On the maximum of the CβE field

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The model

- Consider the following probability distribution for n-point configurations on the unit circle:

$$(C\beta E) \qquad \frac{1}{Z_{n,\beta}} \prod_{1\leq k<l\leq n} \left|e^{i\theta_k} - e^{i\theta_l}\right|^\beta d\theta = \frac{1}{Z_{n,\beta}} \left|\Delta(\theta)\right|^\beta d\theta$$

- For $\beta = 2$, one recognizes the Weyl integration formula for class functions on the compact group $U(n)$. It gives the distribution of the spectrum of a Haar distributed random matrix $U_n \in U(n)$.
- The characteristic polynomial is:

$$X_n(z) := \det\left(\mathrm{id} - zU_n^*\right) = \prod_{1\leq j\leq n} \left(1 - ze^{-i\theta_j}\right)$$

- Choose the determination of the logarithm such that $\log X_n(z=0) = 1$. Defined on a maximal simply-connected domain containing the open disc.

Conjecture

Based on calculations of moments thanks to Toeplitz determinants (whose symbols have Fisher-Hartwig singularities):

Conjecture (Fyodorov, Hiairy, Keating formulated for $\beta = 2$)

As $n \to \infty$, we have the convergence in distribution:

$$\max_{|z|=1} \log |X_n(z)| - \left[\log n - \frac{3}{4} \log\log n\right] \xrightarrow{\mathcal{L}} -\frac{1}{2} (G_1 + G_2)$$

where G_1 and G_2 are two Gumbel random variables.

Recent progress ($\beta = 2$):

- Arguin, Belius, Bourgade showed first order:

$$\frac{\max_{|z|=1} \log |X_n(z)|}{\log n} \xrightarrow{\mathbb{P}} 1$$

- Paquette, Zeitouni showed the second order:

$$\frac{\max_{|z|=1} \log |X_n(z)| - \log n}{\log\log n} \xrightarrow{\mathbb{P}} -\frac{3}{4}$$

Result

Theorem (C., Najnudel, Madaule)

For all $\beta > 0$ and $\sigma \in \{1, i, -i\}$:

$$\sqrt{\frac{\beta}{2}} \max_{|z|=1} \Re\left(\sigma \log X_n(z)\right) = \log n - \frac{3}{4} \log\log n + \mathcal{O}(1)$$

where $\mathcal{O}(1)$ is a tight sequence of random variables.

A few remarks:

- It would be possible (but tedious) to obtain tail estimates for the remainder $\mathcal{O}(1)$.
- Statements on the imaginary part give information about extremal clustering of zeroes.
- I shall only present the ideas behind the upper bound (much easier).

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Intuition for first order

- Fact (Bourgade, Najnudel, Nikeghbali, Rouault, Yor ...): The one-point marginal $\log X_n(\theta)$ splits as a sum of n independent complex random variables. Thanks to the CLT, for every fixed $z \in S^1$:

$$\log X_n(z) \approx \sqrt{\frac{1}{2}\log n}\, \mathcal{N}^{\mathbb{C}} \ .$$

- By Lagrange interpolation, the field $\left(\log X_n(z)\right)_{|z| \in S^1}$ is prescribed by its values on n points. These are heuristically (!) approximated by n independent Gaussians of variance $\frac{1}{2}\log n$.
- Then:

$$\max_{|z|=1} \log |X_n(z)| \approx \sqrt{\frac{1}{2}\log n} \max_{1 \leq i \leq n} \Re \mathcal{N}_i^{\mathbb{C}} \approx \log n$$

Intuition for second order I: Log-correlation

- Without any correlation structure, by computing the maximum of $\mathcal{O}(n)$ independent Gaussians, one would expect:

$$\max_{|z|=1} \log |X_n(z)| \approx \log n - \frac{1}{4}(1+o(1)) \log \log n$$

  WRONG! Correlation is strong!

- We have for $z = e^{i\theta}, z' = e^{i\theta'}$ on the circle:

$$Cov(\theta, \theta') := Cov\left(\log X_n(\theta), \log X_n(\theta')\right) \approx -\log\left|1 - e^{i(\theta-\theta')}\right|.$$

  Hence deep links between characteristic polynomials and Gaussian multiplicative chaos from log-correlated fields (Lambert and Simm's talk).

Intuition for second order II: Branching intuitions

Log-correlation appears naturally in the framework of branching structures (e.g BBM, BRW).

- Consider a dyadic tree of height N (2^N leaves $\subset [0,1]$). Each node v comes with an independent random variable $\mathcal{N}_v^{\mathbb{C}}$, a "displacement" for particle v from its ancestor. Define the "displacement field":

$$X_v = \sum_{u \in v} \mathcal{N}_u^{\mathbb{C}} \ ,$$

  where the sum $u \in v$ refers to all possible ancestors.
- Covariance from explicit branching structure:

$$\begin{aligned} Cov\left(X_v, X_{v'}\right) = & \text{Distance to common ancestor} \\ \approx & -\log|v - v'|_{\mathbb{Q}_2} \end{aligned}$$

- In the class of BRW, we have (Bramson, Shi, Aidékon, Zeitouni ...):

$$\max_{|v|=N} X_v = \log 2^N - \frac{3}{4} \log\log 2^N + \mathcal{O}(1)$$

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OPUC and Szegö recurrence

- Consider a measure μ on the circle and apply the Gram-Schmidt orthonormalization procedure:

$$\{1, z, z^2, \dots\} \rightsquigarrow \{\Phi_0(z), \Phi_1(z), \Phi_2(z), \dots\}$$

- Szegö recurrence:

$$\begin{cases} \Phi_{j+1}(z) & = & z\Phi_j(z) - \overline{\alpha_j}\Phi_j^*(z) \\ \Phi_{j+1}^*(z) & = & -\alpha_j z\Phi_j(z) + \Phi_j^*(z) \end{cases} .$$

Here:

$$\Phi_j^*(z) := z^j \overline{\Phi_j(1/\bar{z})}$$

is the polynomial with reversed and conjugated coefficients and α_j are a sequence of coefficients in the unit disk. The latter coefficients are called Verblunsky coefficients.

Theorem (Verblunsky)

There is a one-to-one correspondence between measures μ on S^1 and sequences of Verblunsky coefficients. Moreover, if $n = |supp(\mu)| < \infty$, then $|\alpha_n| = 1$ and $\alpha_j = 0$ for $j > n$.

Killip, Nenciu and Stoiciu's work

Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the CβE. This was further studied by Killip and Stoiciu.

Theorem (Killip, Nenciu)

- *Let $(\alpha_j)_{j\geq 0}$, η be independent complex random variables, rotationally invariant, such that $|\alpha_j|^2$ is Beta distributed with parameters $(1, \beta_j := \frac{\beta}{2}(j+1))$ and $|\eta| = 1$.*
- *Let $(\Phi_j, \Phi_j^*)_{j\geq 0}$ be the sequence of polynomials obtained from the Verblunsky coefficients $(\alpha_j)_{j\geq 0}$ and the Szegö recursion.*

Then, we have the equality in distribution for the characteristic polynomial of the CβE:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

Laying the ground

- Fact: The sequence of random variables

$$\sup_{|z|=1} \left|\log X_n(z) - \log \Phi_{n-1}^*(z)\right|$$

  is tight. Therefore, we can study extrema of $\log \Phi_{n-1}^*(z)$ only.
- The recursion can be rewritten by using the *deformed Verblunsky coefficients* $(\gamma_j)_{j\geq 0}$, which have the same modulii as $(\alpha_j)_{j\geq 0}$ and the same joint distribution. We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log\left(1 - \gamma_j e^{i\psi_j(\theta)}\right) \ .$$

- The so-called *relative Prüfer phases* $(\psi_k)_{k\geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2\sum_{j=0}^{k-1} \log\left(\frac{1 - \gamma_j e^{i\psi_j(\theta)}}{1 - \gamma_j}\right) \ .$$

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A reduction I

Let us start by a sequence of approximations

- By $\mathcal{E}$, Γ and Θ we denote exponential, gamma and uniform random variables. All are independent. Recall that $\beta_j = \frac{\beta}{2}(j+1)$. For j large, we have the approximation:

$$\begin{aligned}\gamma_j =& \sqrt{\frac{\mathcal{E}_j}{\mathcal{E}_j + \Gamma_j(\beta_j)}} e^{i\Theta_j} \quad \text{"Beta gamma algebra identity"} \\ \approx& \sqrt{\frac{2}{\beta}} \sqrt{\frac{\mathcal{E}_j}{j+1}} e^{i\Theta_j} = \sqrt{\frac{2}{\beta(j+1)}} \mathcal{N}_j^{\mathbb{C}} \ .\end{aligned}$$

- Also

$$\log\left(1 - \gamma_j e^{i\psi_j(\theta)}\right) \approx -\gamma_j e^{i\psi_j(\theta)} \ ,$$

A reduction II

Thus it is believable that:

Proposition

Consider the following field with Gaussian one point marginals:

$$Z_n(\theta) := \sum_{j=0}^{n-1} \frac{\mathcal{N}_j^{\mathbb{C}}}{\sqrt{j+1}} e^{i\psi_j(\theta)} \ .$$

We have:

$$\log \Phi_{n-1}^*(e^{i\theta}) = -\sqrt{\frac{2}{\beta}} Z_n(\theta) + \mathcal{O}(1) \ ,$$

where $\mathcal{O}(1)$ is a tight family of functions.

Remark: Now, for every fixed θ, $\left(Z_n(e^{i\theta})\right)_{n\geq 1}$ is a Gaussian random walk (with inhomogenous increments).

Upper bound

We want ($\log\log = \log_2$):

$$(UB)\ \lim_{C\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{\theta}\Re Z_n(\theta)\geq \log n-\frac{3}{4}\log_2 n+C\right)=0$$

- The crucial point is to add the barrier event:

$$(*)=\left\{\forall k\leq n,\ \sup_{\theta}\Re Z_{\lfloor e^k\rfloor}(\theta)\leq k+...\right\}$$

  which is true with overwhelming probability (via a crude upper bound estimate).

- By approximating $\log\Phi_n^*\approx-\sqrt{\frac{\beta}{2}}Z_n$ by its values on $\mathcal{O}(n)$ points, (UB) is implied by:

$$\lim_{C\to\infty}\limsup_{n\to\infty} n\mathbb{P}(\Re Z_n(\theta=0)\geq \log n-\frac{3}{4}\log_2 n+C,$$
$$\forall k\leq \log n,\ \Re Z_{\lfloor e^k\rfloor}(\theta=0)\leq k)=0$$

Upper bound II

- By writing $W_k = \Re Z_{\lfloor e^k \rfloor}(\theta = 0)$, which is a Gaussian random walk, we want:

$$\lim_{C\to\infty} \limsup_{n\to\infty} n\mathbb{P}\left(W_{\log n} \geq \log n - \frac{3}{4}\log_2 n + C, \forall k \leq \log n,\ W_k \leq k \right) = 0$$

- Via Girsanov, this is implied by:

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(W_{\log n} \geq -\frac{3}{4}\log_2 n + C \text{ and } \forall k \leq \log n,\ W_k \leq 0 \right) = 0 \ .$$

- We conclude by a refined version of the reflection principle for Gaussian walks (known for BM):

$$\mathbb{P}\left(\forall k \leq N, W_k \leq 0 \text{ and } W_N \geq \kappa + C \right) \sim \frac{e^{-2\kappa - 2C}}{N^{3/2}}$$

with

$$\kappa = -\frac{3}{4}\log\log n$$

$$N = \log n \ .$$

A word on lower bound

- The control of the lower bound uses a second moment method. As such it depends crucially on two-point correlations of the field $(Z_n(\theta))_{\theta \in [0,2\pi)}$.
- Understanding two point correlation amounts to understanding Prüfer phases and how they behave at all scales.

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Acknowledgments

Thank you for your attention!