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SINGULAR SHOCKS RETROSPECTIVE AND PROSPECTIVE

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Singular shocks were first devised over 20 years ago as a tool to resolve some otherwise intractable Riemann problems for hyperbolic conservation laws. Although they appeared at first to be merely a mathematical curiosity, new applications suggest that they may have some greater significance. In this paper, I recount the story of their discovery, which owes much to Michelle Schatzmann, describe some of their old and new appearances, and suggest intriguing possible connections with change of type in conservation law systems.

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AMS Subject Classification: Primary 35L65, 35L67, 35M10; Secondary 76T20, 76T99

1. Background: Origins of the Story

Because this paper originated in a talk at a conference in honor of Michelle Schatzman, I would like to start with a story. The story concerns hyperbolic systems of conservation laws, equations of the form

$$U_t + F(U)_x \equiv U_t + A(U)U_x = 0, \tag{1.1}$$

with real characteristic speeds (eigenvalues $\lambda^i(A(U))$ real). Such systems have the interesting property that the characteristic speeds of propagation depend on the state variable U, and this leads to various, sometimes paradoxical, phenomena. The most famous example, Burgers equation, $u_t + uu_x = 0$, where $\lambda = u$, already has the property that solutions, even with smooth or analytic Cauchy data, lose all smoothness after a finite amount of time, and can be described only in the framework of weak solutions. In the case of Burgers equation, the correct space for Cauchy data turns out to be bounded measurable functions, and the solutions themselves are not only bounded and measurable but have bounded variation, as first proved by Kružkov [28].

Systems, more intricate than a scalar equation, have the possibility of exhibiting more complicated dependence of the characteristic speeds on the state variable. In 1965, Glimm [13] published a famous result: A system like (1.1), if the Cauchy data are close to constant and have small total variation, will have a weak (bounded, measurable) solution of small total variation for all time. The proof required a couple of additional conditions, most significantly that the characteristics were separated (strict hyperbolicity) and that each was either genuinely nonlinear or linearly degenerate. ("Genuine nonlinearity" can be given a precise definition which we will not need in this paper. It amounts to a requirement that the characteristic speed vary monotonically when the state variable varies in the corresponding characteristic direction.) When strict hyperbolicity is coupled with data close to a constant state, one consequence is that the characteristic speeds are separated. In this case, each characteristic family is isolated, and the system behaves in some respects like a collection of scalar equations. Glimm's theorem represented the state of the art for over 30 years; only in the 1990s did Bressan and co-workers improve the result to include well-posedness for the same class of problems. The classical theory of conservation laws is expounded in Smoller's book [43]. A modern treatment, including the recent well-posedness results, can be found in Bressan's treatise [6].

In the 1970s, Herb Kranzer and I became interested in systems where waves from different characteristic families interact strongly with each other, possibly in a nonlinear manner. One example occurs in elastic strings, where the speeds of longitudinal (nonlinear) and transverse (linearly degenerate) waves typically differ by an order of magnitude, but there is no theoretical reason for this, and in some materials they may coincide. (The basic system is given by Carrier [7], among others.) Such systems could be called "nonstrictly hyperbolic"; that is, they fail to be strictly hyperbolic. Further classification is possible, as one might have characteristics of constant multiplicity, a case we did not consider, but which has been studied by Freistühler [12]. In addition, at an eigenvalue coincidence, one might have either a full set of eigenvectors or an eigenvector deficiency.

It is standard practice, when faced with a new type of conservation law, to formulate and solve a particularly simple type of initial value problem, the Riemann problem, consisting of data containing only two constant states, separated by a discontinuity at the origin:

$$U(x,0) = \begin{cases} U_L, & x < 0, \\ U_R, & x \geq 0. \end{cases} \tag{1.2}$$

Understanding Riemann problems helps uncover the structure of discontinuities; Riemann problems give rise to particularly simple, self-similar solutions that depend on x/t only; in addition, Riemann solutions are the basis of the existence theorems of Glimm and Bressan; finally, Riemann solutions often describe the asymptotic behavior of solutions to the Cauchy problem.

We solved the Riemann problem for some examples [20]. Our initial examples mimicked the elastic string property of coupling a nonlinear with a linearly degenerate family. We found the rather undramatic answer that when the wave speeds exchange order, an intermediate state in a Riemann solution may change by a large amount; however, it does so in a tiny region, so the solution is no longer continuous in L^∞ but remains continuous in L^1.

Motivated by this, we began to look at other examples. A simple case is a nonlinear wave equation,

$$\begin{aligned} u_t &= v_x, \\ v_t &= \left(\frac{1}{3}u^3\right)_x \end{aligned} \tag{1.3}$$

or $\partial_t^2 u - \partial_x^2 u^3/3 = 0$, whose characteristic speeds are $\lambda = \pm|u|$. Again, we were able to solve Riemann problems for a family of equations that included this example [21], and again the solutions were not dramatic. However, we noticed that our examples had a property that we named "opposite variation": if one characteristic speed increased in a characteristic direction, then the other speed decreased. The first diagram in Fig. 1 sketches the characteristic speeds for the nonlinear wave equation (1.3), which are functions of one component (u) only. In this example, the component u is a satisfactory surrogate for the characteristic direction. Not all the systems we were able to solve were as symmetric as the nonlinear wave equation, but all shared this property. Naturally, we were curious about systems with the opposite property, "same variation". The second diagram gives an example — again, easy to construct with characteristic speeds depending on u only. To our surprise, we were not able to write down solutions to some simple Riemann problems. It was at that point that I showed our example, as pictured in Fig. 1, to Michelle, and she commented that it might be considered as a perturbation of a strictly hyperbolic, "same variation" system. In fact, she drew the superimposed lines in Fig. 1.

It was not difficult to come up with an example whose characteristics looked like that picture. Our model was

$$\begin{aligned} u_t + (u^2 - v)_x &= 0, \\ v_t + \left(\frac{1}{3}u^3 - u\right)_x &= 0. \end{aligned} \tag{1.4}$$

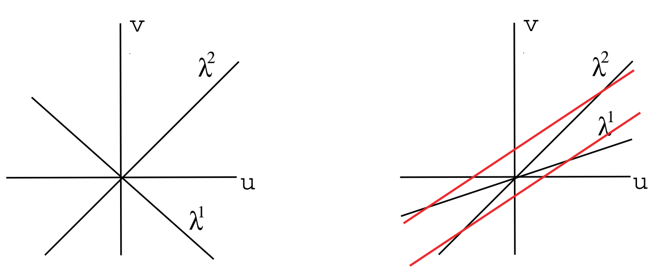

Fig. 1. Opposite and same variation, and the superimposed strictly hyperbolic problem.

Its characteristic speeds are $\lambda^i = u \pm 1$, and to our great surprise we could not solve it either, despite the fact that it is strictly hyperbolic and genuinely nonlinear. For a large set of Riemann data, (1.2), with $U = (u, v)$ and $|u_L - u_R| > 4$, there was no combination of shocks and rarefactions that resolved the discontinuity. (Recall that self-similar shocks look like

$$U(x,t) = \begin{cases} U_0, & x < st, \\ U_1, & x \geq st, \end{cases}$$

where U_0, U_1 and s satisfy the Rankine–Hugoniot relation

$$s(U_1 - U_0) = F(U_1) - F(U_0). \tag{1.5}$$

Self-similar rarefactions connect U_0 and U_1 with $\lambda^i(U_0) < \lambda^i(U_1)$ via a smooth solution $U(x,t) = U(x/t) = U(\xi)$ of $(A(U(\xi)) - \xi I)U' = 0$. Thus,

$$\xi = \lambda^i(U(\xi)) \quad \text{and} \quad (A - \lambda^i)U' = 0$$

for some characteristic family, i.)

It is interesting to note that although we invented the model (1.4) (now often called the "Keyfitz–Kranzer model") from a purely mathematical motivation, it is related to a familiar system. If one begins with the equations of isentropic gas dynamics in a single space dimension with a polytropic equation of state,

$$\rho_t + (\rho u)_x = 0, \tag{1.6}$$

$$(\rho u)_t + (\rho u^2 + \rho^\gamma)_x = 0, \tag{1.7}$$

and manipulates them, following a standard practice in mathematical physics (see, for example, Rosenau [36]), by subtracting u times (1.6) from (1.7), one obtains an equation for "conservation of velocity",

$$u_t + \left(\frac{1}{2}u^2 + q(\rho)\right)_x = 0, \quad q(\rho) = \begin{cases} \dfrac{\gamma}{\gamma - 1}\rho^{\gamma-1}, & \gamma > 1, \\ \log \rho, & \gamma = 1. \end{cases}$$

This replaces the second equation, (1.7), and we introduce another function of u and ρ:

$$v = \frac{1}{2}u^2 - q = \frac{1}{2}u^2 - \log \rho,$$

in the case $\gamma = 1$ (there is a corresponding expression if $\gamma > 1$), to obtain the system (1.4). Thus, (1.4) is simply the equations of isothermal gas dynamics (this is the physical meaning of the limit $\gamma = 1$) with conservation of mass and momentum replaced by conservation of velocity and a quantity that is an entropy for the original system.

To summarize, thanks to Michelle Schatzman's somewhat offhand comment, we were led to consider the example (1.4), which turned out to have interesting new pathologies. These have by now been well-studied. Kranzer and I suggested that by means of a certain new type of wave, which we could describe only vaguely, one could produce a unique resolution for every Riemann problem. We looked at these new waves from a variety of viewpoints, mainly resorting to approximations, including via a self-similar viscosity. Our approximations differ from each other in detail but have common properties of unbounded growth in narrow regions. Unlike classical approximations to conservation laws, they do not converge in L^1, and the limit solutions do not satisfy the equation in the classical weak-solution sense. However, they do converge to limits that can be described precisely, and the residual error in the equations tends to zero in a standard way. It should be noted that enlarging the collection of possible structures for weak solutions does not create problems for classically solvable conservation laws, as the new structures do not exist for them. Note also that the singular shocks described here are not the same as "measure-valued solutions", see for example [11], which are also limits of conventional weak solutions under assumptions of very low regularity, but in a different context. Our results appeared in several places [22, 23, 27]; a few years ago I wrote a summary [18].

It would not be correct to say that the new waves are weak solutions of the system in the standard textbook sense, as the textbook definition requires of weak solutions that they be L^∞ functions, typically with bounded or locally bounded variation. In [41], Sever has given a definition of singular shocks that applies to the example (1.4). Sever's insight is that, despite the intricate detail that appears in the approximations to singular shocks, a definition of the singular shocks themselves can be given in the framework of classical distribution theory. Sever's definition is

Definition 1.1. A singular shock solution of (1.1) is a measure of the form

$$U(x,t) = \widetilde{U}(x,t) + \sum_i M_i(t)\chi_{I_i}(t)\delta(x - x_i(t)),$$

where $\widetilde{U}$ is a weak solution of (1.1) in the complement of $\cup x_i(t)$; the curve segments $\{x_i\}$ supporting the singularities are $W^{1,\infty}$ functions defined on time intervals I_i; and the singular masses M_i are L^∞ (vector-valued) functions. In addition, U is the limit of approximations U^ε satisfying

(1) $U^\varepsilon \in L^1_{\text{loc}}$, uniformly in ε, pointwise in t
(2) $U^\varepsilon(\cdot,t) \rightharpoonup U(\cdot,t)$ weakly in the space of measures on $\mathbb{R}$, pointwise in t
(3) for some positive definite matrix A,

$$U^\varepsilon_t(\cdot,t) + F(U^\varepsilon(\cdot,t))_x - \varepsilon(AU^\varepsilon_x(\cdot,t))_x \rightharpoonup 0$$

weakly in the space of measures on $\mathbb{R}$, pointwise in t.

Thus, singular shock solutions differ from classical solutions by containing delta-function measures supported on line segments. They differ also in the way they can be approximated, as approximations to classical weak solutions converge boundedly almost everywhere.

Returning for a moment to the motivating problem, the model (1.4), one finds that the singular part is confined to the component v. That is, the singular masses M_i are all parallel to the vector $(0,1)$. In the physical derivation of that model, u represents velocity and $v = u^2/2 - \log\rho$ contains the density variable, thus partially justifying the term "mass" for the M_i. (Note however that in the actual singular shocks for (1.4) M appears with a positive coefficient, corresponding to $\rho = 0$, or the appearance of vacuum, rather than mass, in the singular shock.)

It should also be noted that while Sever's theory is more general than the model and its solution studied by Kranzer and myself, he abstracts one important detail from the model: the fact that there is a variable (u in our example) that can be identified with a velocity. This motivates the introduction of a particular type of entropy condition, which is the basis of many of the results in [41].

There remain unanswered questions, such as finding a context in which one might expect to find singular shock solutions to a system. A physical interpretation of their significance is also lacking at present, as is a satisfactory connection between approximations to singular shocks and approximations to exact conservation laws. And, as stated earlier, it is clear that distributions cannot in general satisfy a nonlinear equation like (1.1), and hence an explanation of the sense in which they solve the equation is still lacking.

Finally, it now appears, following the appearance of singular shocks in a model for two-phase flow and, more important, some recent work by Mazzotti [30–32], which is outlined a later section of this paper, that the framework suggested by the example (1.4), upon which Sever's theory is based, may as yet lack the elements needed to describe some wider collection of examples. Specifically, Mazzotti has studied the system of conservation laws arising in two-component chromatography. In this system, the conserved quantities are the masses of the two components. There is no velocity variable and no Newtonian dynamics (that is, the system is kinematic), so a unifying assumption made by Sever appears to be absent. (Of course, since x and t represent space and time, it is always possible to construct variables that have the dimensions of velocity by dividing a component of F by the corresponding component of U. Sever's unifying hypothesis, in [41], that at least one of these variables is constrained by a one-sided bound, may not always be natural, however.)

It is intriguing that Mazzotti's example, as well as the other example we have studied in detail, a two-fluid model for incompressible two-phase flow, appear to be connected with loss of hyperbolicity of the governing equations in some subset of phase space. Some time ago, in a different context, I speculated on the possible relationship between kinematic systems, change of type and pathological behavior of solutions of conservation laws [17]. Mazzotti's new applications of singular shocks raises this issue again.

Thus, it seems timely to review what is known about singular shocks, to outline the common elements of the known examples, and to summarize the techniques presently used for studying them. This we do in the remainder of the paper.

2. Structure and Properties of Singular Shocks

We begin by recalling some details of the model example, (1.4), as a way of illustrating the general statements made in Sec. 1. The salient fact is that although (1.4) is strictly hyperbolic and genuinely nonlinear, the Hugoniot locus $H(U_0)$ — the set of points U such that $s(U - U_0) = F(U) - F(U_0)$ for some value s — is bounded. Its equation is

$$v - v_0 = (u - u_0)\left(\frac{u + u_0}{2} \pm \sqrt{1 - \frac{(u - u_0)^2}{12}}\right).$$

If $U_0 = U_L$ is taken to be the state on the left side of a wave, then the set of states U that can be joined to U_0 via a shock is indicated by the curves S_1 and S_2 in Fig. 2. The subscripts refer to slow (1) or fast (2) waves. The rarefaction curves are parabolas extending to infinity. The Riemann problem can be solved in the standard way for any point U_R within the region bounded by J, J_1 and J_2, and for none outside it. (We were not the first to discover that a system of strictly hyperbolic conservation laws can have a bounded Hugoniot locus. Other examples are given in an early paper of Borovikov [4].)

The philosophy underlying the construction of singular shocks is that the standard formulation for weak solutions — $U \in L^1_{\text{loc}}$ satisfies (1.1) if

$$\int U\phi_t + F(U)\phi_x = 0 \tag{2.1}$$

for all test functions $\phi \in C_0^\infty$ — may not suffice to define all functions one might want to accept as solutions. In particular, (2.1) is the genesis of the Rankine–Hugoniot relation

$$s[U] = [F(U)], \quad [U] = U(x(t)^+, t) - U(x(t)^-, t), \quad s = \frac{dx}{dt}$$

if the limits $U(x(t)^+, t)$ and $U(x(t)^-, t)$ exist. But what if they do not?

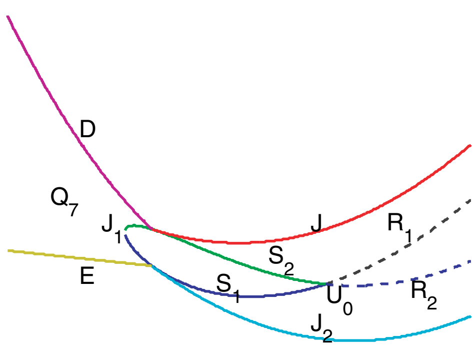

Fig. 2. Components of state space for the Riemann solution for the model problem.

The answer is not obvious and we (and Sever following us) resorted to looking at limits of approximations. We considered several specific approximations and their limits; Sever was able to formulate a more general theory by showing that all that was needed was to consider a reasonable class of approximations. In practice, it may be simplest to prove the existence of a sequence of approximate solutions by choosing a specific approximation, although a concrete choice has both advantages and disadvantages. Let me illustrate, with reference to (1.4), using the self-similar Dafermos–DiPerna viscous regularization:

$$U_t + F(U)_x = \varepsilon t U_{xx}, \tag{2.2}$$

a popular choice when one wants to study self-similar solutions, as it is easily seen to allow solutions of the form $U = U(\xi) = U(x/t)$, which may approach Riemann solutions as $\varepsilon \to 0$. (See [10] for more on this type of approximation.) The insight that led Kranzer and myself to the discovery of singular shocks was to seek solutions of (2.2) for the system (1.4) that were unbounded, as $\varepsilon \to 0$, in the neighborhood of a value, the "singular shock speed", $\xi = s$. A specific ε dependence (this is not unique) that proved fruitful was

$$\widetilde{U}(\xi) = \left(\frac{1}{\varepsilon} \widetilde{u} \left(\frac{\xi - s}{\varepsilon^2} \right), \frac{1}{\varepsilon^2} \widetilde{v} \left(\frac{\xi - s}{\varepsilon^2} \right) \right). \tag{2.3}$$

Now, the point is that such an ε-dependence cannot tell the whole story, but can describe only the singular part of the solution. Specifically, both $\widetilde{u}$ and $\widetilde{v}$ must tend to zero as their arguments tend to $\pm\infty$. Substituting into (1.4), one finds that $(\widetilde{u}, \widetilde{v})$ must be a homoclinic orbit of the planar dynamical system

$$\begin{aligned} x' &= x^2 - y, \\ y' &= \frac{1}{3} x^3. \end{aligned} \tag{2.4}$$

This singular equation does indeed possess homoclinic orbits. Its behavior near the origin is illustrated in Fig. 3. In particular, it becomes clear, comparing the scaling in Eq. (2.3) with the homoclinic orbits of the system (2.4), that a singular part of the approximate solution that appears like an approximate delta-function is plausible: That is what the singular part of the second component, v, looks like. On the other hand, the singular part of the first component, u, tends to zero in the space of measures as $\varepsilon \to 0$. This reasoning — approximating the system with self-similar viscosity and then seeking a singular solution concentrated near the shock — was what led to the theory fleshed out by Sever that culminated in Definition 1.1. However, neither the recourse to a singular approximation nor the abstract definition answers the question of when such approximations, or their limits, actually exist. Nor is there at present a completely satisfactory answer to the question of the sense in which singular solutions "satisfy" the equation.

One simple conclusion can be drawn from regarding the function (2.3) as an inner approximation (in the sense of perturbation theory) to the viscous conservation law. Assuming that (2.3) takes care of the unbounded part of the approximate

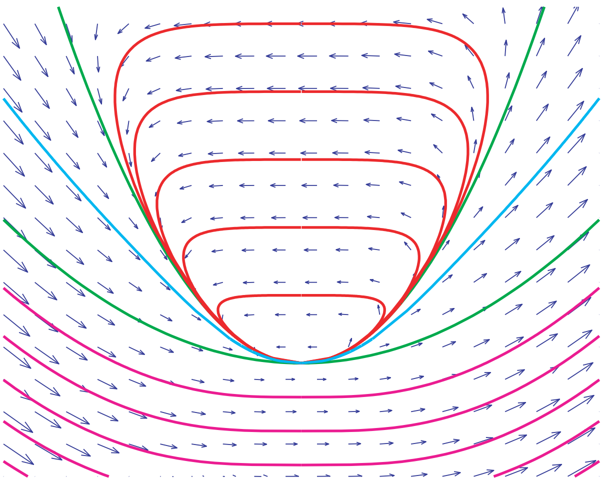

Fig. 3. A rendering, using John Polking's pplane, of orbits of (2.4).

solution, then away from the shock, one anticipates an outer approximation of the form

$$U = \overline{U}\left(\frac{\xi - s}{\epsilon}\right).$$

This is now a bounded function, and applying the usual conservation law reasoning (and the usual perturbation theory reasoning) yields a standard expression:

$$\frac{d\overline{U}}{d\tau} - F(\overline{U}) + s\overline{U} = C_{\mp}, \tag{2.5}$$

where, unlike in the standard conservation law theory, the constant (that comes from integrating (2.2) once) may be different on the right and left sides of the singular shock. In the standard theory, where $C_+ = C_-$, the Rankine–Hugoniot relation for values of U on the two sides of the shock appears to be a consequence of assuming that the approximate solution to (2.2) approaches constant values $U_\pm$ outside a shock profile of width ε. But if singular shocks are present, then integrating across the unbounded part of the shock produces a contribution, and one finds that

$$C_+ - C_- = -\int \tau U_\tau \, d\tau.$$

Again assuming that the approximate solution approaches constant values outside the shock profile, this replaces the Rankine–Hugoniot relation with a generalized relation

$$s[U] - [F(U)] = -\int \tau U_\tau \, d\tau, \tag{2.6}$$

where the integral is taken over the singular part of the solution.

3. Further Developments

At this point, I have described a formal calculus that Kranzer and I found useful in analyzing the model equation. Unsurprisingly, in our example, the first component of the singular contribution to (2.6) is zero — corresponding to the property that the singular part of the first component u is small, and tends to zero as a measure. Thus, the first Rankine–Hugoniot relation is satisfied. What this means is that given any two states, U_- and U_+, there is a unique speed s for which they can be joined by a singular shock. What, then, prevents this from happening for any states, including those for which an ordinary shock could be constructed? Further asymptotic analysis of the singular solution suggests that a necessary condition for approximate solutions of (2.2) of the conjectured form to exist is that the singular shock be overcompressive. That is, we must have

$$\lambda_i(U_-) > s > \lambda_i(U_+) \tag{3.1}$$

for $i = 1$ and $i = 2$. More generally, Sever [41] shows this to be a necessary condition for a general approximation U^ε of the form (3) in Definition 1.1 to converge. This in turn implies that systems that admit singular shocks must have a special characteristic structure, roughly described as overlapping characteristic speeds — something that generalizes the obvious feature of (1.4), with characteristic speeds $u + 1$ and $u - 1$, that the slow speed at some states is faster than the fast speed at others. (One should note, however, that there is more to the story than this, as the system for isothermal gas dynamics — (1.6) and (1.7) with $\gamma = 1$ — has the same characteristic speeds yet all Riemann problems have classical solutions.) One could say that systems that admit singular shocks "fail to be strictly hyperbolic in the large" (with the same caveat as the preceding parenthetical warning).

Kranzer and I verified some of these conjectures for the model system (1.4), showing that a number of different methods of approximating the system lead to approximations to singular shocks, always satisfying the same generalized Rankine–Hugoniot relation, and always admitting a construction as a limit of approximations only under the overcompressive condition (3.1). It should be mentioned that the strict inequalities in (3.1) must be relaxed in order to produce solutions to all possible Riemann problems. For example, only in the open region labeled Q_7 in Fig. 2 are the singular shocks strictly overcompressive; on the boundary curves D and E one obtains equalities in one of the relations in (3.1). And in order to solve Riemann problems for data in the regions between D and J and between E and J_2, where the singular shock is joined to a rarefaction wave, one must use intermediate states on D or E. Obtaining viscous profiles for these limiting waves proves to be more subtle than for states in the open regions.

We formulated our approximations using piecewise constant "box" approximations to the singular perturbation approximations we had found formally, and through this we discovered that the system was not sensitive to "inner" and "outer" approximations. The notion that the singular part of the shock is in some sense narrower than the regular profile is a convenient way to visualize an approximate solution but seems not to be a part of the basic structure of these waves. In addition, we found that we could describe the singular solutions in the language of Colombeau's generalized distribution theory [9], which allows one, for example, to talk about functions whose squares are delta functions. At the time we first thought about this, Colombeau's theory was not very accommodating to applications that required some notion of metric (as we needed to justify our candidates as approximations). However, more recent developments of the theory have greatly extended its usefulness here. See Nedeljkov's recent paper [33], and the references therein for details.

Eventually, we synthesized all these approximations by creating a space of weighted measures, and in [23] we showed that all the approximations were equivalent, and that all of them led to the same conclusion: Approximations to singular shock solutions were indeed approximate solutions to the model equation (1.4), and they converged (weakly or strongly according to how one chose the function space) to measures that could be precisely described. However, the sense in which they satisfy the equation remained (and still remains) mysterious. At that point, we dropped the subject.

Schaeffer, Schecter and Shearer picked it up in [37] to show that the same kind of singular shocks could be used to solve nonstrictly hyperbolic problems of "same variation" type (as in the second picture in Fig. 1), the problem that had motivated this study of the model equation (1.4). Somewhat later, Stephen Schecter [39] was able to prove the existence of the Dafermos profiles we had sought via (2.2) for our model system (1.4). (Recall that we produced approximations in a number of creative ways but we did not show that the system admitted solutions that looked like our formal perturbation approximations.) Michael Sever had begun working on alternative methods of approximation, including via regular viscosity, somewhat earlier [40], and eventually produced a full theory, published in [41]. Sever's book clarifies the distinction between the singular shocks that Kranzer and I invented and the "delta-shocks" that were used by a number of authors (see for example [5, 45]) to solve a number of Riemann problems, notably for the equations of zero pressure gas dynamics, around the time we did our original study [19]. Sever shows that his definition of singular shocks, Definition 1.1, is quite natural under a set of reasonably general assumptions (including, as we mentioned earlier, the assumption that there is a variable that behaves like a velocity in satisfying certain bounds), and establishes the generalized Rankine–Hugoniot relation as a basis for calculating the rate of growth (or possibly decay) of the amplitude, $M_i(t)$, of the singular part of the shocks in Definition 1.1. In fact, it is an interesting feature of singular shocks that once one goes beyond self-similar solutions a new possibility emerges: singular shocks that interact with the flow outside the shock, decrease in amplitude and, eventually, disappear completely, resolving as classical shocks. This behavior, along with other observations on the model equation, is described in Sever's recent paper [42], which succeeds in solving the Cauchy problem for the model equation (1.4).

As another comment on the model equation (1.4) and its relation to Colombeau's theory, let me mention the work of Cauret, Colombeau, and LeRoux [15], which appeared at about the same time as our first investigations. These researchers wrote the same system (derived as a model for a different phenomenon) in a nonconservative form, justified by one of their modeling assumptions. Now, in order to discuss weak solutions they needed to define what is meant by a product of distributions (step functions and their delta-function derivatives), and once again they were able to use the Colombeau calculus to explain their solutions. That some equations modeled by conservation laws (or by their cousins, in which some equations are not in conservation form) lead in different ways to a study of the Colombeau calculus is interesting. It draws attention to the singularities that appear in these problems.

4. The Two-Fluid Model for Two-Phase Flow

Meanwhile, I was pursuing another train of thought, which appeared to be independent and distinct. Only recently are connections with singular shocks beginning to appear. I have long been interested in change of type in conservation laws. The context relevant for this discussion is systems of conservation laws in space and time which are not everywhere hyperbolic. That is to say, the characteristic speeds, which depend on the state variables, are complex for some physically reasonable states. Such systems appear frequently in modeling, where they are often rejected on the grounds that their linearized versions imply catastrophically unstable behavior. Two well-known examples are the two-fluid single-pressure model for two-phase flow [44], and a model for three-phase porous medium flow that at one time was widely used in petroleum reservoir simulation [2]. Neither model is in common use today, but at the time they were used it was noted that the catastrophic instability did not materialize in numerical simulations. There was no general agreement about why this was the case, and current research has moved on to models that avoid averaging over the phases (this is the origin of the "two-fluid" reduction). However, one can argue heuristically that, following some initial growth in amplitude of unstable modes, nonlinear effects will dominate the picture and one will not see exponential growth. Furthermore, one can argue, nonlinear effects, such as shock propagation, can mimic wavelike behavior, and if one allows weak solutions with shocks satisfying some plausible admissibility condition, then one might expect reasonable looking solutions from some nonhyperbolic or change-of-type models. I had been thinking about this for some time, and had published some preliminary results, see [17] for an example, when I was challenged to apply this thinking to a model for incompressible two-phase flow.

A standard equation for the continuum model for two-phase flow takes the form, see [44],

$$\left.\begin{aligned} \partial_t(\alpha_i\rho_i) + \partial_x(\alpha_i\rho_i u_i) &= 0, \\ \partial_t(\alpha_i\rho_i u_i) + \partial_x(\alpha_i\rho_i u_i^2) + \alpha_i\partial_x p_i &= F_i, \end{aligned}\right\} \quad i = 1, 2, \tag{4.1}$$

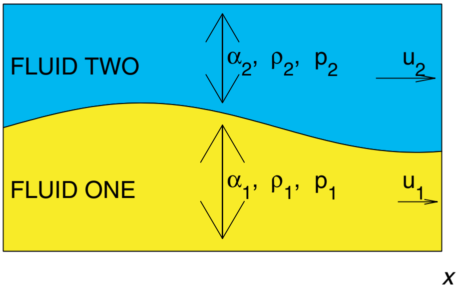

Fig. 4. Variables in the two-fluid model: Separated fluids.

where densities ρ_i, velocities u_i, pressures p_i and volume fractions α_i are illustrated in Fig. 4. In principle, this model applies both to separated fluids and to mixtures. The continuum equations represent conservation of mass, while in the momentum equations the terms F_i take account of momentum transfer between the fluids, with $F_1 = -F_2$ if overall momentum is conserved. If the fluids are separated, then the only contribution to the F_i, neglecting viscosity, is surface tension, a higher-order effect proportional to the curvature of the surface. Of course, in formulating a one-dimensional model one has already made the approximation of neglecting the vertical component of motion. I have also neglected thermal effects by considering the isentropic case; this is merely for simplicity. To obtain a closed system, one requires some additional constraints. For a compressible fluid, one relates density and pressure by constitutive equations, for example

$$\rho_i = \rho_i(p_i) = \left(\frac{p_i}{A_i}\right)^{1/\gamma_i}.$$

For the purposes of this exposition it is sufficient to consider the case that the medium is saturated by the two fluids, so that

$$\alpha_1 + \alpha_2 = 1.$$

The interesting situation occurs when one assumes that each fluid (or component in a mixture) maintains its own velocity, so that

$$u_2 \neq u_1.$$

Finally, there is the question of pressures. This is the most controversial issue (see, for example [8] and references therein), and the classical single-pressure assumption,

$$p_2 \equiv p_1,$$

has aroused strong objections, as there is no doubt that it contributes to difficulties that arise in this model, such as its well-known failure to maintain hyperbolicity throughout state space [44]. However, it is useful to explore the mathematical behavior of the model under this constraint, in view of the fact that the system was at one time widely used in numerical simulations, including those of industrial importance for the design of nuclear reactors, without such bad consequences as instability or blow-up.

4.1. *A system for incompressible flow in simplified coordinates*

It was in looking at the simple model for incompressible two-phase flow that I discovered a connection with singular shocks [19, 24, 25]. To arrive at the simplified equations, one assumes both that the fluids are incompressible and that the density in each phase is constant. The equations immediately reduce to

$$\left.\begin{aligned} \partial_t\alpha_i + \partial_x(\alpha_i u_i) = 0,\\ \rho_i\partial_t u_i + \rho_i u_i\partial_x u_i + \partial_x p = F_i/\alpha_i \end{aligned}\right\} \quad i = 1, 2. \tag{4.2}$$

Recalling that $\alpha_1 + \alpha_2 = 1$, the first two equations can be added and integrated. If one chooses a velocity coordinate moving with the flow, and then introduces new scaled coordinates

$$\begin{aligned} \beta &= \rho_2\alpha_1 + \rho_1\alpha_2,\\ v &= \rho_1 u_1 - \rho_2 u_2, \end{aligned} \tag{4.3}$$

one gets an equivalent system of two equations that represent conservation of mass and transfer of momentum:

$$\beta_t + (vB_1(\beta))_x = 0, \tag{4.4}$$

$$v_t + (v^2B_2(\beta))_x = G, \tag{4.5}$$

with

$$B_1(\beta) = \frac{(\beta-\rho_1)(\beta-\rho_2)}{\beta}, \quad B_2(\beta) = \frac{\beta^2-\rho_1\rho_2}{2\beta^2},$$

and $\rho_2 \le \beta \le \rho_1$. These functions, and the state space, are shown in Fig. 5.

Now, the characteristics of the system (4.4), (4.5) are

$$\lambda = 2vB_2(\beta) \pm v\sqrt{B_1B_2'}$$

and they are clearly complex for $\rho_2 < \beta < \rho_1$, since $B_1 < 0$ and $B_2' > 0$ there. The system is "weakly hyperbolic" only on the H-shaped set $H = \{\beta = \rho_i\} \cup \{v = 0\}$, and linearly unstable everywhere else.

To shed some light on the behavior of this instability in the nonlinear system, we drop the term G, which typically includes lower-order effects of drag and gravity, as well as higher-order terms arising from viscosity or surface tension. Then, when we formulate a Riemann problem for the incompressible system, we discover singular shocks which, in this example, appear to be phase boundaries. Writing the Riemann

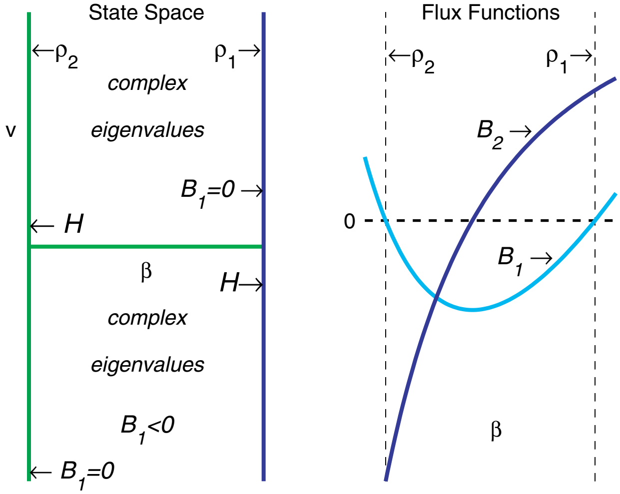

Fig. 5. State space and the functions B_1 and B_2 in the incompressible model.

problem for a system $w_t + q(w)_x = 0$ as

$$w(x,0) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases}$$

taking the self-similar viscosity approximation

$$w_t + q(w)_x = \epsilon t w_{xx}, \quad w = w(\xi) = w\left(x\frac{x}{t}\right) \tag{4.6}$$

and seeking a singular shock near $\xi = s$, one finds a nontrivial solution by choosing an inner expansion of width $\mathcal{O}(\epsilon^2)$ of the form

$$\widetilde{w} = \begin{pmatrix} \beta \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\epsilon^p}\tilde{\beta}\left(\dfrac{\xi - s}{\epsilon^q}\right) \\ \dfrac{1}{\epsilon^r}\tilde{v}\left(\dfrac{\xi - s}{\epsilon^q}\right) \end{pmatrix} = \begin{pmatrix} \tilde{\beta}\left(\dfrac{\xi - s}{\epsilon^2}\right) \\ \dfrac{1}{\epsilon}\tilde{v}\left(\dfrac{\xi - s}{\epsilon^2}\right) \end{pmatrix}.$$

The details of the solution are given in [19, 24, 25]. Note that in this case formal asymptotics give a singularity of much less strength than for the hyperbolic problem we examined earlier. There is no component that contains a delta-function, and the β-component contains no singularity at all. But by way of compensation, the strength of the singularity grows much more rapidly with $1/\epsilon$: Asymptotics show it to be exponential in $1/\epsilon$ as $\epsilon \to 0$.

An illustration of the singular solution is given in Fig. 6, which is presented as though the two fluids were separated (for ease of viewing). In this case, the Riemann problem for an initial state consisting of two mixtures is resolved by the formation of two regions where the flow separates into the two components. Singular shocks

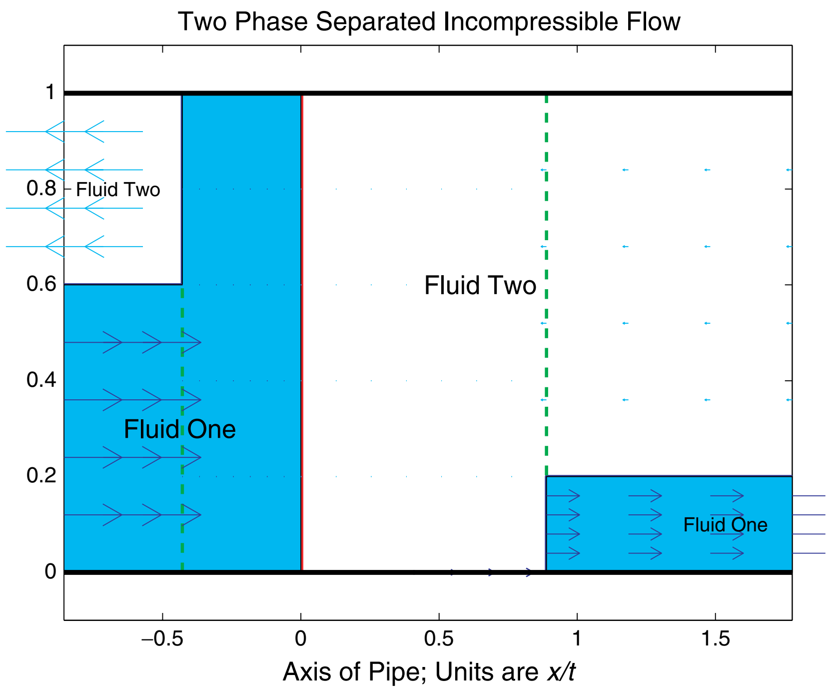

Fig. 6. Illustration of singular shocks in the stratified pipe flow model: Flow separation.

appear at the phase boundaries between mixed and pure states. The separation into pure states can be seen as a manifestation of the Bernoulli effect, which dominates the flow in the absence of surface tension or drag. One could now include those influences as perturbations. I should confess that considering Riemann problems is a mildly dishonest way of avoiding the catastrophic ill-posedness of a non-hyperbolic initial value problem, as the terms "hyperbolic" and "nonhyperbolic" do not retain any significance when one reduces the partial differential equation to an ordinary differential equation. In fact, as evidence of ill-posedness one might note that trivial Riemann data (with $w_- = w_+$) will produce a nonconstant solution similar to the picture above. All this tends to limit the value of the solutions produced by this model. Nonetheless, the mathematical analysis, as far as it goes, appears to be correct. My colleague Richard Sanders at the University of Houston did some careful numerical simulations of the perturbed system (4.6), solving it as a two-point boundary value problem for x/t, and found solutions very much like what the asymptotics predicts, as in Fig. 7, which gives two views of the profiles when $\epsilon = 0.01$. The first graph in Fig. 8 shows the profiles at a larger value of ϵ ($\epsilon = 0.05$), while the second confirms the predicted exponential rate of growth of singular shock strength with $1/\epsilon$ by plotting $\log v(0)$ against $1/\epsilon$ for a range of ϵ between 0.05 and 0.01.

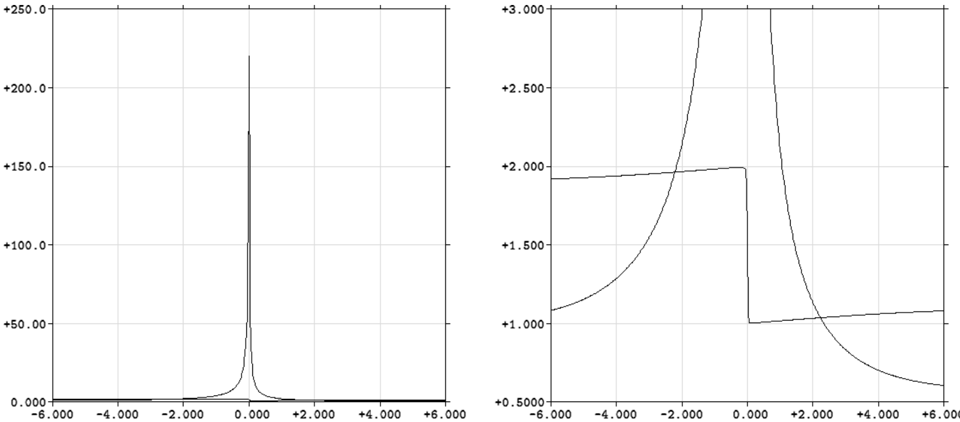

Fig. 7. Profiles of β and v vs. $x/\epsilon t$ for $\epsilon = 0.01$: overview and detail.

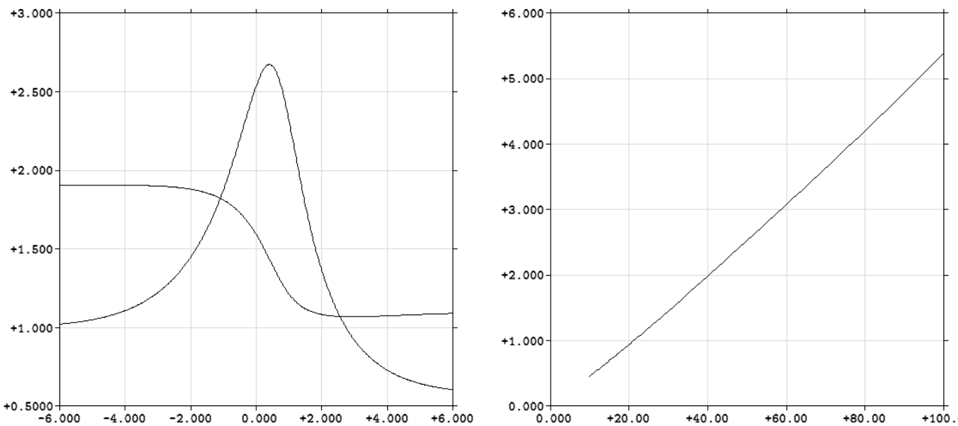

Fig. 8. Profiles of β and v, $\epsilon = 0.05$; $\log(v^\epsilon(0))$ vs. ϵ.

5. A Brief History of Systems that Change Type

The example just discussed, as well as Mazzotti's intriguing example which will be discussed below, suggest that there may be some connection between change of type in conservation laws and solution of Riemann problems using singular shocks. The example (1.4), although it motivated development of such theory as exists, may in fact be atypical in being strictly hyperbolic throughout state space. It is possible that the incompressible two-phase flow example is also atypical in its property that one must assume that nonhyperbolic states have some physical meaning in order to discuss solutions at all. To indicate the variety of examples that have appeared, in this section I give a list, in rough chronological order, of models that have been proposed in which change of type occurs. (Most of these have not been examined for singular shocks.)

5.1. *Two-way traffic flow*

In 1960, Bick and Newell [3] published a continuum model, based on the familiar Lighthill–Richards–Whitham model for one-way traffic, for traffic on a two-way highway. The one-way model is $u_t + (q(u))_x = 0$, where $u(x,t)$ is the density of cars and $q = uv$ is the flux. The modeling assumption is that the velocity at any point depends only on the local density, $v = v(u)$, and the equation simply expresses a "law of conservation of cars". (See Fig. 9.) Bick and Newell suggested that two-way traffic, with the two lanes having densities of $p(x,t)$ and $q(x,t)$, might be similarly modeled, with velocities U and V in either direction, as in the second sketch in Fig. 9. Conserving mass in each direction then yields

$$p_t + (pU)_x = 0,$$
$$q_t + (qV)_x = 0.$$

They further hypothesized that the local velocity should be a decreasing function of both densities. For real traffic, the velocity function would need to be determined empirically, as has been done for the one-way model, but to get a sense of the qualitative properties of the model, the authors assumed a simple linear relation:

$$U = 1 - p - \beta q, \quad V = -1 + q + \beta p.$$

Here the signs are dictated by the flow directions. Some of the coefficients have been taken to be unity by normalizing the density, time and spatial scales, so, assuming symmetry between the two directions, there remains a single non-negative parameter, β, which measures the extent to which traffic density in one direction hinders flow in the other. When $\beta = 0$, the two lanes are uncoupled, and one is dealing with the simple one-way equation again. The uncoupled system is easy to solve, though it is noteworthy that the uncoupled system is, formally, nonstrictly hyperbolic, with an eigenvalue coincidence along $p + q = 1$, a diagonal of the physically feasible region, see Fig. 10. (The feasible region consists of those states (p,q) for which p, q, and U are nonnegative, and V is non-positive.) However, once one considers the case $\beta > 0$, Bick and Newell found a region in state space in which the eigenvalues of the Jacobian matrix are complex, as indicated in Fig. 10. They could not offer a reason for this behavior; after solving Riemann problems for some initial conditions, they recommended abandoning the model.

Vinod studied this model in his thesis, because of its mathematical interest [46], and was able to solve some additional Riemann problems using classical shocks in

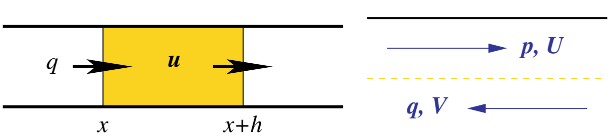

Fig. 9. One-way and two-way continuum traffic models.

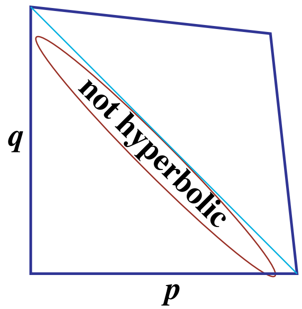

Fig. 10. State space for the two-way traffic system.

some novel configurations. To date, no one has examined this model for possible solutions involving singular shocks.

5.2. *Two-phase compressible flow*

The system of four equations for isentropic flow was given in Sec. 4. This system, as well as the full version with energy equations, has the well-known property that it fails to be hyperbolic for a range of states [44]. Typically, it has been solved numerically, and the assumption is that the addition of viscosity regularizes the system sufficiently to produce stable solutions. In addition, higher- and lower-order terms (representing drag, surface tension and other physical influences) are important and are always included in serious simulations. It is the case that one can produce the two-way traffic model as a naive asymptotic limit of these equations [16]. This might be a reason to try to understand that system. We do not pursue this model further here.

5.3. *Three-phase porous medium flow*

In numerical simulation of petroleum reservoirs, researchers have had to deal with situations where three phases (oil, water and gas) are simultaneously present. A standard approach to porous medium flow has been to take a continuum model for the phases and to assume that the flow velocity in each phase is a function of all three phases and the pressure. In a single space dimension, one can further eliminate the pressure term, and (assuming a saturated medium, so that it is necessary to track only two of the phases) the resulting equations take the form

$$
\begin{aligned}
u^o_t + f^o(u^o, u^g)_x &= D^o,\\
u^g_t + f^g(u^o, u^g)_x &= D^g,
\end{aligned}
$$

where u^i are the saturations of the phases (oil and gas) and f^i the fluxes. The terms D^o and D^g represent dispersion, and in the case of convection-dominated flow they are small, analogous to viscous perturbations in the conservation laws of gas dynamics. Thus, one is led to consider the hyperbolic system, with the D^i

set to zero. The fluxes are modeled semi-empirically. Typically, engineers have a good understanding of these functions when only two phases are present, but they can determine the three-phase versions only by interpolation. In [2] (see also [1] for more detail on the model), the authors examine a particular model and report that change of type occurs here, too, at least for some values of the parameters used in the model. In this case, the equations are hyperbolic in most of phase space, with only a tiny region where the characteristics appear to be complex. Again, the numerics do not indicate any catastrophic instability. On the other hand, in this case dispersion is present and is often included in the numerical schemes. In addition, when the system is simulated in more than one space dimension, the saturation equations are supplemented with an equation for the pressure, and the significance of the change of type is obscured. Because of the almost circular shape of the nonhyperbolic region in phase space, a number of researchers experimented with modeling it as an umbilic point in a system with a quadratic flux function, see [14, 38] and related references. Interesting qualitative behavior was found, in the form of complex shock-rarefaction patterns. There was no evidence of singular shocks; on the other hand, the reported research did not include complete solutions for all Riemann problems.

5.4. *Gas dynamics with $\gamma > 1$, conserving velocity and entropy*

It was mentioned earlier that the same reduction that generates the model system (1.4) can also be applied to the isentropic gas dynamics equations with $\gamma > 1$ — a more realistic parameter range. In this case, the resulting system is not everywhere hyperbolic. The analogue of (1.4) is the system

$$\begin{aligned} u_t + \left(\frac{3-\gamma}{2}u^2 - v\right)_x &= 0, \\ v_t + \left(\frac{(2-\gamma)(5-3\gamma)}{6}u^3 + (\gamma-1)uv\right)_x &= 0, \end{aligned}$$

and it is hyperbolic below the parabola

$$v < \left(1 - \frac{\gamma}{2}\right)u^2,$$

a region that corresponds to positive density. When $\gamma < 5/3$, this system shares with the prototype model (1.4) the property of having a bounded Hugoniot locus, and hence lacks classical Riemann solutions for a large set of data (corresponding to hyperbolic states and hence positive density — we assume that nonhyperbolic data are not interesting). Recently, Charis Tsikkou and I have solved the Riemann problem for this system [26]. By and large, the solutions follow the pattern established by (1.4), although there are some interesting variants. For example, vacuum states appear, as they do in the usual form of the gas dynamics equations. We also find that the singular shocks in this example do not correspond to the appearance of vacuum states. In addition, both viscous approximations to and numerical simulations of Riemann solutions (numerical simulations are of course a type of viscous approximation), take values in the nonhyperbolic region.

5.5. *A model for chemotaxis*

This example appeared in a paper by Levine and Sleeman [29]. I will not explain the derivation of the equations, since it is rather long. In summary, there are a number of models for chemotaxis (the transport of a substance that follows the gradient of an active chemical concentration). Using a model of Othmer and Stevens [34] as their starting point, Levine and Sleeman work with a system

$$\begin{aligned} u_t - a(uv)_x &= u_{xx}, \\ v_v - u_x &= 0, \end{aligned} \tag{5.1}$$

in which $u > 0$ is the density of the substance of interest and v the (scaled) gradient of the active agent. The parameter a is a constant which can be scaled to be ± 1, with sign depending on how the agent affects the substance. Assuming that the term u_{xx} can be neglected (again by scaling), Levine and Sleeman observe that when $a < 0$, system (5.1) exhibits change of type in the region of interest ($u > 0$). This example was the focus of the research in [29]. The authors do not analyze their model from the viewpoint of conservation laws. In fact, getting the correct boundary conditions for (5.1) is quite important and it is not clear what would be the significance of a Riemann problem for this system. However, the phenomena found by the authors, using linearization, Fourier analysis and numerical simulations, include pointwise blow-up. It is thus possible that singular shocks appear in this model. As with the other examples presented in this brief summary, it would be interesting to discover a relationship between change of type and singular solutions.

5.6. *The generalized Langmuir isotherm in chromatography*

In the introductory section of this paper, I mentioned recent work of Marco Mazzotti. To a great extent, his work has revived my interest in the phenomenon. Mazzotti began with a classical model: a binary (two-phase) system in chromatography. The underlying physical situation involves two fluids, with concentrations u_1 and u_2, flowing at constant speed along a column, which are differentially adsorbed, in amounts v_1 and v_2. Under a standard set of assumptions — that convection and adsorption dominate the system, that the mixture is locally in thermodynamic equilibrium, and that v_i depend only on u — one obtains a pair of equations expressing conservation of mass of each component.

$$\frac{\partial}{\partial t}(u_i + v_i) + \frac{\partial u_i}{\partial x} = 0, \quad i = 1, 2. \tag{5.2}$$

(It is natural to process engineers to write the equations in this form: $u_i + v_i$ is the total amount of component i at point x, while u_i is the corresponding flux — the amount of this component still in the fluid phase.) Furthermore, it is also standard to use the *Langmuir isotherm* to describe the equilibrium:

$$v_i = \frac{a_i u_i}{1 + u_1 + u_2}, \quad i = 1, 2. \tag{5.3}$$

The text of Rhee, Aris and Amundson [35], is an excellent reference for background on this topic. In writing (5.2) and (5.3) I have nondimensionalized the system without further comment, setting to unity all constants that do not alter the mathematical properties of the system. All quantities are physically meaningful only when they take non-negative values, so the fact that Eq. (5.2) is not hyperbolic inside a parabola in the second quadrant (unless $a_1 = a_2$, an exceptional case) was not considered important. (In fact, assuming $a_1 < a_2$ for definiteness, the system does fail to be strictly hyperbolic at the point on the u_2-axis. However, the characteristic structure of this system is extraordinarily transparent: It is of what is now called "Blake Temple" type, so the characteristics of different families do not interact, and solutions to the Riemann problem can be written down in a straightforward way.)

Mazzotti, motivated by intellectual curiosity, began to examine (mathematically) systems where the Langmuir isotherm was replaced by different combinations of plus and minus signs in the denominator:

$$v_i = \frac{a_i u_i}{1 \pm u_1 \pm u_2}, \quad i = 1, 2, \;\; a_1 < a_2. \tag{5.4}$$

In principle, this can make physical sense: The plus sign represents a situation where the two substances compete for adsorption sites, and in the case of a minus sign, they cooperate. The most interesting case is a denominator of the form $1 - u_1 + u_2$, for then the nonhyperbolic region lies in the physical region, as in Fig. 11. Since v_i as well as u_i must be positive, one considers only states with $1 - u_1 + u_2 > 0$, and Mazzotti also considered only data in the hyperbolic part of state space, and only in the closure of the open component neighboring the origin. The behavior he found is quite remarkable. For a large set of data, the Riemann problem cannot be solved by

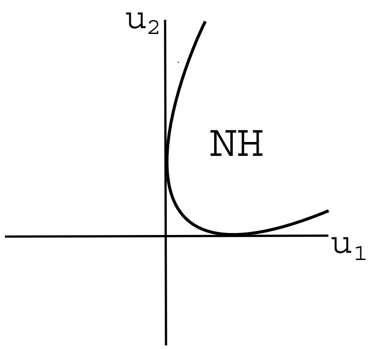

Fig. 11. State space for the chromatography model: Anti-Langmuir kinetics.

classical shocks and rarefactions, and numerical simulations seem to indicate singular shock solutions. In this example, neither of the Rankine–Hugoniot equations is satisfied, but Mazzotti was able to identify a linear combination that satisfied the relation (consistent with Sever's theory) by an ingenious use of approximate delta-functions. His idea, which is to displace the origin of the smoothed delta-function by different amounts for the different components, is consistent with the Colombeau theory of generalized distributions, but is of quite a different character from the choice that Kranzer and I made in [23]. Mazzotti was also able to show that with his choice he obtained approximate solutions to the system, with an error that converged to zero. It is interesting that, up to the present, we have not been able to obtain any useful information from the Dafermos–DiPerna regularization (or from any regularization) of this system.

Mazzotti then set out to confirm this behavior experimentally, and eventually succeeded [32]. It was not trivial to find substances that exhibited the right combination of cooperative and competitive behavior in their adsorption patterns, and in addition actually followed Langmuir kinetics (which are only an approximation to nonlinear equilibrium behavior). However, the experiments, performed by two different groups, exhibit a gratifying spike in concentration. At least as far as these two compounds are concerned, the singular shock really does represent "mass piling up in the shock".

6. Conclusions

The three change-of-type models for which there is now conclusive evidence for the existence of singular shocks are the incompressible two-phase flow model, the velocity-entropy model of isentropic gas dynamics with $1 \leq \gamma < 5/3$, and the binary chromatography model with a particular combination of Langmuir and anti-Langmuir kinetics. The gas dynamics model with $\gamma > 1$ resembles closely the hyperbolic model equation originally studied by Kranzer and myself, while the other two appear quite different. Meanwhile, there are a number of other model systems in which change of type occurs and not all Riemann problems have classical solutions, but not enough research has been done to demonstrate the existence (or not) of singular shock solutions. It is possible that, in the end, "change of type" will have no more to do with the formation of singular shocks than the basic fact that we are considering state variables on a particular domain, outside of which the properties of the system are immaterial. In one of our examples, incompressible flow, it is in fact important to look at data outside the hyperbolic region. In some of the others, such as chromatography, the model for those states (or for the corresponding flux functions) breaks down, and an explanation of what one means by an "approximation" (for example, the solution produced by viscous regularization), which seems to involve a physically impossible situation, is still incomplete.

On returning to study these problems again, I have found Schecter's proof of existence of Dafermos profiles compelling. Not only does Schecter show that our asymptotic expressions are approximations to an actual solution, he also demonstrates how the "inner" and "outer" or "singular" and "nonsingular" or, more precisely, "singular in amplitude" and "singular only in rate of change" parts of the solution relate to each other. Although there is no physical basis for selecting a self-similar artificial viscosity, I would hazard a guess that one could relate the solutions found this way to other viscous approximations, either by adapting the geometric singular perturbation methods to a different system, or by scaling the variables appropriately. Thus it seems worthwhile to explore this technique for other examples. A preliminary examination of the two-phase flow model did not lead to a positive result, but it is probable that one must tailor the coordinate changes to the example at hand. We are not close to a general theory yet.

Acknowledgments

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