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A NOTE ON CANONICAL BASES AND ONE-BASED TYPES IN SUPERSIMPLE THEORIES

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This paper studies the CBP, a model-theoretic property first discovered by Pillay and Ziegler. We first show a general decomposition result of the types of canonical bases, which one can think of as a sort of primary decomposition. This decomposition is then used to show that existentially closed difference fields of any characteristic have the CBP. We also derive consequences of the CBP, and use these results for applications to differential and difference varieties, and algebraic dynamics.

Keywords: Supersimple theories; canonical base; descent of algebraic dynamics.

0. Introduction

In [16], Pillay gives a model-theoretic translation of a property enjoyed by compact complex manifolds (and proved by Campana and by Fujiki). With Ziegler, he then shows in [17] that various algebraic structures enjoy this property (differentially closed fields of characteristic 0; existentially closed difference fields of characteristic 0). As with compact complex manifolds, their proof has as immediate consequence the dichotomy for types of rank 1 in these algebraic structures. This property will later be called the *Canonical Base Property* (CBP for short) by Moosa and Pillay [14]. We will state the precise definition of the CBP later (see Definition 1.5), as it requires several model-theoretic definitions, but here is a rough idea. Let us assume that we have good notions of independence, genericity and dimension, and let $S \subset X \times Y$ be definable. Viewing S as a family of definable subsets S_x of Y, assume that for $x \neq x'$ in X, S_x and $S_{x'}$ do not have the same generics, and have finite dimension. Fix some $a \in X$, a generic b of S_a. The CBP then gives strong restrictions on the set $S^b = \{x \in X \,|\, b \in S_x\}$: for instance in the complex manifold case, it is Moishezon, and in the differential case it is isoconstant.

The aim of this paper is threefold: give reductions to prove the CBP; derive consequences of the CBP; show that existentially closed difference fields of positive characteristic have the CBP. We then give some applications of these results to differential and difference varieties.

We postpone a detailed description of the model-theoretic results of this paper to the middle of Sec. 1 and to the beginning of Sec. 2, but we will now describe two of the algebraic applications. First, an algebraic consequence of Theorems 3.5 and 2.1. We work in some large existentially closed difference field $(\mathcal{U}, \sigma)$, of characteristic p; if $p > 0$, Frob denotes the map $x \mapsto x^p$ and if $p = 0$, the identity map.

Theorem 3.5′. *Let A, B be difference subfields of $\mathcal{U}$ intersecting in C, with algebraic closures intersecting in C^{alg}, and with $\mathrm{tr.deg}(A/C) < \infty$. Let $D \subset B$ be generated over C by all tuples d such that there exist an algebraically closed difference field F containing C and free from B over C, and integers $n > 0$ and m such that $d \in F(e)$ for some tuple e of elements satisfying $\sigma^n \mathrm{Frob}^m(x) = x$. Then A and B are free over D.*

The purely model theoretic result, Proposition 2.10, yields descent results for differential and difference varieties (Proposition 4.3 and Theorem 4.10). We state here a consequence in terms of algebraic dynamics:

Theorem 4.11. *Let K_1, K_2 be fields intersecting in k and with algebraic closures intersecting in k^{alg}; for $i = 1, 2$, let V_i be an absolutely irreducible variety and $\phi_i : V_i \to V_i$ a dominant rational map defined over K_i. Assume that K_2 is a regular extension of k and that there are an integer $r \geq 1$ and a dominant rational map $f : V_1 \to V_2$ such that $f \circ \phi_1 = \phi_2^{(r)} \circ f$ (where $\phi_2^{(r)}$ denotes the map obtained by iterating r times ϕ_2). Then there is a variety V_0 and a dominant rational map $\phi_0 : V_0 \to V_0$, all defined over k, a dominant map $g : V_2 \to V_0$ such that $g \circ \phi_2 = \phi_0 \circ g$ and $\deg(\phi_0) = \deg(\phi_2)$.*

The particular way this result is stated is motivated by a question of Szpiro and Tucker concerning descent for algebraic dynamics, arising out of Northcott's theorem for dynamics over function fields. Assume that K_2 is a function field over k, and that some *limited*[a] subset S of $V_2(K_2)$ satisfies that $\bigcap_{j=0}^{n} \phi^{(j)}(S)$ is Zariski dense in V_2 for every $n > 0$. One can then find (V_1, ϕ_1), r and f as above, so that our result applies to give a quotient (V_0, ϕ_0) of (V_2, ϕ_2) defined over the smaller field k and with $\deg(\phi_0) = \deg(\phi_2)$. Under certain hypotheses, one can even have this g be birational, see [5, 6].

This note originally contained a proof that a type analyzable in terms of one-based types is one-based. However, Wagner [19] found a much nicer proof, working in a more general context, so that this part of the note disappeared. A result of independent interest, Proposition A.3, obtained as a by-product of the study of one-based types and appearing in the Appendix, tells us that if p is a type of SU-rank ω^α for some ordinal α and with algebraically closed base of finite SU-rank, then

[a]See [5] for a definition.

there is a smallest algebraically closed set over which there is a type of SU-rank ω^α non-orthogonal to p. The condition of finite rank of the base is necessary.

The paper is organized as follows. Section 1 contains all definitions and preliminary results on supersimple theories, as well as the proof of the decomposition result Theorem 1.16. Section 2 contains various results which are consequences of the CBP. Section 3 shows that if K is an existentially closed difference field of any characteristic, then $\mathrm{Th}(K)$ has the CBP. Section 4 contains some applications of the CBP to differential and difference varieties. The paper concludes with the Appendix.

Some words on the chronology of the paper and results on the CBP. It all started with the result of Pillay and Ziegler [17], a result inspired by a result of Campana on compact complex spaces (see [16]), and which prompted me to look at the general case. The first version of this paper, which contained only Theorem 3.5, an old version of Theorem 2.1 and Proposition A.3, as well as the proof that a type analyzable in one-based types was one-based, was written in 2002. Almost instantly the result on analyzable one-based types was generalized by Wagner. The paper was submitted, but not accepted for several years. In the meantime, Moosa and Pillay, having read and believed the preprint, further investigated the CBP in [14]. Reading their preprint alerted me to the fact that the CBP might imply other stronger properties, as suggested by the fact that compact complex analytic spaces had the UCBP. Thus the material in Sec. 2 starting from Lemma 2.3, came later (end of 2008, and 2011). Independently, Prerna Juhlin ([11]) has obtained several results on theories with the CBP in her doctoral thesis (2010). Moosa studies in [13] variants of internality in the presence of the CBP. Palacín and Wagner continue and generalize the study of the CBP in [15]. Hrushovski ([9]) gives an example of an $\aleph_1$-categorical theory which does not have the CBP. This example now appears in a paper by Hrushovski, Palacín and Pillay [10].

1. Results on Supersimple Theories

Setting. We work in a model M (sufficiently saturated) of a complete theory T, which is supersimple and eliminates imaginaries. The results given below generalize easily to a simple theory eliminating hyperimaginaries, provided that some of the sets considered are ranked by the SU-rank.

Given (maybe infinite) tuples $a, b \in M$, we denote by $\overline{\mathrm{Cb}}(a/b)$ the smallest algebraically closed subset of M over which $tp(a/b)$ does not fork. Since our theory is supersimple, it coincides with the algebraic closure (in M^{eq}) of the usual canonical basis $\mathrm{Cb}(a/b)$ of $tp(a/b)$, and is contained in $\mathrm{acl}(b)$. For classical results on canonical bases and supersimple theories see e.g. Sec. 3.3 and Secs. 5.1–5.3 of [18].

Remark 1.1. We will use repeatedly the following consequences of our hypotheses on T:

(1) Let $B \subset M$, $a \in M$, and $(a_n)_{n\in\mathbb{N}}$ a sequence of B-independent realizations of $tp(a/B)$. Then for some m, $\overline{\mathrm{Cb}}(a/B)$ is contained in $\mathrm{acl}(a_1 \cdots a_m)$; for any n, $\mathrm{acl}(a_1 \cdots a_n) \cap B \subseteq \overline{\mathrm{Cb}}(a/B)$.

(2) Let $B \subset M$, $a \in M$, and $(a_n)_{n\in\mathbb{N}}$ a sequence of B-independent realizations of $tp(a/B)$. Let m be minimal such that $C = \overline{\mathrm{Cb}}(a/B) \subseteq \mathrm{acl}(a_1 \cdots a_m)$. Then $\mathrm{SU}(a_1/a_2 \cdots a_m) > \mathrm{SU}(a/B)$: otherwise $a_1 \downarrow_{a_2\cdots a_m} C$ would imply $C \subseteq \mathrm{acl}(a_2 \cdots a_m)$, and contradict the minimality of m.
(3) If A and B are algebraically closed subsets of M intersecting in C, and D is independent from AB over C, then $\mathrm{acl}(DA) \cap \mathrm{acl}(DB) = \mathrm{acl}(DC)$ (if $e \in \mathrm{acl}(DA) \cap \mathrm{acl}(DB)$, then $\overline{\mathrm{Cb}}(De/AB) \subseteq A \cap B = C$).

Internality and analyzability. In what follows, we will assume that $\mathcal{S}$ is a set of types with algebraically closed base and which is closed under $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-conjugation. Then non-orthogonality generates an equivalence relation on the regular types in $\mathcal{S}$. For more details, see Sec. 3.4 of [18].

Recall that if $a \in M$ and $A \subseteq M$, then $tp(a/A)$ is *$\mathcal{S}$-internal* [respectively, *almost-$\mathcal{S}$-internal*] if there is some set $B = \mathrm{acl}(B)$ containing A and independent from a over A, and a tuple $b_1, \ldots, b_n$ such that $a \in \mathrm{dcl}(Bb_1 \cdots b_n)$ [respectively, $a \in \mathrm{acl}(Bb_1 \cdots b_n)$], and each b_i realizes a type which is in $\mathcal{S}$ and has base contained in B.

$tp(a/A)$ is *$\mathcal{S}$-analyzable* if there are $a_1, \ldots, a_n$ such that $\mathrm{acl}(Aa_1 \cdots a_n) = \mathrm{acl}(Aa)$ and each $tp(a_i/Aa_1 \cdots a_{i-1})$ is $\mathcal{S}$-internal (or equivalently, each $tp(a_i/Aa_1 \cdots a_{i-1})$ is almost-$\mathcal{S}$-internal).

Observation 1.2. Let $A = \mathrm{acl}(A) \subset M$.

(1) If $tp(a_i/A)$ is almost-$\mathcal{S}$-internal for $i = 1, \ldots, n$, then so is $tp(a_1 \cdots a_n/A)$.
(2) If $tp(a/A)$ is almost-$\mathcal{S}$-internal, and $b \in \mathrm{acl}(Aa)$, then $tp(b/A)$ is almost-$\mathcal{S}$-internal.
(3) If $\mathcal{S}'$ is a set of types which are almost-$\mathcal{S}$-internal, and if p is almost-$\mathcal{S}'$-internal, then p is almost-$\mathcal{S}$-internal.
(4) Similarly for $\mathcal{S}$-analyzability.
(5) Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ be sets of types of SU-rank 1 which are closed under $\mathrm{Aut}(M)$-conjugation. If all types in $\mathcal{S}_1$ are orthogonal to all types in $\mathcal{S}_2$ (denoted by $\mathcal{S}_1 \perp \mathcal{S}_2$) and q_i is $\mathcal{S}_i$-analyzable for $i = 1, 2$, then all extensions of q_1 are orthogonal to all extensions of q_2.

One-basedness. Let $S \subseteq M^k$ be A-invariant. Then S is *one-based* (*over* A) if whenever b is a tuple of elements of S, and $B \supseteq A$ then b is independent from B over $\mathrm{acl}(Ab) \cap \mathrm{acl}(B)$. A type p (over A) is *one-based* if the set of its realizations is one-based over A.

Fact 1.3. (see [19]) (1) Let p be a type, and q a non-forking extension of p. Then p is one-based if and only if q is one-based. One-basedness is preserved under $\mathrm{Aut}(M)$-conjugation.
(2) A union of one-based sets is one-based.
(3) A type analyzable by one-based types is one-based.

Non-orthogonality and internality. Let $A = \mathrm{acl}(A) \subset M$, and a a tuple in M, with $\mathrm{SU}(a/A) = \beta + \omega^\alpha$ for some α, β. Then there is $B = \mathrm{acl}(B)$ containing A and independent from a over A, and b such that $\mathrm{SU}(b/B) = \omega^\alpha$ and $b \not\downarrow_B a$ (see [18], 5.1.12). Then $C = \overline{\mathrm{Cb}}(Bb/Aa)$ is contained in the algebraic closure of independent realizations of $tp(Bb/\mathrm{acl}(Aa))$, and therefore its type over A is almost internal to the set of conjugates of $tp(b/B)$ over A.

If $tp(b/B)$ is one-based, then $tp(C/A)$ is one-based (by Fact 1.3(3)), so that $\mathrm{acl}(Bb)$ and C are independent over their intersection D, and therefore $C = D$. Since $\mathrm{SU}(b/B) = \omega^\alpha$, a standard computation gives $\mathrm{SU}(C/A) = \omega^\alpha$.

Fact 1.4. Every type of finite SU-rank has a *semi-minimal analysis*, i.e. given $A = \mathrm{acl}(A)$ and a of finite SU-rank over A, there are tuples $a_1, \ldots, a_n$ such that $\mathrm{acl}(Aa_1 \cdots a_n) = \mathrm{acl}(Aa)$, and for every i, either $tp(a_i/Aa_1 \cdots a_{i-1})$ is one-based of SU-rank 1, or it is internal to the set of conjugates of some non-one-based type of SU-rank 1.

Definition 1.5. Let T be a simple theory, which eliminates imaginaries and hyperimaginaries. The theory T has the *CBP* if whenever A and B are algebraically closed sets of finite SU-rank over their intersection, and $A = \overline{\mathrm{Cb}}(B/A)$, then $tp(A/B)$ is almost-$\mathcal{S}$-internal, where $\mathcal{S}$ is the set of types of SU-rank 1 with algebraically closed base. (Actually, as we will see in Theorem 1.16, it suffices to take for $\mathcal{S}$ the set of non-one-based types of SU-rank 1 with algebraically closed base.)

One can also restrict this definition to smaller families of types: let $\mathcal{P}$ be a family of types of finite SU-rank and with algebraically closed base. We say that *$\mathcal{P}$ has the CBP* if whenever D is algebraically closed, b is a tuple of realizations of types in $\mathcal{P}$ with base contained in D, and $A = \overline{\mathrm{Cb}}(Db/AD)$, then $tp(A/\mathrm{acl}(Db))$ is almost-$\mathcal{S}$-internal, for the family $\mathcal{S} \subset \mathcal{P}$ of types in $\mathcal{P}$ of SU-rank 1 [and which are not one-based]. Thus Pillay and Ziegler show in [17] that the family of very thin types in separably closed fields of finite degree of imperfection has the CBP. See the concluding remarks at the end of Sec. 2 for a discussion.

Definition 1.6. Let p and q be types. We say that p is *hereditarily orthogonal* to q if every extension of p is orthogonal to q.

Lemma 1.7. *Let $E, B \subset M$ be algebraically closed sets, $b \in M$ a tuple. Assume that $tp(b/B)$ is almost-$\mathcal{S}$-internal, $E = \overline{\mathrm{Cb}}(Bb/E)$, and $\mathcal{S}$ is closed under* $\mathrm{Aut}(M/E)$*-conjugation. If $A = \overline{\mathrm{Cb}}(B/E)$, then $tp(E/A)$ is almost-$\mathcal{S}$-internal.*

Proof. Let $(B_1b_1), \ldots, (B_nb_n)$ be realizations of $tp(Bb/E)$ which are independent over E and such that $E \subseteq \mathrm{acl}(B_1b_1 \cdots B_nb_n)$. Since $B \downarrow_A E$, we get $B_1 \cdots B_n \downarrow_A E$; Observation 1.2(3) then gives the result. $\square$

Lemma 1.8. *Let $E, F \subset M$ be algebraically closed sets, with $\overline{\mathrm{Cb}}(E/F) = F$. If $E_0 = \overline{\mathrm{Cb}}(F/E)$, then $F = \overline{\mathrm{Cb}}(E_0/F)$.*

Proof. Let $F_0 = \overline{\mathrm{Cb}}(E_0/F)$. Then $E_0 \mathop{\smile\!\!\!\!\!\!|}_{F_0} F$ and $E \mathop{\smile\!\!\!\!\!\!|}_{E_0} F$, which imply $E \mathop{\smile\!\!\!\!\!\!|}_{E_0 F_0} F$ (since $F_0 \subseteq F$) and $E \mathop{\smile\!\!\!\!\!\!|}_{F_0} F$ by transitivity. Hence $F_0 = F$. □

Lemma 1.9. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of types of* SU*-rank* 1 *closed under* Aut(M)*-conjugation, with $\mathcal{S}_1 \perp \mathcal{S}_2$. Assume that $tp(E_i)$ is $\mathcal{S}_i$-analyzable for $i = 1, 2$, and that $D = \mathrm{acl}(D) \subseteq \mathrm{acl}(E_1E_2)$. Let $D_i = D \cap \mathrm{acl}(E_i)$ for $i = 1, 2$. Then $D = \mathrm{acl}(D_1D_2)$.*

Proof. Without loss of generality, each E_i is algebraically closed. Since $\overline{\mathrm{Cb}}(E_1/D)$ realizes an $\mathcal{S}_1$-analyzable type, it equals D_1 and hence $D \mathop{\smile\!\!\!\!\!\!|}_{D_1} E_1$. As $D \subseteq \mathrm{acl}(E_1E_2)$, this implies that $tp(D/D_1)$ is $\mathcal{S}_2$-analyzable. Hence so is $tp(D/D_1D_2)$. Similarly, $D \mathop{\smile\!\!\!\!\!\!|}_{D_2} E_2$ and $tp(D/D_1D_2)$ is $\mathcal{S}_1$-analyzable. Our hypothesis on the orthogonality of the members of $\mathcal{S}_1$ and those of $\mathcal{S}_2$ then implies $D \subseteq \mathrm{acl}(D_1D_2)$: a type which is $\mathcal{S}_1$-analyzable and $\mathcal{S}_2$-analyzable must be algebraic. □

Lemma 1.10. *Let $\mathcal{S}$ be a set of types of* SU*-rank* 1*, which is closed under* Aut(M)*-conjugation. Let $B \subset F$ and A be algebraically closed sets such that $tp(A)$ and $tp(B)$ are almost-$\mathcal{S}$-internal (respectively, $\mathcal{S}$-analyzable), and B is maximal contained in F with this property.*

(1) *Then $\mathrm{acl}(AB)$ is the maximal subset of $\mathrm{acl}(AF)$ whose type is almost-$\mathcal{S}$-internal (respectively, $\mathcal{S}$-analyzable).*
(2) *Let G be independent from F. Then $\mathrm{acl}(GB)$ is the maximal subset of $\mathrm{acl}(GF)$ whose type over G is almost-$\mathcal{S}$-internal (respectively, $\mathcal{S}$-analyzable).*

Proof. (1) Let $d \in \mathrm{acl}(AF)$ be such that $tp(d)$ is almost-$\mathcal{S}$-internal. Then so is the type of $\overline{\mathrm{Cb}}(Ad/F)$; hence $\overline{\mathrm{Cb}}(Ad/F) \subseteq B$ and $d \in \mathrm{acl}(AB)$. Same proof for $\mathcal{S}$-analyzable.
(2) Let $e \in \mathrm{acl}(GF)$ realize an almost-$\mathcal{S}$-internal type over G. As $G \mathop{\smile\!\!\!\!\!\!|} F$, $\overline{\mathrm{Cb}}(Ge/F)$ realizes an almost-$\mathcal{S}$-internal type, hence is contained in B. Hence $Ge \mathop{\smile\!\!\!\!\!\!|}_B F$, which implies $e \in \mathrm{acl}(GB)$. Same proof for $\mathcal{S}$-analyzable. □

The following result is well known, but for lack of a reference, we will give the proof.

Lemma 1.11. *Let p and q be types over sets A and B respectively, and assume that $p \not\perp q$. Then for some integer ℓ there are realizations $a_0, \dots, a_\ell$ of p, $b_0, \dots, b_\ell$ of q, such that the tuples a_i are independent over A, the tuples b_j are independent over B,*

$$a_0 \cdots a_\ell \mathop{\smile\!\!\!\!\!\!|}_A B, b_0 \cdots b_\ell \mathop{\smile\!\!\!\!\!\!|}_B A \quad \textit{and} \quad a_0 \cdots a_\ell \not\mathop{\smile\!\!\!\!\!\!|}_{AB} b_0 \cdots b_\ell.$$

Proof. By assumption there are some C containing A and B, and realizations a of p, b of q such that $a \mathop{\smile\!\!\!\!\!\!|}_A C$, $b \mathop{\smile\!\!\!\!\!\!|}_B C$ and $a \not\mathop{\smile\!\!\!\!\!\!|}_C b$. Let $D = \overline{\mathrm{Cb}}(a, b/C)$. Then for some ℓ there are independent realizations (a_i, b_i), $i = 1, \dots, \ell$, of $tp(a, b/C)$ such that

$D \subset \mathrm{acl}(ABa_1 \cdots a_\ell b_1 \cdots b_\ell)$ (by Remark 1.1(1)); we may choose these realizations to be independent from $(a, b) := (a_0, b_0)$ over C. Then

$$a_0 \not\perp_{ABa_1 \cdots a_\ell b_1 \cdots b_\ell} b_0.$$

As $a \perp_A C$, $b \perp_B C$ and C contains AB, the tuples a_i and b_j also satisfy the required first four conditions. Transitivity of independence then implies

$$a_0 \cdots a_\ell \not\perp_{AB} b_0 \cdots b_\ell. \qquad \square$$

Notation 1.12. Let p be a type of SU-rank 1 over some algebraically closed set, and let $C = \mathrm{acl}(C)$. We denote by $\mathcal{S}(p, C)$ the smallest set of types of SU-rank 1 with algebraically closed base, which contains p and is closed under $\mathrm{Aut}(M/C)$-conjugation. We write $\mathcal{S}(p)$ for $\mathcal{S}(p, \emptyset)$.

Remark 1.13. Let p and q be types of SU-rank 1, with algebraically closed base A and B respectively. Certainly if q is almost-$\{p\}$-internal, then $p \not\perp q$. If $A = B$, then the converse holds: $p \not\perp q$ iff q is almost-$\{p\}$-internal (iff p is almost-$\{q\}$-internal). If $A \neq B$, then $p \not\perp q$ implies that q is almost-$\mathcal{S}(p, B)$-internal, since any two realizations of q are in the same $\mathrm{Aut}(M/B)$-orbit; but in general, q will not be almost-$\{p\}$-internal.

In particular, if $q \not\perp p$, then q is almost-$\mathcal{S}(p)$-internal. Hence,

$$\text{either } \mathcal{S}(p) \perp \mathcal{S}(q), \quad \text{or every member of } \mathcal{S}(p) \text{ is } \mathcal{S}(q)\text{-internal}$$

(and every member of $\mathcal{S}(q)$ is $\mathcal{S}(p)$-internal).

In the rest of the first two sections of the paper, the letters $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}_1$, etc. will always denote sets of SU-rank 1 types with algebraically closed base.

We now start towards a proof of Theorem 1.16. It will reduce the problem of showing the CBP to showing it for $\{p\}$-analyzable types when p is a type of SU-rank 1 with algebraically closed base. This reduction is essential in the proof that existentially closed difference fields of positive characteristic have the CBP. We conclude the section with small partial results.

Proposition 1.14. *Let F and E be algebraically closed sets such that $F \cap E = C$, $\mathrm{SU}(E/C)$ and $\mathrm{SU}(F/C)$ are finite, $\overline{\mathrm{Cb}}(E/F) = F$ and $\overline{\mathrm{Cb}}(F/E) = E$. There are non one-based types $p_1, \ldots, p_m$ of SU-rank 1, algebraically closed sets $E_1, \ldots, E_m, F_1, \ldots, F_m$ such that for $i = 1, \ldots, m$ and letting $\mathcal{S}_i = \mathcal{S}(p_i, C)$,*

(i) *$tp(E_i/C)$ and $tp(F_i/C)$ are $\mathcal{S}_i$-analyzable, $\overline{\mathrm{Cb}}(E_i/F_i) = F_i$ and $\overline{\mathrm{Cb}}(F_i/E_i) = E_i$,*
(ii) $\mathrm{acl}(E_1 \cdots E_m) = E$ *and* $\mathrm{acl}(F_1 \cdots F_m) = F$.
(iii) *The sets E_i are independent over C, as well as the sets F_i.*

Proof. Assume the result is false, and take a counterexample with $\mathrm{SU}(EF/C)$ minimal among all possible (E, F, C), and among those, with $\mathrm{SU}(F/C) + \mathrm{SU}(E/C)$ minimal.

Let $p_1, \ldots, p_m$ be types of SU-rank 1 with algebraically closed base, which are pairwise orthogonal, such that each p_i is non-orthogonal to $tp(E/C)$ or to $tp(F/C)$, and such that any SU-rank 1 type which is non-orthogonal to one of $tp(E/C)$, $tp(F/C)$, is non-orthogonal to one of the types p_i (see Sec. 5.2 in [18]). We let $\mathcal{S}_i = \mathcal{S}(p_i, C)$, and E_i and F_i the maximal subsets of E and F respectively such that $tp(E_i/C)$ and $tp(F_i/C)$ are $\mathcal{S}_i$-analyzable. We need to show that no p_i is one-based, the second part of item (i), and item (ii) (item (iii) is immediate since the types p_i are pairwise orthogonal by Observation 1.2(5)).

Adding to the language constants symbols for the elements of C, *we will assume that* $C = \emptyset$.

We say that a set D *satisfies* $(*)$ *over a set* H if $D = \mathrm{acl}(HD_1 \cdots D_m)$, where $tp(D_i/H)$ is $\mathcal{S}_i$-analyzable for each i. If $H = \mathrm{acl}(\emptyset)$, we will simply say that D satisfies $(*)$. Note that by Lemma 1.9, any subset of $\mathrm{acl}(HD)$ whose type over H is $\mathcal{S}_i$-analyzable will be contained in $\mathrm{acl}(HD_i)$.

By Lemma 1.9, an algebraically closed subset of a set satisfying $(*)$ over H also satisfies $(*)$ over H, and the algebraic closure of a union of sets satisfying $(*)$ over H satisfies $(*)$ over H. Hence, if D satisfies $(*)$ over H, $J \supseteq H$, and $\overline{\mathrm{Cb}}(D/J) = J$, then J satisfies $(*)$ over H, as J is contained in the algebraic closure of finitely many realizations of $tp(D/H)$.

Assume that E satisfies $(*)$; then $F = \overline{\mathrm{Cb}}(E/F)$ satisfies $(*)$, and therefore can be written as $\mathrm{acl}(F'_1, \ldots, F'_m)$, where each $tp(F'_i)$ is $\mathcal{S}_i$-analyzable. Each F'_i is contained in F_i, and therefore $\mathrm{acl}(F_1 \cdots F_m) = F$ and $F_i = F'_i$. We know that F is contained in the algebraic closure of F-independent realizations of $tp(E/F)$. Lemma 1.9 then gives us that necessarily F_i is contained in the algebraic closure of F-independent realizations of $tp(E_i/F)$. Then Remark 1.1(1) implies that $\overline{\mathrm{Cb}}(E_i/F) \supseteq F_i$; the reverse inclusion holds since $tp(\overline{\mathrm{Cb}}(E_i/F))$ is $\mathcal{S}_i$-analyzable. By symmetry, $E_i = \overline{\mathrm{Cb}}(F_i/E)$. Furthermore, no p_i is one-based: otherwise, by Fact 1.3(3) $tp(E_i)$ would be one-based, whence $E \cap F = \mathrm{acl}(\emptyset)$ would yield $E_i \mathop{\smile\!\!\!\!\!\!\,|}\, F$, and therefore $E_i = F_i = \mathrm{acl}(\emptyset)$. This shows that if E satisfies $(*)$, then the conclusion of the lemma holds. By symmetry neither E nor F satisfies $(*)$.

Using the semi-minimal analysis of $tp(F)$, there is $B = \mathrm{acl}(B) \subset F$, $B \neq F$, and a type p of SU-rank 1, such that $tp(F/B)$ is almost-$\mathcal{S}(p)$-internal. Note that $B \neq \mathrm{acl}(\emptyset)$: otherwise $tp(F/C)$ would be $\mathcal{S}(p)$-internal, contradicting our assumption that F does not satisfy $(*)$. Let $A = \overline{\mathrm{Cb}}(B/E)$. Then $tp(E/A)$ is almost-$\mathcal{S}(p)$-internal by Lemma 1.7, so that $A \neq \mathrm{acl}(\emptyset)$.

Step 1. A satisfies $(*)$.

Let $B_0 = \overline{\mathrm{Cb}}(A/B)$. Then $B_0 \neq \mathrm{acl}(\emptyset)$ (because otherwise B and A would be independent), and $\overline{\mathrm{Cb}}(B_0/A) = A$ by Lemma 1.8. As $\mathrm{SU}(B_0) < \mathrm{SU}(F)$ and $\mathrm{acl}(AB_0) \subseteq \mathrm{acl}(EF)$, by induction hypothesis A satisfies $(*)$.

Thus $A \neq E$. Since $tp(E/A)$ is almost-$\mathcal{S}(p)$-internal, if $B_1 = \overline{\mathrm{Cb}}(A/F)$, then $tp(F/B_1)$ is almost-$\mathcal{S}(p)$-internal by Lemma 1.7, B_1 satisfies $(*)$, and $\mathrm{acl}(\emptyset) \neq B_1 \neq F$. Let E_p be the largest subset of E realizing an $\mathcal{S}(p)$-analyzable type. If p

is orthogonal to every p_i, then $E_p = \mathrm{acl}(\emptyset)$, by definition of the set $\{p_1, \ldots, p_m\}$. Otherwise, by Remark 1.13, $\mathcal{S}(p) = \mathcal{S}_i$ for some i, and therefore $E_p = E_i$; we will first show in the next two steps that this case is impossible.

Step 2. $\mathrm{acl}(FE_p) \cap E = E_p$.

Let $D = \mathrm{acl}(FE_p) \cap E$. If $D \neq E_p$, then, using the semi-minimal analysis of $tp(D/E_p)$, there is $d \in D \backslash E_p$ with $tp(d/E_p)$ almost-$\mathcal{S}(q)$-internal for some type q of SU-rank 1. By maximality of E_p, we have $\mathcal{S}(q) \perp \mathcal{S}(p)$. Since $tp(F/B_1)$ is almost-$\mathcal{S}(p)$-internal, we obtain $d \in \mathrm{acl}(B_1 E_p)$. Because B_1 and E_p satisfy $(*)$, and $tp(d/E_p) \perp p$, using Lemma 1.9 we may write $\mathrm{acl}(E_p d)$ as $\mathrm{acl}(E_p D_1)$, where $tp(D_1)$ is $\mathcal{S}(q)$-analyzable and hereditarily orthogonal to p. Thus $D_1 \mathop{\smile\!\!\!\!\!\!\mid}\, E_p$; because $D_1 \subseteq \mathrm{acl}(B_1 E_p)$, we obtain $D_1 \subseteq B_1$; as $D_1 \subseteq E$, this implies $D_1 = \mathrm{acl}(\emptyset)$, a contradiction.

Step 3. $E_p = \mathrm{acl}(\emptyset)$.

Let $D = \overline{\mathrm{Cb}}(E/FE_p)$. Then $\overline{\mathrm{Cb}}(E/D) = D$. Then $D \cap E = E_p$ by Step 2. Moreover, $\overline{\mathrm{Cb}}(D/E) = E$: let $E_0 = \overline{\mathrm{Cb}}(D/E)$; from $E \mathop{\smile\!\!\!\!\!\!\mid}_D F$ we deduce $E \mathop{\smile\!\!\!\!\!\!\mid}_{DE_0} F$; since $E \mathop{\smile\!\!\!\!\!\!\mid}_{E_0} D$, transitivity gives $E \mathop{\smile\!\!\!\!\!\!\mid}_{E_0} F$, and therefore $E = E_0$. Thus, if $E_p \neq \mathrm{acl}(\emptyset)$, then $\mathrm{SU}(ED/E_p) < \mathrm{SU}(EF)$ and by induction hypothesis (applied to D and E), E satisfies $(*)$ over E_p. Write $E = \mathrm{acl}(E'_p, E''_p)$, where E'_p satisfies $(*)$ over E_p, $tp(E'_p/E_p)$ is hereditarily orthogonal to all types in $\mathcal{S}(p)$, and $tp(E''_p/E_p)$ is $\mathcal{S}(p)$-analyzable. Then $tp(E''_p)$ is $\mathcal{S}(p)$-analyzable, so that $E''_p = E_p$. On the other hand, $tp(E/A)$ is $\mathcal{S}(p)$-analyzable, and therefore $E'_p \subseteq \mathrm{acl}(AE_p)$ (because $tp(E'_p/E_p)$ is hereditarily orthogonal to all members of $\mathcal{S}(p)$). Hence $E = \mathrm{acl}(AE_p)$ satisfies $(*)$, a contradiction.

By symmetry, if F_p is a subset of F whose type is $\mathcal{S}(p)$-analyzable, then $F_p \subseteq \mathrm{acl}(\emptyset)$.

Step 4. $F \subseteq \mathrm{acl}(B_1 E)$.

Let $D = \overline{\mathrm{Cb}}(F/B_1 E)$. Then $B_1 \subseteq D$, and $tp(D/B_1)$ is almost-$\mathcal{S}(p)$-internal, because it is contained in the algebraic closure of B_1-conjugates of F. Furthermore, we have $\overline{\mathrm{Cb}}(D/E) = E$: let $E_0 = \overline{\mathrm{Cb}}(D/E)$; from $E \mathop{\smile\!\!\!\!\!\!\mid}_D F$ we deduce $E \mathop{\smile\!\!\!\!\!\!\mid}_{DE_0} F$; then $E \mathop{\smile\!\!\!\!\!\!\mid}_{E_0} D$ yields $E \mathop{\smile\!\!\!\!\!\!\mid}_{E_0} F$, whence $E_0 = E$. We now let $D_1 = \overline{\mathrm{Cb}}(E/D)$; then $D_1 \subseteq D \subseteq \mathrm{acl}(B_1 E)$, $tp(D_1/B_1)$ is almost-$\mathcal{S}(p)$-internal, and $\overline{\mathrm{Cb}}(D_1/E) = E$ (by Lemma 1.8). Since E does not satisfy $(*)$, our induction hypothesis implies that either $E \cap D_1 \neq \mathrm{acl}(\emptyset)$ or $\mathrm{acl}(D_1 E) = \mathrm{acl}(EF)$.

Let us assume that $D_1 \cap E \neq \mathrm{acl}(\emptyset)$. Using the semi-minimal analysis of $tp(D_1 \cap E)$, there is $d \in D_1 \cap E$ with $tp(d)$ almost-$\mathcal{S}_i$-internal for some i. Since $E_p = F_p = \mathrm{acl}(\emptyset)$, we know that $\mathcal{S}(p) \perp \mathcal{S}_i$. But $tp(D_1/B_1)$ is almost-$\mathcal{S}(p)$-internal, so that $tp(d/B_1)$ is almost-$\mathcal{S}(p)$-internal, whence $d \in B_1$. Hence $D_1 \cap E \subseteq B_1 \cap E = \mathrm{acl}(\emptyset)$.

Hence $\mathrm{acl}(D_1 E) = \mathrm{acl}(EF)$, which implies $F \subseteq \mathrm{acl}(B_1 E)$.

The proof only used the $\mathcal{S}(p)$-internality of $tp(F/B_1)$, and we reason in the same manner with $\overline{\mathrm{Cb}}(E/AF)$ to get $E \subseteq \mathrm{acl}(AF)$. Since $E_p = F_p = \mathrm{acl}(\emptyset)$, we know that $\mathcal{S}(p) \perp \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_m$. The final contradiction will come from the following lemma, taking $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_m$:

Lemma 1.15. *Let $A \subseteq E$ and $B \subseteq F$ be algebraically closed sets of finite* SU-*rank with $E \cap F = \mathrm{acl}(\emptyset)$, such that E and F are equi-algebraic over AB. Assume that for some set $\mathcal{S}$ of types of* SU-*rank* 1, *which is closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-*conjugation, $tp(A)$ and $tp(B)$ are $\mathcal{S}$-analyzable. Then $tp(E/\mathrm{acl}(\emptyset))$ and $tp(F/\mathrm{acl}(\emptyset))$ are $\mathcal{S}$-analyzable.*

Proof. We may assume that A and B are maximal subsets of E and F respectively whose type are $\mathcal{S}$-analyzable. If $E = A$, then $F \subseteq \mathrm{acl}(AB)$, and we are done; similarly if $F = B$. Assume $E \neq A$, and let p be a type of SU-rank 1 which is non-orthogonal to $tp(E/A)$; we then let $E_0 \subseteq E$ and $F_0 \subseteq F$ be maximal such that $tp(E_0/A)$ and $tp(F_0/B)$ are almost $\mathcal{S}(p)$-internal. Then $E_0 \neq A$ (see the discussion in Fact 1.4) and $p \perp \mathcal{S}$. If $F_1 = \overline{\mathrm{Cb}}(E_0/F)$ and $B_1 = \overline{\mathrm{Cb}}(A/F)$, then $tp(F_1/B_1)$ is almost $\mathcal{S}(p)$-internal by Lemma 1.7, so that $F_1 \subseteq F_0$ and $E_0 \subseteq \mathrm{acl}(AF_0)$. Similarly, $F_0 \subseteq \mathrm{acl}(BE_0)$.

We have therefore shown that if the conclusion of the lemma does not hold, then there is a counterexample (E, F, A, B) where $tp(E/A)$ and $tp(F/B)$ are almost $\mathcal{S}(p)$-internal for some type p of SU-rank 1 which is orthogonal to all members of $\mathcal{S}$. We choose such a counterexample with $r = \mathrm{SU}(B) - \mathrm{SU}(B/A)$ minimal.

Let $E_0 = \overline{\mathrm{Cb}}(F/E)$, and $A_0 = A \cap E_0$. Then $F \subseteq \mathrm{acl}(BE_0)$ and $E_0 \subseteq \mathrm{acl}(AF)$. Also, A_0 is the maximal subset of E_0 with an $\mathcal{S}$-analyzable type (by Lemma 1.10), whence $A \mathop{\smile\!\!\!\!\!\!|}_{A_0} E_0$, and by transitivity $A \mathop{\smile\!\!\!\!\!\!|}_{A_0} E_0 F$, so that $E_0 \subseteq \mathrm{acl}(A_0 F)$. Since $F \neq B$, we have $E_0 \neq A_0$, so that $tp(E_0)$ is not $\mathcal{S}$-analyzable. Replacing E by E_0 and A by A_0, we may therefore assume that $E = \overline{\mathrm{Cb}}(F/E)$. (Note that $\mathrm{SU}(B/A_0) \geq \mathrm{SU}(B/A)$, so that $\mathrm{SU}(B) - \mathrm{SU}(B/A_0) \leq \mathrm{SU}(B) - \mathrm{SU}(B/A)$, and in fact equality holds by minimality of r).

If $r = 0$, then $B \mathop{\smile\!\!\!\!\!\!|} A$; because $tp(E/A)$ is orthogonal to all types in $\mathcal{S}$, we obtain $B \mathop{\smile\!\!\!\!\!\!|} E$; since $tp(F/B)$ is almost-$\mathcal{S}(p)$-internal and $E = \overline{\mathrm{Cb}}(F/E)$, we get that $tp(E)$ is almost-$\mathcal{S}(p)$-internal, a contradiction. Hence $r > 0$.

Since $E = \overline{\mathrm{Cb}}(F/E)$, there are E-independent realizations $F_1, \ldots, F_s$ of $tp(F/E)$ such that $E \subseteq \mathrm{acl}(F_1, \ldots, F_s)$. Let $B_i \subset F_i$ correspond to $B \subset F$. Since $tp(F_1, \ldots, F_s/B_1, \ldots, B_s)$ is orthogonal to all types in $\mathcal{S}$, we necessarily have $A \subset \mathrm{acl}(B_1, \ldots, B_s)$. Furthermore, from $B \mathop{\smile\!\!\!\!\!\!|}_A E$, the sets $B_1, \ldots, B_s$ are independent over A. This implies that $\overline{\mathrm{Cb}}(B/A) = A$ by Remark 1.1(1).

Let $m \leq s$ be minimal such that $A \subset \mathrm{acl}(B_1 \cdots B_m)$. Then $m > 1$ and $\mathrm{SU}(B_m) - \mathrm{SU}\big(B_m/B_1 \cdots B_{m-1}\big) < r$ by Remark 1.1(2). We also have $F_m \cap \mathrm{acl}(F_1 \cdots F_{m-1}) \subseteq F \cap E = \mathrm{acl}(\emptyset)$, and $E \subseteq \mathrm{acl}(B_1 \cdots B_m F_i)$ for every $1 \leq i \leq m$. Hence F_1 and F_m are equi-algebraic over $\mathrm{acl}(B_1 \cdots B_m)$. The induction hypothesis applied to the quadruple $(\mathrm{acl}(F_1 B_2 \ldots B_{m-1}), F_m, \mathrm{acl}(B_1 \cdots B_{m-1}), B_m)$ gives that $tp(F)$ is $\mathcal{S}$-analyzable, a contradiction. $\square$

This concludes the proof of Proposition 1.14. $\square$

Proposition 1.14 has the following immediate consequence:

Theorem 1.16. *Let E, F be algebraically closed sets, and assume that* $\mathrm{SU}(E/E\cap F)$ *is finite and* $F = \overline{\mathrm{Cb}}(E/F)$*. Then there are* $F_1, \ldots, F_m$ *independent over* $E\cap F$*, types* $p_1, \ldots, p_m$ *of* SU*-rank* 1*, such that each* $tp(F_i/E\cap F)$ *is* $\mathcal{S}(p_i, E\cap F)$*-analyzable, and* $\mathrm{acl}(F_1\cdots F_m) = F$*.*

Proof. By Remark 1.1, we know that $\mathrm{SU}(F/E\cap F)$ is also finite. Replace E by $E' = \overline{\mathrm{Cb}}(F/E)$; by Lemma 1.8, $F = \overline{\mathrm{Cb}}(E'/F)$. Then apply Proposition 1.14 to E', F to get the types p_i (which are pairwise orthogonal), and the sets F_i. □

Remark 1.17. Let E, F be as above. Using the semi-minimal analysis of $tp(F/E\cap F)$, there is some $G = \mathrm{acl}(G)$ independent from EF over $C = E\cap F$, and a tuple $a \in \mathrm{acl}(GF)$ of realizations of types of SU-rank 1 over G, such that for any tuple b, whenever $b \not\perp_G F$ then $b \not\perp_G a$ (in other words: *$tp(a/G)$ dominates $tp(F/G)$*, see Sec. 5.2 in [18]). Then working over G, the types p_i of Theorem 1.16 can be taken to be types over G (see the proof of Proposition 1.14), and the subsets F_i of $\mathrm{acl}(GF)$ will then realize $\{p_i\}$-analyzable types over $\mathrm{acl}(GC)$. This is slightly stronger than just saying that the sets F_i realize $\mathcal{S}(p_i)$-analyzable types.

The following result is similar to Proposition 2.2. See also Theorem 1.3 in [14].

Proposition 1.18. *Let $\mathcal{S}$ be a set of types of* SU*-rank* 1*, which is closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$*-conjugation, let B and E be algebraically closed sets of finite* SU*-rank, and assume that $tp(E/B)$ is $\mathcal{S}$-analyzable. Then so is $tp(E/E\cap B)$.*

Proof. Without loss of generality, $B = \overline{\mathrm{Cb}}(E/B)$. Let $C = E\cap B$, and assume that $tp(E/C)$ is not $\mathcal{S}$-analyzable. Let $D \subseteq E$ be maximal such that $tp(D/C)$ is $\mathcal{S}$-analyzable. As $B = \overline{\mathrm{Cb}}(E/B)$, Theorem 1.16 gives us two algebraically closed sets B_1 and B_2 with $\mathrm{acl}(B_1B_2) = B$, $tp(B_1/C)$ $\mathcal{S}$-analyzable, and $tp(B_2/C)$ $\mathcal{S}'$-analyzable for some set $\mathcal{S}'$ of SU-rank 1 types with algebraically closed base and such that $\mathcal{S} \perp \mathcal{S}'$. Then $\overline{\mathrm{Cb}}(D/B) \subseteq B_1$ and $E \perp_D B_1$ because $tp(E/D) \perp \mathcal{S}$ and $tp(B_1/C)$ is $\mathcal{S}$-analyzable. If $B_2 = C$, then $E \perp_D B$, and the $\mathcal{S}$-analyzability of $tp(E/DB)$ implies the $\mathcal{S}$-analyzability of $tp(E/D)$, a contradiction.

Hence $B_2 \neq C$, and if $E_2 = \overline{\mathrm{Cb}}(B_2/E)$, then $E_2 \neq C$ and E_2 realizes an $\mathcal{S}'$-analyzable type over C. As $E_2 \neq C$ and $E\cap B = C$, we have that $tp(E_2/B)$ is non-algebraic and $\mathcal{S}'$-analyzable. On the other hand, $tp(E_2/B)$ is also $\mathcal{S}$-analyzable because $E_2 \subseteq E$, which gives the final contradiction. □

Corollary 1.19. *Let $\mathcal{S}$ be a set of types of rank* 1 *closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$*-conjugation and let* $E = \mathrm{acl}(E)$ *have finite* SU*-rank. Then there is* $A = \mathrm{acl}(A) \subseteq \mathrm{acl}(E)$ *such that $tp(E/A)$ is $\mathcal{S}$-analyzable, and whenever* $B = \mathrm{acl}(B)$ *is such that $tp(E/B)$ is $\mathcal{S}$-analyzable, then $A \subseteq B$.*

Proof. It is enough to show that if A_1, A_2 are algebraically closed subsets of E such that $tp(E/A_i)$ is $\mathcal{S}$-analyzable, then so is $tp(E/A_1 \cap A_2)$: but this is obvious, as $tp(A_1/A_1 \cap A_2)$ is $\mathcal{S}$-analyzable, by Proposition 1.18. $\square$

Remark 1.20. Let $\mathcal{S}$ be a set of types of rank 1 closed under $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-conjugation, and let $E = \mathrm{acl}(E)$ have finite SU-rank. Then one can find $\mathcal{S}' \perp \mathcal{S}$, closed under $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-conjugation and such that $tp(E/\mathrm{acl}(\emptyset))$ is $(\mathcal{S} \cup \mathcal{S}')$-analyzable. It follows that for a set $B = \mathrm{acl}(B)$, $tp(E/B)$ will be hereditarily orthogonal to $\mathcal{S}'$ if and only if it is $\mathcal{S}$-analyzable. Thus the above two results can be stated in terms of hereditary orthogonality to $\mathcal{S}'$ instead of $\mathcal{S}$-analyzability.

We now state an easy lemma reducing further the problem of showing the CBP:

Lemma 1.21. *Let $\mathcal{S}$ be a set of types of* SU*-rank* 1, *which is closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$*-conjugation. Assume that there are algebraically closed sets E and F whose types over $C = E \cap F$ are $\mathcal{S}$-analyzable, such that $\overline{\mathrm{Cb}}(F/E) = E$ and $tp(E/C)$ is not almost $\mathcal{S}$-internal. Then there are such sets E and F whose types over C are $\mathcal{S}$-analyzable in at most two steps, i.e. there is $A \subset E$ such that $tp(A/C)$ and $tp(E/A)$ are almost-$\mathcal{S}$-internal, and similarly for F. Furthermore, $\overline{\mathrm{Cb}}(E/F) = F$.*

Proof. We take such a triple (E, F, C) with $r = \mathrm{SU}(E/C) + \mathrm{SU}(F/C)$ minimal, whence $F = \overline{\mathrm{Cb}}(E/F)$.

By the semi-minimal analysis of $tp(F/C)$, there is a proper algebraically closed subset B of F such that $tp(F/B)$ is almost-$\mathcal{S}$-internal. By Lemma 1.7, if $A = \overline{\mathrm{Cb}}(B/E)$ then $tp(E/A)$ is almost-$\mathcal{S}$-internal. As $\mathrm{SU}(B/C) < \mathrm{SU}(F/C)$, the minimality of r implies that $tp(A/C)$ is almost-$\mathcal{S}$-internal. Hence, $A \neq C, E$ because $tp(E/C)$ is not almost-$\mathcal{S}$-internal, and $tp(E/C)$ is $\mathcal{S}$-analyzable in two steps. Since F is contained in the algebraic closure of realizations of $tp(E/C)$, $tp(F/C)$ will also be $\mathcal{S}$-analyzable in two steps. $\square$

We conclude this section with a partial internality result:

Lemma 1.22. *Let $A \subseteq E$ and $B \subseteq F$ be algebraically closed sets of finite* SU*-rank with $E \cap F = \mathrm{acl}(\emptyset)$, such that E and F are equi-algebraic over AB. Assume that for some set $\mathcal{S}$ of types of* SU*-rank* 1, *which is closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$*-conjugation, $tp(A/\mathrm{acl}(\emptyset))$ and $tp(B/\mathrm{acl}(\emptyset))$ are almost-$\mathcal{S}$-internal. Then $tp(E/\mathrm{acl}(\emptyset))$ and $tp(F/\mathrm{acl}(\emptyset))$ are almost-$\mathcal{S}$-internal.*

Proof. We work over $\mathrm{acl}(\emptyset)$. By Lemma 1.15, we already know that $tp(E)$ and $tp(F)$ are $\mathcal{S}$-analyzable. Hence, reasoning as in the first paragraph of the proof of Lemma 1.15, we may assume that A, B are maximal subsets of E and F respectively which realize almost-$\mathcal{S}$-internal-types, and that $tp(E/A)$ and $tp(F/B)$ are almost-$\mathcal{S}$-internal, but neither $tp(E)$ nor $tp(F)$ is almost-$\mathcal{S}$-internal. The maximality of A and B implies that $A \perp\!\!\!\perp_B F$ and $E \perp\!\!\!\perp_A B$.

Working over some $C = \mathrm{acl}(C)$, independent from EF, and using Remark 1.1(3) and Lemma 1.10, we may assume that A is the algebraic closure of a tuple of realizations of types in $\mathcal{S}$. We choose a counterexample (E, F, A, B) with $r = \mathrm{SU}(A) - \mathrm{SU}(A/B)$ minimal. If $r = 0$, then $A \perp B$ so that $A \perp F$. Letting $F_0 = \overline{\mathrm{Cb}}(E/F)$, this implies that F_0 realizes an almost-$\mathcal{S}$-internal-type, hence is contained in B. But $E \perp_{F_0} F$ then implies $F \subseteq B$, which is absurd. So we may assume that $r > 0$.

Let $(F_i B_i)_{i>0}$ be a sequence of E-independent realizations of $tp(FB/E)$. If $A_0 = \overline{\mathrm{Cb}}(B/A)$, then for some $s > 1$, we have $\mathrm{acl}(B_1 \cdots B_s) \supseteq A_0$, and we take a minimal such s. As A is the algebraic closure of realizations of types of SU-rank 1, there is a finite tuple $a \subset A$ such that $a \perp A_0$ and $\mathrm{acl}(A_0 a) = A$. Then $a \perp A_0 B$, which implies $a \perp A_0 F$ (by transitivity and because $A \perp_B F$).

Let $A' = \mathrm{acl}(aB_1)$, $B' = \mathrm{acl}(B_2 \cdots B_s)$, $E' = \mathrm{acl}(A'F_1)$ and $F' = \mathrm{acl}(B'F_s)$. Then $a \perp A_0 F$, and $a \perp B_1 F'$. Hence in particular, $A' \cap F' = \mathrm{acl}(\emptyset)$ (use Remark 1.1(3) and the fact that $B_1 \cap B' = \mathrm{acl}(\emptyset)$). Moreover,

$$\mathrm{SU}(A') - \mathrm{SU}(A'/B') = \mathrm{SU}(B_1) - \mathrm{SU}(B_1/B_2 \cdots B_s) < r$$

by Remark 1.1(2), and E' and F' are equi-algebraic over $A'B'$. In order to reach a contradiction, it therefore suffices to show that $E' \cap F' = \mathrm{acl}(\emptyset)$: our induction hypothesis gives that $tp(E/\mathrm{acl}(\emptyset))$ is almost-$\mathcal{S}$-internal, which implies that $tp(F_1/\mathrm{acl}(\emptyset)) = tp(F/\mathrm{acl}(\emptyset))$ is also almost-$\mathcal{S}$-internal.

By Lemma 1.10, A' and B' are maximal subsets of E', F' respectively which realize almost-$\mathcal{S}$-internal-types. Assume $E' \cap F' \neq \mathrm{acl}(\emptyset)$; by the semi-minimal analysis of $tp((E' \cap F')/\mathrm{acl}(\emptyset))$, there is $d \in E' \cap F'$ realizing an almost-$\mathcal{S}$-internal-type. Then

$$d \in B' \cap A' \subseteq F' \cap A' = \mathrm{acl}(\emptyset),$$

which gives us the desired contradiction. □

2. Further Properties of Theories with the CBP

Description of the results of this section. Assumptions on M and T are as in the previous section: T is supersimple and eliminates imaginaries, M is sufficiently saturated. Most results are proved under the additional hypothesis of the CBP. We start by proving one of the main results of the paper:

Theorem 2.1. (CBP) *If E and F are algebraically closed sets of finite* SU*-rank over their intersection C and are such that $E = \overline{\mathrm{Cb}}(F/E)$, then $tp(E/C)$ is almost-$\mathcal{S}$-internal for some family $\mathcal{S}$ of types of* SU*-rank* 1.

Note that under the same hypotheses, if $tp(E/F)$ is $\mathcal{S}'$-analyzable for some set $\mathcal{S}'$ of types of SU-rank 1 with algebraically closed base, then $tp(E/C)$ is almost-$\mathcal{S}'$-internal. We then show

Theorem 2.4. (CBP) *Assume that $E = \mathrm{acl}(E)$ has finite* SU*-rank, and let $\mathcal{S}$ be a collection of types of* SU*-rank* 1, *closed under conjugation. Then there is $A =$*

$\mathrm{acl}(A) \subseteq E$ *such that* $tp(E/A)$ *is almost-*$\mathcal{S}$*-internal, and whenever* $B = \mathrm{acl}(B)$ *is such that* $tp(E/B)$ *is almost-*$\mathcal{S}$*-internal, then* $B \supseteq A$.

An immediate consequence of Theorem 2.4 is that the CBP implies the Uniform CBP (UCBP); this answers a question of Moosa and Pillay [14]. We end the section with three results, which were proved with an eye towards geometric applications. The first result is valid in a general setting (as will be clear from the proof), and can be viewed as showing the existence of a "largest internal quotient"; the second can be viewed as showing the existence of a "maximal internal fiber", and the third one as a descent result.

Theorem 2.1. (CBP) *If* E *and* F *are algebraically closed sets of finite* SU*-rank over their intersection* C *and are such that* $E = \overline{\mathrm{Cb}}(F/E)$, *then* $tp(E/C)$ *is almost-*$\mathcal{S}$*-internal for some family* $\mathcal{S}$ *of types of* SU*-rank* 1.

Proof. We assume the result false. By Lemma 1.21, there is a counterexample (E, F, C) with $tp(E/C)$ and $tp(F/C)$ $\mathcal{S}$-analyzable in two steps, and which also satisfies $\overline{\mathrm{Cb}}(E/F) = F$. By Theorem 1.16 (see also Remarks 1.17 and 1.1(3)), working over a larger set $G = \mathrm{acl}(G)$, we can write E as $\mathrm{acl}(E_1 \cdots E_m)$ for some sets E_i which are independent over G, realize $\{p_i\}$-analyzable types over G, and some E_i will not realize an almost-$\{p_i\}$-internal type over G.

Hence, we may assume that $\mathcal{S} = \{p\}$ for some type p of SU-rank 1. *For ease of notation we will assume that the language contains constant symbols for the elements of* G.

Let $A_0 \subset E$ and $B \subset F$ be maximal realizing almost-$\mathcal{S}$-internal types, so that $tp(E/A_0)$ and $tp(F/B)$ are almost-$\mathcal{S}$-internal. Then $E \mathop{\smile\!\!\!\!\!\!\mid}_{A_0} B$ since $\overline{\mathrm{Cb}}(B/E)$ is almost-$\mathcal{S}$-internal and therefore contained in A_0, and similarly $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_B F$. Enlarging G (and using Remark 1.1(3)), we may assume that A_0 and B are the algebraic closures of tuples of realizations of p. Let $A = \overline{\mathrm{Cb}}(B/E)$. Then $A \subseteq A_0$, and $tp(E/A)$ is almost-$\mathcal{S}$-internal (by Lemma 1.7).

The proof is by induction on $r = \mathrm{SU}(B) - \mathrm{SU}(B/A_0)$ $(= \mathrm{SU}(A_0) - \mathrm{SU}(A_0/B))$. If $r = 0$, then $A_0 \mathop{\smile\!\!\!\!\!\!\mid} B$ and from $tp(F/B)$ almost-$\mathcal{S}$-internal and $\overline{\mathrm{Cb}}(F/E) = E$ we deduce that $tp(E)$ is almost-$\mathcal{S}$-internal, a contradiction. Hence $r > 0$.

Step 1. We may assume $A_0 = A$.

We know that $A_0 = \mathrm{acl}(a_0)$ for some tuple a_0 of realizations of p; take $a \subseteq a_0$ maximal independent over A and such that $a \mathop{\smile\!\!\!\!\!\!\mid} A$. Then

$$\mathrm{acl}(Aa) = A_0, \quad a \mathop{\smile\!\!\!\!\!\!\mid} AF \quad \text{and} \quad \mathrm{SU}(B/a) - \mathrm{SU}(B/A_0) = r.$$

Furthermore, $tp(E/a)$ is not almost-$\mathcal{S}$-internal: otherwise, $\overline{\mathrm{Cb}}(E/Fa)$ would also be almost-$\mathcal{S}$-internal, hence contained in $\mathrm{acl}(Ba)$ by maximality of B (see Lemma 1.10(1)); from $E \mathop{\smile\!\!\!\!\!\!\mid}_{Ba} F$ and $F \mathop{\smile\!\!\!\!\!\!\mid}_B A_0$ we would then deduce $E \mathop{\smile\!\!\!\!\!\!\mid}_B F$, i.e. $F = B$, which is absurd. We will now show that $E \cap \mathrm{acl}(Fa) = \mathrm{acl}(a)$. Enlarging G, this will allow us to assume $A = A_0$.

Let $D = E \cap \mathrm{acl}(Fa)$. Since $a \mathop{\smile\!\!\!\!\!\!\mid} AF$, $A_0 \cap \mathrm{acl}(Fa) = \mathrm{acl}(a)$ by Remark 1.1(3). The set $\overline{\mathrm{Cb}}(B/D)$ is almost-$\mathcal{S}$-internal, hence contained in $A_0 \cap \mathrm{acl}(Fa) = \mathrm{acl}(a)$, so that $B \downarrow D$ because $F \downarrow a$. From $D \subset \operatorname{acl}(Fa)$ and the almost-$\mathcal{S}$-internality of $tp(Fa/B)$ we obtain that $tp(D/B)$ is almost-$\mathcal{S}$-internal, and therefore also $tp(D)$, so that $D \subseteq A_0 \cap \operatorname{acl}(Fa) = \operatorname{acl}(a)$.

Step 2. We may assume $E \subseteq \operatorname{acl}(AF)$.

By assumption, there is an algebraically closed set J containing F, such that $J \downarrow_F E$, and a tuple g of realizations of p such that $E \subseteq \operatorname{acl}(Jg)$. Then there is a subset e of g, consisting of independent tuples over AJ, and such that $E \subseteq \operatorname{acl}(AJe)$ and $e \downarrow AJ$. Since $J \supseteq F$ and $e \downarrow J$, we then have $\overline{\operatorname{Cb}}(Je/Ee) = \operatorname{acl}(Ee)$ and $\overline{\operatorname{Cb}}(Be/Ae) = \overline{\operatorname{Cb}}(Be/Ee) = \operatorname{acl}(Ae)$ (use $\overline{\operatorname{Cb}}(J/E) = E$ and Remark 1.1(1)).

Claim. $\operatorname{acl}(Ee) \cap \operatorname{acl}(Je) = \operatorname{acl}(e)$.

Let $D = \operatorname{acl}(Ee) \cap \operatorname{acl}(Je)$. Since $e \downarrow AJ$, we obtain $\operatorname{acl}(Ae) \cap \operatorname{acl}(Je) = \operatorname{acl}(e)$ by Remark 1.1(3).

We know that if $D_0 = \overline{\operatorname{Cb}}(A/D)$, then $tp(D_0)$ is almost-$\mathcal{S}$-internal; the maximal almost-$\mathcal{S}$-internal subset of $\operatorname{acl}(Ee)$ is $\operatorname{acl}(Ae)$ by Lemma 1.10(1), and therefore $D_0 \subseteq \operatorname{acl}(Ae) \cap \operatorname{acl}(Je) = \operatorname{acl}(e)$. Hence $A \downarrow D$, and $tp(D)$ is almost-$\mathcal{S}$-internal (because $D \subset \operatorname{acl}(Ee)$ and $tp(Ee/A)$ is almost-$\mathcal{S}$-internal). Reasoning as we did for D_0, we obtain $D \subseteq \operatorname{acl}(Ae) \cap \operatorname{acl}(Je) = \operatorname{acl}(e)$.

From $E \downarrow_F J$, $e \downarrow AJ$ and $A \downarrow_B F$ we deduce

$$\operatorname{SU}(A/e) = \operatorname{SU}(A) \quad \text{and} \quad \operatorname{SU}(A/Je) = \operatorname{SU}(A/J) = \operatorname{SU}(A/F) = \operatorname{SU}(A/B),$$

so that

$$\operatorname{SU}(A/e) - \operatorname{SU}(A/Je) = \operatorname{SU}(A) - \operatorname{SU}(A/B) = \operatorname{SU}(B) - \operatorname{SU}(B/A) = r.$$

Because $e \downarrow A$ and by maximality of A, we get $e \downarrow E$; thus $tp(E/e)$ is not almost-$\mathcal{S}$-internal. As we saw above, we have $\overline{\operatorname{Cb}}(Je/Ee) = \operatorname{acl}(Ee)$. Hence, working over $\operatorname{acl}(e)$ and replacing F by J, we may assume $E \subseteq \operatorname{acl}(AF)$.

Step 3. The final contradiction.

Let $(F_nB_n)_{n\in\mathbb{N}}$ be a sequence of E-independent realizations of $tp(FB/E)$. From $B \downarrow_A E$, it follows that the sets B_n are independent over A. By Remark 1.1(1), and because $A = \overline{\operatorname{Cb}}(B/A)$, there is m such that $A \subset \operatorname{acl}(B_1 \cdots B_m)$; take the minimal such m. Then $E \subset \operatorname{acl}(B_1 \cdots B_m F_i)$ for every i, so that in particular $F_1 \not\downarrow_{B_1 \cdots B_m} F_m$. On the other hand, we know that

$$F_1 \cap \operatorname{acl}(B_2 \cdots B_{m-1}F_m) \subseteq F \cap E = \operatorname{acl}(\emptyset),$$

and $\operatorname{SU}(B_1) - \operatorname{SU}(B_1/(B_2 \cdots B_m)) < r$ by minimality of m. We apply the induction hypothesis to (F_1, B_1) and $(\operatorname{acl}(B_2 \cdots B_{m-1}F_m), \operatorname{acl}(B_2 \cdots B_m))$: if $J = \overline{\operatorname{Cb}}((B_2 \cdots B_{m-1}F_m)/F_1)$, then $J \not\subseteq B_1$ and $tp(J)$ is almost-$\mathcal{S}$-internal. This contradicts the maximality of B, and finishes the proof. $\square$

We will now prove some more results for supersimple theories with the CBP. Note that Proposition 2.2 below implies Theorem 2.1 and is therefore equivalent to it. It was first proved by Moosa and Pillay in the stable context, see [14].

Proposition 2.2. (CBP) *Let B and E be algebraically closed sets, with* $\mathrm{SU}(E) < \infty$ *and assume that $tp(E/B)$ is almost-$\mathcal{S}$-internal, for some collection $\mathcal{S}$ of types of* SU-*rank* 1, *which is closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-*conjugation. Then $tp(E/(E \cap B))$ is almost-$\mathcal{S}$-internal.*

Proof. Let $C = B \cap E$, and let $A \subseteq E$ be maximal such that $tp(A/C)$ is almost-$\mathcal{S}$-internal. If $B_0 = \overline{\mathrm{Cb}}(E/B)$, then $tp(E/B_0)$ is also almost-$\mathcal{S}$-internal, and we may therefore assume that $B = \overline{\mathrm{Cb}}(E/B)$. By Proposition 1.18, we know that $tp(E/C)$ is $\mathcal{S}$-analyzable, and this implies that $tp(B/C)$ is also $\mathcal{S}$-analyzable. On the other hand, by Theorem 2.1, $tp(B/C)$ is almost-$\mathcal{S}'$-internal, for some collection $\mathcal{S}'$ of types of SU-rank 1 containing $\mathcal{S}$, and these two facts imply that $tp(B/C)$ is almost-$\mathcal{S}$-internal.

Assume $E \neq A$. By assumption, there is some $F = \mathrm{acl}(F)$, independent from E over B, and such that E is equi-algebraic over F with some finite tuple of realizations of types in $\mathcal{S}$.

Claim. $\mathrm{acl}(AF) \neq \mathrm{acl}(EF)$.

Otherwise, $A \subseteq E$ and $E \mathop{\smile\!\!\!\!\!\!\;|}_B F$ would imply $E \subseteq \mathrm{acl}(AB)$. As $tp(B/C)$ is almost-$\mathcal{S}$-internal, this would imply that also $tp(E/C)$ is almost-$\mathcal{S}$-internal, a contradiction.

We may therefore choose some $e \in \mathrm{acl}(EF) \backslash \mathrm{acl}(AF)$ which realizes a type in $\mathcal{S}$. Then $E_0 = \overline{\mathrm{Cb}}(Fe/E) \not\subseteq A$, since $e \in \mathrm{acl}(FE_0) \backslash \mathrm{acl}(FA)$. Note that $E \cap F = E \cap B = C$.

Let $D = \mathrm{acl}(Fe) \cap E$. Then $D \cap F = C$, and by Theorem 2.1 $tp(E_0/D)$ is almost-$\mathcal{S}$-internal (because $E_0 \subseteq E$ and $tp(E/C)$ is $\mathcal{S}$-analyzable). If $D = C$, this gives us the desired contradiction, as $E_0 \not\subseteq A$, and A was maximal contained in E with $tp(A/C)$ almost-$\mathcal{S}$-internal.

Assume therefore that $D \neq C$. Then $\mathrm{SU}(D/F) = 1$, because $\mathrm{SU}(e/F) = 1$ and $D \subset \mathrm{acl}(Fe)$. If $D \mathop{\smile\!\!\!\!\!\!\;|}_C F$, then $\mathrm{SU}(D/C) = 1$, which implies that $tp(D/C)$ is almost-$\mathcal{S}$-internal. In that case we let $D_0 = D$. If $D \not\mathop{\smile\!\!\!\!\!\!\;|}_C F$, we define $D_0 = \overline{\mathrm{Cb}}(F/D)$; then $tp(D_0/(D \cap F))$ is almost-$\mathcal{S}$-internal by Theorem 2.1. Hence, as $D_0 \subseteq \mathrm{acl}(Fe)$, and $D_0 \not\subseteq F$, we have that $e \in \mathrm{acl}(FD_0)$, and $tp(D_0/C)$ is almost-$\mathcal{S}$-internal. As $D_0 \subseteq D \subseteq E$, we obtain $D_0 \subseteq A$, whence $e \in \mathrm{acl}(FA)$, which gives us the desired contradiction and finishes the proof. □

Lemma 2.3. (CBP) *Let* $E = \mathrm{acl}(E)$ *be of finite* SU-*rank over some* $C = \mathrm{acl}(C)$, *and let $\mathcal{S}$ be a collection of types of* SU-*rank* 1, *closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-*conjugation. Assume that* $A_i = \mathrm{acl}(A_i) \subset E$, $i = 1, 2$, *are such that* $A_1 \cap A_2 = C$, *and $tp(E/A_i)$ is almost-$\mathcal{S}$-internal for $i = 1, 2$. Then $tp(E/C)$ is almost-$\mathcal{S}$-internal.*

Proof. Let $A \subseteq E$ be maximal such that $tp(A/C)$ is almost-$\mathcal{S}$-internal. Then $A_1 \subseteq A$: by hypothesis, $tp(A_1/A_2)$ is almost-$\mathcal{S}$-internal, and by Proposition 2.2, $tp(A_1/C)$

is almost-$\mathcal{S}$-internal. Reasoning similarly with A_2, we obtain that $A_1A_2 \subseteq A$. If $F = \mathrm{acl}(F) \supset C$ is independent from E over C, and $tp(E/F)$ is almost-$\mathcal{S}$-internal, then so is $tp(E/C)$, and we may therefore extend C, to a larger set over which A is equi-algebraic with a tuple of realizations of types in $\mathcal{S}$ (by Lemma 1.10(2), we will not lose the maximality of A). Hence, we may assume that in A there is a tuple a of realizations of types in $\mathcal{S}$ such that $a \mathop{\smile\!\!\!\!\!\!|}_C A_1A_2$, and $A = \mathrm{acl}(CA_1A_2a)$. Note that we still have $\mathrm{acl}(A_1a) \cap A_2 = C$: since $a \mathop{\smile\!\!\!\!\!\!|}_C A_1A_2$, we know by Remark 1.1(3) that $\mathrm{acl}(A_1a) \cap \mathrm{acl}(A_2a) = \mathrm{acl}(Ca)$; hence $\mathrm{acl}(A_1a) \cap A_2 \subseteq \mathrm{acl}(Ca) \cap A_2 = C$. Thus, replacing A_1 by $\mathrm{acl}(A_1a)$ we may assume that $\mathrm{acl}(A_1A_2) = A$.

By assumption, for $i = 1, 2$, there are $F_i = \mathrm{acl}(F_i)$ containing A_i, independent from E over A_i, and such that E is equi-algebraic over F_i with some tuple b_i of realizations of types in $\mathcal{S}$. We may choose F_2 independent from EF_1 over A_2; then $F_1 \mathop{\smile\!\!\!\!\!\!|}_E F_2$, whence also F_1 is independent from EF_2 over A_1, and

$$C = A_1 \cap A_2 = F_1 \cap F_2; \quad \mathrm{acl}(F_1b_1) \cap F_2 = A_2; \quad F_1 \cap \mathrm{acl}(F_2b_2) = A_1$$

(use $\mathrm{acl}(F_ib_i) = \mathrm{acl}(F_iE)$, $E \cap F_j = A_j$). For $i = 1, 2$, choose $e_i \subset b_i$ maximal independent over F_iA. Then $E \subseteq \mathrm{acl}(F_iAe_i)$, and $A \cap \mathrm{acl}(F_ie_i) = A_i$. Furthermore

$$\mathrm{acl}(F_1e_1) \cap F_2 = F_1 \cap \mathrm{acl}(F_2e_2) = A_1 \cap A_2 = C.$$

Let $D_0 = \mathrm{acl}(F_1e_1) \cap \mathrm{acl}(F_2e_2)$. As $D_0 \subseteq \mathrm{acl}(F_1e_1)$, $tp(D_0/F_1)$ is almost-$\mathcal{S}$-internal; by Proposition 2.2, $tp(D_0/(D_0 \cap F_1))$ is almost-$\mathcal{S}$-internal; hence $tp(D_0/C)$ is almost-$\mathcal{S}$-internal because $\mathrm{acl}(F_2e_2) \cap F_1 = C$, and this implies that $D_0 \cap E \subseteq A$. Therefore

$$D_0 \cap E = D_0 \cap A = \mathrm{acl}(F_1e_1) \cap \mathrm{acl}(F_2e_2) \cap A = A_1 \cap A_2 = C.$$

Let $D_1 = \overline{\mathrm{Cb}}(F_1e_1/F_2e_2)$. Then $tp(D_1/D_0)$ is almost-$\mathcal{S}$-internal, by Theorem 2.1. We know that F_1e_1 and F_2e_2 are independent over D_1, and therefore

$$F_1e_1 \mathop{\smile\!\!\!\!\!\!|}_{D_1A_1A_2} F_2e_2.$$

Since $\mathrm{acl}(A_1A_2) = A$ and $E \subseteq \mathrm{acl}(F_iAe_i)$, we get $E \subseteq \mathrm{acl}(D_1A)$. Hence $tp(E/D_0)$ is almost-$\mathcal{S}$-internal, and so is $tp(E/D_0 \cap E)$ (by Theorem 2.2). As $D_0 \cap E = C$, we get the result. □

Theorem 2.4. (CBP) *Assume that* $E = \mathrm{acl}(E)$ *has finite* SU-*rank, and let* $\mathcal{S}$ *be a collection of types of* SU-*rank* 1, *closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-*conjugation. Then there is* $A = \mathrm{acl}(A) \subseteq E$ *such that* $tp(E/A)$ *is almost-*$\mathcal{S}$*-internal, and whenever* $B = \mathrm{acl}(B)$ *is such that* $tp(E/B)$ *is almost-*$\mathcal{S}$*-internal, then* $B \supseteq A$.

Proof. This follows immediately from Proposition 2.2 and Lemma 2.3. □

Theorem 2.5. (CBP) *Let* $B = \overline{\mathrm{Cb}}(A/B)$, *where* $A = \mathrm{acl}(A)$ *has finite* SU-*rank, and let* $\mathcal{S}$ *be a collection of types of* SU-*rank* 1, *closed under* $\mathrm{Aut}(M/\mathrm{acl}(\emptyset))$-*conjugation, and such that* $tp(B/A)$ *is almost-*$\mathcal{S}$*-internal. If* $C = \mathrm{acl}(C)$ *is such that* $tp(A/C)$ *is almost-*$\mathcal{S}$*-internal, then so is* $tp(AB/C)$. *That is,* T *has the UCBP.*

Proof. Let $D = \overline{\mathrm{Cb}}(B/A)$. Then $tp(D/B)$ is almost-$\mathcal{S}$-internal, and $B = \overline{\mathrm{Cb}}(D/B)$ (by Lemma 1.8). As $D \subseteq A$, $tp(D/C)$ is almost-$\mathcal{S}$-internal, and by Lemma 2.3, so is $tp(D/(B \cap C))$; this implies that $tp(B/C)$ is also almost-$\mathcal{S}$-internal, since B is contained in the algebraic closure of realizations of the almost-$\mathcal{S}$-internal-type $tp(D/(B \cap C))$ (see Remark 1.1(1)). □

Proposition 2.6. (CBP) *Let G be a group of finite* SU*-rank, let p be a type* (*over* $\emptyset$) *realized by $a \in G$, and let $H = \mathrm{Stab}(p)$ be the left stabilizer of p. If d is the code of $H \cdot a$, then $tp(d)$ is almost-$\mathcal{S}$-internal, where $\mathcal{S}$ is the collection of nonlocally modular types of* SU*-rank* 1 *and with algebraically closed base.*

Proof. The proof is essentially identical to the one given in [17], Corollary 3.11, where it was done in the stable case. Let $c \in G$ be a generic of G over a, and let $D = \overline{\mathrm{Cb}}(c/e)$, where $e = a \cdot c$.

We will first show that $d \in \mathrm{acl}(Dc)$. By genericity of c, we know that $e \mathop{\smile\!\!\!\!\!\!\;|}\, a$, and therefore $a \mathop{\smile\!\!\!\!\!\!\;|}\, De$. The set D has the following property: if e_1, e_2 are D-independent realizations of $tp(e/D)$, then there is c' independent from e_1e_2 over D, and such that $tp(c'e_i/D) = tp(ce/D)$ for $i = 1, 2$. If $a_i = e_i \cdot c'^{-1}$, then $e_1 \cdot e_2^{-1} = a_1 \cdot a_2^{-1}$, and a_1, a_2 realize p. We then deduce successively the following relations:

$$c' \mathop{\smile\!\!\!\!\!\!\;|}_D e_1e_2; \quad c' \mathop{\smile\!\!\!\!\!\!\;|}_D (e_1 \cdot e_2^{-1})e_2; \quad c' \mathop{\smile\!\!\!\!\!\!\;|}_{De_2} e_1 \cdot e_2^{-1}; \quad a_2 \mathop{\smile\!\!\!\!\!\!\;|}_{De_2} e_1 \cdot e_2^{-1};$$

since $a_2 \mathop{\smile\!\!\!\!\!\!\;|}\, De_2$, transitivity implies $a_2 \mathop{\smile\!\!\!\!\!\!\;|}\, De_1 \cdot e_2^{-1}$. As both a_1 and a_2 realize p, and $a_1 = e_1 \cdot e_2^{-1} \cdot a_2$, we get that $e_1 \cdot e_2^{-1} \in H$. So we have shown that if e_1, e_2 are any D-independent realizations of $tp(e/D)$, then $e_1 \cdot e_2^{-1} \in H$. Hence, if e_1 and e_2 realize $tp(e/D)$, then $e_1 \cdot e_2^{-1} \in H$.

If $\tau \in \mathrm{Aut}(M/Dc)$, then $\tau(e) \cdot e^{-1} \in H$, and $\tau(a) = \tau(e) \cdot e^{-1} \cdot a \in H \cdot a$. This shows that $d \in \mathrm{acl}(Dc)$.

By the CBP, we know that $tp(D/\mathrm{acl}(c))$ is almost-$\mathcal{S}$-internal, and therefore so is $tp(d/\mathrm{acl}(c))$. But on the other hand, we know that $d \in \mathrm{acl}(a)$ and $a \mathop{\smile\!\!\!\!\!\!\;|}\, c$: hence $d \mathop{\smile\!\!\!\!\!\!\;|}\, c$ and $tp(d)$ is almost-$\mathcal{S}$-internal. □

Corollary 2.7. (CBP) *Let G be a group of finite* SU*-rank, and p a type over $\emptyset$, realized by $a \in G$. Let $b \in \mathrm{dcl}(a)$ be maximal realizing an almost-$\mathcal{S}$-internal-type, let $S = \{g \in G \,|\, tp((g \cdot a)/b) = tp(a/b)\}$, and let N be the subgroup of G generated by S. Then $N \subseteq H$, where H is the left stabiliser of p.*

Proof. If $\pi : G \to H\backslash G$ is the natural projection, then we know that $H \cdot a$ is coded by $\pi(a)$. By Proposition 2.6, $tp(\pi(a))$ is almost-$\mathcal{S}$-internal, and therefore $\pi(a) \in \mathrm{dcl}(b)$. By definition of b, $tp(a'/b) = tp(a/b)$ implies $a' \in H{\cdot}a$ and $a'{\cdot}a^{-1} \in S$, which gives the result. □

The next results allow us in many cases to pass from the algebraic closure of a set to the set itself. In geometric situations, it will allow us to replace correspondences by rational maps. The delicate point is that in general, if $B = \operatorname{acl}(B) \subset \operatorname{acl}(A)$ and $B_0 = B \cap A$, it may happen that $B \neq \operatorname{acl}(B_0)$. The first result, Observation 2.8, does not need the CBP hypothesis.

In what follows, we work over $\emptyset$, and have a set $\mathcal{S}$ of SU-rank 1 types with algebraically closed base, and which is closed under $\operatorname{Aut}(M)$-conjugation.

Observation 2.8. Let a a tuple, let $B = \operatorname{acl}(B)$ be maximal contained in $\operatorname{acl}(a)$ and such that $tp(B)$ is almost-$\mathcal{S}$-internal. Let $B_0 = \operatorname{dcl}(a) \cap B$; then $\operatorname{acl}(B_0) = B$.

Proof. Let $b \in B$ be such that $B = \operatorname{acl}(B)$, and let b' be a conjugate of b over $\operatorname{dcl}(a)$. Then $tp(b') = tp(b)$, and therefore $tp(b')$ is almost-$\mathcal{S}$-internal. Hence, if c is a tuple encoding the set of conjugates of b over $\operatorname{dcl}(a)$, then $c \in \operatorname{dcl}(a)$, and $tp(c)$ is almost-$\mathcal{S}$-internal, so that $c \in B_0$. As $b \in \operatorname{acl}(c)$, we get $\operatorname{acl}(c) = B$. □

Proposition 2.9. (CBP) *Let a be a tuple of finite* SU*-rank, let $B = \operatorname{acl}(B)$ be such that $tp(a/B)$ is almost-$\mathcal{S}$-internal. If $B_0 = B \cap \operatorname{dcl}(a)$, then $tp(a/B_0)$ is almost-$\mathcal{S}$-internal.*

Proof. We may assume that B is minimal algebraically closed such that $tp(a/B)$ is almost-$\mathcal{S}$-internal. Choose a tuple $b \in B$ such that $B = \operatorname{acl}(b)$. If b' is a conjugate of b over $\operatorname{dcl}(a)$, then $tp(a, b) = tp(a, b')$, and therefore $tp(a/b')$ is also almost-$\mathcal{S}$-internal. The minimality of B (and Lemma 2.3) implies that $\operatorname{acl}(b') = \operatorname{acl}(b)$. Hence, if c is a tuple encoding the set of conjugates of b over $\operatorname{dcl}(a)$, then $\operatorname{acl}(c) = \operatorname{acl}(b)$; as $c \in B \cap \operatorname{dcl}(a) = B_0$, we get $B = \operatorname{acl}(B_0)$. □

Proposition 2.10. (CBP) *Let a_1, a_2, b_1, b_2 be tuples of finite* SU*-rank and assume that*

- *$tp(b_2)$ is almost-$\mathcal{S}$-internal,*
- $\operatorname{acl}(b_1) \cap \operatorname{acl}(b_2) = \operatorname{acl}(\emptyset)$,
- *$a_1 \mathop{\smile\!\!\!\!\!\!|}_{b_1} b_2$ and $a_2 \mathop{\smile\!\!\!\!\!\!|}_{b_2} b_1$,*
- $a_2 \in \operatorname{acl}(a_1 b_1 b_2)$.

Then there is $e \subset \operatorname{dcl}(a_2 b_2)$ such that $tp(a_2/e)$ is almost-$\mathcal{S}$-internal and $e \mathop{\smile\!\!\!\!\!\!|} b_2$. In particular, if $tp(a_2/b_2)$ is hereditarily orthogonal to all types in $\mathcal{S}$, then $a_2 \in \operatorname{acl}(eb_2)$.

Proof. If $C = \overline{\operatorname{Cb}}(a_1 b_1/a_2 b_2)$, then $a_2 \in \operatorname{acl}(C b_2)$. Let $D = \operatorname{acl}(a_1 b_1) \cap \operatorname{acl}(a_2 b_2)$. As $D \subset \operatorname{acl}(a_i b_i)$ for $i = 1, 2$, we have $D \mathop{\smile\!\!\!\!\!\!|}_{b_1} b_2$ and $D \mathop{\smile\!\!\!\!\!\!|}_{b_2} b_1$. Hence $D \mathop{\smile\!\!\!\!\!\!|} b_1 b_2$ because $\operatorname{acl}(b_1) \cap \operatorname{acl}(b_2) = \operatorname{acl}(\emptyset)$. Furthermore, we know by Theorem 1.16 that there is a set $\mathcal{S}'$ of SU-rank 1 types orthogonal to all members of $\mathcal{S}$ and such that $tp(C/D)$ is almost-$(\mathcal{S} \cup \mathcal{S}')$-internal. We may write C as $\operatorname{acl}(c_1 c_2)$ where $tp(c_1/D)$ is almost-$\mathcal{S}$-internal, and $tp(c_2/D)$ is almost-$\mathcal{S}'$-internal. Then $\operatorname{acl}(c_2 D) \mathop{\smile\!\!\!\!\!\!|} b_2$ because $tp(c_2/D)$ is hereditarily orthogonal to all members of $\mathcal{S}$ and $tp(b_2)$ is almost-$\mathcal{S}$-internal. Furthermore, as $a_2 \in \operatorname{acl}(D c_1 c_2 b_2)$, it follows that $tp(a_2/\operatorname{acl}(D c_2))$ is almost-$\mathcal{S}$-internal.

Now, $Dc_2 \subseteq \mathrm{acl}(a_2b_2)$, and Proposition 2.9 implies that if $e = \mathrm{acl}(Dc_2) \cap \mathrm{dcl}(a_2b_2)$, then $tp(a_2/e)$ is almost-$\mathcal{S}$-internal.

The last assertion is clear: $tp(a_2/e)$ almost-$\mathcal{S}$-internal implies $tp(a_2/eb_2)$ almost-$\mathcal{S}$-internal, and our assumption of hereditary orthogonality implies that $a_2 \in \mathrm{acl}(eb_2)$.

□

Concluding remarks. Inspection of the proofs shows that our assumption of supersimplicity on the ambient theory is unnecessary, as long as one restricts one's attention to types ranked by the SU-rank, and the relevant hyperimaginaries and imaginaries are eliminated. Thus, the results of Sec. 1 do apply to types of finite U-rank in separably closed fields of finite degree of imperfection. It is unknown whether this family of types enjoys the CBP, we will explain now what one needs to prove. Let K be a separably closed field of characteristic $p > 0$ and finite (positive) degree of imperfection. It follows from results of Messmer, Hrushovski and Delon (see e.g. [2]), that a type of finite U-rank which is not one-based is non-orthogonal to the generic type q of $\bigcap_n K^{p^n}$. By Theorem 1.16, it is therefore enough to show that the family of all $\{q\}$-analyzable types has the CBP. A partial result in this direction is obtained by Pillay and Ziegler in [17]: they show that the family of very thin types has the CBP. Thus, the results of Sec. 2 apply for the family $\mathcal{P}$ of very thin types. Unfortunately, Pillay and Ziegler also give an example of a $\{q\}$-analyzable type (of U-rank 2) which is not very thin.

The result of Pillay and Ziegler on types in differentially closed fields of characteristic 0 is stronger than the CBP: indeed, if Cb denotes the usual canonical base, then they show that given two tuples a and b of finite rank such that $b = \mathrm{Cb}(a/b)$, then $tp(b/a)$ is internal to the constants. It would be interesting to know whether this implies that $tp(b/C)$ is also internal to the constants (as opposed to almost-internal to the constants), under some reasonable conditions on a, b, and with $C = \mathrm{acl}(a) \cap \mathrm{acl}(b)$, or even $C = \mathrm{dcl}(a) \cap \mathrm{dcl}(b)$.

3. Existentially Closed Difference Fields Have the CBP

Recall that a difference field is a field with a distinguished endomorphism (usually denoted by σ), which we study in the language of rings augmented by a symbol for σ. A difference field K is *inversive* if $\sigma(K) = K$. We refer to [8] for basic algebraic results on difference fields, and to [4] for basic model-theoretic results. Any completion of the theory ACFA of existentially closed difference fields is supersimple and eliminates imaginaries. Moreover, if K is an existentially closed difference field, and $A \subseteq K$, then $\mathrm{acl}(A)$ is the smallest algebraically closed subfield B of K satisfying $\sigma(B) = B$ and containing A. Independence of algebraically closed sets coincides with independence in the sense of the theory of algebraically closed fields, i.e. if $C \subseteq A, B$ are algebraically closed difference subfields of K, then A and B are independent over C if and only if A and B are linearly disjoint over C.

As an immediate corollary of the results of Pillay and Ziegler and of Proposition 2.1, we then obtain

Proposition 3.1. *Let* (K, D) [*respectively,* (K, σ)] *be a differentially closed field* (*respectively, an existentially closed difference field*) *of characteristic* 0. *Let* $C \subseteq A, B$ *be algebraically closed differential* (*respectively, difference*) *subfields of* K, *with* $\mathrm{SU}(B/C) < \omega$. *Assume that* $A = \overline{\mathrm{Cb}}(B/A)$. *Then* $tp(A/A \cap B)$ *is almost internal to* $Dx = 0$ [*respectively,* $\sigma(x) = x$].

Notation 3.2. We denote by A^{alg} the field-theoretic algebraic closure of a field A, and by A^s its separable closure. If $\sigma(E) = E$ is a difference subfield of the inversive difference field K, and a is a tuple of elements of K, then $E(a)_\sigma$ denotes the (inversive) difference subfield $E(\sigma^i(a) \,|\, i \in \mathbb{Z})$ of K. If τ is an automorphism of K, we denote by $\mathrm{Fix}(\tau)$ the subfield of K consisting of elements fixed by τ. We denote by Frob the Frobenius map $x \mapsto x^p$.

p-bases and degree of imperfection. For details and proofs, see [1, §13]. Let $E \subseteq K \subseteq L \subseteq K^{\mathrm{alg}}$ be fields of characteristic $p > 0$, with E perfect and $\mathrm{tr.deg}(K/E) = d < \infty$. Then $[K : K^p] = p^e$ for some $e \leq d$, and there is an e-tuple c of elements of K such that $K = K^p[c]$. Such a tuple is called a *p-basis of* K and its elements are algebraically independent over E. Moreover, if $e = d$, then c is a *separating transcendence basis* of K over E, i.e. $K \subseteq E(c)^s$. The integer e is called the *degree of imperfection of* K.

We also have: $[L : L^p]$ divides p^e, and $[L : L^p] = p^e$ if $L \subseteq K^s$ or if $[L : K] < \infty$.

Lemma 3.3. *Let* (K, σ) *be an existentially closed difference field of characteristic* $p > 0$, *let* $E = \mathrm{acl}(E) \subset K$, *a finite tuple in* K, *and assume that* $tp(a/E)$ *is* $\mathrm{Fix}(\sigma)$-*analyzable. Then there is a finite tuple* b *such that* $E(a)_\sigma = E(b)_\sigma$, *and* $\sigma(b), \sigma^{-1}(b) \in E(b)^s$.

Proof. We will show that if $d = \mathrm{tr.deg}(E(a)_\sigma/E)$, then $[E(a)_\sigma : E(a^p)_\sigma] = p^d$ and $d < \infty$. This will yield the result: let c be a p-basis of $E(a)_\sigma$. Then $E(a)_\sigma \subseteq E(c)^s$, and therefore $E(a)_\sigma = E(c, a)_\sigma \subseteq E(c, a)^s$.

The proof is by induction on the length of a semi-minimal analysis in $\mathrm{Fix}(\sigma)$ of $tp(a/E)$. Assume first that $tp(a/E)$ is almost-$\mathrm{Fix}(\sigma)$-internal. Let $F = \mathrm{acl}(F)$ be independent from a over E, and such that a is equi-algebraic over F with some finite tuple b of $\mathrm{Fix}(\sigma)$. We may assume that $a \in F(b)^s$ (we replace b by b^{1/p^n} if necessary). From $\sigma(b) = b$, we deduce that $F(a)_\sigma \subseteq F(b)^s$, and therefore

$$p^d \geq [F(a)_\sigma : F(a^p)_\sigma] \geq [F(b) : F(b^p)] = p^d.$$

As F was linearly disjoint from $E(a)_\sigma$ over E, this shows $[E(a)_\sigma : E(a^p)_\sigma] = p^d$, with $d < \infty$.

For the general case, choose $a_1, \ldots, a_n \in \mathrm{acl}(Ea)$ such that $a \in E(a_1, \ldots, a_n)_\sigma$, and for every i, $tp(a_i/\mathrm{acl}(Ea_1, \ldots, a_{i-1}))$ is almost-$\mathrm{Fix}(\sigma)$-internal. Let $F_i =$

$\mathrm{acl}(Ea_1 \cdots a_i)$ for $i = 1, \ldots, n$. By reverse induction, we may enlarge $a_n, \ldots, a_1$ so that for every $i < n$:

(a) a_{i+1} contains a p-basis of $F_i(a_{i+1})_\sigma$ and a transcendence basis of $F_i(a_{i+1})_\sigma$ over F_i.
(b) The σ-ideal of difference equations satisfied by a_{i+1} over F_i is generated by the difference equations satisfied by a_{i+1} over $E_i = E(a_1, \ldots, a_i)_\sigma$ (this is possible, since this σ-ideal is finitely generated as a σ-ideal, see e.g. [8]).

Condition (b) then implies that for every $i < n$, $E_i(a_{i+1})_\sigma$ and F_i are linearly disjoint over E_i. By the first case and (a), $F_i(a_{i+1})_\sigma \subseteq F_i(a_{i+1})^s$, and the linear disjointness of F_i and $E_i(a_{i+1})_\sigma$ over E_i then implies that $E_i(a_{i+1})_\sigma \subseteq E_i(a_{i+1})^s$, so that

$$E(a)_\sigma \subseteq E(a_1, \ldots, a_n)_\sigma \subseteq E(a_1, \ldots, a_n)^s.$$

Then $[E(a_1, \ldots, a_n) : E(a_1^p, \cdots a_n^p)] = p^d$ where $d = \mathrm{tr.deg}(E(a_1, \ldots, a_n)/E) < \infty$. Reasoning as in the first case, we deduce $[E(a)_\sigma : E(a^p)_\sigma] = p^d$. ◻

Remark 3.4. Let (K, σ) be an existentially closed difference field of characteristic $p > 0$. We give a description of the classes $\mathcal{S}(q)$, for q a non-one-based type of SU-rank 1.

Let I be the set of pairs $(n, m) \in \mathbb{N}^{>0} \times \mathbb{Z}$, with $(n, m) = 1$ if $m \neq 0$ and $n = 1$ if $m = 0$. For each pair $(n, m) \in I$, choose a non-algebraic type $q_{n,m}$ (over $\mathbb{F}_p^{\mathrm{alg}}$) containing the formula $\sigma^n(x^{p^m}) = x$, and let $\mathcal{S}_{n,m} = \mathcal{S}(q_{n,m})$. Then $q_{n,m}$ is not one-based.

By [7, (7.1)(1)], $\mathrm{SU}(\sigma^n(x^{p^m}) = x) = 1$; as the formula $\sigma^n(x^{p^m}) = x$ defines a subfield of K, this implies that any two non-algebraic types containing this formula are non-orthogonal. This observation, together with the main result of [7] (see the theorem in Sec. 6), shows that any type of SU-rank 1 which is not one-based is non-orthogonal to some $q_{n,m}$. We define $\mathcal{S} = \bigcup \mathcal{S}_{n,m}$.

We will now show that if $(n, m) \neq (n', m')$ are in I, then $\mathcal{S}_{m,n} \cap \mathcal{S}_{m',n'} = \emptyset$.

Indeed, let $F = \mathrm{acl}(F)$, and $a, b \in K \backslash F$ with $\sigma^n(a^{p^m}) = a$, $\sigma^{n'}(b^{p^{m'}}) = b$, and assume that a, b are equi-algebraic over F. Then clearly

$$n = n' = \mathrm{tr.deg}(F(a)_\sigma/F) = \mathrm{tr.deg}(F(b)_\sigma/F).$$

Taking a p^ℓ-power of b, we may assume that $b \in F(a, \ldots, \sigma^{n-1}(a))^s$. Let $\tau = \sigma^n \mathrm{Frob}^m$. Then $F(a, \ldots, \sigma^{n-1}(a))^s$ is closed under τ and τ^{-1} (because $\tau\sigma^i = \sigma^i\tau$ and $\tau(a) = a$), and has degree of imperfection n. On the other hand, if $m \neq m'$, then the closure under τ and τ^{-1} of $F(b)$ is perfect because $\tau(b) = b^{p^{m-m'}}$. This contradicts $b \in F(a, \ldots, \sigma^{n-1}(a))^s$.

Theorem 3.5. *Let (K, σ) be an existentially closed difference field of characteristic $p > 0$, let $C \subseteq A, B$ be algebraically closed difference fields, with $\mathrm{SU}(B/C) < \omega$. Assume that $\overline{\mathrm{Cb}}(B/A) = A$. Then $tp(A/A \cap B)$ is almost-$\mathcal{S}$-internal.*

Proof. By Theorem 2.1, it suffices to show that whenever A and B satisfy the hypotheses of the theorem, then $tp(A/B)$ is almost-$\mathcal{S}$-internal. Fix such A, B, with $C = A \cap B$. We may assume $B = \overline{\mathrm{Cb}}(A/B)$; observe that by Remark 1.1(1), $A = \overline{\mathrm{Cb}}(B/A)$ implies $\mathrm{SU}(A/C) < \omega$.

By Proposition 1.14 and the discussion in Remark 3.4, we already know that $A = \mathrm{acl}(A_1 \cdots A_j)$, where each $tp(A_i/C)$ is $\mathcal{S}_{n,m}$-analyzable for some $(n, m) \in I$, and $B = \mathrm{acl}(B_1 \cdots B_j)$, where $B_i = \overline{\mathrm{Cb}}(A_i/B)$, $A_i = \overline{\mathrm{Cb}}(B_i/A)$. If there is a counterexample to our assertion, then there is one where $tp(A/C)$ and $tp(B/C)$ are $\mathcal{S}_{n,m}$-analyzable for some $(n, m) \in I$, and this is what we will assume. We will also assume that K is sufficiently saturated.

Let $\tau = \sigma^n \mathrm{Frob}^m$. Let b be a (finite) tuple of elements of B such that $B = C(b)^{\mathrm{alg}}$. Then A is the smallest algebraically closed field containing C and the field of definition of the algebraic locus of b over A.

We now work in the difference field (K, τ), which is a reduct of (K, σ), and is also a model of ACFA by Corollary 1.12(1) in [4]. In the reduct (K, τ) we also have $A = \overline{\mathrm{Cb}}(Cb/A)$. By Lemma 3.3, we may assume that $\tau(b)$ and $\tau^{-1}(b)$ are in $C(b)^s$. Hence, there are varieties V, W defined over C, with generics b and $(b, \tau(b))$ respectively, and with $W \subseteq V \times \tau(V)$, and such that the projection maps $W \to V$ and $W \to \tau(V)$ are separable and generically finite. These maps therefore induce isomorphisms between the jet spaces $J^k_{(b,\tau(b))}(W)$ and $J^k_b(V)$, $J^k_{\tau(b)}(\tau(V))$ for every $k > 0$. The proof of Pillay and Ziegler then goes through (see Chap. 3 of [17]), and shows that $tp(A/B)$ is almost-$\mathrm{Fix}(\tau)$-internal (in (K, τ)). Hence there is $M = \tau(M)^{\mathrm{alg}} \supseteq B$, linearly disjoint from AB over B, and some tuple $a \in \mathrm{Fix}(\tau)$ such that $A \subseteq M(a)^{\mathrm{alg}}$. Since the elements of a have SU-rank 1 in the difference field (K, τ), we may assume that a and A are equi-algebraic over M.

If $n = 1$, then $M = \sigma(M)^{\mathrm{alg}}$, and we are done. Assume that $n > 1$; then M is closed under σ^n and σ^{-n}, but not necessarily under σ, σ^{-1}. We need to show that there is a difference field (N, σ) extending (B, σ), containing M and linearly disjoint from AM over M, and such that (N, σ^n) extends $(M, \tau\mathrm{Frob}^{-m})$. This is done as in [4], Lemma 1.12. The saturation of (K, σ) then implies that K contains (a copy of) (N, σ), and shows that $tp(A/B)$ is almost-$\mathrm{Fix}(\tau)$-internal. □

Theorem 3.5′. *Let A, B be difference subfields of $\mathcal{U}$ intersecting in C, such that $A^{\mathrm{alg}} \cap B^{\mathrm{alg}} = C^{\mathrm{alg}}$ and $\mathrm{tr.deg}(A/C) < \infty$. Let $D \subset B$ be generated over C by all tuples d such that there exist an algebraically closed difference field F containing C and free from B over C, and integers $n > 0$ and m such that $d \in F(e)$ for some tuple e of elements satisfying $\sigma^n \mathrm{Frob}^m(x) = x$. Then A and B are free over D.*

Proof. When A and B are algebraically closed, this is a direct consequence of Theorems 3.5 and 2.1: we know that $\overline{\mathrm{Cb}}(A/B)$ realizes a type over $A \cap B$ which is almost-$\mathcal{S}$-internal, where $\mathcal{S}$ is the family of SU-rank 1 types realized in some $\mathrm{Fix}(\tau)$. Hence $\overline{\mathrm{Cb}}(A/B)$ is contained in D^{alg}, which implies that A and B are free over D.

Assume now that A and B are not algebraically closed, and work over their intersection C. Again, we know that $\overline{\mathrm{Cb}}(A/B)$ realizes a type over C^{alg} which is almost-$\mathcal{S}$-internal. Hence $\overline{\mathrm{Cb}}(A/B)$ is contained in the maximal subset D_0 of $\mathrm{acl}(A)$ which realizes an almost-$\mathcal{S}$-internal-type over C^{alg}. By Remark 4.7, we have $D_0 = D^{\mathrm{alg}}$, which gives the result. □

4. Applications of the CBP to Differential and Difference Varieties

Differential fields. We will now apply some of the results of Sec. 2 to the study of (affine) differential varieties. For an introduction to the model theory of differential fields of characteristic 0, see e.g. [12].

Known facts. We work in some large differentially closed field $(\mathcal{U}, \delta)$ of characteristic 0. In analogy with the Zariski topology, we define the *Kolchin topology* on each Cartesian power $\mathcal{U}^n$, as the topology with basic closed sets the zero-sets of differential polynomials, which are called *Kolchin closed sets*. This topology is Noetherian. A *differential* (*affine*) *variety* V is an irreducible Kolchin closed set.

If $A \subset \mathcal{U}$ is a differential field, then $A = \mathrm{dcl}(A)$ and $\mathrm{acl}(A) = A^{\mathrm{alg}}$. The theory of differentially closed fields of characteristic 0 eliminates quantifiers and imaginaries.

Since our results concern differential fields, we first define the analogues of function fields and birational morphisms. The definitions are straightforward.

If a differential variety V is defined over the differential field K, we define the coordinate ring $K[V]_D$ and function field $K(V)_D$ of V as follows: let $K[\bar{X}]_D$ be the ring of differential polynomials in $\bar{X} = (X_1, \ldots, X_n)$, and I the ideal of differential polynomials vanishing on V. Then

$$K[V]_D = K[\bar{X}]_D/I \quad \text{and} \quad K(V)_D = \mathrm{Frac}(K[V]_D).$$

A differential variety V has *finite order* if the transcendence degree of $K(V)_D$ over K is finite. If V, W are differential varieties, a *differential-rational map* $f : V \to W$ is simply a map whose coordinate functions are given by elements of $K(V)_D$; it is therefore defined on some Kolchin-open subset U of V. If $f(U)$ is dense in W for the Kolchin topology, then we will say that f is *dominant*, and the map f induces a K-embedding of $K(W)_D$ into $K(V)_D$. Conversely, any K-embedding of $K(W)_D$ into $K(V)_D$ is induced by some dominant differential-rational $f : V \to W$. A *finite cover* of V is a dominant differential-rational map $f : W \to V$ such that the generic fiber of f is finite. It corresponds to a finite algebraic extension $K(W)_D$ of $K(V)_D$.

The *constant field* is $\mathcal{C} = \{x \in \mathcal{U} \mid Dx = 0\}$. Any non-one-based type is non-orthogonal to the generic type of $\mathcal{C}$. We let $\mathcal{S}$ be this generic type (over $\mathrm{acl}(\emptyset)$).

If $A \subset \mathcal{U}$, then $K(A)_D$ denotes the differential field generated by A over K. If a is a finite tuple, then a is a *generic of the differential variety* V *over* K if $a \in V$ and the specialization map $K[V]_D \to K(a)_D$ is injective.

We say that a differential variety V is *$\mathcal{C}$-internal*[b] if there is a birational $f: V \to \bar{W}(\mathcal{C})$ for some algebraic variety $\bar{W}$. We say that V is *almost-$\mathcal{C}$-internal* if it is a finite cover of a $\mathcal{C}$-internal differential variety. This is equivalent to: if a is a generic of V over K, then $tp(a/K)$ is almost-$\mathcal{S}$-internal.

Proposition 4.1. *Let V be a differential variety of finite order defined over the differential subfield K of $\mathcal{U}$. Then V has a* maximal almost-$\mathcal{C}$-internal quotient $V^{\#}$, *i.e. $V^{\#}$ is almost-$\mathcal{C}$-internal, and if π is a dominant differential-rational map from V to an almost-$\mathcal{C}$-internal variety V_1, then π factors through $V^{\#}$. Furthermore, if $f: W \to V$ is a finite cover of V, then there is a generically finite map $W^{\#} \to V^{\#}$.*

Proof. Let K be a differential field over which everything is defined. Translated into terms of elements, this becomes: let a be a generic of V over K, let $A = \mathrm{acl}(A)$ be the maximal subfield of $\mathrm{acl}(Ka)$ whose type over K is almost-$\mathcal{S}$-internal, and let $A_0 = A \cap K(a)_D$. Then A_0 is finitely generated over K (as a differential field or as a field), say by a tuple b, and we let $V^{\#}$ be the differential variety with generic b over K, and $V \to V^{\#}$ the birational map dual to the inclusion $K(b)_D \to K(a)_D$. Note that this defines $V^{\#}$ uniquely up to a differential birational correspondence, and by definition, $V^{\#}$ is almost-$\mathcal{C}$-internal.

Assume that $\pi : V \to V_1$ is dominant differential-rational, and let $c = \pi(a)$. The almost-$\mathcal{C}$-internality of V_1 is equivalent to the almost-$\mathcal{C}$-internality of $tp(c/K)$, and this implies that $c \in A_0 = K(b)_D$, and shows that the map π factors through $V^{\#}$.

For the last assertion, let $g: W \to V$ be a finite cover of V, let c be a generic of W such that $g(c) = a$, and let $A_1 = A \cap K(c)_D$. As $c \in K(a)_D^{\mathrm{alg}}$, we know that A_1 is the maximal subfield of $K(c)_D$ which realizes an almost-$\mathcal{C}$-internal type over K, i.e. we can take $W^{\#}$ to be the differential variety of which a generator of A_1 over K is a generic. We clearly have $A_0 \subseteq A_1 \subseteq A$, and we need to show that this extension is algebraic: but Observation 2.8 tells us that $A = A_0^{\mathrm{alg}}$. □

Proposition 4.2. *Let V be a differential variety of finite order defined over the differential subfield K of $\mathcal{U}$. Then V has a* maximal almost-$\mathcal{C}$-internal fiber, *i.e. a smallest quotient $V^{\flat}$ with generic fiber an almost-$\mathcal{C}$-internal differential variety.*[c] *Furthermore, if $f: W \to V$ is a finite cover of V, then there is a generically finite map $W^{\flat} \to V^{\flat}$.*

Proof. The translation in terms of differential extensions is similar to the one done in Proposition 4.1, and reduces the problem to the following:

Let $B = \mathrm{acl}(B)$ be minimal such that $tp(a/B)$ is almost-$\mathcal{S}$-internal (cf. Theorem 2.4 for the existence), and let $B_0 = B \cap K(a)_D$. Then $B_0^{\mathrm{alg}} = B$ and

[b] Some authors say that V is *iso-constant*.

[c] In other words, if $\pi : V \to V_1$ is dominant with generic fiber almost-$\mathcal{C}$-internal, then $V^{\flat}$ is a quotient of V_1.

$tp(a/B_0)$ is almost-$\mathcal{S}$-internal. But this last statement is given by (the proof of) Proposition 2.9. □

Proposition 4.3. (Descent) *For $i = 1, 2$, let V_i be a differential variety of finite order defined over the differential subfield K_i, of $\mathcal{U}$, and let $k = K_1 \cap K_2$. Assume that $K_1^{\mathrm{alg}} \cap K_2^{\mathrm{alg}} = k^{\mathrm{alg}}$, that K_2 is a regular extension of k, that there is a differential rational dominant map $f : V_1 \to V_2$ defined over $(K_1K_2)^{\mathrm{alg}}$, and that $tp(K_2/k)$ is almost-$\mathcal{S}$-internal. Then there is a differential variety V_3 defined over k, and a dominant differential rational map $g : V_2 \to V_3$ such that the generic fiber of g is almost-$\mathcal{C}$-internal.*

Proof. Use Proposition 2.10 with $\mathrm{dcl}(\emptyset) = k$, $b_i = K_i$, and a_i a generic of V_i over K_1K_2, $a_2 = f(a_1)$ to get $e \in K_2(a_2)_D$ such that $e \mathop{\smile\!\!\!\!\!\!\mid}_k K_2$ and $tp(a_2/e)$ is almost-$\mathcal{S}$-internal. Since the property of almost-$\mathcal{S}$-internality only depends on $tp(a_2/e)$, we may take for e a finite tuple. Our hypothesis on the extension K_2 of K implies that $k(e)_D$ is a regular extension of k. If V_3 is the differential locus of e over k, and $g : V_2 \to V_3$ is the dominant map induced by the inclusion $K_2(e)_D \subset K_2(a_2)_D$, then the generic fiber of g realizes an almost-$\mathcal{S}$-internal-type (over $K_2(e)_D$ or $k(e)_D$). □

Difference fields. In the same vein, we now apply the results of Sec. 2 to the study of (affine) difference varieties. Again, we have to define the analogues of function fields and birational morphisms. The definitions are straightforward.

We work in some large existentially closed difference field $\mathcal{U}$. In analogy with the Zariski topology, we define the *σ-topology* on each cartesian power $\mathcal{U}^n$, as the topology with basic closed sets the zero-sets of difference polynomials, which are called *σ-closed sets*. This topology is Noetherian. A *difference (affine) variety* is an irreducible σ-closed set, and if this variety is defined over the difference field K, we define its coordinate ring $K[V]_{\sigma+}$ and function field $K(V)_{\sigma+}$ as follows: let $K[\bar{X}]_\sigma$ be the ring of difference polynomials in $\bar{X} = (X_1, \dots, X_n)$, and I the ideal of difference polynomials vanishing on V. Then

$$K[V]_{\sigma+} = K[\bar{X}]_\sigma/I, \quad K(V)_{\sigma+} = \mathrm{Frac}(K[V]_{\sigma+}).$$

The *order* of a difference variety V is the transcendence degree of $K(V)_{\sigma+}$ over K. If V, W are difference varieties, a *σ-rational map* $f : V \to W$ is simply a map whose coordinate functions are given by elements of $K(V)_{\sigma+}$; it is therefore defined on some σ-open subset U of V. If $f(U)$ is dense in W for the σ-topology, then we will say that f is *dominant*, and the map f induces a K-embedding of $K(W)_{\sigma+}$ into $K(V)_{\sigma+}$. Conversely, any K-embedding of $K(W)_{\sigma+}$ into $K(V)_{\sigma+}$ is induced by some dominant σ-rational map $f : V \to W$. A *finite cover* of V is a dominant σ-rational map $f : W \to V$ such that the generic fiber of f is finite. It corresponds to a finite algebraic extension $K(W)_{\sigma+}$ of $K(V)_{\sigma+}$.

If a is a tuple in $\mathcal{U}$, we let $K(a)_{\sigma+} = K(\sigma^i(a) \,|\, i \geq 0)$; if $\sigma(K) = K$, then $K(a)_\sigma = K(\sigma^i(a) \,|\, i \in \mathbb{Z})$ as in Sec. 3. We say that a tuple a is a *generic of the* *difference variety* V *over* K if $a \in V$ and the natural specialisation map $K[V]_{\sigma+} \to K(a)_{\sigma+}$ is injective.

We will often use the following result (see [8], 5.23.18): If $K \subset L \subset M$ are difference fields, with M finitely generated over K (as a difference field), then L is finitely generated over K.

Internality. The definable closure of a difference field K, $\mathrm{dcl}(K)$, is usually much larger than the perfect closure of K. The notion of internality to $\mathrm{Fix}(\sigma)$ therefore does not have a natural geometric interpretation. The right notion to consider is the one of qf-internality: one replaces dcl by "difference field generated by".

Definition 4.4. Let K be a difference field, a a tuple in $\mathcal{U}$, such that $K(a)_\sigma/K$ is regular, let V be the *difference locus* of a over K (i.e. the smallest σ-closed set containing a and defined over K), and let $\mathcal{S}$ be a set of types with algebraically closed base, which is closed under conjugation by $\mathrm{Aut}(\mathcal{U}/K)$. We say that $tp(a/K)$ is *qf-internal to* $\mathcal{S}$, or *qf-*$\mathcal{S}$*-internal*, if for some $L = \mathrm{acl}(L)$ containing K and free from $K(a)_\sigma$ over K, and some tuple b of realizations of types in $\mathcal{S}$, $a \in L(b)_\sigma$. In that case, we also say that the extension $K(a)_\sigma/K$, and the difference variety V are *qf-internal to* $\mathcal{S}$, or *qf-*$\mathcal{S}$*-internal*. (For the difference variety, we should really speak of "generic" qf-internality.) And similarly we will speak of *almost-*$\mathcal{S}$*-internal extensions*, and *almost-*$\mathcal{S}$*-internal difference varieties*. Let $\tau = \sigma^n \mathrm{Frob}^m$ for some $(m, n) \in I$ (see Remark 3.4). If $\mathcal{S}$ consists of all types realized in $\mathrm{Fix}(\tau)$, then we will also speak of *qf-*$\mathrm{Fix}(\tau)$*-internality*, or *qf-internality to* $\mathrm{Fix}(\tau)$.

Internality to fixed fields. Let $\tau = \sigma^n \mathrm{Frob}^m$ for some $(m, n) \in I$, and assume that $tp(a/K)$ is qf-internal to $\mathrm{Fix}(\tau)$. Then one can find L and b as above, such that $L(a)_\sigma = L(b)_\sigma$: take b such that $L(a)_\sigma \cap \mathrm{Fix}(\tau) = (\mathrm{Fix}(\tau) \cap L)(b)_\sigma$; since $L(a)_\sigma$ and $\mathrm{Fix}(\tau)$ are linearly disjoint over their intersection, it follows that $L(a)_\sigma$ and $L\mathrm{Fix}(\tau)$ are linearly disjoint over $L(b)_\sigma$, and therefore $L(a)_\sigma = L(b)_\sigma$. Note that if $m \geq 0$, then $L(b)_\sigma = L(b)_{\sigma+}$ and therefore also $L(a)_{\sigma+} = L(a)_\sigma$. If $m < 0$, then $L(b)_\sigma$ is the perfect hull of $L(b)_{\sigma+}$, and this implies that, choosing b so that $(L \cap \mathrm{Fix}(\tau))(b)_{\sigma+} = L(a)_{\sigma+} \cap \mathrm{Fix}(\tau)$, we have $L(a)_{\sigma+} \supseteq L(b)_{\sigma+} \supseteq L(\sigma^j(a))_{\sigma+}$ for some $j \geq 0$. If $\bar{W}$ is the algebraic locus of b over L, then there is a purely inseparable map π such that $\pi(V)$ is σ-birationally isomorphic (over L) to $\bar{W}(\mathrm{Fix}(\tau))$, the difference variety defined by $x \in \bar{W} \wedge \tau(x) = x$.[d]

Fact 4.5. Let $\tau = \sigma^m \mathrm{Frob}^n$ for some $(m, n) \in I$, let $\ell \geq 1$ be an integer. Let K be a difference subfield of $\mathcal{U}$, and K' a difference field isomorphic to K by an isomorphism φ_0, and $\mathcal{U}'$ an existentially closed difference field containing K'. We will work in the σ^ℓ-difference field $\mathcal{U}[\ell] = (\mathcal{U}, \sigma^\ell)$, and denote by $qftp(-)[\ell]$, $tp(-)[\ell]$, $\mathrm{acl}_{\sigma^\ell}$ the

[d]Because $\mathrm{Fix}(\tau)$ is stably embedded (see [7, Sec. 7.1]), it follows that $\bar{W}$ is defined over $L \cap \mathrm{Fix}(\tau)$. If $m \geq 0$ then $j = 0$ and one does not need the map π.

quantifier-free types, types, and algebraic closure respectively, with superscript $\mathcal{U}$ or $\mathcal{U}'$ if necessary. We will use the following results:

(1) ([7, 1.12]) Assume that $a \in K^{\rm alg}$, and let a' be a field-conjugate of a over K. Then $qftp(a/K)[m] = qftp(a'/K)[m]$ for some $m \geq 1$.
(2) ([6, 2.9]) Let $a \in \mathcal{U}$, $a' \in \mathcal{U}$, and assume that there is an isomorphism of σ^ℓ-difference fields between $K(a)_{\sigma^\ell}$ and $K'(a')_{\sigma^\ell}$ which extends φ_0 and sends a to a'. Then $tp^{\mathcal{U}}(a/K)$ is qf-Fix(τ)-internal if and only if $tp^{\mathcal{U}'}(a'/K')[\ell]$ is qf-Fix(τ^ℓ)-internal.
(3) ([6, 2.11]). Let a and a' be as in (2). Then $tp^{\mathcal{U}}(a/K)$ is one-based if and only if $tp^{\mathcal{U}'}(a'/K')[\ell]$ is one-based.

Conditions on the set $\mathcal{S}$. We fix a set $\mathcal{S}$ of types of SU-rank 1 with algebraically closed base, which is closed under Aut$(\mathcal{U})$-conjugation. If $p \in \mathcal{S}$ is not one-based, then for some τ as above, p is non-orthogonal to any non-algebraic type realized in Fix(τ). If $\mathcal{S}$ consists only of non-one-based types, then we do not impose any additional condition.

If $\mathcal{S}$ contains some one-based type, for convenience we will impose that $\mathcal{S}$ contains all one-based types of SU-rank 1. By abuse of language, we will speak about almost-$\mathcal{S}$-internality even when working in $\mathcal{U}[\ell]$.

Proposition 4.6. *Let V be a difference variety of finite order defined over the difference subfield K of $\mathcal{U}$, and $\mathcal{S}$ as above. Then V has a* maximal almost-$\mathcal{S}$-internal quotient *$V^\#$. Furthermore, if W is a finite cover of V, then $W^\#$ is a finite cover of $V^\#$ via a map $\sigma^{-n} f$ for some integer n and tuple f of rational difference functions on $W^\#$.*

Proof. Let a be a generic of V over K, and let $A = \mathrm{acl}(A) \subseteq \mathrm{acl}(Ka)$ be maximal realizing an almost-$\mathcal{S}$-internal-type over K. Let $A_0 = A \cap K(a)_{\sigma+}$ and let b be a finite tuple such that $A_0 = K(b)_{\sigma+}$. Then $tp(b/K)$ is almost-$\mathcal{S}$-internal, which translates into: if $V^\#$ is the difference variety of which b is a generic, then $V^\#$ is almost-$\mathcal{S}$-internal, is a quotient of V by a difference rational map, and is maximal such (up to birational difference equivalence). This is immediate observing that $K(a)_{\sigma+} = K(V)_{\sigma+}$, and $K(b)_{\sigma+} = K(V^\#)_{\sigma+}$.

As in the proof of Proposition 4.1, the statement about W and $W^\#$ reduces to showing that $A = A_0^{\rm alg}$. First note that because b realizes an almost-$\mathcal{S}$-internal type over K, we have $K(b)_\sigma \subset A_0^{\rm alg}$.

As in Proposition 4.1, we argue that if $c \in A$ and c' is a field conjugate of c over $K(b)_\sigma$, then $tp(c'/K)$ is almost-$\mathcal{S}$-internal because $c' \in A$. Hence if c is such that $A = K(c)^{\rm alg}$, then c and c' are equi-algebraic over K; it then follows that the code d of the set of field conjugates of c over $K(b)_\sigma$ is equi-algebraic with c over K, and therefore that $A = A_0^{\rm alg}$: if the characteristic is 0, then $d \in K(b)_\sigma$, and if the characteristic is $p > 0$, some p^m-power of d is in $K(b)_\sigma$. $\square$

Remark 4.7. A similar statement could be obtained with maximal qf-$\mathcal{S}$-internal quotients instead: Replace A_0 by its maximal subset A_1 realizing a qf-$\mathcal{S}$-internal type over K; then A_0/A_1 is algebraic.

Observe also the following direct consequence of the proof of Proposition 4.6: Let a be a tuple in $\mathcal{U}$, K a difference subfield of $\mathcal{U}$ and A the maximal subset of $\mathrm{acl}(Ka)$ realizing an almost-$\mathcal{S}$-internal type over K^{alg}. If $A = A_0 \cap K(a)_{\sigma+}$, then $A = A_0^{\mathrm{alg}}$.

Proposition 4.8. *Let V be a difference variety of finite order defined over the difference field K. Then, up to composition with a power of Frobenius, V has a* maximal almost-$\mathcal{S}$-internal fiber, *i.e. a unique minimal σ-rational quotient $V^\flat$ with the property that the generic fiber of the quotient map is irreducible and almost-$\mathcal{S}$-internal. Furthermore, if W is a finite cover of V, then $W^\flat$ is a finite cover of $V^\flat$ via a map f, for some tuple f of rational difference functions on $W^\flat$.*

Proof. Let a be a generic of V over K, and $A = \mathrm{acl}(A) \subset \mathrm{acl}(Ka)$ be minimal such that $tp(a/A)$ is almost-$\mathcal{S}$-internal, let $A_0 = A \cap K(a)_{\sigma+}$, and let c be a finite tuple such that $A_0 = K(c)_{\sigma+}$. We now let $V^\flat$ be the difference variety defined over K of which c is a generic and $f : V \to V^\flat$ the map induced by the inclusion $K(c)_{\sigma+} \subseteq K(a)_{\sigma+}$.

As in the proof of Proposition 4.2, the assertion about W and $W^\flat$ reduces to showing that $A = A_0^{\mathrm{alg}}$. Let $b \in A_0^s$ be such that $A = K(b)^{\mathrm{alg}}$, and let $b_2, \ldots, b_m$ be the field-conjugates of $b = b_1$ over $K(a)_\sigma$. By Fact 4.5, there is $\ell \geq 1$ such that, for each $i \geq 2$, there is a σ^ℓ-$K(a)_\sigma$-isomorphism $f_i : K(a)_\sigma(b)_{\sigma^\ell} \to K(a)_\sigma(b_i)_{\sigma^\ell}$ sending b to b_i. Since $\sigma(b) \in K(b)^{\mathrm{alg}}$, we know that $qftp(a, \ldots, \sigma^{\ell-1}(a)/K(b)_{\sigma^\ell})[\ell]$ is almost-$\mathcal{S}$-internal, and therefore so are the types $tp(a, \ldots, \sigma^{\ell-1}(a)/K(\sigma^j(b_i)_{\sigma^\ell}))[\ell]$ for $0 \leq j < \ell$ (it is clear for $j = 0$; then apply powers of σ to get the result for the other values of j). Letting $B = \bigcap_{i=1}^m \bigcap_{j=0}^{\ell-1} \mathrm{acl}_{\sigma^\ell}(K\sigma^j(b_i))$ and noting that $B = \sigma(B)$, Fact 4.5 and Lemma 2.3 imply that $tp(a/B)$ is almost-$\mathcal{S}$-internal. The minimality of A and the fact that $b_1 \in A$ now imply $A = B$. It follows that all tuples b_i belong to A, since $\mathrm{tr.deg}(K(b_i)/K) = \mathrm{tr.deg}(K(b)/K)$. Hence, if d is the tuple encoding the set $\{b_1, \ldots, b_m\}$, then $K(d)^{\mathrm{alg}} = K(b)^{\mathrm{alg}}$ and $tp(a/K(d)_\sigma)$ is almost-$\mathcal{S}$-internal. For some $n, m \geq 0$ we then have $\sigma^n(d^{p^m}) \in K(a)_{\sigma+}$, which shows $A = A_0^{\mathrm{alg}}$. □

Remark 4.9. The proof gives the following: let a be a tuple in $\mathcal{U}$, K a difference subfield of $\mathcal{U}$ and A an algebraically closed difference subfield of $\mathrm{acl}(Ka)$ such that $tp(a/A)$ is almost-$\mathcal{S}$-internal. If $A_0 = A \cap K(a)_{\sigma+}$ then $tp(a/A_0)$ is almost-$\mathcal{S}$-internal.

Descent of difference varieties. The main application of our results are given by Theorems 4.10 and 4.11. Theorem 4.11 is an almost optimal generalization of Theorem 3.3 of [6].

Theorem 4.10. *Let K_i, $i = 1, 2$, be difference subfields of $\mathcal{U}$ with intersection k, and V_i difference varieties of finite order defined over K_i, and assume that $k^{\mathrm{alg}} = K_1^{\mathrm{alg}} \cap K_2^{\mathrm{alg}}$. Assume that there is a σ-rational dominant $f : V_1 \to V_2$ defined over $(K_1K_2)^{\mathrm{alg}}$, that $tp(K_2/k)$ is almost-$\mathcal{S}$-internal and that K_2 is a regular extension of k. Then there is a dominant map $g : V_2 \to V_3$, with V_3 a difference variety defined over k, such that the generic fiber of g is almost-$\mathcal{S}$-internal.*

Proof. Let a_1 be a generic of V_1 over K_1K_2, and $a_2 = f(a_1)$. Letting $b_1 = K_1$ and $b_2 = K_2$, applying Proposition 2.10 and using Proposition 4.8, we obtain $e \in K_2(a_2)_\sigma$ such that $k(e)_\sigma$ and K_2 are free over k, and $tp(a_2/k(e)_\sigma)$ is almost-$\mathcal{S}$-internal. Moreover, $k(e)_\sigma$, being a subfield of $K_2(a_2)_\sigma$, is a regular extension of k and is therefore linearly disjoint from K_2 over k; we may assume that $k(e)_{\sigma+} = K_2(a_2)_{\sigma+} \cap k(e)_\sigma^{\mathrm{alg}}$. If V_3 is the difference variety of which e is a generic (over K_2), then V_3 is defined over k, and the inclusion $K_2(e)_{\sigma+} \subset K_2(a_2)_{\sigma+}$ gives a dominant rational difference map $g : V_2 \to V_3$ (defined over K_2) such that $g(a_2) = e$, and with generic fiber almost-$\mathcal{S}$-internal. $\square$

Theorem 4.11. *Let K_1, K_2 be fields intersecting in k and with algebraic closures intersecting in k^{alg}; for $i = 1, 2$, let V_i be an absolutely irreducible variety and $\phi_i : V_i \to V_i$ a dominant rational map defined over K_i. Assume that K_2 is a regular extension of k and that there are an integer $r \geq 1$ and a dominant rational map $f : V_1 \to V_2$ such that $f \circ \phi_1 = \phi_2^{(r)} \circ f$, where $\phi_2^{(r)}$ denotes the function obtained by iterating r times ϕ_2. Then there is a variety V_0 and a dominant rational map $\phi_0 : V_0 \to V_0$, all defined over k, a dominant map $g : V_2 \to V_0$ such that $g \circ \phi_2 = \phi_0 \circ g$, and* $\deg(\phi_0) = \deg(\phi_2)$.

Proof. Observe that the rational map f will be defined over $(K_1K_2)^{\mathrm{alg}}$, because this is a statement about algebraic varieties and rational morphisms.

Let a_1 be a generic of V_1 over K_1K_2, and let $a_2 = f(a_1)$. Then a_2 is a generic of V_2 over K_1K_2. We fix an existentially closed difference field $(\mathcal{U}, \sigma)$ containing $K_2(a_2)$ and such that σ is the identity on K_2 and $\sigma(a_2) = \phi_2(a_2)$. We fix another existentially closed field $(\mathcal{U}', \tau)$ containing $K_1K_2(a_1)$, such that τ is the identity on K_1K_2, and $\tau(a_1) = \phi_1(a_1)$. Note that τ and σ^r agree on $K_2(a_2)$. By abuse of notation, we let $\mathcal{S}$ denote the set of non-algebraic types of rank 1 realized in $\mathrm{Fix}(\sigma)$ when working in $\mathcal{U}$, in $\mathrm{Fix}(\tau)$ when working in $\mathcal{U}'$, and in $\mathrm{Fix}(\sigma^r)$ when working in $\mathcal{U}[r]$.

Working in $\mathcal{U}'$, by Proposition 2.10, there is an algebraically closed τ-difference field E contained in $K_2(a_2)^{\mathrm{alg}}$ and free from K_2 over k, such that $tp^{\mathcal{U}',\tau}(K_2a_2/E)$ is almost-$\mathcal{S}$-internal. By Remark 4.9 and Fact 4.5, we obtain that $tp^{\mathcal{U}}(K_2a_2/E \cap K_2(a_2))[r]$ is almost-$\mathcal{S}$-internal, and therefore so is $tp^{\mathcal{U}}(K_2a_2/E)[r]$. Applying σ^i for $i \geq 0$, we get that $tp(K_2\sigma^i(a_2)/\sigma^i(E))[r]$ is almost-$\mathcal{S}$-internal, and because $a_2 \in K_2(\sigma^i(a_2))^{\mathrm{alg}}$ so is $tp(a_2/\sigma^i(E))[r]$.

Observe now that because $E = (E \cap K_2(a_2))^{\mathrm{alg}}$ and τ agrees with σ^r on $K_2(a_2)$, we have $\sigma^r(E) = E$. Hence, by Lemma 2.3, we may replace E by $\bigcap_i \sigma^i(E)$ and assume that $\sigma(E) = E$. We now reason as in Proposition 4.8 to show that if a_3 is such that $K_2(a_2) \cap E = k(a_3)$, then $tp(a_2/k(a_3)_\sigma)$ is almost-$\mathcal{S}$-internal. Note that as $K_2(a_2)$ and E are closed under σ, so is $k(a_3)$. Hence, $\sigma(a_3) \in k(a_3)$. As K_2 is a regular extension of k, and $k(a_3) \subset E$, it follows that $k(a_3)_\sigma$ and K_2 are linearly disjoint over k. Letting V_0 be the algebraic locus of a_3 over k, and ϕ_0 the rational endomorphism of V_0 such that $\sigma(a_3) = \phi_0(a_3)$, we get the desired (V_0, ϕ_0). The rational map g is the one given by the inclusion $K_2(a_3) \subset K_2(a_2)$.

It remains the assertion about the degrees of the maps. By Lemma 1.11 of [6], we have $1 = \mathrm{ld}(a_2/K_2(a_3)_\sigma) = \mathrm{ild}(a_2/K_2(a_3)_\sigma)$, which implies $\deg(\phi_2) = \deg(\phi_0)$ and finishes the proof. □

Remarks 4.12. (1) As stated, the theorem says nothing when $\deg(\phi_2) = 1$, since one can take V_0 of dimension 0.
(2) The assertion on the degrees of the map ϕ_2 and ϕ_0 is weaker than the assertion that $tp(a_2/k(a_3)_\sigma)$ is almost-$\mathcal{S}$-internal. Note that for instance if $c \in K_2$ is a finite tuple which generates over k the field of definition of (V_2, ϕ_2), then $c \in k(a_2)_\sigma$, and therefore $tp(c, a_2/k(a_3)_\sigma)$ is almost-$\mathcal{S}$-internal. This should have consequences on the data (V_2, ϕ_2).
(3) One can in fact show that the generic fiber of g is qf-Fix(σ)-internal. The proof goes as follows: we know that there is some $a_4 \in K_2(a_2)$ such that $tp(a_4/k(a_3))$ is qf-Fix(σ)-internal, and $a_2 \in K_2(a_4)^{\mathrm{alg}}$. Observe that because $\mathrm{ld}(a_2/K_2) = \mathrm{ld}(a_4/K_2)$ (=1), the field $K_2(a_2)_\sigma$ is a finite extension of $K_2(a_4)_\sigma$. Let L be a difference field containing $k(a_3)_\sigma$, linearly disjoint from $K_2(a_2)_\sigma$ over $k(a_3)_\sigma$, and such that $L(a_4)_\sigma = L(b)$ for some tuple b in Fix(σ). Enlarging L if necessary, we will assume that L is algebraically closed and that Fix$(\sigma) \cap L$ has absolute Galois group isomorphic to $\hat{\mathbb{Z}}$, so that Fix$(\sigma)L$ contains the algebraic closure of Fix(σ). It then follows by Lemma 4.2 of [3] that $L(a_2) \subset L\mathrm{Fix}(\sigma)$, which shows that $tp(a_2/k(a_3))$ is qf-Fix(σ)-internal.

Appendix

Proposition A.1. *Let E and B be algebraically closed subsets of M, b a tuple in M. Assume that $\mathrm{SU}(B/B \cap E) < \omega$, that $tp(b/B)$ is one-based, and that $B \cap E = \mathrm{acl}(Bb) \cap E$. Then b is independent from E over B.*

Proof. Assume the result is false, and take a counterexample with $r = \mathrm{SU}(B/B \cap E) - \mathrm{SU}(B/E)$ minimal among all such (B, E, b). We may assume that $B \cap E = \mathrm{acl}(\emptyset)$, and $E = \overline{\mathrm{Cb}}(Bb/E)$. Since $tp(b/B)$ is one-based, $\mathrm{acl}(Bb) \cap \mathrm{acl}(BE) \neq B$, and we may therefore assume that $b \in \mathrm{acl}(BE)$.

If $r = 0$, then $B \mathop{\smile\!\!\!\!\!\!\mid} E$, so that $\overline{\mathrm{Cb}}(Bb/E)$ realizes a one-based type over $B \cap E$ (by Lemma 1.7 with $\mathcal{S}$ the set of one-based types with algebraically closed base), and therefore $Bb \mathop{\smile\!\!\!\!\!\!\mid} E$. This contradicts $b \in \mathrm{acl}(BE)$. Hence $r > 0$.

Let $A = \overline{\mathrm{Cb}}(B/E)$. We may then assume that E and b are equi-algebraic over AB: by Lemma 1.7 $tp(E/A)$ is one-based, and if $D = \mathrm{acl}(ABb) \cap E$, then

$b \in \mathrm{acl}(BD)$. Replace E by D. Reasoning as in Step 3 of Theorem 2.1, there is $m \geq 2$, and E-independent realizations $(B_1b_1), \ldots, (B_mb_m)$ of $tp(Bb/E)$ with $A \subset \mathrm{acl}(B_1 \cdots B_m)$ and $\mathrm{SU}(B_m/B_1 \cdots B_{m-1}) > \mathrm{SU}(B/A)$. The induction hypothesis implies $b_1 \mathop{\smile\!\!\!\!\!\!\downarrow}_{B_1} B_2 \cdots B_m$, since

$$\mathrm{acl}(B_1b_1) \cap \mathrm{acl}(B_2 \cdots B_m) \subseteq \mathrm{acl}(Bb) \cap E = \mathrm{acl}(\emptyset).$$

Similarly $b_m \mathop{\smile\!\!\!\!\!\!\downarrow}_{B_m} B_1 \cdots B_{m-1}$, and therefore $B_1 \mathop{\smile\!\!\!\!\!\!\downarrow}_{B_2 \cdots B_m} b_m$.

If $E = A$, then $b \in \mathrm{acl}(AB)$, and $b_1 \in \mathrm{acl}(B_1 \cdots B_m)$; by the above we get $b_1 \in B_1$ which is absurd.

If $E \neq A$, then $E \not\subseteq \mathrm{acl}(B_1 \cdots B_m)$ because $E \mathop{\smile\!\!\!\!\!\!\downarrow}_A B$, and each b_i is equi-algebraic with E over $B_1 \cdots B_m$. Hence b_1 and b_m are equialgebraic over $B_1 \cdots B_m$. However,

$$\mathrm{SU}(B_1/B_2 \cdots B_mb_m) = \mathrm{SU}(B_1/B_2 \cdots B_m) > \mathrm{SU}(B/E).$$

The induction hypothesis, together with the fact that

$$\mathrm{acl}(B_1b_1) \cap \mathrm{acl}(B_2 \cdots B_mb_m) \subseteq \mathrm{acl}(B_1b_1) \cap E = \mathrm{acl}(\emptyset),$$

gives $b_1 \mathop{\smile\!\!\!\!\!\!\downarrow}_{B_1} B_2 \cdots B_mb_m$, a contradiction. $\square$

Remark A.2. This result does not hold when $\mathrm{SU}(B/B \cap E)$ is infinite. Here is a counterexample for T a completion of ACFA in characteristic 0. Let a, b, c be generics and independent over $\mathbb{Q}^{\mathrm{alg}}$, and consider $d = ac + b$, and $e = \sigma(b) - b^2$. Then $\overline{\mathrm{Cb}}(c, d/a, b) = \mathrm{acl}(\mathbb{Q}, a, b)$. Moreover, $tp(b/e)$ is one-based (by Example 6.1 of [4]) and has SU-rank 1. One also has

$$\mathrm{acl}(a, b) \cap \mathrm{acl}(c, d) = \mathbb{Q}^{\mathrm{alg}} = \mathrm{acl}(\emptyset).$$

Take for (B, b, E) the triple $(\mathbb{Q}(a, e)_\sigma^{\mathrm{alg}}, b, \mathbb{Q}(c, d)_\sigma^{\mathrm{alg}})$.

Proposition A.3. *Let $tp(a/A)$ be a one-based type of* SU*-rank ω^α for some ordinal α, with $\mathrm{SU}(A) < \omega$, $A = \mathrm{acl}(A)$, and consider the class $\mathcal{P}$ of all types of* SU*-rank ω^α with algebraically closed base, which are non-orthogonal to $tp(a/A)$. Then $\mathcal{P}$ contains a type q whose base C is contained in all bases of elements of $\mathcal{P}$. If $tp(a/A)$ has* SU*-rank 1 and is trivial, then there is c such that $\mathrm{SU}(c/C) = 1$ and $a \in \mathrm{acl}(Ac)$.*

Proof. Assume that $tp(b/B) \in \mathcal{P}$. Moving a, we may assume that $a \mathop{\smile\!\!\!\!\!\!\downarrow}_A B$. By Lemma 1.11, there are realizations $a_1, \ldots, a_n$ of $tp(a/A)$ which are independent from Ba over A, and realizations $b_1, \ldots, b_m$ of $tp(b/B)$ which are independent from

A over B, such that

$$\mathrm{SU}(a/ABa_1 \cdots a_n b_1 \cdots b_m) < \omega^\alpha.$$

Choose such m, n minimal. Then

$$\mathrm{SU}((a_1, \ldots, a_n, b_1, \ldots, b_m)/AB) = \omega^\alpha(n+m)$$

and

$$\mathrm{acl}(Aa_1 \cdots a_n) \cap \mathrm{acl}(Bb_1 \cdots b_m) = A \cap B = C.$$

By Proposition A.1, we know that $\mathrm{acl}(Aaa_1 \cdots a_n) \cap \mathrm{acl}(Bb_1 \cdots b_m)$ contains some element $d \notin C$. The usual routine arguments then give $tp(d/C) \not\perp tp(a/A)$ and $\mathrm{SU}(d/C) = \omega^\alpha$.

Let $p_1, p_2 \in \mathcal{P}$, with bases A_1, A_2 contained in A. Because $\mathrm{SU}(p) = \omega^\alpha$, the type p has weight 1. Hence the inclusions $A_1, A_2 \subseteq A$ and the non-orthogonality of p_1, p_2 to p imply $p_1 \not\perp p_2$.

Thus the set of bases of types in $\mathcal{P}$ is closed under intersection, and has a smallest element, since one cannot have an infinite decreasing sequence of algebraically closed sets of finite SU-rank.

The last assertion follows immediately from triviality, as non-orthogonality then implies non-almost-orthogonality. $\square$

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