TP7

Exercice 1

```
> restart:
> with(Groebner);
```

[*Basis, FGLM, HilbertDimension, HilbertPolynomial, HilbertSeries, Homogenize, InitialForm, InterReduce, IsBasis, IsProper, IsZeroDimensional, LeadingCoefficient, LeadingMonomial, LeadingTerm, MatrixOrder, MaximalIndependentSet, MonomialOrder, MultiplicationMatrix, MultivariateCyclicVector, NormalForm, NormalSet, RationalUnivariateRepresentation, Reduce, RememberBasis, SPolynomial, Solve, SuggestVariableOrder, Support, TestOrder, ToricIdealBasis, TrailingTerm, UnivariatePolynomial, Walk, WeightedDegree*]

```
> n:=8:
> A:=[[1,2],[1,5],[1,6],[2,3],[2,4],[2,8],[3,4],[3,8],[4,5],
        [4,7],[5,6],[5,7],[6,7],[7,8]];
```

$A := [[1, 2], [1, 5], [1, 6], [2, 3], [2, 4], [2, 8], [3, 4], [3, 8], [4, 5], [4, 7], [5, 6], [5, 7], [6, 7], [7, 8]]$

```
> F0:=seq(X[i]^3-1,i=1..n);
```

$$F0 := X_1^3-1, X_2^3-1, X_3^3-1, X_4^3-1, X_5^3-1, X_6^3-1, X_7^3-1, X_8^3-1$$

```
> F1:=seq(X[a[1]]^2+X[a[1]]*X[a[2]]+X[a[2]]^2,a in A);
```

$F1 := X_1^2+X_1X_2+X_2^2, X_1^2+X_1X_5+X_5^2, X_1^2+X_1X_6+X_6^2, X_2^2+X_2X_3+X_3^2, X_2^2+X_2X_4+X_4^2, X_2^2+X_2X_8+X_8^2, X_3^2+X_3X_4+X_4^2, X_3^2+X_3X_8+X_8^2, X_4^2+X_4X_5+X_5^2, X_4^2+X_4X_7+X_7^2, X_5^2+X_5X_6+X_6^2, X_5^2+X_5X_7+X_7^2, X_6^2+X_6X_7+X_7^2, X_7^2+X_7X_8+X_8^2$

```
> F:=[F0,F1];
```

$F := [X_1^3-1, X_2^3-1, X_3^3-1, X_4^3-1, X_5^3-1, X_6^3-1, X_7^3-1, X_8^3-1, X_1^2+X_1X_2+X_2^2, X_1^2+X_1X_5+X_5^2, X_1^2+X_1X_6+X_6^2, X_2^2+X_2X_3+X_3^2, X_2^2+X_2X_4+X_4^2, X_2^2+X_2X_8+X_8^2, X_3^2+X_3X_4+X_4^2, X_3^2+X_3X_8+X_8^2, X_4^2+X_4X_5+X_5^2, X_4^2+X_4X_7+X_7^2, X_5^2+X_5X_6+X_6^2, X_5^2+X_5X_7+X_7^2, X_6^2+X_6X_7+X_7^2, X_7^2+X_7X_8+X_8^2]$

```
> G:=Basis(F,plex(seq(X[i],i=1..n)));
```

$G := [X_8^3-1, X_7^2+X_7X_8+X_8^2, -X_8+X_6, X_7+X_8+X_5, -X_8+X_4, -X_7+X_3, X_7+X_8+X_2, -X_7+X_1]$

```
> map(g->LeadingMonomial(g,plex(seq(X[i],i=1..n))),G);
```

$$[X_8^3, X_7^2, X_6, X_5, X_4, X_3, X_2, X_1] \quad (1)$$

```
> IsZeroDimensional(F);
```

$$true \quad (2)$$

```
> for i from 1 to n do
>     p[i]:=UnivariatePolynomial(X[i],F)
>      end do;
```

$$p_1 := X_1^3-1$$

$$p_2 := X_2^3 - 1$$

$$p_3 := X_3^3 - 1$$

$$p_4 := X_4^3 - 1$$

$$p_5 := X_5^3 - 1$$

$$p_6 := X_6^3 - 1$$

$$p_7 := X_7^3 - 1$$

$$p_8 := X_8^3 - 1 \qquad (3)$$

```
> sol:=[solve({op(G)},{seq(X[i],i=1..n)})];
```

$$sol := [\{X_1 = RootOf(\_Z^2 + \_Z + 1), X_2 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_3 = RootOf(\_Z^2 + \_Z + 1), X_4 = 1, X_5 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_6 = 1, X_7 = RootOf(\_Z^2 + \_Z + 1), X_8 = 1\}, \{X_1 = 1, X_2 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_3 = 1, X_4 = RootOf(\_Z^2 + \_Z + 1), X_5 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_6 = RootOf(\_Z^2 + \_Z + 1), X_7 = 1, X_8 = RootOf(\_Z^2 + \_Z + 1)\}, \{X_1 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_2 = 1, X_3 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_4 = RootOf(\_Z^2 + \_Z + 1), X_5 = 1, X_6 = RootOf(\_Z^2 + \_Z + 1), X_7 = -RootOf(\_Z^2 + \_Z + 1) - 1, X_8 = RootOf(\_Z^2 + \_Z + 1)\}] \qquad (4)$$

```
> toutessol:=map(allvalues,sol);
```

$$toutessol := [\{X_1 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_2 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_3 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_4 = 1, X_5 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_6 = 1, X_7 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_8 = 1\}, \{X_1 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_2 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_3 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_4 = 1, X_5 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_6 = 1, X_7 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_8 = 1\}, \{X_1 = 1, X_2 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_3 = 1, X_4 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_5 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_6 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_7 = 1, X_8 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}\}, \{X_1 = 1, X_2 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_3 = 1, X_4 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_5 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_6 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_7 = 1, X_8 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}\}, \{X_1 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_2 = 1, X_3 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_4 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_5 = 1, X_6 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_7 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_8 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}\}, \{X_1 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_2 = 1, X_3 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_4 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_5 = 1, X_6 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}, X_7 = -\frac{1}{2} + \frac{1}{2} I\sqrt{3}, X_8 = -\frac{1}{2} - \frac{1}{2} I\sqrt{3}\}] \qquad (5)$$

```
> nops(toutessol);
```

(6)

$$6 \quad (6)$$

Exercice 2

```
> restart;
> with(Groebner):
> f1:=X^2*Y-Y+Z;
```

$$f1 := X^2 Y - Y + Z$$

```
> f2:=X*Y^2-X+Z;
```

$$f2 := X Y^2 - X + Z$$

```
> f3:=X+Y+Z-1;
```

$$f3 := X + Y + Z - 1$$

on calcule la base de Gröbner réduite pour l'ordre lexicographique

```
> G:=Basis([f1,f2,f3],plex(X,Y,Z));
```

$$G := [2 Z^4 - 7 Z^3 - 15 Z^2 + 15 Z - 3, 4 Y Z + 2 Z^2 - 2 Y - 3 Z + 1, 2 Z^3 + 4 Y^2 - 7 Z^2 - 2 Y - 17 Z + 6, X + Y + Z - 1]$$

puis les monômes dominants de chaque élément de G

```
> lmG:=map(e->LeadingMonomial(e,plex(X,Y,Z)),G);
```

$$lmG := [Z^4, Z Y, Y^2, X]$$

les monômes standard sont les monômes qui ne sont divisilbes par aucub des éléments de lmG de sorte que l'on a:

```
> B := [1, Y, Z, Z^2, Z^3];
```

$$B := [1, Y, Z, Z^2, Z^3]$$

ces monômes forment une base du quotient Q[X,Y,Z]/I qui esr donc de dimension 5

```
> u_ := X^5+Y^5+Z^5-5*X^2*Y-5*Y^2*Z-5*Z*X^2-1;
```

$$u\_ := X^5 + Y^5 + Z^5 - 5 X^2 Y - 5 X^2 Z - 5 Y^2 Z - 1$$

```
> u:=NormalForm(u_,G,plex(X,Y,Z));
```

$$u := \frac{15}{2} + \frac{35}{2} Z^3 + \frac{45}{2} Z^2 - 5 Y - \frac{65}{2} Z$$

u est inversible si et seulement s'il existe v avec uv=1; de plus si v existe il est unique.
on cherche v avec des coefficients indéterminés. x_1,x_2,x_3,x_3,x_5
on travaille donc dans l'extension Q(x_1,x_2,x_3,x_3,x_5) ;

```
> unassign('x'):v:=add(x[i]*B[i],i=1..nops(B));
```

$$v := Z^3 x_5 + Z^2 x_4 + Y x_2 + Z x_3 + x_1$$

```
> p:=NormalForm(expand(u*v)-1,G,plex(X,Y,Z));;
```

$$p := \frac{1}{32} \left(-160 x_1 - 110 x_2 - 80 x_3 - 40 x_4 - 20 x_5\right) Y + \frac{1}{32} \left(560 x_1 - 980 x_2 + 2680 x_3 + 12620 x_4 + 60230 x_5\right) Z^3 + \frac{1}{32} \left(720 x_1 - 1500 x_2 + 3240 x_3 + 16060 x_4 + 75390 x_5\right) Z^2 + \frac{1}{32} \left(-1040 x_1 + 765 x_2 - 4080 x_3 - 19280 x_4 - 90640 x_5\right) Z + \frac{15}{2} x_1 - \frac{45}{32} x_2 + \frac{55}{2} x_3 + \frac{505}{4} x_4 + \frac{4735}{8} x_5 - 1$$

```
> sol:=solve({coeffs(p,{X,Y,Z})},{seq(x[i],i=1..nops(B))});
```

$$sol := \left\{x_1 = -\frac{1738851}{1119460},\ x_2 = -\frac{256}{1115},\ x_3 = \frac{1643977}{559730},\ x_4 = \frac{257691}{223892},\ x_5 = -\frac{202143}{559730}\right\}$$

```
> assign(sol);v;
```

$$-\frac{202143}{559730} Z^3 + \frac{257691}{223892} Z^2 - \frac{256}{1115} Y + \frac{1643977}{559730} Z - \frac{1738851}{1119460}$$

sur cet exemple i lest possible de résoudre explicitement le système

```
> S1:=[solve({f1,f2,f3},{X,Y,Z})]:
> S2:=map(allvalues,S1):
> S3:=map(evalc,S2);
```

$$S3 := \left[\left\{X = \frac{1}{4} - \frac{1}{4}\sqrt{17},\ Y = \frac{1}{4} + \frac{1}{4}\sqrt{17},\ Z = \frac{1}{2}\right\}, \left\{X = \frac{1}{4} + \frac{1}{4}\sqrt{17},\ Y = \frac{1}{4}\right.\right.$$
$$\left. - \frac{1}{4}\sqrt{17},\ Z = \frac{1}{2}\right\}, \left\{X = -\frac{1}{4} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{8} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right.$$
$$+ \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right) + I\left(-\frac{1}{4} 8^{1/3}\sin\left(\frac{2}{9}\pi\right)\right.$$
$$\left. + \frac{1}{8} 8^{2/3}\sin\left(\frac{2}{9}\pi\right) - \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right)\right),\ Y =$$
$$-\frac{1}{4} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{8} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) + \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right)\right.$$
$$\left. + \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right) + I\left(-\frac{1}{4} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{8} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right.$$
$$\left. - \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right)\right),\ Z = \frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right)$$
$$+ \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) - \sqrt{3}\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right) + 1$$
$$+ I\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right) + \sqrt{3}\left(\frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right)\right.\right.$$
$$\left.\left.\left. - \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right)\right)\right\}, \left\{X = \frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) + \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right.$$
$$+ I\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right),\ Y = \frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right)$$
$$+ \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) + I\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right),\ Z =$$
$$\left. -8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{2} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) + 1 + I\left(-8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{2} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right)\right\}$$
$$, \left\{X = -\frac{1}{4} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{8} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right)\right.\right.$$
$$\left. + \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right) + I\left(-\frac{1}{4} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{8} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right.$$
$$\left. + \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{4} 8^{2/3}\cos\left(\frac{2}{9}\pi\right)\right)\right),\ Y = -\frac{1}{4} 8^{1/3}\cos\left(\frac{2}{9}\pi\right)$$
$$- \frac{1}{8} 8^{2/3}\cos\left(\frac{2}{9}\pi\right) - \frac{1}{2}\sqrt{3}\left(\frac{1}{2} 8^{1/3}\sin\left(\frac{2}{9}\pi\right) + \frac{1}{4} 8^{2/3}\sin\left(\frac{2}{9}\pi\right)\right) + I\Big($$

$$- \frac{1}{4}\, 8^{1/3} \sin\left(\frac{2}{9}\, \pi\right) + \frac{1}{8}\, 8^{2/3} \sin\left(\frac{2}{9}\, \pi\right) + \frac{1}{2}\, \sqrt{3}\, \left(\frac{1}{2}\, 8^{1/3} \cos\left(\frac{2}{9}\, \pi\right)\right.$$

$$\left.\left. - \frac{1}{4}\, 8^{2/3} \cos\left(\frac{2}{9}\, \pi\right)\right)\right), Z = \frac{1}{2}\, 8^{1/3} \cos\left(\frac{2}{9}\, \pi\right) + \frac{1}{4}\, 8^{2/3} \cos\left(\frac{2}{9}\, \pi\right)$$

$$+ \sqrt{3}\, \left(\frac{1}{2}\, 8^{1/3} \sin\left(\frac{2}{9}\, \pi\right) + \frac{1}{4}\, 8^{2/3} \sin\left(\frac{2}{9}\, \pi\right)\right) + 1 + \mathrm{I}\, \left(\frac{1}{2}\, 8^{1/3} \sin\left(\frac{2}{9}\, \pi\right)\right.$$

$$\left.\left. - \frac{1}{4}\, 8^{2/3} \sin\left(\frac{2}{9}\, \pi\right) - \sqrt{3}\, \left(\frac{1}{2}\, 8^{1/3} \cos\left(\frac{2}{9}\, \pi\right) - \frac{1}{4}\, 8^{2/3} \cos\left(\frac{2}{9}\, \pi\right)\right)\right)\right\}\Big]$$

```
> nops(S3);
```

5

```
>
```