Eigenvalues of Random Matrices and Multiplicative Chaos

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Gaussian Multiplicative Chaos

Log-correlated Processes

Formally, a a log-correlated Gaussian process is a centered Gaussian random field whose correlation kernel is of the form

$$\mathbb{E}\left[H(u)H(v)\right] := \mathrm{G}(u,v) = \log \frac{1}{|u-v|} + g(u,v).$$

Motivation: Quantum Gravity and the Gaussian Free Field

Give a rigorous meaning to Liouville Quantum Gravity (Polyakov '81).

$$\text{"}e^{\gamma H(z)}\text{"}\, dg(z)$$

where $\gamma > 0$, H is a Gaussian Free Field (GFF) on the Riemann sphere $\hat{\mathbb{C}}$ equipped with the metric $dg(z) = \frac{4}{(1+|z|^2)} dA(z)$.

The GFF is a Gaussian process whose covariance kernel is given by the Green function G of the Laplacian on $\hat{\mathbb{C}}$. Namely define $\hat{\nabla} f = \frac{1}{g} \nabla f$ and for all $z \in \mathbb{C}$,

$$-\hat{\nabla}^2 \mathrm{G}(z, \cdot) = 2\pi\delta_z \qquad \text{and} \qquad \int \mathrm{G}(z, w) dg(w) = 0.$$

It turns out that the Green function of the Laplacian on $\hat{\mathbb{C}}$ has an explicit form:

$$\mathrm{G}(z, w) = \log\left(\frac{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}{|z-w|}\right).$$

So, the GFF H is merely a random distribution on $\hat{\mathbb{C}}$ so that

$$\mathbb{E}\left[\langle H, \varphi\rangle \langle H, \psi\rangle\right] = \iint \varphi(z)\varphi(w)\mathrm{G}(z, w) dg(z) dg(w)$$

for all $\varphi, \psi \in C_0^\infty(\hat{\mathbb{C}} \to \mathbb{R})$.

Gaussian Multiplicative Chaos

Let H be a Gaussian process on an interval I with covariance kernel

$$\mathbb{E}\left[H(u)H(v)\right] := \mathrm{G}(u,v) = \log\frac{1}{|u-v|} + g(u,v).$$

Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \geq 0$ and $\displaystyle\int \phi(x)dx = 1$. If $\gamma > 0$ and $\epsilon > 0$ is small, define for all $u \in [0,1]$,

$$H_\epsilon(u) := \int H(u+\epsilon x)\phi(x)dx$$

$$\frac{d\mu_\epsilon^\gamma}{du} := \exp\left(\sqrt{2\gamma}H_\epsilon(u) - \gamma\mathbb{E}\left[H_\epsilon(u)^2\right]\right).$$

Remark. The normalization of the random measure μ_ϵ^γ is such that $\mathbb{E}\left[\frac{d\mu_\epsilon^\gamma}{du}\right] = 1$. Moreover, an elementary computation shows that

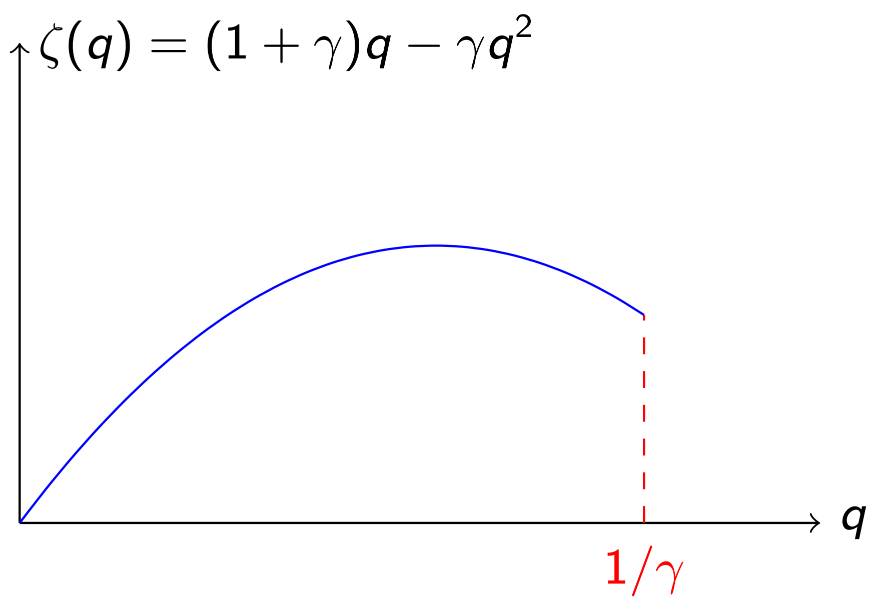
$$\mathbb{E}\left[H_\epsilon(u)^2\right] = \log 1/\epsilon + \underset{\epsilon\to 0}{O}(1).$$

Gaussian Multiplicative Chaos

Theorem [Robert-Vargas '10, Berestycki '15]

If $\gamma < \sqrt{2}$, the measure μ_ϵ^γ converges in probability and in L^1 to a measure μ^γ. Moreover, the measure μ^γ does not depend on the mollifier ϕ.

Multifractality. For any $q < 1/\gamma$, we have $\mathbb{E}\left[\left(\mu^\gamma[0,r]\right)^q\right] \sim C_q r^{\zeta(q)}$ as $r \to 0$.



L^2 - phase

When $\gamma < 1/2$, for any Borel set $A \subseteq [0,1]$, we have $\mathbb{E}\left[\mu^\gamma(A)^2\right] < \infty$ and

$$\mu_\epsilon^\gamma(A) = \int_A e^{\widetilde{H}_\epsilon(u)} du \xrightarrow{L^2} \mu^\gamma(A) \qquad \text{as} \quad \epsilon \to 0,$$

where $\widetilde{H}_\epsilon(u) = \sqrt{2\gamma} H_\epsilon(u) - \gamma \mathbb{E}\left[H_\epsilon(u)^2\right]$.

Proof. It is possible to show that

$$\mathbb{E}\left[H_\epsilon(u) H_\epsilon(v)\right] = \log_+\left(\frac{\epsilon}{|u-v|}\right) + \mho_\epsilon(u,v).$$

where $|\mho_\epsilon(u,v)| \leq C$ and $\mho_\epsilon(u,v) \to 0$ as $\epsilon \to 0$ for almost all $u, v \in [0,1]$.

$$\begin{aligned}\mathbb{E}\left[\mu_\epsilon^\gamma(A)\mu_\delta^\gamma(A)\right] &= \iint_{A^2} \mathbb{E}\left[\exp\left(\widetilde{H}_\epsilon(u) + \widetilde{H}_\delta(v)\right)\right] dudv \\ &= \iint_{A^2} \exp\left(2\gamma \mathbb{E}\left[H_\epsilon(u) H_\delta(v)\right]\right) dudv.\end{aligned}$$

In particular, this implies that $\lim_{\epsilon\to 0} \mathbb{E}\left[\mu_\epsilon^\gamma(A)^2\right] = \iint_{A^2} |u-v|^{2\gamma} dudv < \infty$.

Thick points

To prove convergence in L^2, it is enough to prove that

$$\liminf_{\epsilon,\delta\to 0} \mathbb{E}\left[\mu_\epsilon^\gamma(A)\mu_\delta^\gamma(A)\right] \geq \iint_{A^2} |u-v|^{2\gamma} dudv.$$

Indeed, this implies that

$$\mathbb{E}\left[\left|\mu_\epsilon^\gamma(A)-\mu_\delta^\gamma(A)\right|^2\right] = \mathbb{E}\left[\mu_\epsilon^\gamma(A)^2\right] + \mathbb{E}\left[\mu_\delta^\gamma(A)^2\right] - 2\mathbb{E}\left[\mu_\epsilon^\gamma(A)\mu_\delta^\gamma(A)\right]$$

converges to 0 as $\epsilon, \delta \to 0$. By Fatou's lemma, this reduces the problem to show that for almost all $u, v \in [0,1]$,

$$\liminf_{\epsilon,\delta\to 0} \mathbb{E}\left[H_\epsilon(u)H_\delta(v)\right] \geq \log\frac{1}{|u-v|}.$$

□

We say that a point $u \in [0,1]$ is a α-**thick point** if

$$\liminf_{\epsilon\to 0} \frac{H_\epsilon(u)}{\log \epsilon^{-1}} = \alpha.$$

Proposition

The set $\mathcal{T}_\alpha$ of α-thick points has Hausdorff dimension $\left(1-\alpha^2/2\right)^+$ and the random measure μ^γ *lives on* the set $\mathcal{T}_{\sqrt{2\gamma}}$.

Random Matrix Theory

Dyson's Circular Unitary Ensemble '62

Let $U \in \mathcal{U}_N$ be distributed according to the Haar measure $\mathbb{E}_{\mathcal{U}_N}$. We are interested in the empirical spectral measure:

$$\Xi_N = \sum_{j=1}^{N} \delta_{z_j}$$

where $\left\{z_1 = e^{2\pi i\theta_1}, \dots, z_1 = e^{2\pi i\theta_N}\right\}$ are the eigenvalues of the random matrix U.

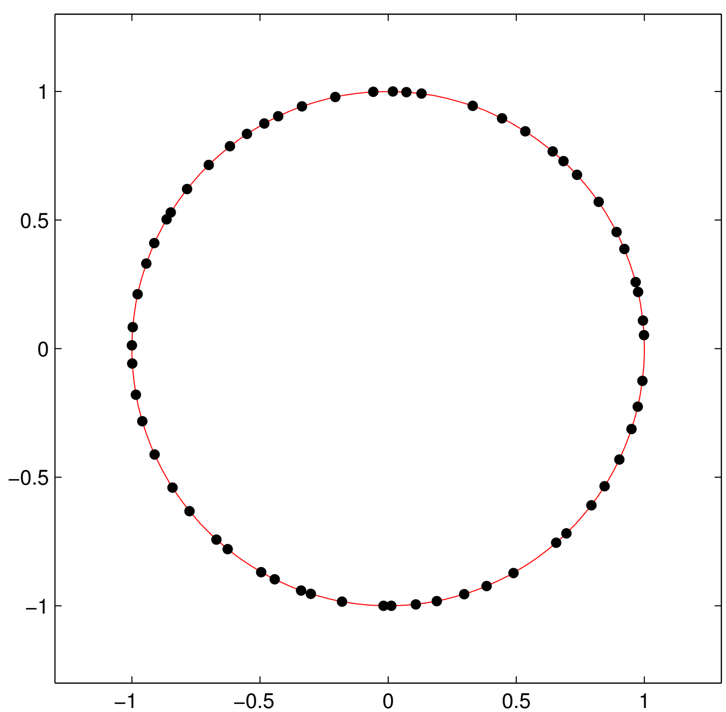


Figure: Circular Unitary Ensemble (60 points)

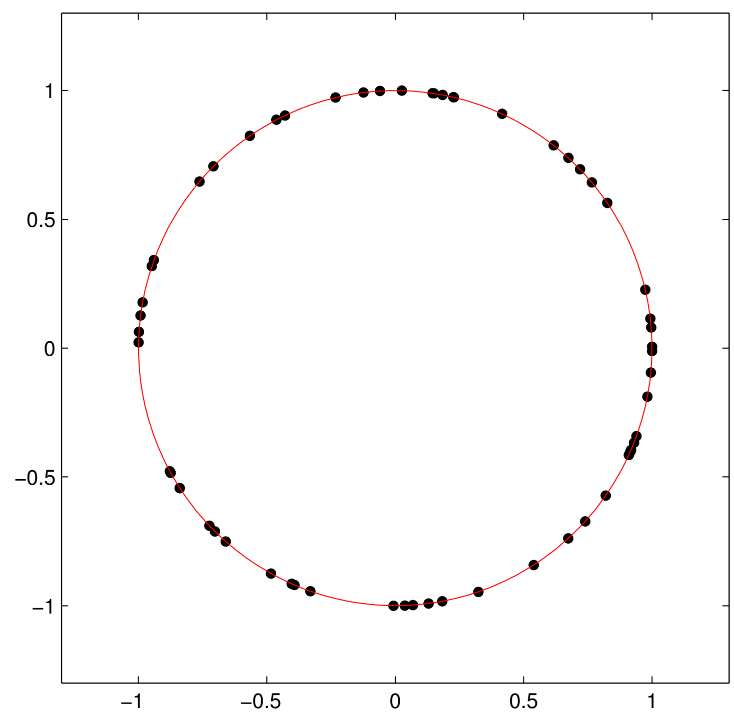


Figure: Independent points (60 points)

Joint distribution of the eigenvalues

By Weyl's integration formula, for any class function $g : \mathcal{U}_N \to \mathbb{R}_+$

$$\mathbb{E}_{\mathcal{U}_N}\left[g(U)\right] = Z_N^{-1} \int_{[0,1]^N} g(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_N}) \left|\triangle(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_N})\right|^2 d^N\boldsymbol{\theta}$$

where $\triangle(z_1, \ldots z_n) = \prod_{1\leq k<j\leq n} (z_j - z_k) = \det_{n\times n}[z_k^{j-1}]$ is the Vandermonde determinant.

In particular if $g(U) = \exp(\mathrm{Tr}\, f(U))$ where f is an integrable function on $\{|z| = 1\}$, we obtain:

$$\mathbb{E}_{\mathcal{U}_N}\left[e^{\mathrm{Tr}\, f(U)}\right] = Z_N^{-1} \int_{[0,1]^N} \prod_{j=1}^{N} e^f(e^{2\pi i\theta_j}) \det_{N\times N}\left[\sum_{n=0}^{N-1} e^{2\pi i n(\theta_k-\theta_j)}\right] d^N\boldsymbol{\theta} \tag{1}$$

$$= Z_N^{-1} N! \det_{N\times N}\left[\int_0^1 e^f(e^{2\pi i\theta}) e^{2\pi i(k-j)\theta} d\theta\right]. \tag{2}$$

Determinantal structure

Formula (1) shows that the eigenvalues of the matrix U form a determinantal process with correlation kernel:

$$K_{\mathcal{U}_N}(z,w) = \sum_{n=0}^{N-1} z^n \overline{w}^n$$

on $\{|z|=1\}^2$. This means that the correlation function of the process are given by, for any $n = 1, \dots, N$,

$$\rho_n(z_1, \dots, z_n) = \det_{n\times n}\left[K_{\mathcal{U}_N}(z_k, z_j)\right].$$

For instance, this implies that

$$\mathbb{E}\left[\Xi_N f\right] = \int_0^1 f(e^{2\pi i\theta}) K_{\mathcal{U}_N}(e^{2\pi i\theta}, e^{2\pi i\theta}) d\theta = N\hat{f}_0,$$

so that the mean empirical measure $N^{-1}\Xi_N$ converges to the uniform probability measure on $\mathbb{T} = \{|z|=1\}$.

As another application, we have

$$\mathrm{Var}\left[\Xi_N f\right] = \sum_{k=1}^{\infty} N \wedge k \, \left|\hat{f}_k\right|^2.$$

Toeplitz determinant

The second formula shows that

$$\mathbb{E}_{\mathcal{U}_N}\left[e^{\operatorname{Tr} f(U)}\right] = \det_{N\times N}\left[\widehat{e^f}_{k-j}\right] := \mathrm{D}_N[e^f].$$

Strong Szegő Theorem [Szegő '52, Kac '54, Ibragimov '68, Johansson '88, Deift '99]

Suppose that $f \in L^1(\mathbb{T})$ and that $\Sigma^2(f) = \sum_{n=1}^{\infty} n|\hat{f}_n|^2 < \infty$. Then

$$\log \mathrm{D}_N[e^f] = N\hat{f}_0 + \sum_{n=1}^{\infty} n|\hat{f}_n|^2 + \underset{N\to\infty}{o(1)}.$$

In particular, since $\operatorname{Tr} f(U) = \Xi_N f$, this implies that the centered linear statistics

$$\Xi_N f - \mathbb{E}\left[\Xi_N f\right]$$

converges in distribution to a Gaussian random variable with variance $\Sigma^2(f)$.

The inner-product $\langle f, g\rangle_\alpha := \sum_{n\in\mathbb{Z}} |n|^{2\alpha} \hat{f}_n \overline{\hat{g}_n}$ defines a Hilbert space (modulo constant) which is usually denoted by H^α.

What happens to the linear statistic $\Xi_N f$ when the test function $f \notin \mathrm{H}^{1/2}$?

For instance, this is the case if $f_u(e^{2\pi i\theta}) = \mathbb{1}_{|\theta|\leq u}$ so that $\Xi_N f_u = \#\{j : |\theta_j| \leq u\}$. We have

$$\widehat{f}_{u\,k} = \frac{\sin(2\pi ku)}{\pi k},$$

and for any $u > 0$,

$$\begin{aligned}\operatorname{Var}\left[\Xi_N f_u\right] &= \frac{1}{\pi^2}\left(\sum_{k=1}^{N}\frac{\sin^2(2\pi ku)}{k} + N\sum_{k>N}\frac{\sin^2(2\pi ku)}{k^2}\right)\\ &= \frac{\log N}{2\pi^2} + \underset{N\to\infty}{O}(1).\end{aligned}$$

Theorem [Costin-Lebowitz '95, Wieand '00, Soshnikov '00]

$$\frac{\Xi_N f_u - 2uN}{\sqrt{\log N}/\sqrt{2}\pi} \Rightarrow \mathcal{N}(0,1)$$

The characteristic polynomial

Let $\mathrm{Z}(z) = \det\left(I - Uz\right)$ on $\mathbb{T}$. Formally,

$$\log \mathrm{Z}(z) \sim -\sum_{n=1}^{\infty} \frac{\mathrm{Tr}\, U^n}{n} z^n$$

Theorem [Diaconis-Shahshahani '94]

The collection of random variables $\left\{ \frac{\mathrm{Tr}\, U^n}{\sqrt{n/2}} \right\}_{n=1}^{\infty}$ converges in distribution to $\{\xi_n\}_{n=1}^{\infty}$ i.i.d. standard complex Gaussian random variables.

Then

$$\log \mathrm{Z}(z) \sim H(z) := \sum_{n=1}^{\infty} \frac{\xi_n}{\sqrt{2n}} z^n \qquad \text{as} \quad N \to \infty.$$

The Gaussian process H is understood as a random distribution:

$$\langle H, f \rangle := \sum_{n=1}^{\infty} \frac{\xi_n}{\sqrt{2n}} \overline{\hat{f}_n}$$

for any real-valued function $f \in \mathrm{H}^{\epsilon}$ for any $\epsilon > 0$.

Rigorous result about convergence

Theorem [Hughes-Keating-O'Connell '01]

The random process $\log \mathrm{Z}(z)$ converges weakly in the Sobolev space $\mathrm{H}^{-\epsilon}$ to $H(z)$.

Let us sketch the argument for $X_N(z) := \Re \log \mathrm{Z}(z) = \mathrm{Tr} \log |I - Uz|$. First, observe that from the definition of the random distribution H,

$$\left\langle \Re H(e^{2\pi i\theta}), \Re H(e^{2\pi i\vartheta}) \right\rangle = \sum_{k=1}^{\infty} \frac{\cos\left(2\pi k(\theta - \vartheta)\right)}{2k} = \log \left| e^{2\pi i\theta} - e^{2\pi i\vartheta} \right|^{-1/2}.$$

It is enough to know the precise asymptotics of the Laplace transform of the random variable

$$\alpha_1 X_N(z_1) + \cdots + \alpha_q X_N(z_q) = \Xi_N f$$

for almost every $\mathbf{z} \in \mathbb{T}^q$. Here $e^f(w) = \prod_{j=1}^q |1 - wz_j|^{\alpha_j}$ (Fisher-Hartwig symbol) and by Heine's formula:

$$\mathbb{E}_{\mathcal{U}_N}\left[e^{\alpha_1 X_N(z_1) + \cdots + \alpha_q X_N(z_q)}\right] = \mathrm{D}_N(e^f).$$

Fisher-Hartwig asymptotics

Theorem [Fisher-Hartwig '68, Widom '73, Deift-Its-Krasovsky '11]

If $\alpha_1, \dots, \alpha_q > -1$ and $z_1, \dots, z_q$ are distinct points on the unit circle, then

$$\log \mathrm{D}_N(e^f) = \sum_{j=1}^{q} \alpha_j^2 \frac{\log N}{4} - \frac{1}{2} \sum_{k<j} \alpha_k \alpha_j \log |z_k - z_j| + \sum_{j=1}^{q} \Upsilon(\alpha_j) + \underset{N\to\infty}{o(1)}.$$

where $e^{\Upsilon(\alpha)} = \dfrac{G(1+\alpha/2)^2}{G(1+\alpha)}$.

This implies that $\mathbb{E}_{\mathcal{U}_N}\left[X_N(z)^2\right] = \dfrac{\log N}{2} + \Upsilon''(0) + \underset{N\to\infty}{o(1)}$ and

$\mathbb{E}_{\mathcal{U}_N}\left[e^{\alpha_1 X_N(z_1) + \cdots + \alpha_q X_N(z_q)}\right]$

$$\sim \exp\left(\frac{1}{2} \sum_{j=1}^{q} \alpha_j^2 \mathbb{E}_{\mathcal{U}_N}\left[X_N(z_j)^2\right] + \sum_{k<j} \alpha_k \alpha_j \log |z_k - z_j|^{-1/2} + \sum_{j=1}^{q} \widetilde{\Upsilon}(\alpha_j) + \underset{N\to\infty}{o(1)} \right)$$

where $\widetilde{\Upsilon}(\alpha) = \Upsilon(\alpha) - \frac{\Upsilon''(0)}{2}\alpha^2 \sim \sum_{n=3}^{\infty} \kappa_n \frac{\alpha^n}{n!}$. □

Non-Gaussian Multiplicative Chaos

Let

$$\frac{d\nu_N^\gamma}{d\theta} = |\mathrm{Z}(e^{2\pi i\theta})|^{2\gamma}, \qquad \theta \in [0,1].$$

Theorem [Webb '15]

For any $0 < \gamma < 1/2$, the random measure ν_N^γ converges in probability and in L^2 to the GMC measure μ^γ.

Proof. Use the **uniform** Fisher-Hartwig asymptotics obtained by [Claeys-Krasovsky '15] when $\mathbf{q} = \mathbf{1}, \mathbf{2}$, the Diaconis-Shahshahani theorem, and the L^2 computation. □

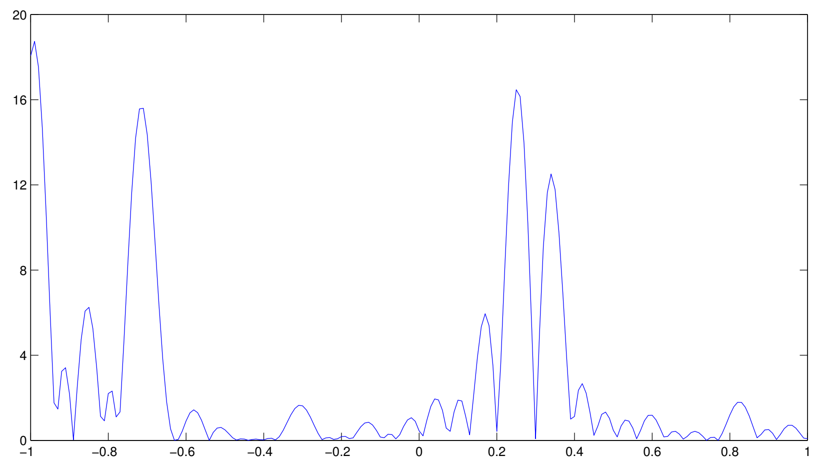


Figure: Sample of the density of ν_N^γ with parameters $N = 100$ and $\gamma = 1/2$.

Numerical simulation

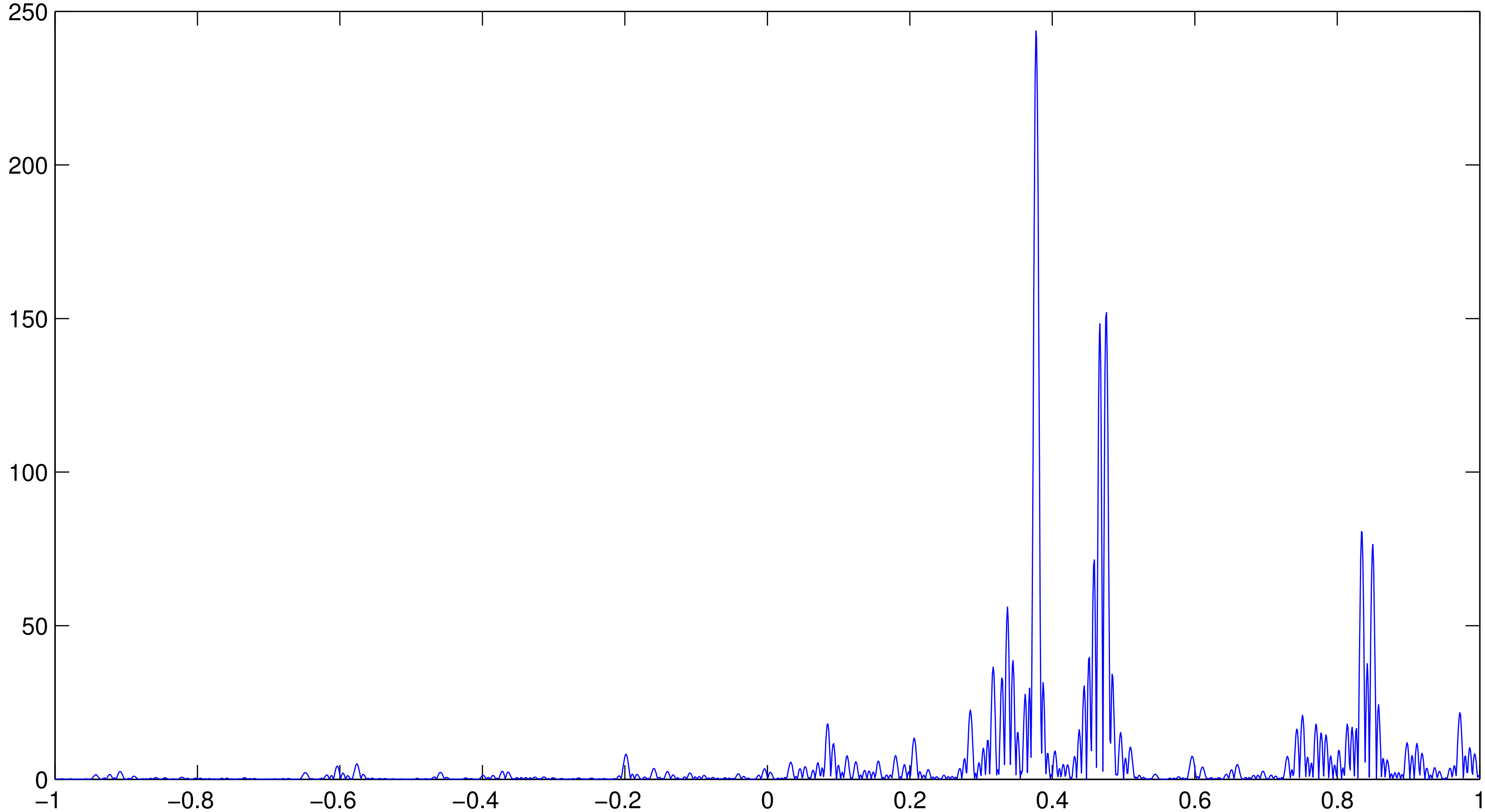


Figure: Sample of the density of ν_N^γ with parameters $N = 1000$ and $\gamma = 1$.

Numerical simulation

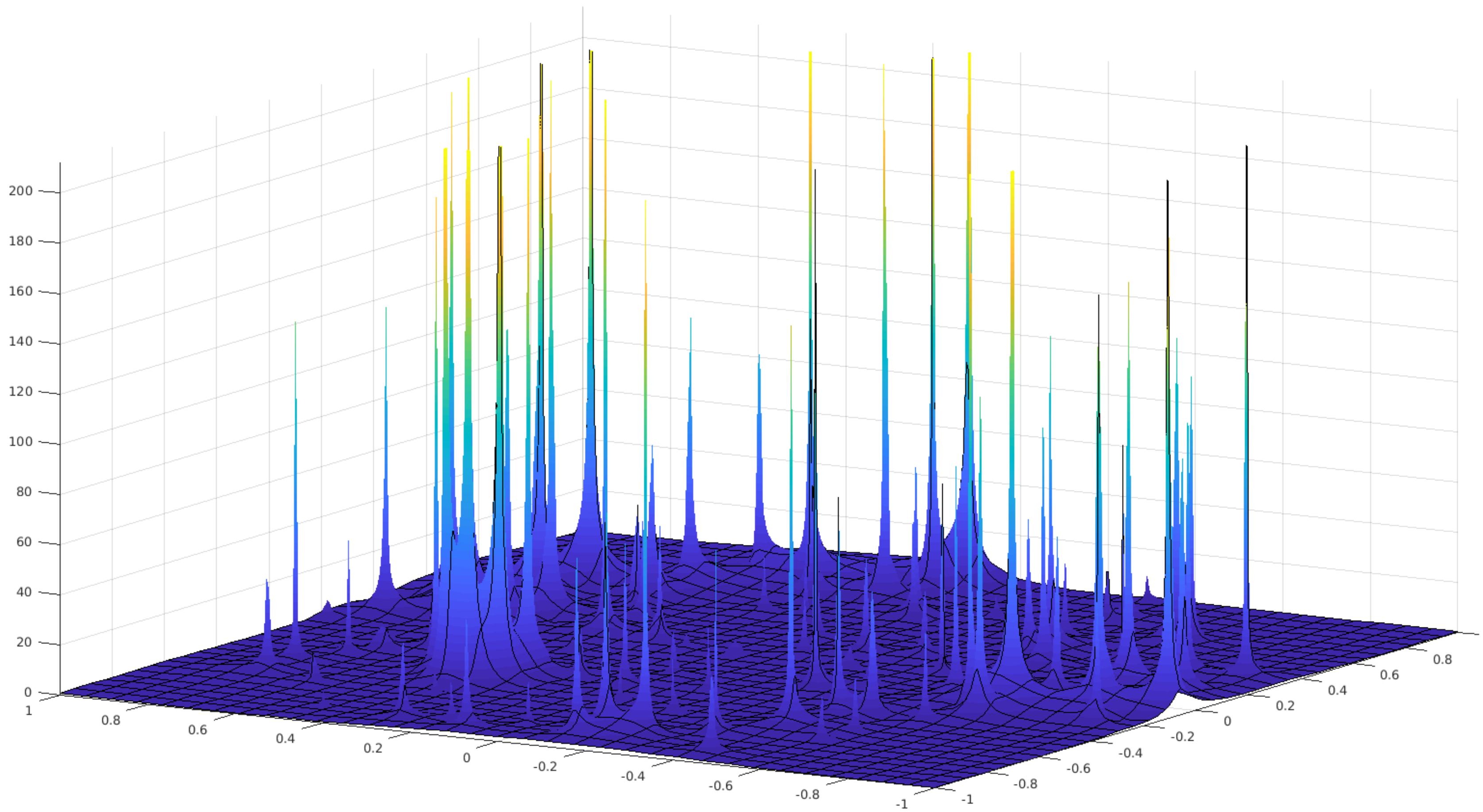


Figure: Sample of the density of ν_N^γ with parameters $N = 1000$ and $\gamma = 1$.

Numerical simulation

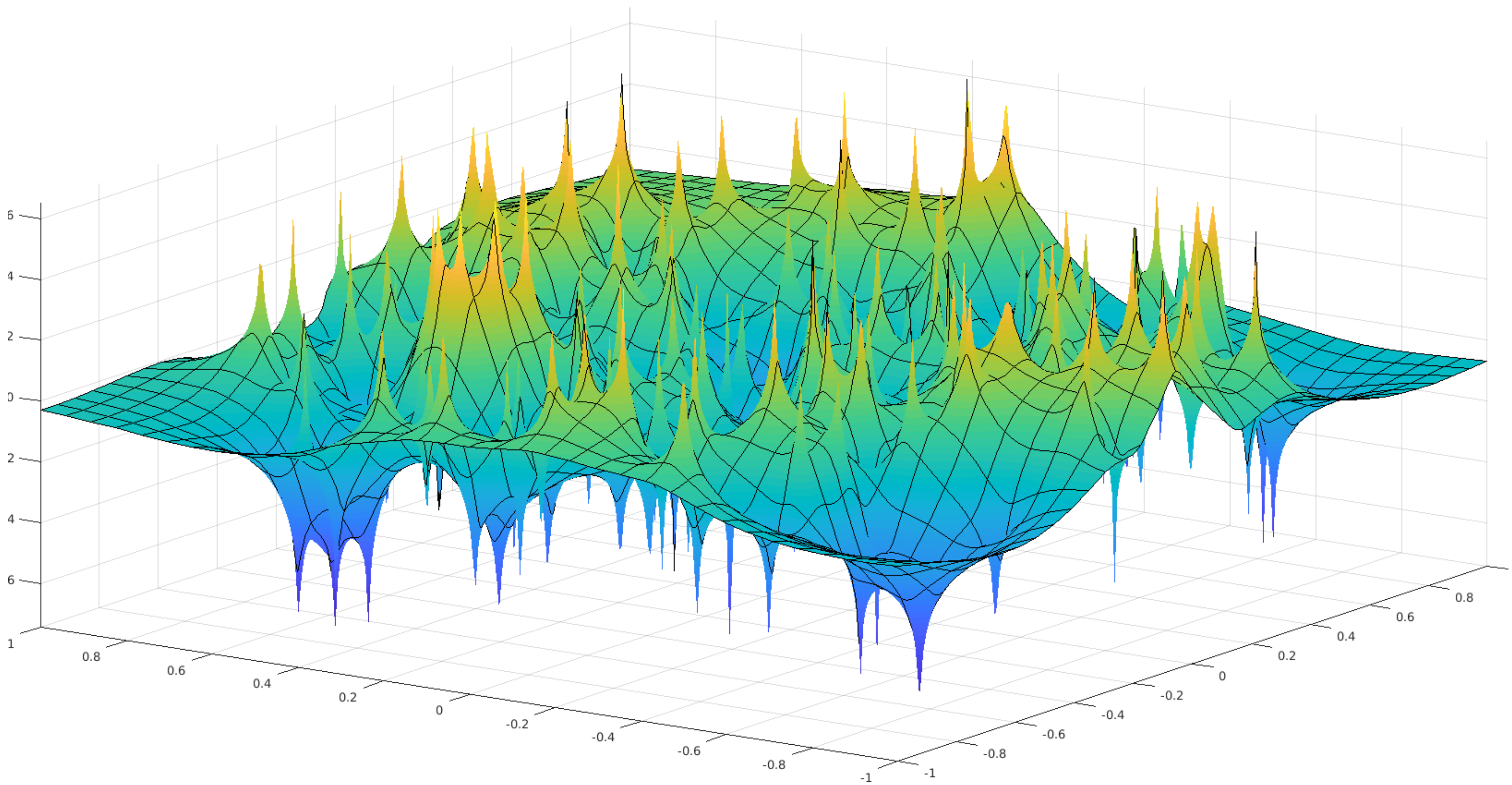


Figure: Sample of the density of ν_N^γ with parameters $N = 1000$ and $\gamma = 1$.

The sine process

Recall that the CUE is a determinantal process on $\{|z| = 1\}^2$ with correlation kernel:

$$K_{\mathcal{U}_N}(z, w) = \sum_{n=0}^{N-1} z^n \overline{w}^n.$$

An equivalent correlation kernel for the CUE eigenvalue process is

$$K'_{\mathcal{U}_N}(z, w) := \frac{z^{(N-1)/2}}{w^{(N-1)/2}} K_{\mathcal{U}_N}(z, w) = \frac{\Im(z^{N/2}\overline{w}^{N/2})}{\Im(z^{1/2}\overline{w}^{1/2})}$$

so that

$$K'_{\mathcal{U}_N}(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) = \frac{\sin\left(\pi N(\theta_1 - \theta_2)\right)}{\sin\left(\pi(\theta_1 - \theta_2)\right)}.$$

In particular, for any sequence $L_N \to \infty$, we obtain

$$K'_{\mathcal{U}_N}\left(e^{2\pi i(\theta_0 + x/L_N)}, e^{2\pi i(\theta_0 + y/L_N)}\right) = \frac{\sin\left(\nu_N(x - y)\right)}{\pi(x - y)} + \underset{N\to\infty}{O}(1/L_N)$$

where $\nu_N = \pi N / L_N$.

CLT for the sine process

Theorem [Soshnikov '00]

Let $\{\lambda_j\}_{j\in\mathbb{Z}}$ be a point configuration of the sine process with density ν_N. For any function $f \in L^1(\mathbb{R})$ such that $\int_0^\infty k\left|\hat{f}(k)\right|^2 dk < \infty$, we have

$$\sum_{j\in\mathbb{Z}} f(\lambda_j) - \nu_N \int f(\lambda)d\lambda \Rightarrow \mathcal{N}\left(0, \int_0^\infty k\left|\hat{f}(k)\right|^2 dk\right) .$$

Let $\phi \geq 0$ such that $\int \phi(x)dx = 1$, and consider the linear statistics:

$$X_{N,u} := \sum_{j\in\mathbb{Z}} f_{u;\epsilon}(\lambda_j) \qquad \text{where} \qquad f_{u;\epsilon} := \mathbb{1}_{[u,1+u]} * \phi_\epsilon$$

Some computations show that

$$\langle X_{N,u}, X_{N,v}\rangle = 2\int_0^\infty k \wedge \nu_N \ \widehat{f_{u;\epsilon}}(k)\overline{\widehat{f_{v;\epsilon}}(k)}dk$$

$$\simeq \frac{1}{2\pi^2}\int_0^\infty \left(e^{2\pi i(1+u)k} - e^{-2\pi iuk}\right)\left(e^{-2\pi i(1+v)k} - e^{-2\pi ivk}\right)\left|\hat{\phi}(k\epsilon)\right|^2 \frac{dk}{k} .$$

Asymptotics

Thus

$$\langle X_{N,u}, X_{N,v} \rangle \simeq \frac{1}{2\pi^2} \log \frac{1}{|u-v| \vee \epsilon} \qquad \text{as } |u-v| \to 0 .$$

Let $\gamma > 0$ and define

$$\widetilde{X}_{N,u} = 2\pi\sqrt{\gamma}\big(X_{N,u} - \mathbb{E}\left[X_{N,u}\right]\big) - 2\pi^2\gamma\mathbb{E}\big[X_{N,u}^2\big].$$

Theorem

Assume that $1/\nu_N \ll \epsilon(N) \ll 1$. For any $q \in \mathbb{N}$, we have

$$\log \mathbb{E}\left[\exp\left(\widetilde{X}_{N,u_1} + \cdots + \widetilde{X}_{N,u_q}\right)\right] = \gamma \sum_{i \neq j} \mathrm{Q}_N(u_i, u_j) + \mho_N(\mathbf{u})$$

where

$$\mathrm{Q}_N(u,v) = \log \frac{1}{|u-v| \vee \epsilon(N)} + \mathrm{G}_N(u,v)$$

where there exists a function $\mathrm{G} : [0,1]^2 \to \mathbb{R}$ so that $\mathrm{G}_N(u,v) \to \mathrm{G}(u,v)$ as $N \to \infty$. Moreover, the error term satisfies:

$$\sup_{\mathbf{u} \in [0,1]^q} \left|\mho_N(\mathbf{u})\right| \leq C \qquad \text{and} \qquad \mho_N(\mathbf{u}) \to 0 \qquad \text{for all } \mathbf{u} \in (0,1)^q.$$

Results

Define the random measure

$$\frac{d\mu_N^\gamma}{du} = \exp\left(\widetilde{X}_{N,u}\right).$$

Theorem

For any $0 < \gamma < 1/2$, the random measure μ_N^γ converges in probability and in L^2 to the GMC measure μ^γ.

Theorem

For any $q \in \mathbb{N}$ such that $\mu q < 1$ and for any $0 < r < 1$,

$$\mathbb{E}\left[\left(\mu_N^\gamma[0,r]\right)^q\right] = \int_{[0,r]^q} \prod_{i<j} |u_i - u_j|^{-2\gamma} \prod_{i<j} |1 + u_i - u_j|^{2\gamma} d^q\mathbf{u}$$

In particular, if $\zeta(q) = (1+\gamma)q - \gamma q^2$, we see that

$$\mathbb{E}\left[\left(\mu_N^\gamma[0,r]\right)^q\right] = r^{\zeta(q)} \int_{[0,1]^q} \prod_{i<j} |u_i - u_j|^{-2\gamma} \prod_{i<j} |1 + r(u_i - u_j)|^{2\gamma} d^q\mathbf{u}$$

$$\simeq r^{\zeta(q)} \prod_{k=0}^{q-1} \frac{\Gamma(1+k\gamma)^2\Gamma(1+(k+1)\gamma)}{\Gamma(2+(q+k-1)\gamma)\Gamma(1+\gamma)} \qquad \text{as } r \to 0 .$$

Thank you!