Non commutative notions of Independence and Large Random Matrices

Camille Male
ProbabLY ON Random matrices

Université de Bordeaux

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Independence : absence of relation between entities.

In classical probability: no ambiguity, a single notion.

In non commutative probability, several notions:

- Free independence (Voiculescu)
- Tensor independence (=classical)
- Boolean independence (von Waldenfels)

Speicher: there is no more universal notions.
Muraki: + 2 other quasi-universal notions.

In traffic probability: extension of free probability with a single independence that unifies in some sense the three above notions (M. + Gabriel) Question of existence of other notions in this context (Speicher) not investigated.

Presentation

(1) Notions of non commutative independence

(2) Traffic distributions and associated notion of independence

(3) Abstract traffics, LLN and CLT

Definition

A Non Commutative Probability Space in a pair $(\mathcal{A}, \Phi)$ where

1. $\mathcal{A}$ *is a unital algebra over* $\mathbb{C}$,
2. Φ *is a unital linear* $(\Phi(\mathbb{1}) = 1)$

Examples

- Commutative space: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider $(L^\infty(\Omega, \mu), \mathbb{E})$
- Matrix spaces: $(\mathrm{M}_N(\mathbb{C}), \frac{1}{N}\mathrm{Tr})$
- on an algebra spanned by random matrices, one consider the state $\Phi_N = \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\big]$.

Distribution of $\mathbf{a} = (a_j)_{j \in J} \in \mathcal{A}^J$:

$$\Phi_{\mathbf{a}} : P \mapsto \Phi\big[P(\mathbf{a})\big]$$

for all non commutative polynomials in a_j.

Examples

- Commutative space: moments of random variables
- Matrix spaces:
  - For a single self-adjoint matrix A_N, moments of the empirical eigenvalues distribution: data of $\mathbb{E}\big[\frac{1}{N}\mathrm{Tr} A_N^K\big]$ for all $K \geq 1$.
  - For several matrices $\mathbf{A}_N = (A_j)_{j \in J}$, generalized moments: data of $\mathbb{E}\big[\frac{1}{N}\mathrm{Tr} A_{j_1} \dots A_{j_K}\big]$ for all $K \geq 1, j_1, \dots, j_K \in J$.

Interest: convergence of $\mathbf{A}_N \Rightarrow$ convergence of $H_N = Q(\mathbf{A}_N)$ for any non commutative polynomial Q.

A notion of independence: associative computation rule for mixed moments.

Definition

Unital sub-algebra $\mathcal{A}_1 \dots \mathcal{A}_L$ of $(\mathcal{A}, \Phi)$ are tensor independent whenever the $\mathcal{A}_l$ commute (i.e. $ab = ba$ for any $a \in \mathcal{A}_\ell, b \in \mathcal{A}_{\ell'}$ with $\ell \neq \ell'$) and for any $a_\ell \in \mathcal{A}_\ell$, $\ell = 1, \dots, L$,

$$\Phi(a_1 \dots a_L) = \Phi(a_1) \dots \Phi(a_L).$$

Definition

Sub-algebra $\mathcal{A}_1 \dots \mathcal{A}_L$ of $(\mathcal{A}, \Phi)$ (non-unital in general) are Boolean independent whenever for any $n \geq 1$ and any $a_j \in \mathcal{A}_{\ell_j}$ where $\ell_j \neq \ell_{j+1}$ in $\{1, \dots, L\}$,

$$\Phi(a_1 \dots a_n) = \Phi(a_1) \dots \Phi(a_n).$$

Remark: if a is Boolean independent with the unit $\mathbb{1}$, then

$$\Phi(a^k) = \Phi(a \mathbb{1} a \mathbb{1} \dots \mathbb{1} a) = \Phi(a)^k.$$

Definition

Unital sub-algebra $\mathcal{A}_1 \dots \mathcal{A}_L$ *of* $(\mathcal{A}, \Phi)$ *are free independent whenever alternated products of centered elements are centered, namely: for any* $n \geq 1$ *and any* $a_j \in \mathcal{A}_{\ell_j}$ *where* $\ell_j \neq \ell_{j+1}$ *in* $\{1, \dots, L\}$ *and* $\Phi(a_j) = 0$*, then*

$$\Phi(a_1 \dots a_n) = 0.$$

Computation rule: for an arbitrary alternated product of freely independent variables

$$\Phi(a_1 \dots a_n) = \underbrace{\Phi(\mathring{a}_1 \dots \mathring{a}_n)}_{=0} + \text{other terms}$$

where $\mathring{a} = a - \Phi(a)\mathbb{1}$ and the other terms are computed by induction.

Remark: mimics the definition of freeness of sub-groups, namely $\Gamma_1, \dots, \Gamma_L$ subgroups of Γ are free if and only if an alternated product of non trivial elements is non trivial.

Let $\mathcal{P}(n)/\mathcal{I}(n)/\mathcal{NC}(n)$ denote the set of partitions/interval partitions/non crossing partitions respectively.

Definition

Let $(\mathcal{A}, \Phi)$ be a non commutative probability space. The tensor/Boolean/free cumulants are the families of n-linear maps $\kappa_n^{tens}/\kappa_n^{Bool}/\kappa_n^{free}$ defined implicitly by

$$\Phi(a_1 \dots a_n) = \sum_{\pi \in \dots} \prod_{\{j_1, \dots, j_\ell\} \in \pi} \kappa_\ell^{\mathcal{X}}(a_{j_1}, \dots, a_{j_\ell})$$

where the sum is over $\pi \in \mathcal{P}(n)$ for $\mathcal{X} = tens$, $\pi \in \mathcal{I}(n)$ for $\mathcal{X} = Bool$ and $\pi \in \mathcal{NC}(n)$ for $\mathcal{X} = free$.

Theorem

$\mathcal{A}_1, \dots, \mathcal{A}_L$ are tensor/Boolean/free independent if and only if mixed cumulants vanishes, namely $\kappa^{\mathcal{X}}(a_1, \dots, a_n) = 0$ as soon as $\exists a_j \in \mathcal{A}_\ell, a_{j'} \in \mathcal{A}_{\ell'}$ for $\ell \neq \ell'$.

Theorem (Voiculescu (91), Collins and Śniady (04))

$\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

(1) *each family is unitarily invariant,*

(2) *each family converges in N.C. Distribution, i.e.* $\mathbb{E}\big[\frac{1}{N}\mathrm{Tr} P(\mathbf{A}_N^{(\ell)})\big]$ *converges*

(3) $\|\mathbf{A}_N^{(\ell)}\|$ *is uniformly bounded and* $\frac{1}{N}\mathrm{Tr} P(\mathbf{A}_N^{(\ell)})$ *converges a.s.*

Then $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *are asymptotically freely independent.*

The conclusion remains valid if one family consists in independent Wigner matrices.

Remark: replace (1) by "the matrices are diagonal and permutation invariant" $\Rightarrow$ the matrices are tensor independent.

Lenczewski found examples of asymptotically Boolean independent matrices (not associated to a distributional symmetry).

(2) Traffic distributions and associated notion of independence

M. and Gabriel found independently a way to extend this statement:

Theorem

$\mathbf{A}_N^{(1)}, \ldots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

(1) *each family is permutation invariant,*

(2) *each family converges in a stronger sense, i.e.* $\mathbb{E}\big[\frac{1}{N}\mathrm{Trg}(\mathbf{A}_N^{(\ell)}\big]$ *converges for more observables g than polynomials. Called convergence in traffic distribution*

(3) *+ concentration assumption*

Then $\mathbf{A}_N^{(1)}, \ldots, \mathbf{A}_N^{(L)}$ *are asymptotically independent in a generalized sense.*

Motivations: [M.] Adjacency matrices of random graphs [Gabriel] permutation invariant Matricial Lévy process

In [Gabriel]: definition of the associated notion of cumulants, same kind of objects as in P. Biane's talk

A graph monomial g in matrices $\mathbf{A}_N = (A_j)_{j\in J}$:

- a finite connected oriented graph (V, E),
- with an input and an output in V,
- a labeling $\gamma : E \to J$ of edges by matrices.

For N by N matrices $A_1, \dots, A_K$,

$$g(A_1,\dots,A_K)(i,j) = \sum_{\substack{\phi:V\to\{1,\dots,N\}\\ \phi(in)=j,\phi(out)=i}} \prod_{e=(v,w)\in E} A_{\gamma(e)}\big(\phi(w),\phi(v)\big)$$

Traffic distribution: data of $\mathbb{E}\big[\frac{1}{N}\mathrm{Tr}g(\mathbf{A}_N)\big]$ for all graph monomial.

To be compared with

$$A_{\gamma_1}\times\cdots\times A_{\gamma_K}(i,j) = \sum_{i_2,\dots,i_{K-1}=1}^{N}\prod_{k=1}^{K} A_{\gamma_k}(i_k, i_{k+1})$$

Traffic distribution: data of

$$\tau_N\big[T(\mathbf{A}_N)\big] := \mathbb{E}\Big[\frac{1}{N}\sum_{\phi:V\to\{1,\dots,N\}}\prod_{e=(v,w)\in E} A_{\gamma(e)}\big(\phi(w),\phi(v)\big)\Big]$$

where $T=(V,E,\gamma)$, finite connected graph with labeling.
Define τ_N^0 the injective version of τ_N by the same formula with ϕ injective. Then

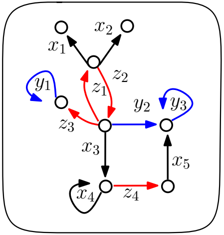
$$\tau_N\big[T\big] = \sum_{\pi\in\mathcal{P}(V)} \tau_N^0\big[T^\pi\big]$$

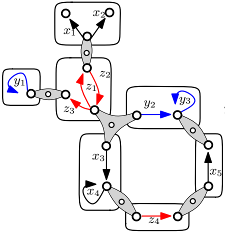
where T^π is the quotient graph where vertices in a same block are identified.

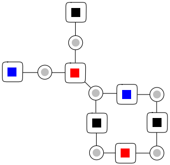
Looks like cumulants, but Gabriel shows we can define

$$\tau_N^0\big[T\big] = \sum_{\pi\in\mathcal{P}(T)}\prod_{B\in\pi}\kappa^{traf}(\pi)$$

where κ^{traf} has the mixed cumulants vanishing property

$T =$ 

$=$ 

$, \mathcal{GCC}(T) =$ 

The families of matrices $\mathbf{A}_1, \dots, \mathbf{A}_L$ are asymptotically traffic independent iff

$$\tau_N^0\big[T(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}\big] \underset{N\to\infty}{\longrightarrow} \mathbf{1}\big(\mathcal{GCC}(T) \text{ is a tree}\big) \prod_{S \in \mathcal{CC}(T)} \tau^0[S].$$

In practice $\tau_N^0\big[T(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}\big]$ is often the quantity easy to compute.

Case 1: Traffic independence $\Rightarrow$ tensor independence for diagonal matrices

Case 2: Traffic independence $\Rightarrow$ free independence for this class of matrices

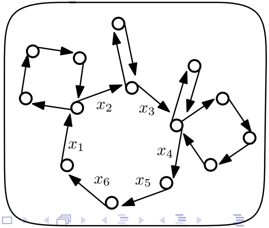
Proposition (Cebron, Dahlqvist, M.)

A family $\mathbf{A}_N$ *of unitary invariant matrices converges in distribution of traffics iff it converges in* **-distribution:*

$$\tau_N^0\big[T(\mathbf{A}_N)\big] = \mathbf{1}(T \ \ cactus) \prod_{S \ \ cycle} \kappa^{free}(x_1, x_2, \dots, x_n)$$

where the variables are those along the cycles.

Applications: [M. Péché] Adjacency matrix of uniform regular graph with large degree d_N has asymptotically a cactus type traffic distribution ($|\frac{N}{2} - \frac{d_N}{N} - \eta\sqrt{d_n}| \underset{N\to\infty}{\longrightarrow} \infty$).



Case 3: Traffic independence $\Rightarrow$ Boolean independence if the limit traffic distribution is supported on trees, but...

instead of the trace Φ, one consider instead the expectation

$$\Psi_N(A_N) = \mathbb{E}\big[\frac{1}{N}\sum_{i,j} A_N(i,j)\big]$$

Tree like distribution implies $\Phi(A_N^K) \underset{N\to\infty}{\longrightarrow} 0$ for any $K \geq 1$.

Examples:

- the matrix $\mathbb{J}_N$ whose entries are $\frac{1}{N}$ converges to the constant one $(\mathcal{A}, \Psi)$.
- The distribution of matrix $Y_N = \big(\frac{Y_i + \bar{Y}_j}{N}\big)$ w.r.t. Ψ_N, for Y_i iid centered with $\mathbb{V}\mathrm{ar}\ 1$, converges to the Rademacher distribution.
- $\mathbb{J}_N$ and independent copies of Y_N are asymptotically Boolean independent w.r.t. Ψ_N.

(3) Abstract traffics, LLN and CLT

Traffic space: $(\mathcal{A}, \tau)$ where $\mathcal{A}$ enriches the notion of algebra and τ enriches the notion of expectation

Tool: operad algebra. Susbtitution: we can replace the edges of g by graphs monomial $g_1, \ldots, g_K$ to get a new graph monomial $g(g_1, \ldots, g_K)$.

$\mathcal{G}$-algebra: a vector space $\mathcal{A}$ over $\mathbb{C}$ with $\forall g$ graph operation with K edges, a K linear operation $Z_g : \mathcal{A}^K \to \mathcal{A}$ s.t.

- Unity: $Z_{(\cdot)} = \mathbb{I}$ is a fixed element
- Identity: $Z_{(\cdot \leftarrow \cdot)} = id_{\mathcal{A}}$
- Substitution: $Z_g\big(Z_{g_1}, \ldots, Z_{g_K}\big) = Z_{g(g_1, \ldots, g_K)}$

τ: a linear map on the space of finite connected graphs whose edges are labeled by elements of $\mathcal{A}$ + Compatibility with Substitution. Define to expectation

$$\Phi(a) = \tau[{}^a\circlearrowleft], \quad \Phi(a) = \tau[\cdot \overset{a}{\leftarrow} \cdot]$$

In an abstract context we can state the classical limits theorems that interpolates the three worlds:

1. Let $(a_n)_{n\geq 1}$ be i.i.d. self-adjoint traffics. Then $\frac{a_1+\dots+a_n}{n}$ converges to $\Phi(a)\mathbb{I}+\Psi(a)\mathbb{J}$, where $\mathbb{J}$ is the limit $\mathbb{J}_N$.
2. If moreover $\Phi(a)=\Psi(a)=0$ then $\frac{a_1+\dots+a_n}{\sqrt{n}}$ converges to a sum $x+y+z$, each element representing the free, Boolean, or tensor world.

Example: $M_{N,n}=\frac{1}{\sqrt{2n}}\sum_{i=1}^n\left(V_N^{(i)}+V_N^{(i)t}-2\mathbb{J}_N\right)$ standardized sum of i.i.d. permutation matrices and their transpose. Then
$M_{N,n}\underset{N\to\infty}{\longrightarrow} m_n=\frac{1}{\sqrt{2n}}\sum_{i=1}^n\left(v^{(i)}+v^{(i)t}-2\mathbb{J}\right)\underset{n\to\infty}{\longrightarrow} m$
Then m has the distribution of the limit of $X_N-\left(X_N\mathbb{J}_N+\mathbb{J}_NX_N\right)$.

Thank you for your attention !