Blowup issues for water wave propagation in shallow water

Manuel Fernando Cortez

Institut Camille
UMR 5208 du CNRS

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Université Claude Bernard Lyon 1

The Rod Equation

Introduction

Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be the unit circle. The Cauchy problem for the periodic rod equation is written as follows:

$$\begin{cases} u_t + \gamma u u_x = -\partial_x p * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right), & t \in (0,T),\ x \in \mathbb{S}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}$ and p is the kernel of the convolution operator $(1-\partial_x^2)^{-1}$. It is the continuous 1-periodic function given by

$$p(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)},$$

where $[\cdot]$ denotes the integer part.

Remark

- If u satisfies (1.1), then u is a periodic solutions of the rod equation.
- The real parameter γ is related to the Finger deformation tensor of the material.
- The Cauchy problem (1.1) also models shallow water waves inside channels.
- When $\gamma = 1$ then (1.1) became in the Camassa–Holm equation. (dispertionless case)

Well-posedness

- If $u_0 \in H^s(\mathbb{S})$, with $s > 3/2$, then, $\forall \gamma \in \mathbb{R}$, the Cauchy problem (1.1) is locally well-posed: $\exists$ a maximal time $0 < T^* \leq \infty$ and a unique solution $u \in C([0,T^*), H^s(\mathbb{S})) \cap C^1([0,T^*), H^{s-1}(\mathbb{S}))$.
- The solution u depends continuously on u_0. It is also known that u admits several invariant integrals, among which the energy integral,

$$E(u) = \int_{\mathbb{S}} (u^2 + u_x^2)\, dx.$$

- In particular, because of the conservation of the Sobolev H^1-norm, the solution $u(x,t)$ remains uniformly bounded up to the time T^*.

Global Existence

- In the case of the Camassa–Holm equation on the real line, a striking necessary and sufficient condition for the global existence of strong solution can be given in terms on the initial potential $y_0 = u_0 - u_{0,xx}$, see (H. McKean 2004).
- On the other hand, very little is known on the global existence of strong solutions when $\gamma \neq 1$.
- Smooth solitary waves that are global strong solutions were constructed at least for some γ see (J. Lenells,2006).

Blowup Phenomena

Blowup Scenario

If $T^* < \infty$ then $\limsup_{t\to T^*} \|u(t)\|_{H^s} = \infty$ ($s > 3/2$) and more precisely (breaking mechanism):

$$T^* < \infty \iff \liminf_{t\to T^*}\Big(\inf_{x\in\mathbb{S}} \gamma u_x(t,x)\Big) = -\infty,$$

see (A.Constantin, W. Strauss 2000).

Quick overview of the main results

First Theorem

- Loosely, we show that if $|\gamma|$ is *not too small*, then there exist a constant $\beta_\gamma > 0$ such that if

  $$u_0'(x_0) > \beta_\gamma |u_0(x_0)| \text{ if } \gamma < 0, \text{ or } u_0'(x_0) < -\beta_\gamma |u_0(x_0)| \text{ if } \gamma > 0$$

  in at least one point $x_0 \in \mathbb{S}$, then the solution arising from $u_0 \in H^s(\mathbb{S})$ must blow up in finite time.

Second theorem

- We make precise what "$|\gamma|$ not too small" means, addressing also the delicate issue of finding sharp estimates for β_γ.

Second theorem

- In the particular case of the periodic (C-H) equation we get that a sufficient condition for the blowup is:

$$\exists\, x_0 \in \mathbb{S}\ u_0'(x_0) < -\sqrt{\frac{5}{2} - \frac{3}{2} \cdot \frac{\cosh\frac{1}{2}\cosh\frac{3}{2} - 1}{\sinh\frac{1}{2}\sinh\frac{3}{2}}}\,|u_0(x_0)|.$$

Corollary

- $\exists$ an absolute constant β_∞ ($\beta_\infty = 0.295\ldots$)with the following property: If $u_0 \in H^s(\mathbb{S})$, is such that for some $x_0 \in \mathbb{S}$, $u_0'(x_0) > \beta_\infty|u_0(x_0)|$, or otherwise $u_0'(x_0) < -\beta_\infty|u_0(x_0)| \Rightarrow$ the solutions (depending on γ) of (1.1) arising from u_0 blow up in finite time respectively if $\gamma \ll -1$ or $\gamma \gg 1$.

Difference between in the behavior of periodic and non-periodic case

- The analogue blowup result for the rod equation on $\mathbb{R}$ (L., Brandolese 2012), could be established only in the range $1 \leq \gamma \leq 4$ in the non-periodic case.
- The relevant estimates on $\mathbb{S}$ that we will establish turn out to be much stronger.
- It may be that if $u_0 \in H^s(\mathbb{S})$ and $\widetilde{u}_0 \in H^s(\mathbb{R})$ agree on an arbitrarily large finite interval, and that periodic solution arising from u_0 blows up, whereas the solution arising from $\widetilde{u}_0$ and vanishing at infinity exists globally.

Difference between previous blowup criteria and ours

- A huge number of previous papers addressed the blowup issue of solutions to equation (1.1), (R. Camassa, L. Holm,J. Hyman 93,A. Constantin, J. Escher 98).
- Typically, conditions of the form $u_0'(x_0) < -c_\gamma \|u_0\|_{H^1(\mathbb{S}^1)}$ or some other integral conditions on u_0, or otherwise antisymmetry conditions, etc.
- Our blowup criteria is that it they are *local-in-space*. This means that these criteria involve a condition *only on a small neighborhood of a single point* of the datum.

Some Notations

- For any real constant α and β, let $I(\alpha,\beta) \geq -\infty$ defined by

$$I(\alpha,\beta) = \inf\left\{ \int_0^1 \left(p + \beta p_x\right)\left(\alpha u^2 + u_x^2\right) dx, u \in H^1(\mathbb{S}) \right\}.$$

- For $\gamma \in \mathbb{R}^*$, the quantity $\beta_\gamma \in [0,+\infty]$ defined by

$$\beta_\gamma = \inf\left\{ \beta \in \mathbb{R}^+ : \beta^2 + I\left(\frac{3-\gamma}{\gamma}, \beta\right) - \frac{3-\gamma}{\gamma} \geq 0 \right\},$$

with the usual convention that $\beta_\gamma = +\infty$ if the infimum is taken on the empty set.

First Theorem

- Let $\gamma \in \mathbb{R}^*$ be such that $\beta_\gamma < +\infty$. Let $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$ and assume that there exists $x_0 \in \mathbb{S}$, such that

  $u_0'(x_0) > \beta_\gamma |u_0(x_0)|$ if $\gamma < 0$, *or* $u_0'(x_0) < -\beta_\gamma |u_0(x_0)|$ if $\gamma > 0$,

  then maximal time T^* is estimated by

$$T^* \leq \frac{2}{\gamma \sqrt{u_0'(x_0)^2 - \beta_\gamma^2 u_0(x_0)^2}} < \infty$$

Second Theorem

- In order to state our next theorem, let us introduce the complex number

$$\mu = \frac{1}{2}\sqrt{1 + 4(3-\gamma)/\gamma}, \qquad \gamma \neq 0,$$

where $\sqrt{1 + 4(3-\gamma)/\gamma}$ denotes any of the two complex square roots. We also consider the four constants:

$$\begin{aligned} \gamma_1^- &= -1.036\ldots \qquad & \gamma_1^+ &= 0.269\ldots \\ \gamma_2^- &= -1.508\ldots \qquad & \gamma_2^+ &= 0.575\ldots \end{aligned}$$

Second Theorem

- For any $\gamma \in (-\infty, \gamma_1^-] \cup [\gamma_1^+, +\infty)$, we have $\beta_\gamma < +\infty$, so that Theorem 1 applies in such range. More precisely, if $\gamma \in (-\infty, \gamma_2^-] \cup [\gamma_2^+, \infty)$, then

$$\beta_\gamma \leq \sqrt{\frac{3}{\gamma} - \frac{1}{2} - \mu \cdot \frac{\cosh\frac{1}{2}\cosh\mu - 1}{\sinh\frac{1}{2}\sinh\mu}}.$$

- The limit $\beta_\infty = \lim_{\gamma\to\pm\infty}\beta_\gamma$ does exist and

$$\beta_\infty \leq \sqrt{\frac{\sqrt{3}\left(1 - \cosh\frac{1}{2}\cos\frac{\sqrt{3}}{2}\right)}{2\sinh\frac{1}{2}\sin\frac{\sqrt{3}}{2}} - \frac{1}{2}} = 0.296\ldots.$$

Second Theorem

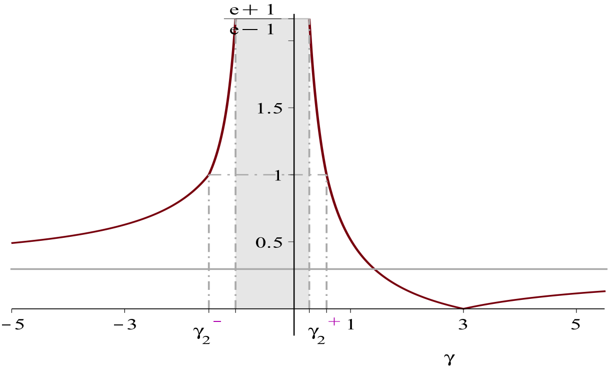

Figure: The upper-bound estimate of β_γ given by Theorem 1. The estimate is valid outside the interval $[\gamma_1^-, \gamma_1^+]$ (gray region).

Main Tools

First properties of $I(\alpha, \beta)$

- For any real β, let us consider the $1-$ periodic function
$$\omega(x) = p(x) + \beta p'(x),$$
- The non-negativity condition $\omega \geq 0$ is equivalent to the inequality $\cosh(1/2) \geq \pm\beta \sinh(1/2)$, *i.e.*, to the condition
$$-\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}.$$
- Throughout this section, we will work under the above condition on β.

- Let us introduce the weight Sobolev space

$$E_\beta = \left\{ u \in L^1_{\text{loc}}(0,1) : \; \|u\|^2_{E_\beta} = \int_0^1 \omega(x)(u^2 + u_x^2)(x)\, dx < \infty \right\}$$

- Let us consider the closed subspace $E_{\beta,0}$ of E_β defined as closure of $C_c^\infty(0,1)$ in E_β.
- Notice that, with slightly abusive notation :

$$E_{\beta,0} = \left\{ u \in E_\beta : u(0) = u(1) = 0 \right\}$$

Remark

If $|\beta| < \frac{e+1}{e-1}$

- Notice that E_β agrees with the classical Sobolev space $H^1(0,1)$ when $|\beta| < \frac{e+1}{e-1}$
- $E_{\beta,0} = H_0^1(0,1)$

If $|\beta| = \frac{e+1}{e-1}$

- E_β is strictly larger that $H^1(0,1)$. For instance, $|\log(x/2)|^3 \in E_{\frac{e+1}{e-1}}$ but does not belong to $H^1(0,1)$.

If $|\beta| = \frac{e+1}{e-1}$

- If $u \in E_{\frac{e+1}{e-1},0}$, then $u(x) = O(\sqrt{|\log x|})$ as $x \to 0^+$ and $u(1) = 0$.
- If $u \in E_{-(\frac{e+1}{e-1}),0}$, then $u(x) = O(\sqrt{|\log(1-x)|})$ as $x \to 1^-$ and $u(0) = 0$.

Lemma

- For all $-\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}$, there exists a constant $C > 0$ such that

$$\forall\, v \in E_{\beta,0} : \int_0^1 \omega(x) v(x)^2 \, dx \leq C \int_0^1 \omega(x) v_x(x)^2 \, dx. \tag{1.2}$$

- We denote $C(\beta) > 0$ the *best constant*, which verifies the weighted Poincaré inequality (1.2).

Proposition

We have

$$I(\alpha,\beta) > -\infty \iff \begin{cases} -\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}, \\ \alpha > -1/C(\beta), \end{cases}$$

Moreover, if $|\beta| < \frac{e+1}{e-1}$*, then* $I(\alpha,\beta)$ *is in fact a minimum and there is only one minimizer* $u \in H^1(0,1)$ *with* $u(0) = u(1) = 1$.

Proposition

The function $(\alpha,\beta) \mapsto I(\alpha,\beta) \in \mathbb{R} \cup \{-\infty\}$*, defined for all* $(\alpha,\beta) \in \mathbb{R}^2$*, is concave with respect to each one of its variables and is even with respect to the variable* β*. Moreover,*

$$\forall\, \alpha \in \mathbb{R},\ \forall\, |\beta| \le \tfrac{e+1}{e-1},\ -\infty \le I(\alpha, \tfrac{e+1}{e-1}) \le I(\alpha,\beta) \le I(\alpha,0) \le \alpha.$$

Lemma

For any $\alpha, \beta \in \mathbb{R}$ and all $u \in H^1(\mathbb{S})$ the following convolution estimate holds:

$$\forall\, x \in \mathbb{S}, \qquad (p + \beta p') * (\alpha u^2 + u_x^2)(x) \ge I(\alpha,\beta)\, u(x)^2 \qquad (1.3)$$

and $I(\alpha,\beta)$ is the best possible constant.

Outline of proof Theorem 1

The starting point is the analysis of the flow map $q(t,x)$, defined by

$$\begin{cases} q_t(t,x) = \gamma u(t, q(t,x)), & t \in (0,T),\ x \in \mathbb{R}, \\ q(0,x) = x. \end{cases}$$

From the rod equation

$$u_t + \gamma u u_x = -\partial_x p * \left(\tfrac{3-\gamma}{2} u^2 + \tfrac{\gamma}{2} u_x^2\right), \tag{1.4}$$

differentiating with respect to the x variable and applying the identity $\partial_x^2 p * f = p * f - f$, we get

$$u_{tx} + \gamma u u_{xx} = \frac{3}{2(\alpha+1)}\Big[\alpha u^2 - u_x^2 - p * (\alpha u^2 + u_x^2)\Big]. \qquad (1.5)$$

Here we set

$$\alpha = \frac{3-\gamma}{\gamma}.$$

Let us introduce the two C^1-functions of the time variable, depending on β,

$$f(t) = (-u_x + \beta u)(t, q(t, x_0)), \text{ and } g(t) = -(u_x + \beta u)(t, q(t, x_0))$$

Computing the time derivative using the definition of the flow q, next using equations (1.4)-(1.5), we get

$$\begin{aligned}\frac{df}{dt}(t) &= \Big[(-u_{tx} - \gamma u u_{xx}) + \beta(u_t + \gamma u u_x)\Big](t, q(t, x_0)) \\ &= \frac{3}{2(\alpha+1)}\Big[-\alpha u^2 + u_x^2 + (p - \beta p') * (\alpha u^2 + u_x^2)\Big](t, q(t, x_0)),\end{aligned}$$

and

$$\frac{dg}{dt}(t) = \frac{3}{2(\alpha+1)}\Big[-\alpha u^2 + u_x^2 + (p + \beta p') * (\alpha u^2 + u_x^2)\Big](t, q(t, x_0)).$$

- case $\gamma > 0$. Then $\alpha > -1$.From the condition $\beta_\gamma < \infty$, we deduce that $\exists\, \beta \geq 0$ s. t.

$$\beta^2 \geq \alpha - I(\alpha, \beta). \tag{1.6}$$

Applying the estimate (1.3) and as $I(\alpha,-\beta) = I(\alpha,\beta)$, we get, for all $\beta \geq 0$ satisfying (1.6),

$$\begin{aligned}\frac{df}{dt}(t) &\geq \frac{3}{2(\alpha+1)}\Big[u_x^2 - \big(\alpha - I(\alpha,-\beta)\big)u^2\Big](t,q(t,x_0))\\ &\geq \frac{3}{2(\alpha+1)}\Big[u_x^2 - \beta^2 u^2\Big](t,q(t,x_0))\\ &= \frac{3}{2(\alpha+1)}f(t)g(t).\end{aligned}$$

In the same way,

$$\begin{aligned}\frac{dg}{dt}(t) &\geq \frac{3}{2(\alpha+1)}\Big[u_x^2 - \big(\alpha - I(\alpha,\beta)\big)u^2\Big](t,q(t,x_0))\\ &\geq \frac{3}{2(\alpha+1)}\Big[u_x^2 - \beta^2 u^2\Big](t,q(t,x_0))\\ &= \frac{3}{2(\alpha+1)}f(t)g(t).\end{aligned}$$

As $u_0'(x_0) < -\beta_\gamma|u(x_0)|$ there $\exists\, \beta$ satisfying (1.6), with $\beta - \beta_\gamma > 0$ and $u_0'(x_0) < -\beta|u_0(x_0)|$ For such a choice of β we have

$$f(0) > 0 \quad \text{and} \quad g(0) > 0.$$

The blowup of u will rely on the following basic property:

Lemma

Let $0 < T^* \le \infty$ and $f, g \in C^1([0, T^*), \mathbb{R})$ be such that, for some constant $c > 0$ and all $t \in [0, T^*)$,

$$\frac{df}{dt}(t) \ge cf(t)g(t)$$
$$\frac{dg}{dt}(t) \ge cf(t)g(t).$$

If $f(0) > 0$ and $g(0) > 0$, then

$$T^* \le \frac{1}{c\sqrt{f(0)g(0)}} < \infty.$$

The minimization problem with $\beta = \frac{e+1}{e-1}$ and $\alpha > -1/C(\frac{e+1}{e-1})$

In this case the weight function becomes

$$w(x) = \frac{2e}{(e-1)^2}\sinh(x) \quad x \in (0,1).$$

Now, we call $\nu(\alpha)$:

$$\nu(\alpha) = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\alpha} \in \{z \in \mathbb{C}\ \Im m(z) \geq 0\}.$$

By our computations (variational calculation), we get

$$I(\alpha, \tfrac{e+1}{e-1}) \geq \frac{(e+1)^2}{2e}\frac{P'_{\nu(\alpha)}}{P_{\nu(\alpha)}}(\cosh 1).$$

where $P_\nu(\alpha)$ is the associate Legendre function of the first kind, of degree $\nu(\alpha)$ and $P_\nu(\alpha)$ is bounded as $y \to 1^+$. Moreover, P_ν is a polynomial when ν is an integer.

case $\beta = 1$

- The associated Euler–Lagrange boundary value problem can be explicitly solved.
- The computation below will be valid for $\alpha > -1/C(1)$. As a byproduct of our calculations, we will find the explicit expression

$$C(1) = 4/(1 + 4\pi^2)$$

and the weight function becomes

$$w(x) = \frac{e^x}{(e-1)}.$$

Now, if we call $\mu(\alpha)$ as

$$\mu(\alpha) = \frac{1}{2}\sqrt{1+4\alpha} \in \{z \in \mathbb{C}\ \Im\mathfrak{m}(z) \geq 0\}.$$

Thus by our computations (variational calculation), we get

$$I(\alpha, 1) = -\frac{1}{2} + \mu(\alpha) \cdot \frac{\cosh\frac{1}{2}\cosh(\mu(\alpha))}{\sinh\frac{1}{2}\sinh(\mu(\alpha))}.$$

Outline of proof Theorem 2

Let us recall the definition of β_γ

$$\beta_\gamma = \inf\Big\{\beta \in \mathbb{R}^+ : \beta^2 + I(\alpha,\beta) - \alpha \geq 0\Big\},$$

where the one-to-one relation between α and γ is

$$\alpha = \frac{3-\gamma}{\gamma}, \qquad \text{or} \qquad \gamma = \frac{3}{1+\alpha}.$$

Using the results of the previous section we can now give explicit bounds from below for $I(\alpha,\beta)$ that can be used for the estimate of β_γ.

Using the results of the previous section we get

$$\begin{cases} I(\alpha,\beta) \geq I(\alpha,1), \text{ if } 0 \leq \beta \leq 1 \\ I(\alpha,\beta) \geq R(\alpha,\beta), \text{ if } 1 \leq \beta \leq \frac{e+1}{e-1}. \end{cases}$$

where

$$R(\alpha,\beta) = \frac{e-1}{2}\Big(I(\alpha,\tfrac{e+1}{e-1}) - I(\alpha,1)\Big)\beta + \tfrac{e+1}{2}I(\alpha,1) - \tfrac{e-1}{2}I(\alpha,\tfrac{e+1}{e-1})$$

Under our assumptions we have that

$$I(\alpha,1) \geq I(\alpha,\tfrac{e+1}{e-1}) > -\infty.$$

Thus, by our computations we have that: a sufficient condition on α guaranteeing $\beta_\gamma < +\infty$ is :

$$1 + I(\alpha, 1) - \alpha \geq 0 \quad \text{or} \quad \left(\frac{e+1}{e-1}\right)^2 + I\left(\alpha, \frac{e+1}{e-1}\right) - \alpha \geq 0.$$

- Therefore, there exists $\alpha_1^- < 0 < \alpha_1^+$ such that

$$\left(\frac{e+1}{e-1}\right)^2 + I\left(\alpha, \frac{e+1}{e-1}\right) - \alpha \geq 0 \iff \alpha_1^- \leq \alpha \leq \alpha_1^+.$$

- For the same reason, there exists $\alpha_2^- < 0 < \alpha_2^+$ such that

$$1 + I(\alpha, 1) - \alpha \geq 0 \iff \alpha_2^- \leq \alpha \leq \alpha_2^+.$$

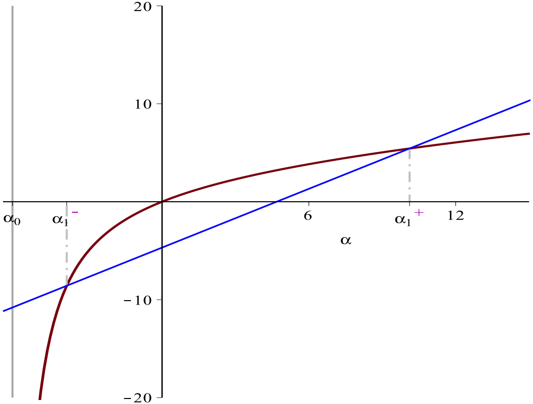

Figure: The plot of $\alpha \mapsto I(\alpha, \frac{e+1}{e-1})$ and of the straight line $\alpha \mapsto \alpha - \left(\frac{e+1}{e-1}\right)^2$, intersecting the curve at α_1^- and α_1^+.

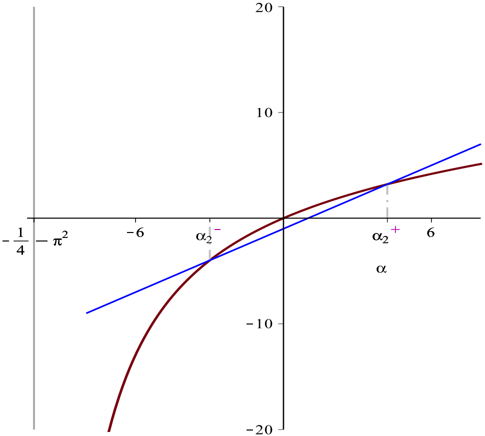

Figure: Plot of $\alpha \mapsto I(\alpha, 1)$ and of the straight line $\alpha \mapsto \alpha - 1$, intersecting the curve at α_2^- and α_2^+.

- The above zeros can be easily estimated via Newton's method. We find in this way

$$\alpha_1^- < \alpha_2^- < 0 < \alpha_2^+ < \alpha_1^+,$$

- According to (1), let us introduce the four constants

$$\gamma_1^- = \frac{3}{1+\alpha_1^-} = -1.036\ldots \qquad \gamma_1^+ = \frac{3}{1+\alpha_1^+} = 0.269\ldots$$
$$\gamma_2^- = \frac{3}{1+\alpha_2^-} = -1.508\ldots \qquad \gamma_2^+ = \frac{3}{1+\alpha_2^+} = 0.575\ldots$$

- The above constants $\gamma_1^-, \gamma_1^+, \gamma_2^-, \gamma_2^+$ are precisely the constants arising in the statement of theorem 2

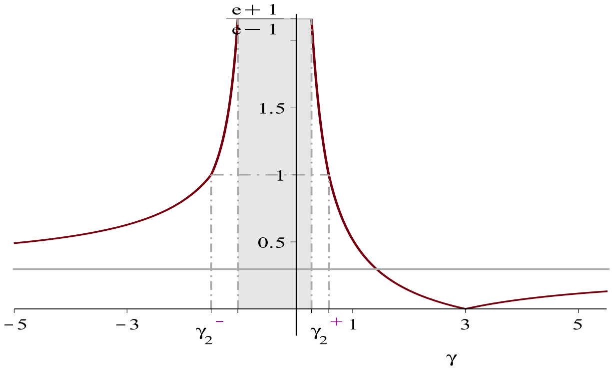

Figure: The upper-bound estimate of β_γ given by Theorem 1. The estimate is valid outside the interval $[\gamma_1^-, \gamma_1^+]$ (gray region).

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Thanks for your attention