega^k(\mathrm{d}, \delta, M)$ the space $H\Omega^k(\mathrm{d}, M) \cap H\Omega^k(\delta, M)$ equipped with the norm

$$\|\omega\|_{H\Omega^k(\mathrm{d},\delta,M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\mathrm{d}\omega\|_{L^2\Omega^{k+1}(M)} + \|\delta\omega\|_{L^2\Omega^{k-1}(M)}.$$

$H^s(M)$ and $L^2(M)$ denotes the respective spaces $H^s\Omega^0(M)$ and $L^2\Omega^0(M)$. Observe that for $k = 0$ (i.e. for functions), the space $H\Omega^0(\mathrm{d}, \delta, M)$ is exactly the usual Sobolev space $H^1(M)$, while $H\Omega^1(\mathrm{d}, \delta, M)$ cannot be identified to $(H^1(M))^3$.

2.1. *Statement of the problem*

Let Γ be a compact oriented surface of $\mathbb{R}^3$ without boundary. Consider the smooth connected bounded domain $\mathcal{O}_{\mathsf{c}}$ enclosed by Γ; $\mathcal{O}_{\mathsf{c}}$ is surrounded by a thin layer $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ with constant thickness ε. This material with thin layer is embedded in an ambient smooth connected domain $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ with compact oriented boundary. We denote by $\mathcal{O}$ the ε-independent domain defined by

$$\mathcal{O} = \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \overline{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \cup \mathcal{O}_{\mathsf{c}}.$$

Moreover, we denote by Γ_ε the boundary of $\overline{\mathcal{O}_{\mathsf{c}} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon}}$ (see Fig. 1). Let μ_{c}, μ_{m} and μ_{e} be three positive constants and let q_{e}, q_{c} and q_{m} be three complex numbers. Define the two piecewise functions μ and q on $\mathcal{O}$ by

$$\forall\, x \in \mathcal{O}, \quad \mu(x) = \begin{cases} \mu_{\mathsf{e}}, & \text{in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ \mu_{\mathsf{m}}, & \text{in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ \mu_{\mathsf{c}}, & \text{in } \mathcal{O}_{\mathsf{c}}, \end{cases} \quad \forall\, x \in \mathcal{O}, \quad q(x) = \begin{cases} q_{\mathsf{e}}, & \text{in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ q_{\mathsf{m}}, & \text{in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ q_{\mathsf{c}}, & \text{in } \mathcal{O}_{\mathsf{c}}. \end{cases}$$

The function μ is the dimensionless permeability of $\mathcal{O}$ while the function q denotes its dimensionless complex permittivity.[b]

Let $d_0 > 0$ be such that for each point q of Γ, the normal lines of Γ passing through q, with center at q and length $2d_0$ are disjoints. In the following, we assume that $\varepsilon \in (0, d_0)$. We denote by $\mathcal{O}_{\mathsf{e}}^{d_0}$ the set of points $\mathbf{x} \in \mathcal{O}_{\mathsf{e}}^{\varepsilon}$ at distance greater than d_0 of Γ. We assume that the current density $\mathbb{J}$ is imposed to the ambient medium, $\mathbb{J}$

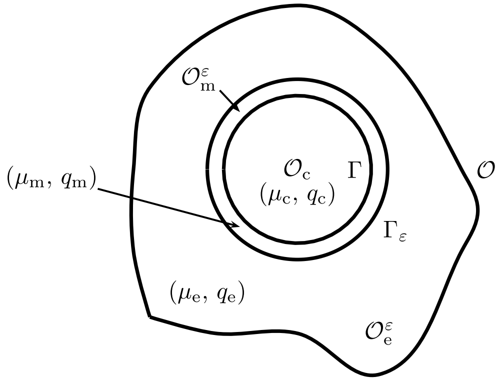

Fig. 1. Geometry of the model.

[b]Using the notations of the electrical engineering community, $q = \omega^2(\epsilon - \mathrm{i}\frac{\sigma}{\omega})$, where ω is the frequency, ϵ the permittivity and σ the conductivity of the domain [3].

being compactly supported in $\mathcal{O}_{\mathsf{e}}^{d_0}$. Throughout the paper the following hypothesis holds.

Hypothesis 2.2. (i) There exist $c_1, c_2 > 0$ such that for all $x \in \mathcal{O}$,

$$c_1 \leq -\Im(q(x)) \leq c_2, \quad 0 < \Re(q(x)) \leq c_2. \tag{2.3}$$

(ii) The source current density $\mathbb{J}$ is a 1-form that satisfies

$$\operatorname{supp}(\mathbb{J}) \Subset \mathcal{O}_{\mathsf{e}}^{d_0}, \quad \mathbb{J} \in L^2\Omega^1(\mathcal{O}), \quad \delta\mathbb{J} = 0, \text{ in } \mathcal{O}.$$

Maxwell's equations describe the behavior of the electromagnetic field in $\mathcal{O}$. Denote by $\mathbb{E}$ and $\mathbb{H}$ the 1-forms representing respectively the electric and the magnetic fields in $\mathcal{O}$ in time-harmonic regime. Denote by $N_{\partial\mathcal{O}}$ the normal vector field of $\partial\mathcal{O}$ outwardly directed from $\mathcal{O}$. In the following, the normal vector field and the corresponding normal 1-form are identified. Maxwell's equations in the time-harmonic regime read [20, 21, 39, 4] (with $\mathrm{i}^2 = -1$)

$$\mathrm{d}\mathbb{E} = \mathrm{i} \star (\mu\mathbb{H}), \quad \mathrm{d}\mathbb{H} = -\mathrm{i} \star (q\mathbb{E} + \mathbb{J}), \text{ in } \mathcal{O}, \tag{2.4a}$$

$$N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}. \tag{2.4b}$$

Using the idempotence of $\star$ in $\mathbb{R}^3$, we may infer the vector wave equation on $\mathbb{E}$

$$\star\,\mathrm{d}\left(\frac{1}{\mu} \star \mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \quad \text{in } \mathcal{O}, \qquad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}.$$

Since μ is a scalar function[c] of $\mathcal{O}$, we infer

$$\delta\left(\frac{1}{\mu}\mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \quad \text{in } \mathcal{O}, \qquad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}. \tag{2.5}$$

Problem (2.5) is the so-called vector wave equation in the time-harmonic regime [3]. Observe the power of the differential form formalism. In Eq. (2.5) suppose now that $\mathbb{E}$ and $\mathbb{J}$ are functions. Since the coderivative applied to the functions identically vanishes, Eq. (2.5) is nothing but the well-known Helmholtz equation:

$$-\operatorname{div}\left(\frac{1}{\mu}\nabla\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \quad \text{in } \mathcal{O}, \qquad \mathbb{E}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O},$$

therefore using differential forms enables us to link the Helmholtz equation and the vector wave equations in one formalism.

Remark 2.3. Denote $\mathbb{E}$ in Euclidean coordinates by $E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z$ and similarly for $\mathbb{H}$ and $\mathbb{J}$. Problem (2.4) and problem (2.5) write now

$$\operatorname{curl}\mathbb{E} = \mathrm{i}\mu\mathbb{H}, \qquad \operatorname{curl}\mathbb{H} = -\mathrm{i}(q\mathbb{E} + \mathbb{J}), \quad \text{in } \mathcal{O}, \qquad N_{\partial\mathcal{O}} \times \mathbb{E}|_{\partial\mathcal{O}} = O, \quad \text{on } \partial\mathcal{O},$$

[c]If μ is a tensor, the previous equation (2.5) becomes $\delta(\star\mu^{-1} \star \mathrm{d}\mathbb{E}) - q\mathbb{E} = \mathbb{J}$.

and

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \quad \text{in } \mathcal{O}, \qquad N_{\partial\mathcal{O}} \times \mathbb{E}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O},$$

which is the tensorial formulation of the vector wave equation in the time-harmonic regime.

The aim of this paper is to derive transmission conditions equivalent to $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ in order to avoid its meshing. Hereafter, it is demonstrated that writing these conditions with differential forms enables us to consider similarly the Helmholtz equation and the vector wave equations. For the sake of clarity, and since the case of functions is much simpler, we only provide the detailed proofs of the results for 1-forms (i.e. for the vector wave equation), and we let the reader verify that the corresponding results hold for the Helmholtz equation.

2.2. *Regularized variational formulation*

Our functional space $\mathbf{X}(q)$ is defined as

$$\mathbf{X}(q) = \{u \in H\Omega^1(\mathrm{d}, \mathcal{O}),\ \delta(qu) \in L^2(\mathcal{O}),\ N_{\partial\mathcal{O}} \wedge u|_{\partial\mathcal{O}} = 0\},$$

associated with its graph norm

$$\|u\|_{\mathbf{X}(q)} = \|u\|_{H\Omega^1(\mathrm{d},\mathcal{O})} + \|\delta(qu)\|_{L^2(\mathcal{O})}.$$

Define the sesquilinear form a_q in $\mathbf{X}(q)$ adapted to a regularized variational formulation of the problem (2.5) by

$$a_q(u, v) = \int_{\mathcal{O}} \left(\frac{1}{\mu}\langle \mathrm{d}u, \mathrm{d}\overline{v}\rangle_{\Omega^2} + \langle \delta(qu), \delta(\overline{qv})\rangle_{\Omega^0} - q\langle u, \overline{v}\rangle_{\Omega^1}\right) \mathrm{dvol}_{\mathcal{O}}.$$

Using inequalities (2.3), the following lemma holds.

Lemma 2.4. *There exist a constant $c_0 > 0$ and $\alpha \in \mathbb{C}$ such that for all $\varepsilon \in (0, d_0)$,*

$$\Re(\alpha a_q(u, u)) \geq c_0 \|u\|^2_{\mathbf{X}(q)}. \tag{2.6}$$

For all $\varepsilon \in (0, d_0)$, we consider the variational problem: find $\mathbb{E} \in \mathbf{X}(q)$ such that

$$\forall\, u \in \mathbf{X}(q), \quad a_q(\mathbb{E}, u) = \int_{\mathcal{O}} \langle \mathbb{J}, \overline{u}\rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}}\,. \tag{2.7}$$

Using Hypothesis 2.2 the following theorem holds.

Theorem 2.5. (Equivalent problems) *Let Hypothesis 2.2 hold.*

(i) *There is at most one solution $\mathbb{E} \in \mathbf{X}(q)$ to problem (2.7).*

(ii) *The solution* $\mathbb{E}$ *satisfies* (2.5) *in a weak sense*

$$\delta \mathrm{d}\mathbb{E} - \mu q \mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0,$$

with the divergence condition

$$\delta(q\mathbb{E}) = 0, \text{ in } \mathcal{O} \tag{2.8}$$

and the following equalities[d] *hold for* $\mathscr{S} \in \{\Gamma, \Gamma_\varepsilon\}$

$$\left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{E}\right]_{\mathscr{S}} = 0, \quad [N_{\mathscr{S}} \wedge \mathbb{E}]_{\mathscr{S}} = 0, \quad [q\,\mathsf{int}(N_{\mathscr{S}})\mathbb{E}]_{\mathscr{S}} = 0. \tag{2.9}$$

(iii) *If* $(\mathbb{E}, \mathbb{H}) \in (L^2\Omega^1(\mathcal{O}))^2$ *is a solution to problem* (2.4), *then* $\mathbb{E} \in \mathbf{X}(q)$ *satisfies* (2.5). *Conversely, if* $\mathbb{E} \in \mathbf{X}(q)$ *satisfies* (2.5) *then the couple of* 1*-forms* $(\mathbb{E}, -(\mathrm{i}/\mu) \star \mathrm{d}\mathbb{E})$ *belongs to* $(L^2\Omega^1(\mathcal{O}))^2$ *and satisfies problem* (2.4).

Remark 2.6. For the Helmholtz equation, the appropriate space is $H^1(\mathcal{O})$. Since $\delta f \equiv 0$ for any function f, Eq. (2.7) is exactly the variational formulation of (2.5) applied to 0-form. Therefore the Lax–Milgram lemma ensures straightforwardly the equivalences of the above theorem, replacing 1-forms by 0-forms.

Proof. Unlike Remark 2.6, when dealing with 1-forms, Eq. (2.7) is not the variational formulation of Eq. (2.5), hence the theorem is not obvious. Its proof is based on an idea of Costabel *et al.*

(i) According to estimate (2.6), a straightforward application of the well-known Lax–Milgram theorem leads to the existence and uniqueness of the solution $\mathbb{E}$ to the regularized variational problem (2.7).

(ii) The proof is precisely worked out in full details in [7, 8] in a very slightly different configuration. We just give here the sketch of the proof. The first transmission condition of (2.9) comes easily from the Green formula (see Schwarz [35]) and since $\mathbb{E} \in \mathbf{X}(q)$, then $N_{\mathscr{S}} \wedge \mathbb{E}$ and $q\,\mathsf{int}(N_{\mathscr{S}})\mathbb{E}$ are continuous across $\mathscr{S} \in \{\Gamma, \Gamma_\varepsilon\}$.

It remains to prove that $\mathbb{E}$ satisfies $\delta(q\mathbb{E}) = 0$. Denote by $H\Delta(\mathcal{O})$ the space of functions $\phi \in H_0^1(\mathcal{O})$ such that $\delta(q\mathrm{d}\phi)$ belongs to $L^2(\mathcal{O})$. Integrations by parts imply

$$\forall\, \phi \in H\Delta(\mathcal{O}), \quad a_q(\mathbb{E}, \mathrm{d}\phi) = \int_{\mathcal{O}} \langle \delta(q\mathbb{E}), \overline{\delta(q\mathrm{d}\phi) + \phi} \rangle_{\Omega^0} \,\mathrm{dvol}_{\mathcal{O}}.$$

Since $\Im(q) \le -c_1 < 0$, the function $\delta(q\mathrm{d}\phi) + \phi$ runs through the whole $L^2(\mathcal{O})$ space as ϕ runs through $H\Delta(\mathcal{O})$. Moreover, since $\delta(\mathbb{J})$ vanishes we have

$$\int_{\mathcal{O}} \langle \mathbb{J}, \overline{\mathrm{d}\phi} \rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}} = 0,$$

from which we infer that $\delta(q\mathbb{E})$ identically vanishes in $L^2(\mathcal{O})$ according to (2.7). Therefore the solution $\mathbb{E}$ of problem (2.7) solves problem (2.5).

[d]For an oriented surface $\mathscr{S}$ without boundary and for a differential form u defined in a neighborhood of $\mathscr{S}$ we denote by $[u]_{\mathscr{S}}$ the jump across $\mathscr{S}$. $N_{\mathscr{S}}$ denotes the normal of $\mathscr{S}$ outwardly directed from the domain enclosed by $\mathscr{S}$ to the exterior.

(iii) If $(\mathbb{E},\mathbb{H})$ solves problem (2.4) we straightforwardly infer (2.5), since $\star$ is idempotent and since μ is a scalar function. Conversely, defining $\mathbb{H}$ by

$$\mathbb{H} = -\frac{\mathrm{i}}{\mu}\star \mathrm{d}\mathbb{E},$$

we infer that $(\mathbb{E},\mathbb{H})$ solves problem (2.4). □

Denote by $\mathcal{O}_{\mathsf{e}}$ the domain $\mathcal{O}_{\mathsf{e}} = \mathcal{O}\backslash\overline{\mathcal{O}_{\mathsf{c}}}$. Define $\tilde{\mu}$ and $\tilde{q}$ by

$$\forall\, x \in \mathcal{O},\quad \tilde{\mu}(x) = \begin{cases}\mu_{\mathsf{c}}, & \text{in } \mathcal{O}_{\mathsf{c}},\\ \mu_{\mathsf{e}}, & \text{in } \mathcal{O}_{\mathsf{e}},\end{cases}\qquad \forall\, x \in \mathcal{O},\quad \tilde{q}(x) = \begin{cases}q_{\mathsf{c}}, & \text{in } \mathcal{O}_{\mathsf{c}},\\ q_{\mathsf{e}}, & \text{in } \mathcal{O}_{\mathsf{e}}.\end{cases}$$

Let $\mathbb{E}^0 \in \mathbf{X}(\tilde{q})$ be the "background" solution defined by

$$\forall\, u \in \mathbf{X}(\tilde{q}),\quad a_{\tilde{q}}(\mathbb{E}^0, u) = \int_{\mathcal{O}} \left\langle \mathbb{J}, \overline{u}\right\rangle_{\Omega^1} \mathrm{dvol}_{\mathcal{O}},$$

which means in a weak sense

$$\delta\left(\frac{1}{\tilde{\mu}}\mathrm{d}\mathbb{E}^0\right) - \tilde{q}\,\mathbb{E}^0 = \mathbb{J},\ \text{in } \mathcal{O},\quad N_{\partial\mathcal{O}} \wedge \mathbb{E}^0|_{\partial\mathcal{O}} = 0. \tag{2.10}$$

We have the following regularity result.

Proposition 2.7. *Let Hypothesis 2.2 hold. Moreover, let $s \geq 0$ and $\mathbb{J}$ belong to $H^s\Omega^1(\mathcal{O}_{\mathsf{e}}^{d_0})$. Then the 1-form $\mathbb{E}^0$ exists and is unique in $\mathbf{X}(\tilde{q})$. Moreover, denoting by $\mathbb{E}^{\mathsf{c},0}$ and $\tilde{\mathbb{E}}^{\mathsf{e},0}$ its respective restrictions to $\mathcal{O}_{\mathsf{c}}$ and $\mathcal{O}_{\mathsf{e}}$, we have*

$$\tilde{\mathbb{E}}^{\mathsf{e},0} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{e}}),\quad \mathbb{E}^{\mathsf{c},0} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{c}}).$$

Proof. The 1-form $\mathbb{E}^0$ satisfies (2.10). The proof of the existence and the uniqueness of $\mathbb{E}^0$ in $\mathbf{X}(\tilde{q})$ is very similar to the one performed in Theorem 2.5, by replacing $\mathbf{X}(q)$ by $\mathbf{X}(\tilde{q})$ and a_q by $a_{\tilde{q}}$. Since $\delta\mathbb{J}$ vanishes, we infer $\delta(\tilde{q}\mathbb{E}^0) = 0$ and therefore $\mathbb{E}^0$ satisfies

$$-\Delta\mathbb{E}^0 - \tilde{\mu}\tilde{q}\,\mathbb{E}^0 = \mathbb{J},\quad \text{in } \mathcal{O}_{\mathsf{e}} \cup \mathcal{O}_{\mathsf{c}},\qquad N_{\partial\mathcal{O}} \wedge \mathbb{E}^0|_{\partial\mathcal{O}} = 0,$$

with transmission conditions

$$[N_\Gamma \wedge \mathrm{d}\mathbb{E}^0]_\Gamma = 0,\quad [\tilde{q}\,\mathsf{int}(N_\Gamma)\mathbb{E}^0]_\Gamma = 0,$$
$$\left[\frac{1}{\tilde{\mu}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^0\right]_\Gamma = 0,\quad [\delta(\tilde{q}\mathbb{E}^0)]_\Gamma = 0.$$

The same calculations as performed in Proposition 2.1 of Costabel *et al.* [8] imply that the set of the above transmission and boundary conditions covers[e] the Laplacian in $\mathcal{O}_{\mathsf{c}}$ and in $\mathcal{O}_{\mathsf{e}}$, in the sense of Definition 1.5 on p. 125 of Lions and

[e]According to the Appendix of the paper of Li and Vogelius [22] the regularity of $\mathbb{E}^0$ may also be obtained by using a reflection to reduce the problem to an elliptic system with complementing boundary conditions in the sense of Agmon *et al.* [1, 2].

Magenes [23]. Therefore we infer the piecewise elliptic regularity of $\mathbb{E}^0$, which ends the proof of the lemma. $\square$

The following estimates, which ensure that $\mathbb{E}^0$ is the zeroth order approximation of $\mathbb{E}$, hold.

Proposition 2.8. *Under Hypothesis 2.2, there exists $C > 0$ such that for any small parameter $\varepsilon \in (0, d_0)$*

$$\|\mathbb{E}\|_{\mathbf{X}(q)} \leq C, \tag{2.11}$$

$$\|\mathbb{E} - \mathbb{E}^0\|_{H\Omega^1(\mathrm{d},\mathcal{O})} \leq C\sqrt{\varepsilon}. \tag{2.12}$$

Proof. Using (2.6), estimates (2.11) are obvious since $\Im(q) \leq -c_1 < 0$. Now prove (2.12). We first mention that $\mathbb{E}^0$ belongs to $H^2\Omega^1(\varpi)$ for $\varpi \in \{\mathcal{O}_\mathsf{e}, \mathcal{O}_\mathsf{c}\}$, according to Proposition 2.7; hence $\mathbb{E}^0 \in L^\infty\Omega^1(\varpi)$ and $\mathrm{d}\mathbb{E}^0 \in L^\infty\Omega^2(\varpi)$. Denoting by $\mathbb{U} = \mathbb{E} - \mathbb{E}^0$ we infer

$$\begin{aligned}\int_{\mathcal{O}} \frac{1}{\mu}\langle \mathrm{d}\mathbb{U}, \overline{\mathrm{d}\mathbb{U}}\rangle_{\Omega^2} - q\langle \mathbb{U}, \overline{\mathbb{U}}\rangle_{\Omega^1}\, \mathrm{dvol}_{\mathcal{O}} \\ = q_\mathsf{m} \int_{\mathcal{O}^\varepsilon_\mathsf{m}} \langle \mathbb{E}^0, \overline{\mathbb{U}}\rangle_{\Omega^1}\, \mathrm{dvol}_{\mathcal{O}^\varepsilon_\mathsf{m}} - \frac{1}{\mu_\mathsf{m}} \int_{\mathcal{O}^\varepsilon_\mathsf{m}} \langle \mathrm{d}\mathbb{E}^0, \overline{\mathrm{d}\mathbb{U}}\rangle_{\Omega^2}\, \mathrm{dvol}_{\mathcal{O}^\varepsilon_\mathsf{m}}\,.\end{aligned}$$

Therefore using (2.11) and using the assumption (2.3) on q, we infer

$$\|\mathrm{d}\mathbb{U}\|_{L^2\Omega^2(\mathcal{O})} + \|\mathbb{U}\|_{L^2\Omega^1(\mathcal{O})} \leq C\sqrt{\varepsilon}. \qquad \square$$

2.3. *Main result*

Consider the inclusion $\mathcal{J} : \Gamma \to \mathcal{O}$, and $\mathcal{J}^*$ its pull-back $\mathcal{J}^* : \Omega^k(\mathcal{O}) \to \Omega^k(\Gamma)$, for $k \in \{0, 1, 2, 3\}$. Denote by d_Γ and δ_Γ the exterior differential and the codifferential operators defined on $\Omega^k(\Gamma)$. Define $\mathbb{S}$ and $\mathbb{T}$ by[f]

$$\mathbb{S} = (q_\mathsf{m} - q_\mathsf{e})\mathcal{J}^*(\mathbb{E}^0) + \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right) \delta_\Gamma \mathrm{d}_\Gamma(\mathcal{J}^*(\mathbb{E}^0)), \tag{2.13}$$

$$\mathbb{T} = \left(\frac{1}{q_\mathsf{m}} - \frac{1}{q_\mathsf{e}}\right) \mathrm{d}(\mathsf{int}(N_\Gamma)(\tilde{q}\mathbb{E}^0)|_\Gamma) + (\mu_\mathsf{m} - \mu_\mathsf{e})\, \mathsf{int}(N_\Gamma)\left(\frac{1}{\tilde{\mu}}\mathrm{d}\mathbb{E}^0\right)\Big|_\Gamma. \tag{2.14}$$

The explicit expressions of $\mathbb{S}$ and $\mathbb{T}$ in local coordinates are given in Sec. 6. Let $\mathbb{E}^1$ be the 1-forms defined by

$$\delta\mathrm{d}\mathbb{E}^1 - \tilde{\mu}\tilde{q}\,\mathbb{E}^1 = 0, \quad \text{in } \mathcal{O}_\mathsf{e} \cup \mathcal{O}_\mathsf{c}, \qquad N_{\partial\mathcal{O}} \wedge \mathbb{E}^1|_{\partial\mathcal{O}} = 0,$$

[f] For a sufficiently smooth k-form ϕ defined in $\mathcal{O}$, we denote by $\phi|_\Gamma$ its restriction to Γ. In addition, if ϕ is regular in $\mathcal{O}_e$ and $\mathcal{O}_c$ but not in $\mathcal{O}$, we denote by $\phi|_{\Gamma+}$ (respectively $\phi|_{\Gamma-}$) the restriction to Γ of ϕ from the domain $\mathcal{O}_e$ (respectively $\mathcal{O}_c$).

with the following transmission conditions on Γ

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^1|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^1|_{\Gamma^-} = \mathbb{S}, \tag{2.15}$$

$$N_\Gamma \wedge \mathbb{E}^1|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^1|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}. \tag{2.16}$$

The aim of this paper is to prove the following theorem.

Theorem 2.9. *Under Hypothesis 2.2, if moreover the current density $\mathbb{J}$ belongs to $H^3\Omega^1(\mathcal{O}_{\mathsf{e}}^{d_0})$, there exist $\varepsilon_0 > 0$ and a constant C, independent of ε such that*

$$\forall\, \varepsilon \in (0, \varepsilon_0), \quad \|\mathbb{E} - (\mathbb{E}^0 + \varepsilon\mathbb{E}^1)\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \leq C\varepsilon^2,$$

and for any domain ϖ compactly embedded in $\mathcal{O}_{\mathsf{e}}$, there exist $\varepsilon_\varpi > 0$ and a constant $C_\varpi > 0$ independent of ε such that

$$\forall\, \varepsilon \in (0, \varepsilon_\varpi), \quad \|\mathbb{E} - (\mathbb{E}^0 + \varepsilon\mathbb{E}^1)\|_{H\Omega^1(\mathrm{d},\delta,\varpi)} \leq C_\varpi\varepsilon^2.$$

Remark 2.10. It is possible to give a precise behavior of $\mathbb{E}$ in a neighborhood of Γ by defining a 1-form in the thin membrane (see Theorem 6.3).

In this paper we choose to deal with differential forms, in accordance with Flanders [18]. This point of view has the convenience of considering both electric and magnetic fields as 1-forms, i.e. they are physically similar in accordance with electrical engineering considerations [3]. We point out a few arguments to enlighten the convenience of the differential calculus formalism.

(i) *Anisotropy.* For the sake of simplicity, we deal here with isotropic materials, although the anisotropic case may be interesting for applications. In this case, μ and q are matrices and the vector wave equation becomes

$$\delta((\star\mu^{-1}\star)\mathrm{d}\mathbb{E}) - q\mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O} \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O},$$

and the following transmission conditions hold on $\mathscr{S} \in \{\Gamma, \Gamma_\varepsilon\}$

$$[\mathsf{int}(N_{\mathscr{S}})(\star\mu^{-1}\star\mathrm{d}\mathbb{E})]_{\mathscr{S}} = 0, \quad [N_{\mathscr{S}} \wedge \mathbb{E}]_{\mathscr{S}} = 0.$$

To obtain the approximate transmission conditions equivalent to the thin layer, we just have to write the tensor $\star\mu^{-1}\star$ in local coordinates, with the help of the explicit formulas given in the Appendix. The calculations are more tedious but we are confident that the reader has all the tools to perform the analysis.

(ii) *Non-constant thickness.* We consider here a thin layer with constant thickness. As mentioned in Sec. 1 a high electric field may destabilize the cell membrane, possibly leading to the apparition of pores. Hence the thickness of the membrane is no longer constant with respect to the tangential variable. As performed in [31], the change of variables would lead to additional terms in the transmission conditions. These terms would come from the fact that the coefficients g_{i3} of the matrix (g_{ij}) given in Sec. 4 by (4.1) do not vanish. The derivation of the asymptotics would be more tedious but, once again, we are confident that all the tools are given in this paper to perform the calculation. In the case of a rough thin layer, the present analysis may not be applied. We have to introduce appropriate correctors as performed in [6].

(iii) *Link with the Helmholtz equation.* As previously mentioned, Eqs. (2.5) are well-defined if $\mathbb{E}$ and $\mathbb{J}$ are functions, since operators d and δ are defined for k-forms and the exterior product between a 1-form and a function is also well-defined. Moreover, since δ acting on functions is zero, the operator $-\delta\mathrm{d}$ coincides with Laplace–Beltrami operator Δ. In addition, the above differential forms $\mathbb{S}$ and $\mathbb{T}$ are well-defined even if $\mathbb{E}^0$ is a function, and in this case we have

$$\mathbb{S} = (q_{\mathsf{m}} - q_{\mathsf{e}})\mathbb{E}^0|_\Gamma + \left(\frac{1}{\mu_{\mathsf{m}}} - \frac{1}{\mu_{\mathsf{e}}}\right)\delta_\Gamma \mathrm{d}_\Gamma(\mathbb{E}^0)|_\Gamma,$$

$$\mathbb{T} = \frac{\mu_{\mathsf{m}} - \mu_{\mathsf{e}}}{\mu_{\mathsf{c}}}\,\mathsf{int}(N_\Gamma)(\mathrm{d}\mathbb{E}^0)\big|_{\Gamma^-},$$

since the interior product $\mathsf{int}(N_\Gamma)$ acting on functions is zero. Writing our asymptotic transmission conditions for functions in tensor calculus formalism, we infer that the function u solution to

$$-\nabla\cdot\left(\frac{1}{\mu}\nabla u\right) - qu = j, \quad \text{in } \mathcal{O}, \qquad u|_{\partial\mathcal{O}} = 0,$$

is approached by $u^0 + \varepsilon u^1$ where $(u^k)_{k=0,1}$ satisfy

$$-\Delta u^k - \tilde{\mu}\tilde{q}u^k = \delta_0^k j, \quad \text{in } \mathcal{O}_{\mathsf{c}} \cup \mathcal{O}_{\mathsf{e}}, \qquad u^k|_{\partial\mathcal{O}} = 0,$$

with the following transmission conditions

$$[u^0]_\Gamma = 0, \quad \left[\frac{1}{\tilde{\mu}}\partial_n u^0\right]_\Gamma = 0, \quad u^1|_{\Gamma^+} - u^1|_{\Gamma^-} = \frac{\mu_{\mathsf{m}} - \mu_{\mathsf{e}}}{\mu_{\mathsf{c}}}\partial_n u^0|_{\Gamma^-},$$

$$\frac{1}{\mu_{\mathsf{e}}}\partial_n u^1|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}}\partial_n u^1|_{\Gamma^-} = (q_{\mathsf{m}} - q_{\mathsf{e}})u^0|_\Gamma - \left(\frac{1}{\mu_{\mathsf{m}}} - \frac{1}{\mu_{\mathsf{e}}}\right)\Delta_\Gamma u^0|_\Gamma.$$

This approximation is rigorously proved in [29] (see Eqs. (4) on p. 4 of [29]). Therefore the differential calculus provides transmission conditions that are valid for the Helmholtz equation and the Maxwell equations. It is also possible to derive our asymptotics by tensor calculus considerations, as used in linear elasticity of thin shells [9, 15, 16]. This approach is worked out in full details in the thesis [28] of the second author and in [5, 10].

Remark 2.11. (The tensor calculus formulation) Since we are confident that our result might be useful for bioelectromagnetic computations, and since the electrical engineering community may feel uncomfortable with the differential calculus formalism, we translate our result with the help of the "usual" differential operators. Denote by ∇_Γ and $\nabla_\Gamma\cdot$ the respective gradient and divergence operators on

Γ. Define Rot_Γ and rot_Γ by

$$\forall\, f \in C^\infty(\Gamma), \quad \mathrm{Rot}_\Gamma\, f = (\nabla_\Gamma f) \times N_\Gamma,$$
$$\forall\, \mathbf{f} \in (C^\infty(\Gamma))^3, \quad \mathrm{rot}_\Gamma\, \mathbf{f} = \nabla_\Gamma \cdot (\mathbf{f} \times N_\Gamma).$$

Then $(\mathbb{E}^k)_{k=0,1}$ (seen as vector field) satisfies the following equations

$$\mathrm{curl}\,\mathrm{curl}\,\mathbb{E}^k - \tilde{\mu}\tilde{q}\mathbb{E}^k = \delta_0^k \mathbb{J}, \quad \text{in } \mathcal{O}_\mathsf{e} \cup \mathcal{O}_\mathsf{c}, \quad N_{\partial\mathcal{O}} \times \mathbb{E}^k|_{\partial\mathcal{O}} = 0,$$

with the following transmission conditions on Γ

$$N_\Gamma \times \mathbb{E}^0|_{\Gamma^+} = N_\Gamma \times \mathbb{E}^0|_{\Gamma^-}, \ \frac{1}{\mu_\mathsf{e}}(N_\Gamma \times \mathrm{curl}\,\mathbb{E}^0)|_{\Gamma^+} = \frac{1}{\mu_\mathsf{c}}(N_\Gamma \times \mathrm{curl}\,\mathbb{E}^0)|_{\Gamma^-}, \quad (2.17)$$

$$\begin{aligned} N_\Gamma \times \mathbb{E}^1|_{\Gamma^+} \times N_\Gamma &= N_\Gamma \times \mathbb{E}^1|_{\Gamma^-} \times N_\Gamma + q_\mathsf{c}\left(\frac{1}{q_\mathsf{m}} - \frac{1}{q_\mathsf{e}}\right)\nabla_\Gamma(\mathbb{E}^0|_{\Gamma^-} \cdot N_\Gamma) \\ &\quad + \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}}(\mathrm{curl}\,\mathbb{E}^0 \times N_\Gamma)|_{\Gamma^-}, \\ \frac{1}{\mu_\mathsf{e}}(\mathrm{curl}\,\mathbb{E}^1 \times N_\Gamma)|_{\Gamma^+} &= \frac{1}{\mu_\mathsf{c}}(\mathrm{curl}\,\mathbb{E}^1 \times N_\Gamma)|_{\Gamma^-} + (q_\mathsf{m} - q_\mathsf{e})(\mathrm{N}_\Gamma \times \mathbb{E}^0 \times N_\Gamma)|_\Gamma \\ &\quad + \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right)\mathrm{Rot}_\Gamma\,\mathrm{rot}_\Gamma(\mathrm{N}_\Gamma \times \mathbb{E}^0 \times N_\Gamma)|_\Gamma. \end{aligned} \quad (2.18)$$

Remark 2.12. (The impedance boundary condition of Engquist–Nédélec [14]) Let $\mathbb{J}$ be supported in $\mathcal{O}_\mathsf{c}$ (and be divergence-free) and suppose that $\mathcal{O}_\mathsf{e}^\varepsilon$ is a perfectly conducting domain. Therefore $q_\mathsf{e} = +\infty$ and $\mu_\mathsf{e} = 0$. A homogeneous Dirichlet condition is then imposed on Γ_ε

$$N_{\Gamma_\varepsilon} \times \mathbb{E}|_{\Gamma_\varepsilon} = 0.$$

We are now in the same configuration as the problem studied by Engquist and Nédélec [14], p. 18. According to (2.17) and (2.18), writing the condition satisfied by $\mathbb{E}^0 + \varepsilon\mathbb{E}^1$ and neglecting the terms of order ε^2, we infer the following boundary condition for the first-order approximation $\mathbb{E}_a$ of the field

$$N_\Gamma \times \mathbb{E}_a|_{\Gamma^-} \times N_\Gamma = -\varepsilon\left(\frac{q_\mathsf{c}}{q_\mathsf{m}}\nabla_\Gamma\left(\mathbb{E}_a|_{\Gamma^-} \cdot N_\Gamma\right) + \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}}(\mathrm{curl}\,\mathbb{E}_a \times N_\Gamma)|_{\Gamma^-}\right).$$

According to Maxwell's equations, $\mathrm{curl}\,\mathbb{E} = \mathrm{i}\mu_\mathsf{c}\mathbb{H}$ and $\mathrm{curl}\,\mathbb{H} = -\mathrm{i}q_\mathsf{c}\mathbb{E}$. Therefore $q_\mathsf{c}\mathbb{E} \cdot N_\Gamma = \mathrm{i}\,\mathrm{curl}\,\mathbb{H} \cdot N_\Gamma$. The definition of $\nabla_\Gamma\cdot$ (see, for example, Eq. (2.22) p. 5 of [14]) leads to[g]

$$\nabla_\Gamma \cdot (\mathbb{H} \times N_\Gamma) = \mathrm{curl}\,\mathbb{H} \cdot N_\Gamma = -\mathrm{i}q_\mathsf{c}\mathbb{E} \cdot N_\Gamma, \quad (2.19)$$

[g] Using differential forms and since $\mathrm{d}N = 0$, equality (8.1) implies

$\mathsf{int}(N_\Gamma)\mathbb{E}^0|_{\Gamma^-} = -\frac{1}{\mathrm{i}q_\mathsf{c}}\mathsf{int}(N_\Gamma)\delta(\star\mathbb{H}^0) = -\frac{1}{\mathrm{i}q_\mathsf{c}}\delta_\Gamma(\mathsf{int}(N)\star\mathbb{H}^0|_\Gamma)$, which is exactly equality (2.19).

and the impedance boundary condition follows

$$N_\Gamma \times \mathbb{E}_a|_{\Gamma^-} \times N_\Gamma = -\mathrm{i}\varepsilon \left(\frac{1}{q_{\mathsf{m}}} \nabla_\Gamma (\nabla_\Gamma \cdot (\mathbb{H}_a \times N_\Gamma)) + \mu_{\mathsf{m}} (\mathbb{H}_a \times N_\Gamma)|_{\Gamma^-} \right).$$

Observe that this is the impedance boundary condition given in [14] p. 19, since they took the normal interior to their domain Ω_∞, hence $n = -N_\Gamma$.

3. Numerical Simulations

We have tested the model when Γ is a sphere of radius 0.04. The outside boundary of $\mathcal{O}$ is a sphere of radius 0.08. We impose a Silver-Muller condition on this outer boundary. Hexahedral mesh has been used for experiments, as presented in Fig. 2. The current source is a Gaussian source polarized along x-coordinate and centered around the point (0, 0, 0.06). The exact solution is computed numerically on a similar mesh, where a thin layer made of hexahedra is inserted between the two domains. Edge finite elements of fourth order (Nedelec's first family) are used with curved elements in order to correctly approximate the geometry. We have observed that the numerical error between fourth-order and fifth-order is below 0.1%. According to [17], we chose the biological electrical parameters:

$$\varepsilon_m = 10, \quad \varepsilon_e = \varepsilon_c = 80, \qquad \sigma_m = 10^{-5}, \quad \sigma_e = \sigma_c = 0.5, \tag{3.1}$$

and the frequency is equal to 1.2 GHz. The numerical values of $\mathbb{E}_0$ and $\mathbb{E}_1$ are displayed in Fig. 3. We have displayed the convergence of the model in Fig. 4. Observe that the numerical convergence rate, which is of order ε^2, coincides with the theory for small values of ε only. This is in accordance with the assumption

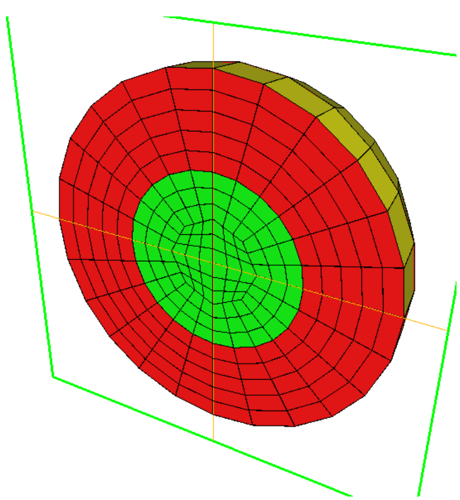

Fig. 2. Hexahedral mesh used for experiments.

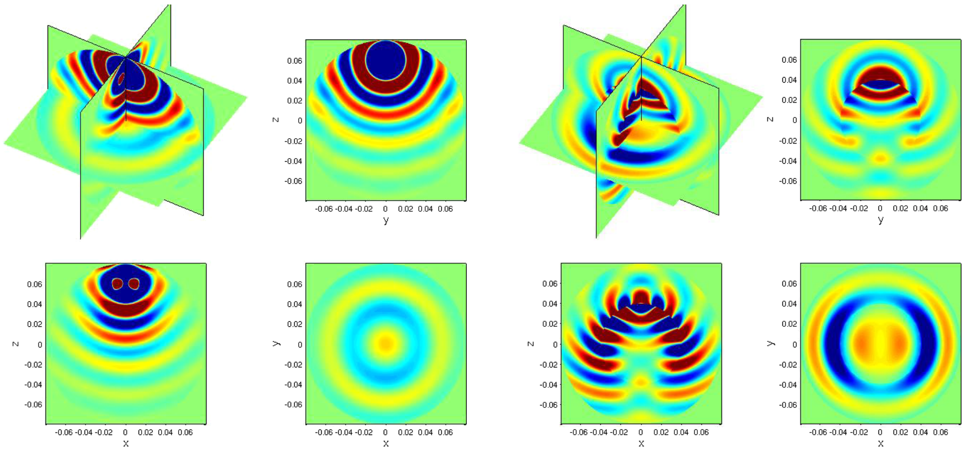

Fig. 3. Real part of the electric field (x-component) for $\mathbb{E}^0$ (left) and $\mathbb{E}^1$ (right).

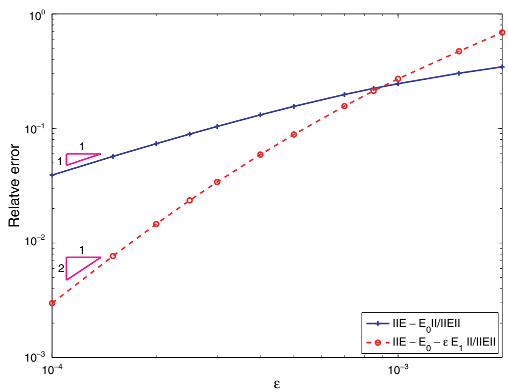

Fig. 4. Relative error between the model and the exact solution.

"ε goes to zero" to be imposed, since at the crossingpoint of Fig. 4, ε equal 0.001 which is not small compared with the sphere radius of 0.04.

In addition, the frequency range for which the thin layer model is valid has been studied. Actually, observe that in (3.1), the cell membrane conductivity is very low compared with the outer and inner conductivities, while the permittivity of the three domains are quite similar, compared with the membrane thickness. Moreover, for large frequency, the displacement currents are dominant, meaning that the permittivities have to be mainly considered. Therefore, for large frequencies, the cell is a soft contrast material with a thin layer, and the theoretical results presented in this paper hold. However, if the frequency dramatically decreases, the conduction currents dominate. In this case, the conductivities have to be used, and since the membrane conductivity is very low, the cell is then a high contrast

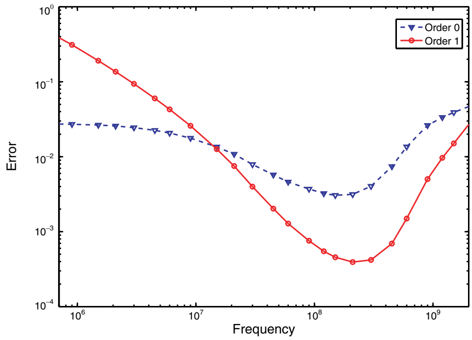

Fig. 5. Relative error between the model and the exact solution versus frequency.

medium with a thin layer: two small parameters are then involved in the equation, and the asymptotic analysis presented here is no longer valid. This phenomenon is illustrated in Fig. 5, where we have checked the accuracy of the model versus the frequency when ε is chosen constant, and equal to 0.0002: above 100 MHz, the approximate transmission conditions precisely replace the membrane but below 10 MHz, the conditions are no longer valid and another analysis has to be performed. Observe that above 2.10^8 Hz both errors increase: this is due to the fact that the membrane thickness ϵ remains constant while the wavelength decreases.

4. Geometry

Let $\mathscr{V}_\Gamma$ be the tubular open neighborhood of Γ composed by the points at distance d_0 of Γ. In the following, it will be convenient to write the involved differential form $\mathbb{E}$ in local coordinates in the tubular neighborhood $\mathscr{V}_\Gamma$ of Γ. We denote by $\mathscr{V}_\mathsf{e}^\varepsilon$ and $\mathscr{V}_\mathsf{c}$ the respective intersections $\mathscr{V}_\Gamma \cap \mathcal{O}_\mathsf{e}^\varepsilon$ and $\mathscr{V}_\Gamma \cap \mathcal{O}_\mathsf{c}$.

4.1. *Parametrization of* Γ

Let $\mathbf{x}_\mathsf{T} = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathbf{x}_\mathsf{T})\}$. By abuse of notations, we denote by $\mathbf{x}_\mathsf{T} \in \Gamma$ the point of Γ equal to $\psi(\mathbf{x}_\mathsf{T})$. In the (x_1, x_2)-coordinates, we denote by N_Γ the outward vector normal to Γ defined by

$$N_\Gamma = \frac{\partial_1\psi \wedge \partial_2\psi}{\|\partial_1\psi \wedge \partial_2\psi\|}$$

and we define by Φ the following map

$$\forall\,(\mathbf{x}_\mathsf{T}, x_3) \in \Gamma \times \mathbb{R}, \quad \Phi(\mathbf{x}_\mathsf{T}, x_3) = \psi(\mathbf{x}_\mathsf{T}) + x_3 N_\Gamma(\mathbf{x}_\mathsf{T}).$$

Notation 4.1. In the following ∂_j stands for ∂_{x_j} for $j = 1, 2, 3$. Moreover, we use the summation indices convention $a_i b_i = \sum_{i=1,2,3} a_i b_i$. Observe that according to our change of variables, $\mathbf{x}_\mathsf{T}$ denotes the tangential variables and x_3 is the normal direction. In order to stress the difference between $\mathsf{x_T}$ and x_3, the Greek letters α and β (and possibly γ, ι, κ and λ) denote the indices in $\{1,2\}$, while the letters i,j,k denote the indices in $\{1,2,3\}$. Eventually it is convenient to introduce the Levi–Cività symbol ϵ_{ijk} defined by

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1, & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ 0, & \text{if any two labels are the same.} \end{cases}$$

According to the definition of d_0, the tubular neighborhood $\mathscr{V}_\Gamma$ of Γ may be parametrized by

$$\mathscr{V}_\Gamma = \{\Phi(\mathsf{x_T}, x_3),\ (\mathsf{x_T}, x_3) \in \Gamma \times (-d_0, d_0)\}.$$

The $(\mathsf{x_T}, x_3)$-system of coordinates is the so-called local coordinates of $\mathscr{V}_\Gamma$. The Euclidean metric of $\mathscr{V}_\Gamma$ written in $(\mathsf{x_T}, x_3)$-coordinates is given by the following matrix $(g_{ij})_{i,j=1,2,3}$

$$(g_{ij})_{i,j=1,2,3} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where the coefficient $g_{\alpha\beta}$ equals $g_{\alpha\beta} = \langle \partial_\alpha \Phi, \partial_\beta \Phi \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^3$. Denote by (g^{ij}) the inverse matrix of (g_{ij}), and by g the determinant of (g_{ij}). The coefficients $g_{\alpha\beta}$ might be written with the help of the coefficients of the first, the second and of the third fundamental forms of Γ in the basis $(\partial_1 \psi, \partial_2 \psi)$ (see Do Carmo [11])

$$g_{\alpha\beta}(\mathsf{x_T}, x_3) = g^0_{\alpha\beta}(\mathsf{x_T}) - 2x_3 b_{\alpha\beta}(\mathsf{x_T}) + x_3^2 c_{\alpha\beta}(\mathsf{x_T}).$$

The mean curvature $\mathscr{H}$ of Γ equals

$$\mathscr{H} = -\frac{1}{2} \left. \frac{\partial_3(\sqrt{g})}{\sqrt{g}} \right|_{x_3=0}. \tag{4.2}$$

4.2. *The transmission conditions in local coordinates*

In the $(\mathsf{x_T}, x_3)$-coordinates, write $\mathbb{E} = E_i \mathrm{d}x^i$. N_Γ is the outward normal field of Γ, which is identified to the 1-form $\mathrm{d}x^3$. Applying straightforward the formulas of the Appendix we infer

$$N_\Gamma \wedge \mathbb{E} = E_\alpha \mathrm{d}x^3 \mathrm{d}x^\alpha, \quad \mathsf{int}(N_\Gamma)\mathbb{E} = E_3, \quad \mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E} = (\partial_3 E_\alpha - \partial_\alpha E_3)\mathrm{d}x^\alpha.$$

Hence transmission conditions (2.9) write for $h \in \{0, \varepsilon\}$

$$[E_\alpha]_{x_3=h} = 0, \quad \left[\frac{1}{\mu}(\partial_3 E_\alpha - \partial_\alpha E_3)\right]_{x_3=h} = 0, \quad [q\, E_3]_{x_3=h} = 0. \tag{4.3}$$

4.3. *Rescaling in the thin layer*

Denote by E_j^{e} and by E_j^{c} the respective restrictions of E_j to $\mathscr{V}_{\mathsf{e}}^{\varepsilon}$ and to $\mathscr{V}_{\mathsf{c}}$. In $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ we perform the rescaling $x_3 = \varepsilon\eta$, $\eta \in (0,1)$, and we denote by E_j^{m}, by g_{ij}^{m} and by g^{m} the following functions

$$\forall\, \eta \in (0,1), \quad \begin{cases} \mathcal{E}_j^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = E_j(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta), \\ g_{ij}^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g_{ij}(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta), \quad \text{for } i,j = 1,2,3. \\ g^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta), \end{cases}$$

Observe that $g_{\alpha\beta}^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g_{\alpha\beta}^{0}(\mathbf{x}_{\mathsf{T}}) - 2\varepsilon\eta b_{\alpha\beta}(\mathbf{x}_{\mathsf{T}}) + \varepsilon^2\eta^2 c_{\alpha\beta}(\mathbf{x}_{\mathsf{T}})$, hence for $l \in \mathbb{N}$, $\partial_\eta^l g_{\alpha\beta}^{\mathsf{m}} = O(\varepsilon^l)$, while $\partial_\alpha^l g_{\iota\kappa}^{\mathsf{m}} = O(1)$. Denote by

$$\delta \mathrm{d}\mathbb{E} = a_i^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta)\mathrm{d}x^i, \quad \text{in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}.$$

Applying formula (8.5) with the metric given by (4.1), and performing the rescaling $x_3 = \varepsilon\eta$, we infer,

$$\begin{aligned} a_\lambda^{\mathsf{m}} = &-\frac{1}{\varepsilon^2}\partial_\eta^2 \mathcal{E}_\lambda^{\mathsf{m}} + \frac{1}{\varepsilon}\left(\partial_\eta\partial_\lambda \mathcal{E}_3^{\mathsf{m}} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\frac{\partial_\eta}{\varepsilon}\left(\frac{g_{\alpha\kappa}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\right)\partial_\eta \mathcal{E}_\beta^{\mathsf{m}}\right) \\ &+ \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\left(\partial_\kappa\left(\frac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha \mathcal{E}_\beta^{\mathsf{m}}\right) - \frac{\partial_\eta}{\varepsilon}\left(\frac{g_{\alpha\kappa}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\right)\partial_\beta \mathcal{E}_3^{\mathsf{m}}\right), \end{aligned} \tag{4.4}$$

$$a_3^{\mathsf{m}} = \frac{1}{\varepsilon}\epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\partial_\kappa\left(\frac{g_{\alpha\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\partial_\eta \mathcal{E}_\beta^{\mathsf{m}}\right) + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\partial_\alpha\left(\frac{g_{\beta\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\partial_\kappa \mathcal{E}_3^{\mathsf{m}}\right). \tag{4.5}$$

The divergence-free condition $\delta\mathbb{E}^{\mathsf{m}} = 0$ with equality (8.3) can then be written as:

$$\frac{1}{\varepsilon}\partial_\eta \mathcal{E}_3^{\mathsf{m}} + \frac{1}{\sqrt{g^{\mathsf{m}}}}\frac{\partial_\eta}{\varepsilon}(\sqrt{g^{\mathsf{m}}})\mathcal{E}_3^{\mathsf{m}} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha\left(\frac{g_{\kappa\beta}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\mathcal{E}_\iota^{\mathsf{m}}\right) = 0. \tag{4.6}$$

The transmission conditions (4.3) in $x_3 = \varepsilon$ become

$$\frac{1}{\mu_{\mathsf{e}}}(\partial_3 E_\lambda - \partial_\lambda E_3)|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}\left(\frac{1}{\varepsilon}\partial_\eta \mathcal{E}_\lambda^{\mathsf{m}} - \partial_\lambda \mathcal{E}_3^{\mathsf{m}}\right)\Big|_{\eta=1}, \tag{4.7a}$$

$$E_\lambda|_{x_3=\varepsilon^+} = \mathcal{E}_\lambda^{\mathsf{m}}|_{\eta=1}. \tag{4.7b}$$

The transmission conditions (4.3) in $x_3 = 0$ are:

$$\frac{1}{\mu_{\mathsf{m}}}\left(\frac{1}{\varepsilon}\partial_\eta \mathcal{E}_\lambda^{\mathsf{m}} - \partial_\lambda \mathcal{E}_3^{\mathsf{m}}\right)\Big|_{\eta=0} = \frac{1}{\mu_{\mathsf{c}}}(\partial_3 E_\lambda - \partial_\lambda E_3)|_{x_3=0^-}, \tag{4.8a}$$

$$\mathcal{E}_\lambda^{\mathsf{m}}|_{\eta=0} = E_\lambda|_{x_3=0^-}, \tag{4.8b}$$

and the transmission conditions for the normal components E_3 are

$$q_{\mathsf{e}} E_3|_{x_3=\varepsilon^+} = q_{\mathsf{m}} \mathcal{E}_3^{\mathsf{m}}|_{\eta=1}, \quad q_{\mathsf{m}} \mathcal{E}_3^{\mathsf{m}}|_{\eta=0} = q_{\mathsf{c}} E_3|_{x_3=0^-}. \tag{4.9}$$

5. Ansatz and Formal Expansion

We now set our Ansatz. We look for solutions written as formal series in ε

$$\mathbb{E}|_{\mathcal{O}_\mathsf{e}^\varepsilon} = \tilde{\mathbb{E}}^{\mathsf{e},0}|_{\mathcal{O}_\mathsf{e}^\varepsilon} + \varepsilon\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\mathcal{O}_\mathsf{e}^\varepsilon} + \cdots, \text{ in } \mathcal{O}_\mathsf{e}^\varepsilon, \tag{5.1a}$$

$$\mathbb{E}|_{\mathcal{O}_\mathsf{c}} = \mathbb{E}^{\mathsf{c},0} + \varepsilon\mathbb{E}^{\mathsf{c},1} + \cdots, \text{ in } \mathcal{O}_\mathsf{c}, \tag{5.1b}$$

and in the cylinder $\Gamma \times (0,1)$,

$$\mathbb{E}|_{\mathcal{O}_\mathsf{m}^\varepsilon} \circ \Phi(\mathbf{x}_\mathsf{T}, \varepsilon\eta) = \mathcal{E}^{\mathsf{m},0}(\mathbf{x}_\mathsf{T}, \eta) + \varepsilon\mathcal{E}^{\mathsf{m},1}(\mathbf{x}_\mathsf{T}, \eta) + \cdots, \tag{5.1c}$$

where the 1-forms $(\tilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$, and $(\mathbb{E}^{\mathsf{c},n})_{n\in\mathbb{N}}$ are defined in ε-independent domains. We emphasize that the sequence $(\tilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$ is defined in $(\mathcal{O}_\mathsf{m}^\varepsilon)^\mathbb{N}$ even if its associated series does not approach $\mathbb{E}$ in the thin layer.

Remark 5.1. The 1-forms $(\mathcal{E}^{\mathsf{m},n})_{n\in\mathbb{N}}$ are profiles defined in the cylinder $\Gamma\times(0,1)$; note the difference with the 1-forms $(\mathbb{E}^{\mathsf{c},n})_{n\in\mathbb{N}}$ and $(\tilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$. These profiles are the key-point of the following asymptotic expansion.

In $\mathscr{V}_\Gamma$, for $n \in \mathbb{N}$, we denote by

$$\tilde{\mathbb{E}}^{\mathsf{e},n} = \tilde{E}_i^{\mathsf{e},n}(\mathbf{x}_\mathsf{T}, x_3)\mathrm{d}x^i, \quad \mathbb{E}^{\mathsf{c},n} = E_i^{\mathsf{c},n}(\mathbf{x}_\mathsf{T}, x_3)\mathrm{d}x^i,$$
$$\mathcal{E}^{\mathsf{m},n} = \mathcal{E}_i^{\mathsf{m},n}(\mathbf{x}_\mathsf{T}, \eta)\mathrm{d}x^i, \quad \eta = x_3/\varepsilon.$$

Our aim is to identify the first two terms of the sequences and to estimate the remainder term. Suppose that for $n \in \mathbb{N}$, the forms $(\tilde{E}_k^{\mathsf{e},n})_{k=1,2,3}$ are as regular as necessary. Using formal Taylor expansion, we infer for $l = 0, 1$

$$\partial_j^l \tilde{E}_k^{\mathsf{e},n}|_{x_3=\varepsilon^+} = \partial_j^l \tilde{E}_k^{\mathsf{e},n}|_{x_3=0^+} + \varepsilon\partial_3\partial_j^l \tilde{E}_k^{\mathsf{e},n}|_{x_3=0^+} + \cdots. \tag{5.2}$$

It is convenient to define $\mathbb{E}^n$ for $n \in \mathbb{N}$ by

$$\mathbb{E}^n = \tilde{\mathbb{E}}^{\mathsf{e},n}, \quad \text{in } \mathcal{O}_\mathsf{e}, \qquad \mathbb{E}^n = \mathbb{E}^{\mathsf{c},n}, \quad \text{in } \mathcal{O}_\mathsf{c}.$$

We are now ready to derive formally our asymptotics. Replace the coefficients $(\mathcal{E}_j^\mathsf{m})_{j=1,\ldots,3}$ and $(E_j)_{j=1,\ldots,3}$ in Eqs. (4.4)–(4.6) and in transmission conditions (4.7)–(4.9) by their respective formal expansion (5.1), and use the formal Taylor expansion (5.2). Observe that for any $n \in \mathbb{N}$, we necessarily have

$$\delta\mathrm{d}\mathbb{E}^n - \tilde{\mu}\tilde{q}\mathbb{E}^n = \delta_0^n\,\mathbb{J}, \quad \text{in } \mathcal{O}_\mathsf{e}\cup\mathcal{O}_\mathsf{c}, \qquad N_{\partial\mathcal{O}} \wedge \mathbb{E}^n|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}. \tag{5.3a}$$

$$\text{Observe that} \quad \delta\mathbb{E}^n = 0, \quad \text{in } \mathcal{O}_\mathsf{c}\cup\mathcal{O}_\mathsf{e}, \tag{5.3b}$$

since $\delta\mathbb{J} = 0$. It remains to build the appropriate transmission conditions by identifying the terms with the same power of ε.

5.1. *Order* 0

The term of order -2 in (4.4) vanishes hence $\partial_\eta^2 \mathcal{E}_\alpha^{\mathsf{m},0} = 0$. From the divergence-free condition (4.6) we infer $\partial_\eta \mathcal{E}_3^{\mathsf{m},0} = 0$. Equality (4.7a) implies $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},0} = 0$. Therefore the coefficients $\mathcal{E}_j^{\mathsf{m},0}$ depend only on $\mathbf{x}_\mathsf{T}$. From (4.7b), (4.8b) and (4.9) we infer for $n = 0, 1$

$$\partial_\beta^n \tilde{E}_\alpha^{\mathsf{e},0}|_{x_3=0^+} = \partial_\beta^n E_\alpha^{\mathsf{c},0}|_{x_3=0^-}, \tag{5.4a}$$

$$q_\mathsf{e} \partial_\beta^n \tilde{E}_3^{\mathsf{e},0}|_{x_3=0^+} = q_\mathsf{c} \partial_\beta^n E_3^{\mathsf{c},0}|_{x_3=0^-}. \tag{5.4b}$$

5.2. *Order* 1

Since $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},0}$ and the terms of order -1 in (4.4) vanish, we infer

$$\partial_\eta^2 \mathcal{E}_\alpha^{\mathsf{m},1} = 0. \tag{5.5}$$

Hence $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}$ is constant with respect to η. Therefore, according to (4.7a)

$$\frac{1}{\mu_\mathsf{e}}(\partial_3 \tilde{E}_\alpha^{\mathsf{e},0} - \partial_\alpha \tilde{E}_3^{\mathsf{e},0})|_{x_3=0^+} = \frac{1}{\mu_\mathsf{c}}(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0})|_{x_3=0^-}. \tag{5.6}$$

According to (5.3), (5.4) and (5.6) the 1-forms $\tilde{\mathbb{E}}^{\mathsf{e},0}$ and $\mathbb{E}^{\mathsf{c},0}$ satisfy the elliptic problem (2.10). According to (4.8b) and (4.9), we infer

$$\mathcal{E}_\alpha^{\mathsf{m},0}(\mathbf{x}_\mathsf{T}, \eta) = E_\alpha^{\mathsf{c},0}(\mathbf{x}_\mathsf{T}, 0), \tag{5.7a}$$

$$\mathcal{E}_3^{\mathsf{m},0}(\mathbf{x}_\mathsf{T}, \eta) = \frac{q_\mathsf{c}}{q_\mathsf{m}} E_3^{\mathsf{c},0}(\mathbf{x}_\mathsf{T}, 0). \tag{5.7b}$$

Therefore the terms of order 0 are entirely determined. According to (4.8a), using (5.7) and since $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}$ does not depend on η according to (5.5), we infer

$$\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}(\mathbf{x}_\mathsf{T}, \eta) = \frac{q_\mathsf{c}}{q_\mathsf{m}} \partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} + \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}}(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0})|_{x_3=0^-}. \tag{5.8}$$

The transmission conditions follow

$$\tilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} + \partial_3 \tilde{E}_\alpha^{\mathsf{e},0}|_{x_3=0^+} = \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1} + \mathcal{E}_\alpha^{\mathsf{m},1}|_{\eta=0}$$

and

$$\mathcal{E}_\alpha^{\mathsf{m},1}|_{\eta=0} = E_\alpha^{\mathsf{c},1}|_{x_3=0^-}.$$

Therefore we infer

$$\tilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} - E_\alpha^{\mathsf{c},1}|_{x_3=0^-} = \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1} - \partial_3 \tilde{E}_\alpha^{\mathsf{e},0}|_{x_3=0^+}.$$

Using (5.8) and according to (5.4) and (5.6) we infer

$$\begin{aligned}\tilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} - E_\alpha^{\mathsf{c},1}|_{x_3=0^-} &= \left(\frac{q_\mathsf{c}}{q_\mathsf{m}} - \frac{q_\mathsf{c}}{q_\mathsf{e}}\right) \partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} \\ &\quad + \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}}(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0})|_{x_3=0^-}.\end{aligned} \tag{5.9}$$

The divergence-free condition leads to

$$\partial_\eta \mathcal{E}_3^{\mathsf{m},1} = -\frac{1}{\sqrt{g^0}}\epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\partial_\alpha \left(\frac{g^0_{\kappa\beta}}{\sqrt{g^0}}E_\iota^{\mathsf{c},0}\right)\bigg|_{x_3=0^-} + 2\mathscr{H}\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}}E_3^{\mathsf{c},0}|_{x_3=0^-}, \tag{5.10}$$

where $\mathscr{H}$ is given by (4.2). Transmission condition (4.9) implies

$$q_{\mathsf{e}}\tilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} + q_{\mathsf{e}}\partial_3\tilde{E}_3^{\mathsf{e},0}|_{x_3=0^+} = q_{\mathsf{m}}\partial_\eta\mathcal{E}_3^{\mathsf{m},1} + q_{\mathsf{c}}E_3^{\mathsf{c},1}|_{x_3=0^-}. \tag{5.11}$$

According to (2.10) $\mathbb{E}^{\mathsf{c},0}$ satisfy the divergence-free condition hence

$$-\frac{1}{\sqrt{g^0}}\epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\,\partial_\alpha\left(\frac{g^0_{\kappa\beta}}{\sqrt{g^0}}E_\iota^{\mathsf{c},0}\right)\bigg|_{x_3=0^-} = \partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} - 2\mathscr{H}E_3^{\mathsf{c},0}|_{x_3=0^-}, \tag{5.12}$$

and similarly for $\tilde{\mathbb{E}}^{\mathsf{e},0}$ by replacing $E_i^{\mathsf{c},0}$ by $\tilde{E}_i^{\mathsf{e},0}$. From (5.10)–(5.12), we infer

$$\partial_\eta\mathcal{E}_3^{\mathsf{m},1} = \partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} + 2\mathscr{H}\left(\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} - 1\right)E_3^{\mathsf{c},0}|_{x_3=0^-}. \tag{5.13}$$

Moreover, using (5.4) in (5.12) we infer

$$q_{\mathsf{e}}\partial_3\tilde{E}_3^{\mathsf{e},0}|_{x_3=0^+} = q_{\mathsf{e}}\partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} - 2\mathscr{H}(q_{\mathsf{e}} - q_{\mathsf{c}})E_3^{\mathsf{c},0}|_{x_3=0^-},$$

and therefore (5.11) with equality (4.2) implies

$$q_{\mathsf{e}}\tilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} - q_{\mathsf{c}}E_3^{\mathsf{c},1}|_{x_3=0^-} = (q_{\mathsf{m}} - q_{\mathsf{e}})\,\frac{1}{\sqrt{g|_{x_3=0}}}\,\partial_3(\sqrt{g}\tilde{E}_3^{\mathsf{e},0})\bigg|_{x_3=0^+}. \tag{5.14}$$

5.3. *Order* 2

Since $\partial_\eta\mathcal{E}_\alpha^{\mathsf{m},0} = 0$, we identify the terms in ε^2 in (4.4) to infer

$$\begin{aligned}\partial_\eta^2\mathcal{E}_\lambda^{\mathsf{m},2} = \partial_\eta\partial_\lambda\mathcal{E}_3^{\mathsf{m},1} + \epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\frac{g^0_{\lambda\iota}}{\sqrt{g^0}}\Bigg\{\frac{\partial_\eta}{\varepsilon}\left(\frac{g^{\mathsf{m}}_{\alpha\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\bigg|_{\eta=0}\partial_\eta\mathcal{E}_\beta^{\mathsf{m},1}\\ + \left(\partial_\kappa\left(\frac{1}{\sqrt{g^0}}\partial_\alpha\mathcal{E}_\beta^{\mathsf{m},0}\right) - \frac{\partial_\eta}{\varepsilon}\left(\frac{g^{\mathsf{m}}_{\alpha\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\bigg|_{\eta=0}\partial_\beta\mathcal{E}_3^{\mathsf{m},0}\right)\Bigg\} + \mu_{\mathsf{m}}q_{\mathsf{m}}\sqrt{g^0}\mathcal{E}_\lambda^{\mathsf{m},0}.\end{aligned} \tag{5.15}$$

Since the right-hand side of the previous equality does not depend on η, we have

$$\begin{aligned}\frac{1}{\mu_{\mathsf{e}}}\left(\partial_3\tilde{E}_\lambda^{\mathsf{e},1} - \partial_\lambda\tilde{E}_3^{\mathsf{e},1}\right)\bigg|_{x_3=0^+} - \frac{1}{\mu_{\mathsf{c}}}\left(\partial_3 E_\lambda^{\mathsf{c},1} - \partial_\lambda E_3^{\mathsf{c},1}\right)\bigg|_{x_3=0^-}\\ = \frac{1}{\mu_{\mathsf{m}}}\left(\partial_\eta^2\mathcal{E}_\lambda^{\mathsf{m},2} - \partial_\lambda\partial_\eta\mathcal{E}_3^{\mathsf{m},1}\right) - \frac{1}{\mu_{\mathsf{e}}}\left(\partial_3^2\tilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} - \partial_\lambda\partial_3\tilde{E}_3^{\mathsf{e},0}|_{x_3=0^+}\right).\end{aligned}$$

Since $\delta \mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},0} - \mu_{\mathsf{e}} q_{\mathsf{e}} \tilde{\mathbb{E}}^{\mathsf{e},0} = 0$, explicit formulas of the Appendix imply

$$\partial_3^2 \tilde{E}_j^{\mathsf{e},0}|_{x_3=0^+} = \Bigg[\mu_{\mathsf{e}} q_{\mathsf{e}} \sqrt{g} \tilde{E}_j^{\mathsf{e},0} + \partial_3 \partial_\lambda \tilde{E}_3^{\mathsf{e},0} + \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \frac{g_{\lambda\iota}}{\sqrt{g}} \partial_3 \left(\frac{g_{\alpha\kappa}}{\sqrt{g}} \right) \partial_3 \tilde{E}_\beta^{\mathsf{e},0}$$
$$+ \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \frac{g_{\lambda\iota}}{\sqrt{g}} \left(\partial_\kappa \left(\frac{1}{\sqrt{g}} \partial_\alpha \tilde{E}_\beta^{\mathsf{e},0} \right) - \partial_3 \left(\frac{g_{\alpha\kappa}}{\sqrt{g}} \right) \partial_\alpha \tilde{E}_3^{\mathsf{e},0} \right) \Bigg] \Bigg|_{x_3=0^+} .$$

According to the transmission condition at the order 0, the following equalities hold

$$\frac{1}{\mu_{\mathsf{e}}} (\partial_\lambda \tilde{E}_3^{\mathsf{e},0} - \partial_3 \tilde{E}_\lambda^{\mathsf{e},0})|_{x_3=0^+} = \frac{1}{\mu_{\mathsf{m}}} (\partial_\lambda \mathcal{E}_3^{\mathsf{m},0} - \partial_\eta \mathcal{E}_\lambda^{\mathsf{m},1})|_{\eta=0} , \quad \tilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} = \mathcal{E}_\lambda^{\mathsf{m},0}|_{\eta=0},$$

hence we infer the following transmission conditions

$$\frac{1}{\mu_{\mathsf{e}}} (\partial_3 \tilde{E}_\lambda^{\mathsf{e},1} - \partial_\lambda \tilde{E}_3^{\mathsf{e},1}) - \frac{1}{\mu_{\mathsf{c}}} (\partial_3 E_\lambda^{\mathsf{c},1} - \partial_\lambda E_3^{\mathsf{c},1})$$
$$= (q_{\mathsf{m}} - q_{\mathsf{e}}) \tilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} + \left(\frac{1}{\mu_{\mathsf{m}}} - \frac{1}{\mu_{\mathsf{e}}} \right) \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \frac{g_{\lambda\alpha}}{\sqrt{g}} \partial_\beta \left(\frac{1}{\sqrt{g}} \partial_\iota \tilde{E}_\kappa^{\mathsf{e},0} \right) \Bigg|_{x_3=0^+} . \tag{5.16}$$

Therefore $\mathbb{E}^1$ satisfies (5.3) for $n = 1$ with the transmission conditions (5.9)–(5.16) written in local coordinates. Equalities (5.8)–(5.13) lead to

$$\mathcal{E}_\lambda^{\mathsf{m},1}(\mathbf{x}_{\mathsf{T}}, \eta) = \eta \partial_\eta \mathcal{E}_\lambda^{\mathsf{m},1} + E_\lambda^{\mathsf{c},1}|_{x_3=0^-}, \quad \mathcal{E}_3^{\mathsf{m},1}(\mathbf{x}_{\mathsf{T}}, \eta) = \eta \partial_\eta \mathcal{E}_3^{\mathsf{m},1} + \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} E_3^{\mathsf{c},1}|_{x_3=0^-} .$$

Remark 5.2. The coefficients at order 1 are now uniquely determined. Since

$$\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},2}|_{\eta=0} = \partial_\alpha \mathcal{E}_3^{\mathsf{m},1}|_{\eta=0} - \frac{\mu_{\mathsf{m}}}{\mu_{\mathsf{c}}} (\partial_\alpha E_3^{\mathsf{c},1} - \partial_3 E_\alpha^{\mathsf{c},1})|_{x_3=0^-},$$

$\partial_\eta \mathcal{E}_\lambda^{\mathsf{m},2}$ is uniquely determined by (5.15), namely

$$\partial_\eta \mathcal{E}_\lambda^{\mathsf{m},2} = \eta \partial_\eta^2 \mathcal{E}_\lambda^{\mathsf{m},2} + \partial_\alpha \mathcal{E}_3^{\mathsf{m},1}|_{\eta=0} - \frac{\mu_{\mathsf{m}}}{\mu_{\mathsf{c}}} (\partial_\alpha E_3^{\mathsf{c},1} - \partial_3 E_\alpha^{\mathsf{c},1})|_{x_3=0^-} . \tag{5.17}$$

Remark 5.3. Transmission condition (5.14) might be obtained straightforward from (5.3), (5.9) and (5.16). Writing $\delta \mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},1} = \tilde{a}_i^{\mathsf{e},1} \mathrm{d}x^i$ and $\delta \mathrm{d}\mathbb{E}^{\mathsf{c},1} = a_i^{\mathsf{c},1} \mathrm{d}x^i$ we infer

$$a_3^{\mathsf{c},1} = \frac{1}{\sqrt{g}} \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \partial_\alpha \left(\frac{g_{\beta\kappa}}{\sqrt{g}} \left(\partial_3 E_\iota^{\mathsf{c},1} - \partial_\iota E_3^{\mathsf{c},1} \right) \right),$$

and similarly for $\tilde{a}_3^{\mathsf{e},1}$ by replacing $E^{\mathsf{c},1}$ by $\tilde{E}^{\mathsf{e},1}$. According to (5.16) we have

$$\frac{1}{\mu_{\mathsf{e}}}\tilde{a}_3^{\mathsf{e},1}|_{x_3=0^+} - \frac{1}{\mu_{\mathsf{c}}}a_3^{\mathsf{c},1}|_{x_3=0^-} = \frac{(q_{\mathsf{m}}-q_{\mathsf{e}})}{\sqrt{g}}\epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\partial_\alpha\left(\frac{g_{\beta\kappa}}{\sqrt{g}}\tilde{E}_\iota^{\mathsf{e},1}\right)\Bigg|_{x_3=0^+}.$$

The divergence-free property of $\tilde{\mathbb{E}}^{\mathsf{e},0}$ applied in $x_3 = 0^+$ implies

$$\frac{1}{\mu_{\mathsf{e}}}a_3^{\mathsf{e},1}|_{x_3=0^+} - \frac{1}{\mu_{\mathsf{c}}}a_3^{\mathsf{c},1}|_{x_3=0^-} = -(q_{\mathsf{m}}-q_{\mathsf{e}})\frac{1}{\sqrt{g|_{x_3=0}}}\partial_3(\sqrt{g}\tilde{E}_3^{\mathsf{e},0})|_{x_3=0^+}.$$

Moreover, we have

$$\frac{1}{\mu_{\mathsf{e}}}a_3^{\mathsf{e},1}|_{x_3=0^+} + q_{\mathsf{e}}\tilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} = \frac{1}{\mu_{\mathsf{c}}}a_3^{\mathsf{c},1}|_{x_3=0^-} + q_{\mathsf{c}}E_3^{\mathsf{c},1}|_{x_3=0^-} = 0,$$

therefore, we infer

$$q_{\mathsf{e}}\tilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} - q_{\mathsf{c}}E_3^{\mathsf{c},1}|_{x_3=0^-} = (q_{\mathsf{m}}-q_{\mathsf{e}})\,\frac{1}{\sqrt{g|_{x_3=0}}}\,\partial_3(\sqrt{g}\tilde{E}_3^{\mathsf{e},0})\Bigg|_{x_3=0^+},$$

which is exactly condition (5.14).

6. Justification of the Expansion

Let us rewrite the equations satisfied by the first two terms of the asymptotic expansion of $\mathbb{E}$ in terms of differential forms. Denote by $\mathbb{S}$ and $\mathbb{T}$ the following forms

$$\mathbb{S} = \left((q_{\mathsf{m}}-q_{\mathsf{e}})\tilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} + \left(\frac{1}{\mu_{\mathsf{m}}} - \frac{1}{\mu_{\mathsf{e}}}\right)\epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\frac{g_{\lambda\alpha}}{\sqrt{g}}\partial_\beta\left(\frac{1}{\sqrt{g}}\partial_\iota\tilde{E}_\kappa^{\mathsf{e},0}|_{x_3=0^+}\right)\right)\mathrm{d}x^\lambda,$$

$$\mathbb{T} = \left(\left(\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} - \frac{q_{\mathsf{c}}}{q_{\mathsf{e}}}\right)\partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} + \frac{\mu_{\mathsf{m}}-\mu_{\mathsf{e}}}{\mu_{\mathsf{c}}}(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0})|_{x_3=0^-}\right)\mathrm{d}x^\alpha.$$

The reader easily verifies that the definitions (2.13) and (2.14) coincide with the above expressions of $\mathbb{S}$ and $\mathbb{T}$. The 1-form $\mathbb{E}^0$ satisfies (2.10) in a weak sense and $\mathbb{E}^1$ satisfy (5.3) with the following transmission conditions on Γ according to (5.9)–(5.14)

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_\Gamma)\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = \mathbb{S}, \tag{6.1a}$$

$$N_\Gamma \wedge \tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}. \tag{6.1b}$$

Observe[h] that according to (5.14)

$$\delta\mathbb{S} = -(q_{\mathsf{m}}-q_{\mathsf{e}})\,\frac{1}{\sqrt{g|_{x_3=0}}}\,\partial_3(\sqrt{g}\tilde{E}_3^{\mathsf{e},0})\Bigg|_{x_3=0^+}. \tag{6.2}$$

[h] Since $q_{\mathsf{e}}\,\mathsf{int}(N_\Gamma)\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} = \mathsf{int}(N_\Gamma)((1/\mu_{\mathsf{e}})\delta\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+})$ using (8.1) since $\mathrm{d}N_\Gamma = 0$ we infer $\mathsf{int}(N_\Gamma)((1/\mu_{\mathsf{e}})\delta\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+}) = -\delta((1/\mu_{\mathsf{e}})\mathsf{int}(N_\Gamma)\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+})$, and similarly for $\mathbb{E}^{\mathsf{c},1}$. Therefore according to (6.1a) we infer $q_{\mathsf{e}}\,\mathsf{int}(N_\Gamma)\tilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - q_{\mathsf{c}}\,\mathsf{int}(N_\Gamma)\mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = -\delta\mathbb{S}$, hence (6.2) according to (5.14).

In the cylinder $\Gamma \times (0,1)$, the 1-form $\mathcal{E}^{\mathsf{m},0}$ equals

$$\mathcal{E}^{\mathsf{m},0} = E_\alpha^{\mathsf{c},0}|_{x_3=0^-}\,\mathrm{d}x^\alpha + \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} E_3^{\mathsf{c},0}|_{x_3=0^-}\,\mathrm{d}x^3, \tag{6.3}$$

while the 1-form $\mathcal{E}^{\mathsf{m},1}$ equals

$$\begin{aligned}\mathcal{E}^{\mathsf{m},1} = &\left\{ E_\alpha^{\mathsf{c},1}|_{x_3=0^-} + \eta \left(\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} \partial_\alpha E_3^{\mathsf{c},0} + \frac{\mu_{\mathsf{m}}}{\mu_{\mathsf{c}}} (\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0}) \right)\Bigg|_{x_3=0^-} \right\} \mathrm{d}x^\alpha \\ &+ \left\{ \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} E_3^{\mathsf{c},1}|_{x_3=0^-} + \eta \left(\partial_3 E_3^{\mathsf{c},0} + 2\mathscr{H}\left(\frac{q_{\mathsf{m}}}{q_{\mathsf{c}}} - 1\right) E_3^{\mathsf{c},0} \right)\Bigg|_{x_3=0^-} \right\} \mathrm{d}x^3.\end{aligned} \tag{6.4}$$

6.1. *Regularity results*

We now present the regularity of the 1-forms $\mathbb{E}^0$ and $\mathbb{E}^1$.

Proposition 6.1. *Let Hypothesis 2.2 hold. Moreover, let $s \geq 0$ and $\mathbb{J}$ belong to $H^{1+s}\Omega^1(\mathcal{O}_{\mathsf{e}}^{d_0})$. Then the 1-forms $\mathbb{E}^0$ and $\mathbb{E}^1$ exist and are unique. Moreover, the following regularity results hold*

$$\tilde{\mathbb{E}}^{\mathsf{e},0} \in H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{e}}), \quad \mathbb{E}^{\mathsf{c},0} \in H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{c}}),$$
$$\tilde{\mathbb{E}}^{\mathsf{e},1} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{e}}), \quad \mathbb{E}^{\mathsf{c},1} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{c}}).$$

Proof. All the assertions concerning $\mathbb{E}^0$ are proved in the above Proposition 2.7. Since $\tilde{\mathbb{E}}^{\mathsf{e},0}$ and $\mathbb{E}^{\mathsf{c},0}$ belong respectively to $H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{e}})$ and $H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{c}})$, the forms $\mathbb{S}$ and $\mathbb{T}$ belong to the following Sobolev spaces

$$\mathbb{S} \in H^{1/2+s}\Omega^1(\Gamma), \quad \mathbb{T} \in H^{3/2+s}\Omega^1(\Gamma).$$

Moreover, according to (6.2), $\delta\mathbb{S} \in H^{3/2+s}(\Gamma)$. Let $C \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{c}})$ such that

$$\delta C = 0, \quad \text{in } \mathcal{O}_{\mathsf{c}}, \quad \begin{cases} N_\Gamma \wedge C|_\Gamma = N_\Gamma \wedge \mathbb{T}, \\ \dfrac{1}{\mu_{\mathsf{c}}} \mathsf{int}(N_\Gamma)\mathrm{d}C|_\Gamma = \mathbb{S}, \end{cases} \quad \begin{cases} q_{\mathsf{c}}\,\mathsf{int}(N_\Gamma)C|_\Gamma = \delta\mathbb{S}, \\ \delta(q_{\mathsf{c}} C|_\Gamma) = 0. \end{cases}$$

Observe that $\delta\mathrm{d}C - \mu_{\mathsf{c}} q_{\mathsf{c}} C$ belongs to $H^s\Omega^1(\mathcal{O}_{\mathsf{c}})$. Denote by $\mathbb{U}$ the following 1-form

$$\mathbb{U} = \tilde{\mathbb{E}}^{1,e}, \quad \text{in } \mathcal{O}_{\mathsf{e}}, \qquad \mathbb{U} = \mathbb{E}^{1,c} - C, \quad \text{in } \mathcal{O}_{\mathsf{c}}.$$

Then $\mathbb{U}$ satisfies

$$\begin{aligned}&\delta\mathrm{d}\mathbb{U} - \mu_{\mathsf{e}} q_{\mathsf{e}} \mathbb{U} = 0, \text{ in } \mathcal{O}_{\mathsf{e}}, \\ &\delta\mathrm{d}\mathbb{U} - \mu_{\mathsf{c}} q_{\mathsf{c}} \mathbb{U} = -\delta\mathrm{d}C + \mu_{\mathsf{c}} q_{\mathsf{c}} C, \text{ in } \mathcal{O}_{\mathsf{c}}, \\ &N_{\partial\mathcal{O}} \wedge \mathbb{U}|_{\partial\mathcal{O}} = 0,\end{aligned}$$

with the following homogeneous transmission conditions on Γ

$$[N_\Gamma \wedge \mathbb{U}]_\Gamma = 0, \quad \left[\frac{1}{\tilde{\mu}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{U}\right]_\Gamma = 0, \quad [\tilde{q}\,\mathsf{int}(N_\Gamma)\mathbb{U}]_\Gamma = 0.$$

Arguing as in Proposition 2.7, we infer Proposition 6.1. □

The next proposition gives the regularity of the 1-form $\mathcal{E}^{\mathsf{m},0}$, $\mathcal{E}^{\mathsf{m},1}$ and $\mathcal{E}^{\mathsf{m},2}$. Its proof easily comes from Proposition 6.1 and from the explicit expressions of the components of $\mathcal{E}^{\mathsf{m},n}$, for $n = 0, 1, 2$, given in Sec. 5.

Proposition 6.2. *Let Hypothesis 2.2 hold. Moreover, let $s \geq 0$ and suppose that $\mathbb{J}$ belongs to $H^{1+s}\Omega^1(\mathcal{O}_\mathsf{e}^{d_0})$. By abuse of notations,*[i] *we define $\mathcal{E}^{\mathsf{m},2}$ using (5.17) by*

$$\mathcal{E}^{\mathsf{m},2} = \int_0^{x_3/\varepsilon} \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},2} \mathrm{d}\eta \, \mathrm{d}x^\alpha.$$

Denote by $C^\infty\Omega^1([0,1], H^{5/2+s-n}\Omega^1(\Gamma))$ is the space of the 1-forms, which are smooth in the normal variable η, and which belong to $H^{5/2+s-n}\Omega^1(\Gamma)$ at given $\eta \in [0,1]$. Then for $n = 0, 1, 2$, the profile terms belong to $\mathcal{E}^{\mathsf{m},n} \in C^\infty\Omega^1([0,1], H^{5/2+s-n}\Omega^1(\Gamma))$.

6.2. *Convergence*

Suppose that Hypothesis 2.2 holds, and let the source current density $\mathbb{J}$ belong to $H^3\Omega^1(\mathcal{O}_\mathsf{e}^{d_0})$, with $\delta\mathbb{J} = 0$. It is convenient to define

$$\mathbb{E}_{\text{app}}^{\mathsf{e}} = \tilde{\mathbb{E}}^{\mathsf{e},0} + \varepsilon\tilde{\mathbb{E}}^{\mathsf{e},1}, \quad \text{in } \mathcal{O}_\mathsf{e}^\varepsilon, \qquad \mathbb{E}_{\text{app}}^{\mathsf{c}} = \mathbb{E}^{\mathsf{c},0} + \varepsilon\mathbb{E}^{\mathsf{c},1}, \quad \text{in } \mathcal{O}_\mathsf{c},$$

$$\forall\,(\mathbf{x}_\mathsf{T}, x_3) \in \Gamma \times (0,\varepsilon), \quad \mathbb{E}_{\text{app}}^{\mathsf{m}} \circ \Phi(\mathbf{x}_\mathsf{T}, x_3) = \sum_{n=0}^{2} \varepsilon^n \mathcal{E}^{\mathsf{m},n}(\mathbf{x}_\mathsf{T}, x_3/\varepsilon),$$

and let $\mathbb{E}_{\text{app}}$ equal to $\mathbb{E}_{\text{app}}^{\mathsf{e}}$ in $\mathcal{O}_e^\varepsilon$, $\mathbb{E}_{\text{app}}^{\mathsf{c}}$ in $\mathcal{O}_c$ and to $\mathbb{E}_{\text{app}}^{\mathsf{m}}$ in $\mathcal{O}_m^\varepsilon$. According to the construction of the coefficients $(\mathcal{E}^{\mathsf{m},n})_{n=0,1,2}$ and using Proposition 6.2, there exists a 1-form $\mathbb{G} \in C^\infty\Omega^1([0,1], H^{1/2}\Omega^1(\Gamma))$, such that

$$\delta\mathrm{d}\mathbb{E}_{\text{app}}^{\mathsf{m}} - \mu_\mathsf{m} q_\mathsf{m} \mathbb{E}_{\text{app}}^{\mathsf{m}} = \varepsilon\mathbb{G} \circ \Phi^{-1}, \quad \text{in } \mathcal{O}_\mathsf{m}^\varepsilon,$$

and for an ε-independent constant $C > 0$,

$$\sup_{\eta\in[0,1]} \|\mathbb{G}(\cdot,\eta)\|_{H^{1/2}\Omega^1(\Gamma)} \leq C, \qquad \sup_{\eta\in[0,1]} \|\delta\mathbb{G}(\cdot,\eta)\|_{H^{3/2}(\Gamma)} \leq C.$$

Define $\mathbb{W}$ by $\mathbb{W} = \mathbb{E} - \mathbb{E}_{\text{app}}$ and denote by $\mathbb{W}^\mathsf{e}$, $\mathbb{W}^\mathsf{m}$ and $\mathbb{W}^\mathsf{c}$ the respective restrictions of $\mathbb{W}$ to $\mathcal{O}_\mathsf{e}^\varepsilon$, $\mathcal{O}_\mathsf{m}^\varepsilon$ and $\mathcal{O}_\mathsf{c}$. In local coordinates, $\mathbb{W}^\mathsf{e} = W_i^\mathsf{e}\mathrm{d}x^i$, $\mathbb{W}^\mathsf{m} = W_i^\mathsf{m}\mathrm{d}x^i$ and $\mathbb{W}^\mathsf{c} = W_i^\mathsf{c}\mathrm{d}x^i$. Theorem 2.9 is a straightforward corollary of the following result.

Theorem 6.3. *There exists an ε-independent constant $C > 0$ such that*

$$\|\mathbb{W}^\mathsf{e}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_\mathsf{e}^\varepsilon)} + \sqrt{\varepsilon}\|\mathbb{W}^\mathsf{m}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_\mathsf{m}^\varepsilon)} + \|\mathbb{W}^\mathsf{c}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_\mathsf{c})} \leq C\varepsilon^2.$$

[i]Since $\mathcal{E}^{\mathsf{m},2}$ vanishes in $x_3 = 0$, it is not the third coefficient of the profile in $\Gamma \times (0,1)$.

Proof. The 1-form $\mathbb{W}$ satisfies

$$\delta \mathrm{d}\mathbb{W} - \mu q \mathbb{W} = \varepsilon 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \mathbb{G}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{W}^{\mathsf{e}}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O},$$

with the following transmission conditions for $\mathscr{S} \in \{\Gamma_\varepsilon, \Gamma\}$

$$[N_{\mathscr{S}} \wedge \mathbb{W}]_{\mathscr{S}} = -[N_{\mathscr{S}} \wedge \mathbb{E}_{\mathrm{app}}]_{\mathscr{S}}, \tag{6.5a}$$

$$\left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{W}\right]_{\mathscr{S}} = -\left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{E}_{\mathrm{app}}\right]_{\mathscr{S}}. \tag{6.5b}$$

Let $\mathbb{E}_{\mathrm{app}}^{\mathsf{e}} = E_i^{\mathsf{e},\mathrm{app}} \mathrm{d}x^i$. According to Proposition 6.1, $\mathbb{E}_{\mathrm{app}}^{\mathsf{e}} \in H^4\Omega^1(\mathcal{O}_{\mathsf{e}})$. Hence there exist $f_\alpha \in H^{1/2}(\Gamma)$ and $g_j \in H^{3/2}(\Gamma)$ such that

$$(\partial_3 E_\alpha^{\mathsf{e},\mathrm{app}} - \partial_\alpha E_3^{\mathsf{e},\mathrm{app}})|_{x_3=\varepsilon} = \sum_{l=0,1} \varepsilon^l \partial_3^l (\partial_3 E_\alpha^{\mathsf{e},\mathrm{app}} - \partial_\alpha E_3^{\mathsf{e},\mathrm{app}})|_{x_3=0^+} + \varepsilon^2 f_\alpha,$$

$$E_j^{\mathsf{e},\mathrm{app}}|_{x_3=\varepsilon} = E_j^{\mathsf{e},\mathrm{app}}|_{x_3=0^+} + \varepsilon \partial_3 E_j^{\mathsf{e},\mathrm{app}}|_{x_3=0^+} + \varepsilon^2 g_j.$$

Moreover there exists an ε-independent constant $C > 0$ such that

$$|f_\alpha|_{H^{1/2}(\Gamma)} \leq C, \quad |g_j|_{H^{3/2}(\Gamma)} \leq C. \tag{6.6}$$

After simple calculations involving the explicit expressions of $(\mathcal{E}^{\mathsf{m},n})_{n=0,1,2}$ in local coordinates, transmission conditions (6.5) are written as

$$\begin{aligned}
&\frac{1}{\mu_{\mathsf{e}}}(\partial_3 W_\alpha^{\mathsf{e}} - \partial_\alpha W_3^{\mathsf{e}})|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}(\partial_3 W_\alpha^{\mathsf{m}} - \partial_\alpha W_3^{\mathsf{m}})|_{x_3=\varepsilon^-} + \frac{\varepsilon^2}{\mu_{\mathsf{e}}} f_\alpha,\\
&\frac{1}{\mu_{\mathsf{c}}}(\partial_3 W_\alpha^{\mathsf{c}} - \partial_\alpha W_3^{\mathsf{c}})|_{x_3=0^-} = \frac{1}{\mu_{\mathsf{m}}}(\partial_3 W_\alpha^{\mathsf{m}} - \partial_\alpha W_3^{\mathsf{m}})|_{x_3=0^+},\\
&W_\alpha^{\mathsf{e}}|_{x_3=\varepsilon^+} = W_\alpha^{\mathsf{m}}|_{x_3=\varepsilon^-} + \varepsilon^2 g_\alpha, \quad \text{and} \quad W_\alpha^{\mathsf{c}}|_{x_3=0^-} = W_\alpha^{\mathsf{m}}|_{x_3=0^+}.
\end{aligned}$$

Observe that $\delta\mathbb{W} = -\frac{\varepsilon}{\mu_{\mathsf{m}} q_{\mathsf{m}}} 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \delta\mathbb{G}$, and the following equalities hold

$$q_{\mathsf{e}} W_3^{\mathsf{e}}|_{x_3=\varepsilon^+} = q_{\mathsf{m}} W_3^{\mathsf{m}}|_{x_3=\varepsilon^-} + q_{\mathsf{e}}\varepsilon^2 g_3, \quad q_{\mathsf{c}} W_3^{\mathsf{c}}|_{x_3=0^-} = q_{\mathsf{m}} W_3^{\mathsf{m}}|_{x_3=0^+}.$$

We choose $\mathbb{P} = p_i \mathrm{d}x^i$ in $H^2\Omega^1(\mathcal{O}_{\mathsf{e}}^{\varepsilon})$ such that

$$N_{\partial\mathcal{O}} \wedge \mathbb{P}|_{\partial\mathcal{O}} = 0, \quad \text{and} \quad \mathbb{P}|_{x_3=\varepsilon^+} = g_i(\mathbf{x}_{\mathsf{T}})\mathrm{d}x^i.$$

Since for $\varepsilon \in (0, d_0/2)$, the domain $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ satisfies $\mathcal{O}_{\mathsf{e}} \backslash (\mathscr{V}_\Gamma \cap \mathcal{O}_{\mathsf{e}}) \subset \mathcal{O}_{\mathsf{e}}^{\varepsilon} \subset \mathcal{O}_{\mathsf{e}}$, and according to (6.6), there exists an ε-independent constant $C > 0$ such that

$$\|\mathbb{P}\|_{H^2\Omega^1(\mathcal{O}_{\mathsf{e}}^{\varepsilon})} \leq C.$$

Defining $\tilde{\mathbb{W}} = \mathbb{W} + \varepsilon^2 1_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} \mathbb{P}$, we infer

$$\delta\mathrm{d}\tilde{\mathbb{W}} - \mu q \tilde{\mathbb{W}} = \varepsilon^2 1_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} (\delta\mathrm{d}\mathbb{P} - \mu_{\mathsf{e}} q_{\mathsf{e}} \mathbb{P}) + \varepsilon 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \mathbb{G}, \text{ in } \mathcal{O}, \quad N_{\partial\mathcal{O}} \wedge \tilde{\mathbb{W}}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O},$$

and the following transmission conditions hold

$$\frac{1}{\mu_{\mathsf{e}}}(\partial_3 \tilde{W}^{\mathsf{e}}_\alpha - \partial_\alpha \tilde{W}^{\mathsf{e}}_3)|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}(\partial_3 \tilde{W}^{\mathsf{m}}_\alpha - \partial_\alpha \tilde{W}^{\mathsf{m}}_3)|_{x_3=\varepsilon^-} + \frac{\varepsilon^2}{\mu_{\mathsf{e}}}\widetilde{f_\alpha},$$
$$\frac{1}{\mu_{\mathsf{c}}}(\partial_3 \tilde{W}^{\mathsf{c}}_\alpha - \partial_\alpha \tilde{W}^{\mathsf{c}}_3)|_{x_3=0^-} = \frac{1}{\mu_{\mathsf{m}}}(\partial_3 \tilde{W}^{\mathsf{m}}_\alpha - \partial_\alpha \tilde{W}^{\mathsf{m}}_3)|_{x_3=0^+},$$
$$\tilde{W}^{\mathsf{e}}_\alpha|_{x_3=\varepsilon^+} = \tilde{W}^{\mathsf{m}}_\alpha|_{x_3=\varepsilon^-}, \quad \tilde{W}^{\mathsf{c}}_\alpha|_{x_3=0^-} = \tilde{W}^{\mathsf{m}}_\alpha|_{x_3=0^+},$$

where $\widetilde{f_\alpha} = f_\alpha - (\partial_3 p_\alpha - \partial_\alpha p_3)|_{x_3=\varepsilon^+}$. Moreover,

$$q_{\mathsf{e}}\tilde{W}^{\mathsf{e}}_3|_{x_3=\varepsilon^+} = q_{\mathsf{m}}\tilde{W}^{\mathsf{m}}_3|_{x_3=\varepsilon^-}, \quad q_{\mathsf{c}}\tilde{W}^{\mathsf{c}}_3|_{x_3=0^-} = q_{\mathsf{m}}\tilde{W}^{\mathsf{m}}_3|_{x_3=0^+}.$$

Since the functions $\widetilde{f_\alpha}$ are defined on Γ, it is convenient to define $\widetilde{F_\alpha}$ on Γ_ε by

$$\forall\, \mathbf{x}_{\mathsf{T}} \in \Gamma, \quad \widetilde{F_\alpha} \circ \Phi(\mathbf{x}_{\mathsf{T}}, \varepsilon) = \widetilde{f_\alpha}(\mathbf{x}_{\mathsf{T}}).$$

Denoting by $\tilde{\mathbb{G}}$ and $\tilde{\mathbb{F}}$, the following 1-forms defined by

$$\tilde{\mathbb{G}} = \varepsilon 1_{\mathcal{O}^\varepsilon_{\mathsf{m}}}\mathbb{G} + \varepsilon^2 1_{\mathcal{O}^\varepsilon_{\mathsf{e}}}(\delta \mathrm{d}\mathbb{P} - \mu_{\mathsf{e}} q_{\mathsf{e}}\mathbb{P}), \quad \tilde{\mathbb{F}} = \widetilde{F_\alpha}\mathrm{d}x^\alpha,$$

there exists an ε-independent constant $C > 0$ such that

$$\|\tilde{\mathbb{G}}\|_{L^2\Omega^1(\mathcal{O})} \le C\varepsilon^{3/2}, \quad \|\delta\tilde{\mathbb{G}}\|_{L^2(\mathcal{O})} \le C\varepsilon^{3/2} \quad \text{and} \quad \|\tilde{\mathbb{F}}\|_{H^{-1/2}\Omega^1(\Gamma_\varepsilon)} \le C.$$

The 1-form $\tilde{\mathbb{W}}$ satisfies the following equalities

$$\delta\mathrm{d}\tilde{\mathbb{W}} - \mu q\tilde{\mathbb{W}} = \tilde{\mathbb{G}}, \quad \text{in } \mathcal{O}^\varepsilon_{\mathsf{e}} \cup \mathcal{O}^\varepsilon_{\mathsf{m}} \cup \mathcal{O}_{\mathsf{c}}, \qquad N_{\partial\mathcal{O}} \wedge \tilde{\mathbb{W}}^{\mathsf{e}}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}, \tag{6.7a}$$

with the following transmission conditions on Γ_ε and on Γ

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_{\Gamma_\varepsilon})\mathrm{d}\tilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma^+_\varepsilon} = \frac{1}{\mu_{\mathsf{m}}}\mathsf{int}(N_{\Gamma_\varepsilon})\mathrm{d}\tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^-_\varepsilon} + \frac{\varepsilon^2}{\mu_{\mathsf{e}}}\tilde{\mathbb{F}}, \tag{6.7b}$$

$$\frac{1}{\mu_{\mathsf{m}}}\mathsf{int}(N_\Gamma)\mathrm{d}\tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+} = \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\tilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-}, \tag{6.7c}$$

$$N_{\Gamma_\varepsilon} \wedge \tilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma^+_\varepsilon} = N_{\Gamma_\varepsilon} \wedge \tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^-_\varepsilon}, \quad \text{and} \quad N_\Gamma \wedge \tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+} = N_\Gamma \wedge \tilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-}. \tag{6.7d}$$

Moreover,

$$\delta\tilde{\mathbb{W}} = \frac{1}{\mu q}\delta\tilde{\mathbb{G}}, \quad \text{in } \mathcal{O}^\varepsilon_{\mathsf{e}} \cup \mathcal{O}^\varepsilon_{\mathsf{m}} \cup \mathcal{O}_{\mathsf{c}}, \tag{6.8}$$

and $\tilde{\mathbb{G}}$ and $\tilde{\mathbb{F}}$ are such that

$$q_{\mathsf{e}}\,\mathsf{int}(N_{\Gamma_\varepsilon})\tilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma^+_\varepsilon} = q_{\mathsf{m}}\,\mathsf{int}(N_{\Gamma_\varepsilon})\tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^-_\varepsilon}, \quad q_{\mathsf{c}}\,\mathsf{int}(N_\Gamma)\tilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-} = q_{\mathsf{m}}\,\mathsf{int}(N_\Gamma)\tilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+}.$$

Multiply (6.7) by $\tilde{\mathbb{W}}$ and integrate by parts with the help of (6.8) to infer

$$\|\tilde{\mathbb{W}}^{\mathsf{e}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{e}})} + \sqrt{\varepsilon}\|\tilde{\mathbb{W}}^{\mathsf{m}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{m}})} + \|\tilde{\mathbb{W}}^{\mathsf{c}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \le C\varepsilon^2,$$

for an ε-independent constant C. Morever, $\tilde{\mathbb{W}} = \mathbb{W} + \varepsilon^2 1_{\mathcal{O}^\varepsilon_{\mathsf{e}}}\mathbb{P}$ implies

$$\|\mathbb{W}^{\mathsf{e}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{e}})} + \sqrt{\varepsilon}\|\mathbb{W}^{\mathsf{m}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{m}})} + \|\mathbb{W}^{\mathsf{c}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \le C\varepsilon^2, \tag{6.9}$$

from which we infer Theorem 2.9. □

7. Asymptotic Expansion at any Order

We may extend our derivation principle to obtain asymptotic transmission conditions at any order. Actually, there exists a recurrence formula, which is given in this section. The sketch of the proof of the expansion, which is similar to the proof of Theorem 6.3 is left to the reader. For $(\alpha,\beta,\iota,\kappa)\in\{1,2\}^4$ define the following sequences $(A^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$, $(B^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$, $(C^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$ and $(D^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$ by

$$\begin{cases} A^l_{\alpha\beta\iota\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g^{\mathsf{m}}_{\alpha\iota}}{\sqrt{g^{\mathsf{m}}}}\dfrac{\partial_\eta}{\varepsilon}\left(\dfrac{g^{\mathsf{m}}_{\beta\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}, \\ B^l_{\alpha\beta\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g^{\mathsf{m}}_{\alpha\beta}}{\sqrt{g^{\mathsf{m}}}}\partial_\kappa\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}, \\ C^l_{\alpha\beta} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g_{\alpha\beta}}{g^{\mathsf{m}}}\right)\Bigg|_{\eta=0}. \end{cases} \qquad \begin{cases} D^l = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\dfrac{\partial_\eta}{\varepsilon}\left(\sqrt{g^{\mathsf{m}}}\right)\right)\Bigg|_{\eta=0}, \\ E^l_{\alpha\beta\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha\left(\dfrac{g_{\beta\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}, \end{cases}$$

Using (4.4)–(4.6), for $k\geq 1$ we define $\partial^2_\eta\mathcal{E}^{\mathsf{m},k+2}_\lambda$ and $\partial_\eta\mathcal{E}^{\mathsf{m},k+1}_3$ respectively by

$$\begin{aligned} \partial^2_\eta\mathcal{E}^{\mathsf{m},k+2}_\lambda &= \partial_\eta\partial_\lambda\mathcal{E}^{\mathsf{m},k+1}_3 + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}A^0_{\lambda\alpha\iota\kappa}\partial_\eta\mathcal{E}^{\mathsf{m},k+1}_\beta - \mu_{\mathsf{m}}q_{\mathsf{m}}\mathcal{E}^{\mathsf{m},k}_\lambda \\ &\quad + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\sum_{l=1}^{k}\{(B^l_{\lambda\iota\kappa}\partial_\alpha + C^l_{\lambda\iota}\partial_\kappa\partial_\alpha)\mathcal{E}^{\mathsf{m},k-l}_\beta \\ &\quad + A^l_{\lambda\alpha\iota\kappa}(\partial_\eta\mathcal{E}^{\mathsf{m},k+1-l}_\beta - \partial_\beta\mathcal{E}^{\mathsf{m},k-l}_3)\}, \\ \partial_\eta\mathcal{E}^{\mathsf{m},k+1}_3 &= -\sum_{l=0}^{k}(D^l\mathcal{E}^{\mathsf{m},k-l}_3 + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\left(C^l_{\kappa\beta}\partial_\alpha + E^l_{\alpha\beta\kappa}\right)\mathcal{E}^{\mathsf{m},k-l}_\iota). \end{aligned}$$

Define now the differential forms $\mathbb{S}_{k+1}$ and $\mathbb{T}_{k+1}$ by

$$\begin{aligned} \mathbb{S}_{k+1} &= \Bigg\{\frac{1}{\mu_{\mathsf{m}}}\int_0^1(\partial^2_\eta\mathcal{E}^{\mathsf{m},k+2}_\lambda - \partial_\lambda\partial_\eta\mathcal{E}^{\mathsf{m},k+1}_3)\mathrm{d}\eta \\ &\qquad -\frac{1}{\mu_{\mathsf{e}}}\sum_{l=0}^{k}\partial^l_{x_3}(\partial_3\tilde{\mathbb{E}}^{\mathsf{e},k-l}_\lambda - \partial_\lambda\tilde{\mathbb{E}}^{\mathsf{e},k-l}_3)|_{x_3=0^+}\Bigg\}\mathrm{d}x^\lambda, \\ \mathbb{T}_{k+1} &= \left\{\int_0^1\partial_\eta\mathcal{E}^{\mathsf{m},k+1}_\lambda\mathrm{d}\eta - \sum_{l=0}^{k}\partial^l_{x_3}\tilde{\mathbb{E}}^{\mathsf{e},k-l}_\lambda\right\}\mathrm{d}x^\lambda. \end{aligned}$$

The 1-forms $\tilde{\mathbb{E}}^{\mathsf{e},k+1}$ and $\mathbb{E}^{\mathsf{c},k+1}$ are therefore defined by

$$\begin{aligned} &\delta\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},k+1} - \mu_{\mathsf{e}}q_{\mathsf{e}}\tilde{\mathbb{E}}^{\mathsf{e},k+1} = 0, \quad \text{in } \mathcal{O}_{\mathsf{e}}, \\ &\delta\mathrm{d}\mathbb{E}^{\mathsf{c},k+1} - \mu_{\mathsf{c}}q_{\mathsf{c}}\mathbb{E}^{\mathsf{c},k+1} = 0, \quad \text{in } \mathcal{O}_{\mathsf{c}}, \\ &N_{\partial\mathcal{O}}\wedge\tilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\partial\mathcal{O}} = 0, \end{aligned}$$

with the following transmission conditions on Γ

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_\Gamma)\mathrm{d}\tilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^{\mathsf{c},k+1}|_{\Gamma^-} = \mathbb{S}_{k+1},$$
$$N_\Gamma \wedge \tilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^{\mathsf{c},k+1}|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}_{k+1}.$$

Since for $n = 0, 1$ the 1-forms $(\mathcal{E}^{\mathsf{m},n}, \mathbb{E}^{\mathsf{c},n}, \tilde{\mathbb{E}}^{\mathsf{e},n})_{n=0,1}$ are determined by (2.10)–(6.3)–(6.1)–(6.4), and since $\partial_\eta \mathcal{E}^{\mathsf{m},2}_\lambda$ is also known according to Remark 5.2, the recurrence process is initialized. The reader could prove that outside a neighborhood of $\mathcal{O}^\varepsilon_{\mathsf{m}}$ the following estimate holds $\mathbb{E} = \sum_{k=0}^n \varepsilon^k \mathbb{E}^k + O(\varepsilon^n)$.

Appendix: Explicit Formulas

We refer the reader to [18, 35] for the basic notions of differential calculus for a general compact connected oriented Riemannian manifold $(M, \mathfrak{g})$ of $\mathbb{R}^n$ with smooth compact boundary ∂M. The following property has been used throughout the paper.

Property A.1. (Useful equality) Suppose that M is a compact connected oriented Riemannian manifold without boundary of $\mathbb{R}^n$, and let k be an integer smaller than n. Let ω is a k-form and Y is a smooth 1-form such that $\mathrm{d}Y = 0$. Then applying the above Green formula with the help of the definition of the inner product we infer that for $\omega \in H\Omega^k(\delta, M)$

$$\mathsf{int}(Y)\delta\omega = (-1)^k \delta(\mathsf{int}(Y)\omega). \tag{A.1}$$

Proof. Actually, for any $\eta \in H\Omega^{k-2}(\mathrm{d}, M)$, we have

$$\begin{aligned}
\int_M \langle \mathsf{int}(Y)\delta\omega, \eta\rangle_{\Omega^{k-2}} \,\mathrm{dvol}_M &= \int_M \langle \delta\omega, Y \wedge \eta\rangle_{\Omega^{k-1}} \,\mathrm{dvol}_M \\
&= \int_M \langle \omega, \mathrm{d}(Y \wedge \eta)\rangle_{\Omega^k} \,\mathrm{dvol}_M, \\
&= (-1)^{k-2} \int_M \langle \omega, Y \wedge \mathrm{d}\eta\rangle_{\Omega^k} \,\mathrm{dvol}_M \\
&= (-1)^{k-2} \int_M \langle \mathsf{int}(Y)\omega, \mathrm{d}\eta\rangle_{\Omega^{k-1}} \,\mathrm{dvol}_M \\
&= (-1)^{k-2} \int_M \langle \delta(\mathsf{int}(Y)\omega), \eta\rangle_{\Omega^{k-2}} \,\mathrm{dvol}_M. \qquad \square
\end{aligned}$$

We now present explicit formulas of the differential calculus for a manifold $M \subset \mathbb{R}^3$ endowed with the Euclidean metric. Denote by (x, y, z) the usual Euclidean coordinates of M and let (y_1, y_2, y_3) be another system of coordinates: there exists a C^∞-diffeomorphism ψ such that $\psi(y_1, y_2, y_3) = (x, y, z)$. The Euclidean metric in (y_1, y_2, y_3)-coordinates is given by the matrix $(g_{ij})_{i,j=1,2,3} : g_{ij} = \partial_{y_i}\psi \cdot \partial_{y_j}\psi$, where $\cdot$ denotes the Euclidean scalar product of $\mathbb{R}^3$. The inverse matrix of $(g_{ij})_{ij}$ is denoted by $(g^{ij})_{ij}$ and let g denote by its determinant $g = \det((g_{ij})_{i,j=1,2,3})$.

Denote by $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3)$ the basis of $\Omega^1(M)$ associated to (y_1, y_2, y_3). It is clear that 2-forms $(\mathrm{d}y^2 \wedge \mathrm{d}y^3, \mathrm{d}y^3 \wedge \mathrm{d}y^1, \mathrm{d}y^1 \wedge \mathrm{d}y^2)$ is a basis of $\Omega^2(M)$. Since M is equipped with the Euclidean metric, we perform the change of coordinates $\psi(y_1, y_2, y_3) = (x, y, z)$ to infer that the inner product $\langle \cdot, \cdot \rangle_{\Omega^k}$ for $k = 0, 1, 2$, is determined in (y_1, y_2, y_3)-coordinates by[j] the following equalities

$$\begin{cases} \langle F, G \rangle_{\Omega^0} = FG, \quad \langle \mathrm{d}y^i, \mathrm{d}y^j \rangle_{\Omega^1} = g^{ij}, \\ \langle \mathrm{d}y^i \mathrm{d}y^k, \mathrm{d}y^j \mathrm{d}y^l \rangle_{\Omega^2} = g^{ij} g^{kl} - g^{il} g^{jk}, \\ \langle F \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3, G \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3 \rangle_{\Omega^3} = \dfrac{1}{g} FG, \end{cases} \tag{A.2}$$

where F and G are smooth functions on M, and g is the determinant of (g_{ij}).

- *Exterior products on* $\mathbb{R}^3$*.* The exterior product between a k-form and a l-form equals zero as soon as $k + l > 3$. Moreover, for $k \in \{0, \dots, 3\}$, the exterior product between a 0-form and a k-form is the usual scalar multiplication between a function and a k-form. The following formulas hold (see Flanders [18]).
  - ▷ Exterior product of 1-forms. Let $\lambda = \lambda_i \mathrm{d}y^i$ and $\mu = \mu_i \mathrm{d}y^i$ be two 1-forms, then

    $$\lambda \wedge \mu = \lambda_i \mu_j \mathrm{d}y^i \mathrm{d}y^j = \frac{\epsilon_{ijk}}{2} (\epsilon_{klm} \lambda_l \mu_m) \mathrm{d}y^i \mathrm{d}y^j.$$

  - ▷ Exterior product between a 2-form and a 1-form. Let $\lambda = \frac{\epsilon_{ijk}}{2} \lambda_k \mathrm{d}y^i \mathrm{d}y^j$ and $\mu = \mu_i \mathrm{d}y^i$, then

    $$\lambda \wedge \mu = \lambda_k \mu_k \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3.$$

- *Expression of* d. A straightforward application of the recurrence formula for d given Schwarz [35] implies the following formulas.
  - ▷ d *on* 0*-forms.* Let λ be a 0-form, i.e. λ is a function. Then

    $$\mathrm{d}\lambda = \frac{\partial \lambda}{\partial y_i} \mathrm{d}y^i.$$

  - ▷ d *on* 1*-forms.* Let $\mu = \mu_i \mathrm{d}y^i$, then $\mathrm{d}\mu$ equals

    $$\mathrm{d}\mu = \frac{\partial \mu_j}{\partial y_i} \mathrm{d}y^i \mathrm{d}y^j = \frac{\epsilon_{ijk}}{2} \left(\epsilon_{klm} \frac{\partial \mu_m}{\partial y_l} \right) \mathrm{d}y^i \mathrm{d}y^j.$$

  - ▷ d *on* 2*-forms.* Let $\lambda = \frac{\epsilon_{ijk}}{2} \lambda_k \mathrm{d}y^i \mathrm{d}^j$ be a 2-form, then we have

    $$\mathrm{d}\lambda = \frac{\partial \lambda_k}{\partial y_k} \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3.$$

[j] To simplify notations, we omit the sign $\wedge$ between the differential forms $\mathrm{d}y^i$ and $\mathrm{d}y^j$, for $i, j = 1, 2, 3$.

Proposition A.2. (Star Hodge operator) *Star Hodge operator is defined in* $\mathbb{R}^3$ *by the following formula.*

- *Hodge on functions and* 3*-forms. Let* S *be a* 0*-form and* $T = \tau\, \mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3$ *be a* 3*-form. Then*

$$\star S = \sqrt{g} S\, \mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3, \quad \star T = \frac{1}{\sqrt{g}}\tau.$$

- *Hodge on* 1*-forms. Let* $R = R_i\, \mathrm{d}y^i$ *be a* 1*-form. Then* $\star R$ *is the* 2*-form defined by*

$$\star R = \frac{\epsilon_{ijk}}{2}\sqrt{g} g^{kl} R_l\, \mathrm{d}y^i\mathrm{d}y^j.$$

- *Hodge on* 2*-forms. Let* $S = \frac{\epsilon_{ijk}}{2} S_k\, \mathrm{d}y^i\mathrm{d}y^j$ *be a* 2*-form. Then* $\star S$ *is the* 1*-form equal to*

$$\star S = \frac{1}{\sqrt{g}} g_{ik} S_k\, \mathrm{d}y^i.$$

Proof. If ω is a k-form in $\mathbb{R}^3$, then $\star\omega$ is the $(3-k)$-form such that

$$\forall\, \eta \in \Omega^k(M), \quad \eta \wedge \star\omega = \langle \eta, \omega\rangle_{\Omega^k(M)} \sqrt{g}\, \mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3.$$

Applying the above formulas of the exterior products, and equalities (A.2), we infer the proposition. □

Proposition A.3. (The codifferential operator δ) *According to the codifferential definition* (*see Schwarz* [35]) *the following formulas hold.*

- *Codifferential of* 1*-forms. Let* $\mu = \mu_i \mathrm{d}y^i$, *then*

$$\delta\mu = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}(\sqrt{g} g^{kl}\mu_l). \tag{A.3}$$

- *Codifferential of* 2*-forms. Let* $\lambda = \frac{\epsilon_{ijk}}{2}\lambda_k \mathrm{d}y^i\mathrm{d}y^j$, *then*

$$\delta\lambda = \epsilon_{jkl}\frac{g_{ij}}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\frac{g_{lm}}{\sqrt{g}}\lambda_m\right)\mathrm{d}y^i.$$

Proof. Since the codifferential on k-forms in $\mathbb{R}^3$ is defined by $\delta = (-1)^{3k} \star \mathrm{d}\star$, a straightforward application of the formulas of the differential operator d and the use of Proposition A.2 lead us to the formulas of the codifferential operator. □

Proposition A.3 with the formulas of d differential operator implies the following corollary.

Corollary A.4. ($\delta\mathrm{d}$ and Δ operators on functions and on 1-forms) *Recall that* $\Delta = -(\delta\mathrm{d} + \mathrm{d}\delta)$.

- *Let f be a function. Then*

$$\Delta f = -\delta \mathrm{d} f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_k} \left(\sqrt{g} g^{kl} \frac{\partial}{\partial y_l} f \right). \tag{A.4}$$

- *Let $\lambda = \lambda_i \mathrm{d}y^i$ be a 1-form, then*

$$\delta \mathrm{d}\lambda = \epsilon_{ijk}\epsilon_{lmn} \frac{g_{ri}}{\sqrt{g}} \frac{\partial}{\partial y_j} \left(\frac{g_{kl}}{\sqrt{g}} \frac{\partial}{\partial y_m} \lambda_n \right) \mathrm{d}y^r, \tag{A.5}$$

$$\Delta\lambda = -\left(\epsilon_{ijk}\epsilon_{lmn} \frac{g_{ri}}{\sqrt{g}} \frac{\partial}{\partial y_j} \left(\frac{g_{kl}}{\sqrt{g}} \frac{\partial}{\partial y_m} \lambda_n \right) - \frac{\partial}{\partial y_r} \left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial y_k} (\sqrt{g} g^{kl} \lambda_l) \right) \right) \mathrm{d}y^r. \tag{A.6}$$

Using duality between the interior and the exterior products [35], we infer the following proposition.

Proposition A.5. (Interior product) *Let N be a vector-field identified with the corresponding 1-form $N = N_i \mathrm{d}y^i$.*

- *Interior product of a vector-field on a 1-form. Let $\mu = \mu_i \mathrm{d}y^i$. Then*

$$\mathsf{int}(N)\mu = g^{ij} N_j \mu_i \tag{A.7}$$

- *Interior product of a vector-field on a 2-form. Let $\mu = \mu_{ij}\, \mathrm{d}y^i \mathrm{d}y^j$, then*

$$\mathsf{int}(N)\mu = g_{rl}\mu_{ij} N_k (g^{ik} g^{jl} - g^{il} g^{jk})\, \mathrm{d}y^r. \tag{A.8}$$

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