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ON PSEUDO-SPECTRAL PROBLEMS RELATED TO A TIME-DEPENDENT MODEL IN SUPERCONDUCTIVITY WITH ELECTRIC CURRENT

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This article is initially inspired by a paper of Almog [2] on the effect of the electric current in a problem in superconductivity. Our goal here is to discuss in detail the simplest models which we think are enlightening for understanding the role of the pseudo-spectra in this question and to relate them to recent results obtained together with Almog and Pan. This paper is dedicated to Michelle Schatzman, who has shown her interest for these pseudo-spectral questions by giving a course in 2006 with the french title: "Une visite de la galerie des horreurs: pseudospectres de matrices de Toeplitz".

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1. Introduction

We would like to understand the following problem coming from superconductivity. We consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state. As described for example in our recent book with Fournais [13], this kind of question, without electric fields, can be treated by using accurate semiclassical results on the spectral theory of the Schrödinger operator with magnetic field starting with the analysis of the case with constant magnetic field in the whole space and in the half-space. So it is natural to start an analogous analysis when an electric current is considered.

We will start by describing the motivation of Almog's paper. In the second part, we will present proofs which have some general character and apply in a more physical model, involving for example the non-self-adjoint operator $-\partial_x^2 - (\partial_y - i\frac{x^2}{2})^2 + icy$ ($c \neq 0$) on $\mathbb{R}^2_{x,y}$ or its Dirichlet realization on $\{y > 0\}$, for which we have obtained recently results together with Almog and Pan [3, 4].

After a presentation in the next section of the general problems and of our main results, we will come back to Almog's analysis and will start from a nice "pseudo-spectral" analysis for the complex Airy operator $D_x^2 + ix$ $\big($with $D_x = -i\frac{d}{dx}\big)$ on the line or on $\mathbb{R}^+$ and make a survey of what is known.

A preliminary version of this paper (with presentation of other results) was written for the proceedings of a PDE conference in Evian [16]. Our results are also related to recent results on the Fokker–Planck equation obtained by Helffer, Hérau, Nier ([17] and references therein) or Villani [26].

2. The Model in Superconductivity

2.1. *General context*

The physical problem is posed in a domain Ω with specific boundary conditions. We will only analyze here limiting situations where the domain possibly after a blowing argument becomes the whole space (or the half-space). We will work in dimension 2 for simplification (corresponding to a cylindrical 3D problem). We assume that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample and identified with a function. We denote the Ginzburg–Landau parameter of the superconductor by κ ($\kappa > 0$) and the normal conductivity of the sample by σ. Then the time-dependent Ginzburg–Landau system (also known as the Gorkov–Eliashberg equations) is in $]0, T[\, \times \Omega$:

$$\begin{cases} \partial_t \psi + i\kappa\Phi\psi = \Delta_{\kappa\mathbf{A}}\psi + \kappa^2(1 - |\psi|^2)\psi\,, \\ \kappa^2 \operatorname{curl}^2 \mathbf{A} + \sigma(\partial_t \mathbf{A} + \nabla\Phi) = \kappa \operatorname{Im}(\bar{\psi}\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2 \operatorname{curl}\mathcal{H}^e\,, \end{cases} \tag{2.1}$$

where ψ is the order parameter, $\mathbf{A}$ the magnetic potential, Φ the electric potential, $\nabla_{\kappa\mathbf{A}} = \nabla + i\kappa\mathbf{A}$ and $\Delta_{\kappa\mathbf{A}} = (\nabla + i\kappa\mathbf{A})^2$ is the magnetic Laplacian associated with magnetic potential $\kappa\mathbf{A}$. In addition, $(\psi, \mathbf{A}, \Phi)$ satisfies an initial condition at $t = 0$.

In order to solve this equation, one should also define a gauge (Coulomb, Lorentz, etc.). The orbit of $(\psi, \mathbf{A}, \Phi)$ by the gauge group is

$$\{(\exp(i\kappa q)\psi, \mathbf{A} + \nabla q, \Phi - \partial_t q) \,|\, q \in \mathcal{Q}\},$$

where $\mathcal{Q}$ is a suitable space of regular functions of (t, x, y). We refer to [5] (Paragraph B in the Introduction) for a discussion of this point. We will choose the Coulomb gauge which reads that we can add the condition $\operatorname{div}\mathbf{A} = 0$ for any t. Another possibility could be to take $\operatorname{div}\mathbf{A} + \sigma\Phi = 0$ but this will not be further discussed. As in the analysis of the surface superconductivity, the "normal" solutions will play an important role. We recall that a solution $(\psi, \mathbf{A}, \Phi)$ is called a normal state solution if $\psi = 0$ in the whole sample.

2.2. *Stationary normal solutions*

We now determine the stationary (i.e. time independent) normal solutions of the system. From (2.1), we see that if $(0, \mathbf{A}, \Phi)$ is such a solution, then $(\mathbf{A}, \Phi)$ satisfies the system

$$\kappa^2 \operatorname{curl}(\operatorname{curl} \mathbf{A}) + \sigma \nabla \Phi = \kappa^2 \operatorname{curl} \mathcal{H}^e, \quad \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega. \tag{2.2}$$

Note that, identifying $\mathcal{H}^e$ with a function h, $\operatorname{curl} \mathcal{H}^e = (-\partial_y h, \partial_x h)$. Interpreting these two equations as the Cauchy–Riemann equations, this can be rewritten (in addition to the divergence free condition) as the property that

$$\kappa^2(\operatorname{curl} \mathbf{A} - \mathcal{H}^e) + i\sigma\Phi,$$

is holomorphic function in Ω. In particular, if $\sigma \neq 0$, Φ and $\operatorname{curl} \mathbf{A} - \mathcal{H}^e$ are harmonic.

Special situation: Φ affine

As simple natural example, we observe that, if $\Omega = \mathbb{R}^2$, (2.1) has the following stationary normal state solution

$$\mathbf{A} = \frac{1}{2J}(Jx + h)^2 \hat{\imath}_y, \quad \Phi = \frac{\kappa^2 J}{\sigma} y. \tag{2.3}$$

Note that

$$\operatorname{curl} \mathbf{A} = (Jx + h)\hat{\imath}_z,$$

that is, the induced magnetic field equals the sum of the applied magnetic field $h\hat{\imath}_z$ and the magnetic field produced by the electric current $Jx\hat{\imath}_z$.

For this normal state solution, the linearization of (2.1) with respect to the order parameter is

$$\partial_t \psi + \frac{i\kappa^3 J y}{\sigma}\psi = \Delta\psi + \frac{i\kappa}{J}(Jx + h)^2 \partial_y \psi - \left(\frac{\kappa}{2J}\right)^2 (Jx + h)^4 \psi + \kappa^2 \psi. \tag{2.4}$$

Applying the transformation $x \to x - h/J$ and taking for simplification $\kappa = 1$, the time-dependent linearized Ginzburg–Landau equation takes the form

$$\frac{\partial \psi}{\partial t} + i\frac{J}{\sigma} y \psi = \Delta\psi + iJx^2 \frac{\partial \psi}{\partial y} - \left(\frac{1}{4}J^2 x^4 - 1\right)\psi. \tag{2.5}$$

Rescaling x and t by applying

$$t \to J^{2/3} t, \quad (x, y) \to J^{1/3}(x, y), \tag{2.6}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u, \tag{2.7}$$

where, with $D_x = -i\partial_x, D_y = -i\partial_y$,

$$\mathcal{A}_{0,c} := D_x^2 + \left(D_y + \frac{1}{2}x^2\right)^2 + icy \tag{2.8}$$

and

$$c = 1/\sigma, \quad \lambda = \frac{1}{J^{2/3}}, \quad u(x, y, t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t).$$

Our main problem will be to analyze the long time property of the attached semigroup.

We now apply the transformation

$$u \to u\, e^{icyt}$$

to obtain

$$\partial_t u = -\left(D_x^2 u + \left(D_y + \frac{1}{2} x^2 - ct \right)^2 u - \lambda u \right). \tag{2.9}$$

Note that by considering the partial Fourier transform with respect to the y variable, we obtain for the Fourier transform $\hat{u}$ of u:

$$\partial_t \hat{u} = -D_x^2 \hat{u} - \left[\left(\frac{1}{2} x^2 + (-ct + \omega) \right)^2 - \lambda \right] \hat{u}. \tag{2.10}$$

This can be rewritten as the analysis of a family (depending on $\omega \in \mathbb{R}$) of time-dependent problems on the line

$$\partial_t \hat{u} = -\mathcal{M}_{\beta(t,\omega)} \hat{u} + \lambda \hat{u}, \tag{2.11}$$

with $\mathcal{M}_\beta$ being the well-known anharmonic oscillator (also called the Montgomery operator in other contexts [15] and references therein):

$$\mathcal{M}_\beta = D_x^2 + \left(\frac{1}{2} x^2 + \beta \right)^2 \tag{2.12}$$

and

$$\beta(t, \omega) = -ct + \omega.$$

2.3. *Recent results by Almog–Helffer–Pan* [3]

The main point concerning the previously defined operator $\overline{\mathcal{A}_{0,c}}$ is to obtain an optimal control of the decay of the associated semigroup as $t \to +\infty$.

Theorem 2.1. *If* $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ *has compact resolvent, empty spectrum, and there exists* $C > 0$ *such that*

$$\|\exp(-t\mathcal{A})\| \leq \exp\left(-\frac{2\sqrt{2c}}{3} t^{3/2} + C t^{3/4} \right), \tag{2.13}$$

for any $t \geq 1$ *and*

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \exp\left(\frac{1}{6c} (\operatorname{Re} \lambda)^3 + C\, (\operatorname{Re} \lambda)^{3/2} \right), \tag{2.14}$$

for all λ *such that* $\operatorname{Re} \lambda \geq 1$.

Here a semiclassical analysis of the operator $\mathcal{M}_\beta$ as $|\beta| \to \pm\infty$ plays an important role. We refer to [3] for details and to [14] for the involved semiclassical analysis.

If we consider instead the Dirichlet realization $\mathcal{A}_c^D$ of $\mathcal{A}_{0,c}$ in $\{y > 0\}$, it is easily proven that $\mathcal{A}_c^D$ has compact resolvent if $c \neq 0$. We prove in [4] that if the spectrum of $\mathcal{A}_c^D$ is not empty, then the decay of the semigroup $\exp -t\mathcal{A}_c^D$ is exponential with a rate corresponding to $\inf_{z\in\sigma(\mathcal{A}_c^D)} \operatorname{Re} z$. We will explain the argument in the case of a simpler model: the complex Airy operator. We also conjecture in [4] that $\sigma(\mathcal{A}_c^D)$ is not empty and we give a proof of the statement for $|c|$ large enough.

3. A Simplified Model: No Magnetic Field

We assume, following Almog [2], that a current of constant magnitude J is being flown through the sample in the x axis direction, and that there is no applied magnetic field: $h = 0$. Then (2.1) has (in some asymptotic regime) the following stationary normal state solution

$$\mathbf{A} = 0, \quad \Phi = Jx. \tag{3.1}$$

For this normal state solution, the linearization of (2.1) gives

$$\partial_t \psi + iJx\psi = \Delta_{x,y}\psi + \psi, \tag{3.2}$$

whose analysis is (see ahead) strongly related to the Airy equation.

3.1. *The complex Airy operator in* $\mathbb{R}$

This operator can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$ $\mathcal{A}_0^+ := D_x^2 + ix$. We observe that $\mathcal{A} = (\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^- := D_x^2 - ix$ and that its domain (see [21]) is

$$D(\mathcal{A}) = \{u \in H^2(\mathbb{R}), xu \in L^2(\mathbb{R})\}.$$

In particular, $\mathcal{A}$ has compact resolvent.

It is also easy to see that

$$\operatorname{Re} \langle \mathcal{A}u | u \rangle \geq 0. \tag{3.3}$$

Hence $-\mathcal{A}$ is the generator of a semigroup S_t of contraction,

$$S_t = \exp -t\mathcal{A}. \tag{3.4}$$

Hence all the results of this theory can be applied.

In particular, we have, for $\operatorname{Re}\lambda < 0$

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re}\lambda|}. \tag{3.5}$$

One can also show that the operator is maximally accretive. We can, for example, use the following criterion, which extends the standard criterion of essential self-adjointness:

Theorem 3.1. *For an accretive operator $\mathcal{A}_0$, the following conditions are equivalent for its closed extension $\mathcal{A} = \overline{\mathcal{A}_0}$:*

(i) *$\mathcal{A}$ is maximally accretive.*
(ii) *There exists $\lambda_0 > 0$ such that $\mathcal{A}_0^* + \lambda_0 I$ is injective.*

In our case it is immediate to verify that $D_x^2 - ix + \lambda$ is actually injective in $\mathcal{S}'(\mathbb{R})$ (take the Fourier transform).

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a \mathcal{A} = (\mathcal{A} - ia) T_a, \tag{3.6}$$

where T_a is the translation operator $(T_a u)(x) = u(x-a)$.

As immediate consequence, we obtain that the spectrum is empty and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$\|(\mathcal{A} - \lambda)^{-1}\| = \|(\mathcal{A} - \operatorname{Re}\lambda)^{-1}\|. \tag{3.7}$$

One can also look at the semiclassical question, i.e. consider the operator

$$\mathcal{A}_h = h^2 D_x^2 + ix, \tag{3.8}$$

and observe that it is the toy model for some results of Dencker–Sjöstrand–Zworski [11]. The symbol is $(x,\xi) \mapsto p(x,\xi) = \xi^2 + ix$ and microlocally at $(0,0)$, we have $\{\operatorname{Re} p, \operatorname{Im} p\}(0,0) = 0$ and $\{\operatorname{Im} p, \{\operatorname{Re} p, \operatorname{Im} p\}\}(0,0) \neq 0$.

Of course in such a homogeneous situation one can go from one point of view to the other but it is sometimes good to look at what each theory gives on this very particular model. We refer for example to the lectures by Sjöstrand [23].

The most interesting property is the control of the resolvent for $\operatorname{Re}\lambda \geq 0$.

Proposition 3.1. (W. Bordeaux-Montrieux [6]) *As* $\operatorname{Re}\lambda \to +\infty$, *we have*

$$\|(A - \lambda)^{-1}\| \sim \sqrt{\frac{\pi}{2}} (\operatorname{Re}\lambda)^{-\frac{1}{4}} \exp \frac{4}{3} (\operatorname{Re}\lambda)^{\frac{3}{2}}. \tag{3.9}$$

This improves a previous result by Martinet [21]. The proof of the (rather standard) upper bound is based on the direct analysis of the semigroup in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + ix)\mathcal{F}^{-1} = \xi^2 - \frac{d}{d\xi}. \tag{3.10}$$

Then we have

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp\left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) v(\xi + t), \tag{3.11}$$

and this implies immediately

$$\|S_t\| = \exp \max_{\xi} \left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) = \exp\left(-\frac{t^3}{12}\right). \tag{3.12}$$

Then one can get an estimate of the resolvent by using, for $\lambda \in \mathbb{C}$, the formula

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A} - \lambda)\, dt. \tag{3.13}$$

The right-hand side can be estimated using (3.12) and the Laplace method.

For a closed accretive operator, (3.13) is standard when $\operatorname{Re}\lambda < 0$, but estimate (3.12) on S_t gives immediately a holomorphic extension of the right-hand side to the whole space, showing independently that the spectrum is empty (see Davies [8]) and giving for $\lambda > 0$ the estimate

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \int_0^{+\infty} \exp\left(\lambda t - \frac{t^3}{12}\right) dt. \tag{3.14}$$

The asymptotic behavior as $\lambda \to +\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t = \lambda^{\frac{1}{2}} s$ in the integral.

The proof by [21] of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A}-\lambda)$ in its Fourier representation. We observe (assuming $\lambda > 0$), that

$$\xi \mapsto u(\xi;\lambda) := \exp\left(\frac{\xi^3}{3} - \lambda\xi - \frac{2}{3}\lambda^{\frac{3}{2}}\right) \tag{3.15}$$

is a solution of

$$\left(-\frac{d}{d\xi} + \xi^2 - \lambda\right) u(\xi;\lambda) = 0. \tag{3.16}$$

Multiplying $u(\cdot;\lambda)$ by a cutoff function χ_λ with support in $]-\infty, \sqrt{\lambda}[$ and $\chi_\lambda = 1$ on $]-\infty, \sqrt{\lambda}-1[$, we obtain a very good quasimode, concentrated as $\lambda \to +\infty$, around $-\sqrt{\lambda}$, satisfying

$$\left(-\frac{d}{d\xi} + \xi^2 - \lambda\right) (\chi_\lambda u(\xi;\lambda)) = r_\lambda(\xi), \tag{3.17}$$

where r_λ is with support in $[\sqrt{\lambda}-1, \sqrt{\lambda}]$ and of order $e^{-\frac{4}{3}\lambda^{\frac{3}{2}}}$. This is not far of giving the announced lower bound for the resolvent. The proof by Bordeaux-Montrieux is by introducing a Grushin's problem.

Of course this is a very special case of a result on the pseudo-spectra but this leads to an almost optimal result.

4. Pseudo-Spectra and Semigroups

We arrive now at the analysis of the properties of a contraction semigroup $\exp -t\mathcal{A}$, with $\mathcal{A}$ maximally accretive. As before, we have, for $\operatorname{Re}\lambda < 0$,

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re}\lambda|}. \tag{4.1}$$

If we add the assumption that $\operatorname{Im}\langle \mathcal{A}u, u\rangle \geq 0$ for all u in the domain of $\mathcal{A}$, one gets also for $\operatorname{Im}\lambda < 0$

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \frac{1}{|\operatorname{Im}\lambda|}, \tag{4.2}$$

so the main remaining question is the analysis of the resolvent in the set $\operatorname{Re}\lambda \geq 0, \operatorname{Im}\lambda \geq 0$, which corresponds to the numerical range of the operator.

We recall that, for any $\epsilon > 0$, we define the ϵ-pseudo-spectra by

$$\Sigma_\epsilon(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,|\, \|(\mathcal{A} - \lambda)^{-1}\| > \frac{1}{\epsilon} \right\}, \tag{4.3}$$

with the convention that $\|(\mathcal{A} - \lambda)^{-1}\| = +\infty$ if $\lambda \in \sigma(\mathcal{A})$.

We have

$$\cap_{\epsilon>0} \Sigma_\epsilon(\mathcal{A}) = \sigma(\mathcal{A}). \tag{4.4}$$

We define, for any $\epsilon > 0$, the ϵ-pseudo-spectral abscissa by

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \operatorname{Re} z \tag{4.5}$$

and the growth bound of $\mathcal{A}$ by

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log \|\exp -t\mathcal{A}\|. \tag{4.6}$$

Of course, by considering eigenfunctions, we have

$$-\widehat{\omega}_0(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \operatorname{Re} z, \tag{4.7}$$

but the equality is wrong in general. The right behavior is given by:

Theorem 4.1. (Gearhart–Prüss) *Let $\mathcal{A}$ be a densely defined closed operator in a Hilbert space X such that $-\mathcal{A}$ generates a contraction semigroup, then*

$$\lim_{\epsilon \to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A}). \tag{4.8}$$

We refer to [12] for a proof and to [18] for a more quantitative version of this theorem.

5. The Complex Airy Operator in $\mathbb{R}^+$

5.1. *Spectral analysis*

Here we mainly describe some results presented in [2], who refers to [20]. We consider the Dirichlet realization $\mathcal{A}^D$ of the complex Airy operator $D_x^2 + ix$ on the half-line, whose domain is

$$D(\mathcal{A}^D) = \{u \in H_0^1(\mathbb{R}^+), x^{\frac{1}{2}} u \in L^2(\mathbb{R}^+), (D_x^2 + ix)u \in L^2(\mathbb{R}^+)\}, \tag{5.1}$$

and which is defined (in the sense of distributions) by

$$\mathcal{A}^D u = (D_x^2 + ix)u. \tag{5.2}$$

Moreover, by construction, we have

$$\operatorname{Re} \langle \mathcal{A}^D u \,|\, u \rangle \geq 0, \quad \forall\, u \in D(\mathcal{A}^D). \tag{5.3}$$

Again we have an operator, which is the generator of a semigroup of contraction, whose adjoint is described by replacing in the previous description $(D_x^2 + ix)$ by $(D_x^2 - ix)$, the operator is injective and as its spectrum contained in $\operatorname{Re} \lambda > 0$. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([2]) that the spectrum of $\mathcal{A}^D$ is given by

$$\sigma(\mathcal{A}^D) = \bigcup_{j=1}^{+\infty} \{\lambda_j\} \tag{5.4}$$

with

$$\lambda_j = -\left(\exp i\frac{\pi}{3}\right)\mu_j, \tag{5.5}$$

the μ_j's being real zeros of the Airy function satisfying

$$0 > \mu_1 > \cdots > \mu_j > \mu_{j+1} > \cdots. \tag{5.6}$$

As can be recovered by Weyl's formula, there exists a constant $c \neq 0$ such that $\mu_j \sim cj^{\frac{2}{3}}$. It is also in [2] that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$. But there is no way to normalize these eigenfunctions for getting a good basis of $L^2(\mathbb{R}^+)$. See Almog [2], Davies [9] and Henry [19] who show that the norm of the spectral projector π_n associated with the nth eigenvalue increases exponentially like $\exp \alpha n$ for some $\alpha > 0$. Following Davies [9], we say in this case that $\mathcal{A}^D$ is spectrally wild.

5.2. *Decay of the semigroup*

We now apply Gearhart–Prüss theorem to our operator $\mathcal{A}^D$ and our main theorem is:

Theorem 5.1.

$$\widehat{\omega}_0(\mathcal{A}^D) = -\operatorname{Re}\lambda_1 = \mu_1/2. \tag{5.7}$$

This statement was established by Almog [2] in a much weaker form. Using the first eigenfunction it is easy to see that

$$\|\exp -t\mathcal{A}^D\| \geq \exp -\operatorname{Re}\lambda_1 t. \tag{5.8}$$

Hence we immediately have

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\operatorname{Re}\lambda_1. \tag{5.9}$$

To prove that $-\operatorname{Re}\lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$, it is enough to show the following lemma.

Lemma 5.1. *For any* $\alpha < \operatorname{Re}\lambda_1$, *there exists a constant* C *such that, for all* λ s.t. $\operatorname{Re}\lambda \leq \alpha$

$$\|(\mathcal{A}^D - \lambda)^{-1}\| \leq C. \tag{5.10}$$

Proof. We know that λ is not in the spectrum. Hence the problem is just a control of the resolvent as $|\operatorname{Im}\lambda| \to +\infty$. The case when $\operatorname{Im}\lambda < 0$ has already be considered. Hence it remains to control the norm of the resolvent as $\operatorname{Im}\lambda \to +\infty$ and $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$. □

This is indeed a semiclassical result.[a] The main idea is that when $\operatorname{Im}\lambda \to +\infty$, we have to inverse the operator

$$D_x^2 + i(x - \operatorname{Im}\lambda) - \operatorname{Re}\lambda .$$

If we consider the Dirichlet realization in the interval $]0, \frac{\operatorname{Im}\lambda}{2}[$ of $D_x^2 + i(x - \operatorname{Im}\lambda) - \operatorname{Re}\lambda$, it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse $R_1(\lambda)$ satisfies

$$\|R_1(\lambda)\| \leq \frac{2}{\operatorname{Im}\lambda} .$$

Far from the boundary, we can use the resolvent $R(\lambda)$ of the problem on the line for which we have a uniform control of the norm for $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$.

More precisely (see [4] for details) we approximate the resolvent by

$$\phi_{1,\lambda} R_1(\lambda) \phi_{1,\lambda} + \phi_{2,\lambda} R(\lambda) \phi_{2,\lambda} ,$$

where $\phi_{j,\lambda}(x) = \phi_j(\frac{x}{\operatorname{Im}\lambda})$ (for $j = 1, 2$), $\operatorname{Supp}\phi_1 \subset]-\infty, \frac{1}{2}[$, $\operatorname{Supp}\phi_2 \subset]\frac{1}{4}, +\infty[$ and $\phi_1^2 + \phi_2^2 = 1$ on $[0, +\infty[$.

5.3. *Physical interpretation*

Coming back to the application in superconductivity, one is looking at the semigroup associated with $\mathcal{A}_J := D_x^2 + iJx - 1$ (where $J \geq 0$ is a parameter). The stability analysis leads to a critical value

$$J_c = (\operatorname{Re}\lambda_1)^{-\frac{3}{2}} , \tag{5.11}$$

such that:

- For $J \in [0, J_c[$, $\| \exp -t\mathcal{A}_J \| \to +\infty$ as $t \to +\infty$.
- For $J > J_c$, $\| \exp -t\mathcal{A}_J \| \to 0$ as $t \to +\infty$.

This improves Lemma 2.4 in Almog [2], who gets only this decay for $\|\exp -t\mathcal{A}_J \psi\|$, with ψ in a specific dense subspace in $L^2(\mathbb{R}^+)$.

6. Numerical Computations

There is a classical picture due to Trefethen of the pseudo-spectra of the Davies operator $D_x^2 + ix^2$ on the line. Our picture gives the corresponding one realized for the case of the complex Airy operator by Bordeaux-Montrieux, who has used eigtool.[b] The picture gives the level-curves $\mathcal{C}(\epsilon)$ of the norm of the resolvent $\|(A - z)^{-1}\| = \frac{1}{\epsilon}$ corresponding to the boundary of the ϵ-pseudo-spectra. The right column gives the correspondence between the color and $\log_{10}(\epsilon)$.

[a] After a dilation the operator becomes $\operatorname{Im}\lambda \left(h^2 D_x^2 + i(x-1) - \frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda} \right)$ with $h = |\operatorname{Im}\lambda|^{-\frac{3}{2}}$.

[b] see http://www-pnp.physics.ox.ac.uk/~stokes/courses/scicomp/eigtool/html/eigtool/documentation/menus/airy-demo.html and http://www.comlab.ox.ac.uk/pseudospectra/eigtool/.

As usual for this kind of computation for non-self-adjoint operators, we observe, in addition to the (discrete) spectrum lying on the half-line of argument $\frac{\pi}{4}$ (respectively $\frac{\pi}{3}$), an unexpected spectrum starting from the fifteenth eigenvalue. This was already observed by Davies for the complex harmonic oscillator $D_x^2 + ix^2$. This is immediately connected with the accuracy of the computations of Maple.

If we consider the level curves $\mathcal{C}(\epsilon)$, Bordeaux-Montrieux [6] gets for the Davies operator

$$\operatorname{Im} z = (1 + o(1)) \left(\frac{3}{2}\right)^{\frac{2}{3}} (\operatorname{Re} z)^{\frac{1}{3}} \left(\ln \frac{(\operatorname{Re} z)^{\frac{1}{3}}}{\epsilon}\right)^{\frac{2}{3}}$$

as $\operatorname{Re} z \to +\infty$.

The computation for the picture is done on an interval $[0, L]$ with Dirichlet conditions at 0 and L using 400 "grid points". The figure gives the level-curves of the norm of the resolvent $\|(A - z)^{-1}\| = \frac{1}{\epsilon}$ corresponding for each ϵ to the boundary of the ϵ-pseudo-spectrum. The right column gives the correspondence between the color and $\log_{10}(\epsilon)$.

In the upper part of the Airy-picture in Fig. 1, these level-curves become asymptotically vertical lines corresponding to the fact that each ϵ-pseudo-spectrum of the Airy operator is a left-bounded half-plane.

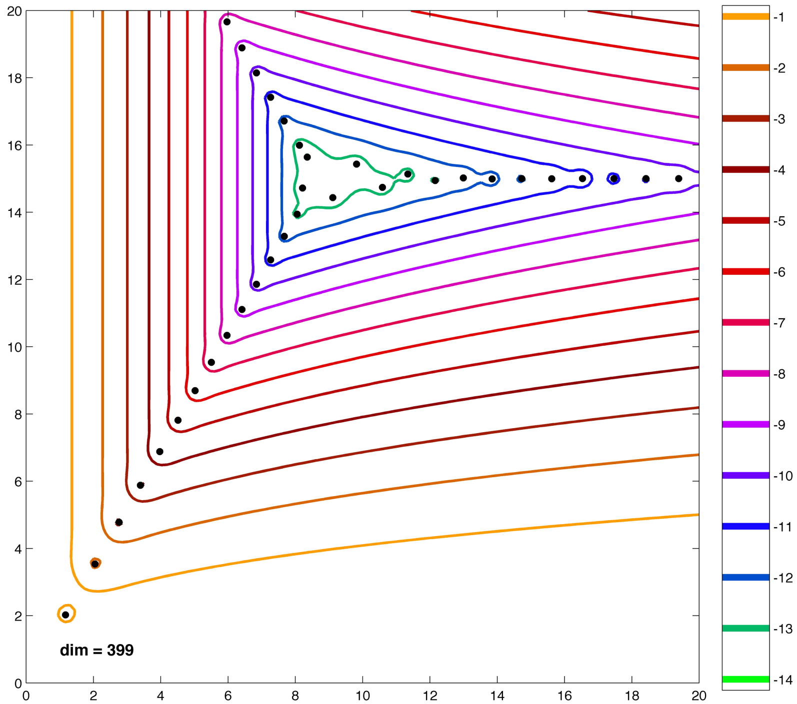

Fig. 1. Pseudo-spectra of the Airy complex operator with Dirichlet condition. (The original, color version of this picture is visible at http://www.math.u-psud.fr/helffer/pseudoairy.jpg.)

A more accurate analysis of the figure also shows that for $\epsilon = 10^{-1}$, the ϵ-pseudo-spectrum has two components, the bounded one containing the first eigenvalue. For $\epsilon = 10^{-2}$, the ϵ-pseudo-spectrum has three components, each bounded one containing one eigenvalue. Note also the property that, for a given k, as $\epsilon \to 0$, the component of the ϵ-pseudo-spectrum containing one eigenvalue μ_k becomes asymptotically a disk centered at μ_k.

7. Higher Dimension Problems Relative to Airy

Here we follow (and extend) Almog [2].

7.1. *The model in* $\mathbb{R}^2$

We consider the operator

$$\mathcal{A}_2 := -\Delta_{x,y} + ix. \tag{7.1}$$

Proposition 7.1.

$$\sigma(\mathcal{A}_2) = \emptyset. \tag{7.2}$$

Proof. After a Fourier transform in the y variable, it is enough to show that

$$(\widehat{\mathcal{A}_2} - \lambda)$$

is invertible with

$$\widehat{\mathcal{A}_2} = D_x^2 + ix + \eta^2. \tag{7.3}$$

We just have to control, for a given $\lambda \in \mathbb{C}$, the norm of $(D_x^2 + ix + \eta^2 - \lambda)^{-1}$ (whose existence is given by the 1D result) in $\mathcal{L}(L^2(\mathbb{R}))$, uniformly with respect to η. □

7.2. *The model in* $\mathbb{R}^2_+$*: perpendicular current*

Here it is useful to reintroduce the parameter J, which is assumed to be positive. Hence we consider the Dirichlet realization

$$\mathcal{A}_2^{D,\perp} := -\Delta_{x,y} + iJx \tag{7.4}$$

in $\mathbb{R}^2_+ = \{x > 0\}$.

Proposition 7.2.

$$\sigma(\mathcal{A}_2^{D,\perp}) = \cup_{r\geq 0, j\in\mathbb{N}^*} (\lambda_j + r), \tag{7.5}$$

where the λ_j*'s have been introduced in* (5.5).

Proof. For the inclusion

$$\bigcup_{r\geq 0, j\in\mathbb{N}^*} (\lambda_j + r) \subset \sigma(\mathcal{A}_2^{D,\perp}),$$

we can use L^∞ eigenfunctions in the form

$$(x, y) \mapsto u_j(x) \exp iy\eta,$$

where u_j is the eigenfunction associated to λ_j. We have then use the fact that L^∞-eigenvalues belong to the spectrum. This can be formulated in the following proposition. □

Proposition 7.3. *Let $\Psi \in L^\infty(\mathbb{R}^2_+) \cap H^1_{\rm loc}(\overline{\mathbb{R}^2_+})$ satisfying, for some $\lambda \in \mathbb{C}$,*

$$-\Delta_{x,y}\Psi + iJx\Psi = \lambda\Psi \tag{7.6}$$

in $\mathbb{R}^2_+$ and

$$\Psi_{x=0} = 0. \tag{7.7}$$

Then either $\Psi = 0$ or $\lambda \in \sigma(\mathcal{A}_2^{D,\perp})$.

The strong relation between the spectrum and the existence of generalized eigenfunctions is well known (under the name of Sch'nol's theorem), see [13] and [3].

For the opposite inclusion, we observe that we have to control uniformly

$$(\mathcal{A}^D - \lambda + \eta^2)^{-1}$$

with respect to η under the condition that

$$\lambda \notin \bigcup_{r\geq 0, j\in\mathbb{N}^*} (\lambda_j + r).$$

It is enough to observe the uniform control as $\eta^2 \to +\infty$ which results from (4.1).

7.3. *The model in $\mathbb{R}_2^+$: Parallel current*

Here the models are the Dirichlet realization in $\mathbb{R}^2_+$:

$$\mathcal{A}_2^{D,\|} = -\Delta_{x,y} + iJy, \tag{7.8}$$

or the Neumann realization

$$\mathcal{A}_2^{N,\|} = -\Delta_{x,y} + iJy. \tag{7.9}$$

Using the reflexion (or antireflexion) trick, we can see the problem as a problem on $\mathbb{R}^2$ restricted to odd (respectively even) functions with respect to $(x,y) \mapsto (-x,y)$. It is clear from Proposition 7.1 that in this case the spectrum is empty.

Remark 7.1. The case when the current is neither parallel nor perpendicular is open.

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