Non-Hermitian random matrices with a variance profile

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joint work with N. Cook, W. Hachem and D. Renfrew

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The matrix model Y_n and the main results

Master equations and description of deterministic equivalents $\mu_n^{\det}$

Elements of proof

Hand waving

The matrix model Y_n

Consider a $n \times n$ matrix Y_n with entries

$$Y_{ij} = \frac{\sigma_{ij}}{\sqrt{n}} X_{ij}$$

where

- the X_{ij}'s are i.i.d. random variables with

$$\mathbb{E} X_{ij} = 0 \; ; \quad \mathbb{E}|X_{ij}|^2 = 1 \; , \quad \mathbb{E}|X_{ij}|^{4+\varepsilon} < \infty \; .$$

- The σ_{ij}'s ($= \sigma_{ij}(n)$) are deterministic, ≥ 0 and account for the **variance** of Y_n's entries as

$$\mathbb{E}|Y_{ij}|^2 = \frac{\sigma_{ij}^2}{n} \qquad (\sigma_{ij}^2(n) \leq \sigma_{\mathbf{max}}^2 < \infty)$$

  Matrix $V_n = \left(\frac{\sigma_{ij}^2}{n} \right)$ is the **normalized variance profile matrix** of Y_n

Associated spectral measure

Denote by μ_n^Y the **spectral distribution** of Y_n

$$\mu_n^Y = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \qquad (\text{a priori } \lambda_i \in \mathbb{C})$$

- The purpose of the talk is to describe the limiting behaviour of μ_n^Y under appropriate assumptions on V_n as $n \to \infty$.

Canonical example: The circular law

- If $\sigma_{ij}^2 := \sigma^2$, then μ_n^Y converges to **Girko's circular law**

$$\text{(almost surely)} \qquad \mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \mu_{\mathrm{circ}}(dx\,dy) = \frac{dx\,dy}{\pi\sigma^2}\mathbf{1}_{\{|z|\le\sigma\}}$$

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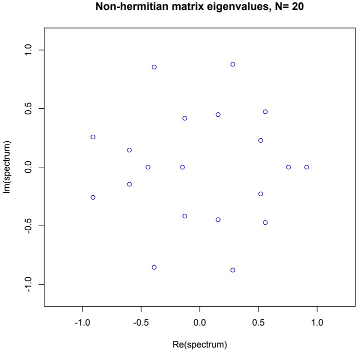

Figure: Distribution of $\mathbf{Y}_N$'s eigenvalues

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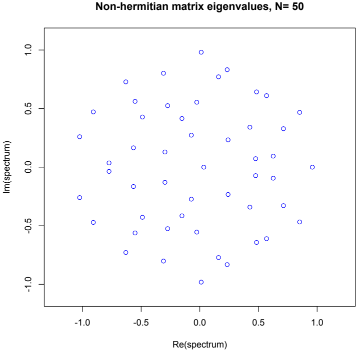

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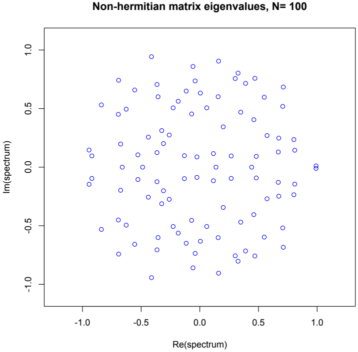

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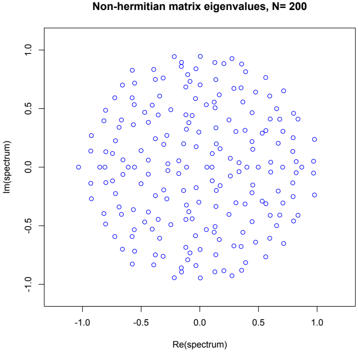

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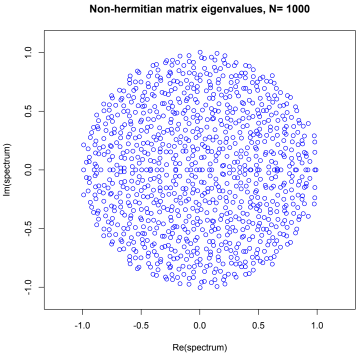

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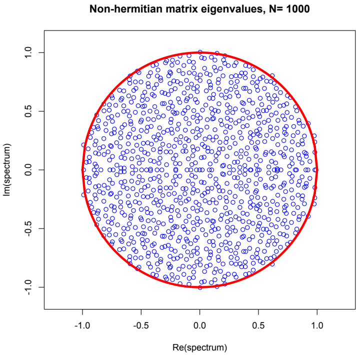

Figure: The circular law (in red)

Doubly stochastic variance profile V_n

Theorem (Cook, Hachem, N., Renfrew)

If $V_n = \left(\frac{1}{n}\sigma_{ij}^2\right)$ is doubly stochastic, i.e.

$$\frac{1}{n}\sum_i \sigma_{ij}^2 = 1 \quad \text{and} \quad \frac{1}{n}\sum_j \sigma_{ij}^2 = 1 \qquad \forall i,j \in [n]$$

Then

$$\mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \mu_{\text{circ}} \quad \text{(in probability)}$$

Remarks

- Some sparcity is allowed
- The number of non-zero entries in V_n in each row/column remains linear in n.

$$\text{Example :} \quad V_n = \frac{1}{n}\begin{pmatrix} 0 & 0 & \mathbf{1}_{n\times n} \\ 0 & \mathbf{1}_{n\times n} & 0 \\ \mathbf{1}_{n\times n} & 0 & 0 \end{pmatrix}$$

Remark 1: deterministic equivalents

- A priori, the sequence of variance profiles (V_n) is arbitrary with no relationship between V_n and V_{n+1}.
- In particular, we cannot always expect the existence of some probability measure μ_∞ such that

$$\mu_n^Y \xrightarrow[n\to\infty]{} \mu_\infty \,.$$

- To describe μ_n^Y as $n \to \infty$, we exhibit **deterministic** probability distributions μ_n^{det} on $\mathbb{C}$ such that

$$\boxed{\mu_n^Y \overset{\mathcal{P}}{\sim} \mu_n^{\mathrm{det}}} \qquad \Leftrightarrow \quad \forall f \in C_b(\mathbb{C})\,, \quad \int f\, d\mu_n^Y - \int f\, d\mu_n^{\mathrm{det}} \xrightarrow[n\to\infty]{\mathcal{P}} 0 \,.$$

(μ_n^{det}) will be called the **deterministic equivalents** of μ_n^Y.

Remark 2: Sparcity

Recall that the variance profile is upper bounded:

$$\sup_{n\geq 1} \sup_{i,j\leq n} \sigma_{ij}^2 \;\leq\; \sigma_{\mathbf{max}}^2\;.$$

We also want to enable some of the σ_{ij}^2's to be equal to zero.

- Matrix V_n cannot be too sparse.

$$\#\{\text{non null entries of } V_n\} \;\propto\; n^2,$$

  otherwise, one easily shows that

$$\mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \delta_0 \qquad \text{(in probability)}$$

Remark 3: Irreducibility of the variance profile V_n

- If matrix V_n is **reducible**, then there exists a permutation matrix P_σ such that

$$P_\sigma^{-1} V_n P_\sigma = \begin{pmatrix} A & \star \\ 0 & B \end{pmatrix}$$

where A, B are square matrices. If not, V_n is **irreducible**.

- If V_n **reducible**, then up to a permutation,

$$Y_n = \begin{pmatrix} Y_n^1 & \star \\ 0 & Y_n^2 \end{pmatrix}$$

and it suffices to study separately the spectral measures of Y_n^1 and Y_n^2.

- We therefore assume that $\boxed{\text{matrix } V_n \text{ is } \textbf{irreducible}}$.

Main result

Theorem (Cook, Hachem, N., Renfrew)

Assume that the variance profile (V_n) is **robustly irreducible** then

- for all $n \geq 1$, there exists a **radial** probability distribution

$$\mu_n^{\det} \in \mathcal{P}(\mathbb{C}) \,, \quad \mathrm{supp}(\mu_n^{\det}) \subset \left\{ z \in \mathbb{C} \,, \ |z| \leq \sqrt{\rho(V_n)} \right\} \,,$$

  defined via a set of 2n **master equations** such that

$$\boxed{\mu_n^Y \overset{\mathcal{P}}{\sim} \mu_n^{\det}}$$

- If $\mu_n^{\det}$ does not depend on n, then

$$\forall n \geq 1 \,, \ \mu_n^{\det} := \mu_\infty \qquad \text{and} \qquad \boxed{\mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \mu_\infty} \quad \text{(in probability)}$$

Remark

Why is $\mu_n^{\det}$ radial? Can be expected if Y_n has complex gaussian entries:

$$\forall \ \theta \in [0, 2\pi] \,, \qquad Y_n^\theta = \left(\frac{\sigma_{ij}}{\sqrt{n}} X_{ij} e^{\boldsymbol{i}\theta} \right) \quad \overset{\mathcal{L}}{=} \quad Y_n = \left(\frac{\sigma_{ij}}{\sqrt{n}} X_{ij} \right)$$

Related work

Subsequent paper by Alt, Erdös and Krüger

"Local Inhomogeneous circular law" arXiv:1612.07776

- local law for the same model (much sharper than the global law)
- **but** for lower bounded variance profiles

$$\sigma_{ij}^{(n)} \geq \sigma_* > 0 \ ,$$

which prevents from any sparcity.

Robust irreducibility assumption I

Through examples

- If the variance profile is uniformily (in n) lower bounded:

$$\sigma_{ij}^2 \geq \sigma_{\min}^2 > 0$$

  then it is **RI**.

- The variance profile is **RI**

$$V_{3n} = \frac{1}{3n} \begin{pmatrix} 0 & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & 0 & 0 \\ \mathbf{1}_{n\times n} & 0 & 0 \end{pmatrix}$$

- The band variance profile, with band of width $\varepsilon_0 n$ is **RI**

$$\sigma_{ij}^2 = \sigma^2\left(\frac{i}{n}, \frac{j}{n}\right) \quad \text{with} \quad \sigma^2(x,y) = \mathbf{1}_{\{|x-y|\leq\varepsilon_0\}} \ .$$

Robust irreducibility assumption (informal)

The sequence (V_n) is RI if

- there exists a threshold $\boldsymbol{\sigma} > 0$ such that the **skeleton matrix**

$$V_n^{\boldsymbol{\sigma}} = \frac{1}{n}\left(\sigma_{ij}^2 \mathbf{1}_{\{\sigma_{ij}^2 > \boldsymbol{\sigma}^2\}}\right)$$

  is irreducible,
- there is a linear proportion of non-null entries in each column/row

$$\begin{array}{rcll} |\{i, V_{ij}^{\boldsymbol{\sigma}} > 0\}| & \geq & \delta n & \forall j \in [n] \\ |\{j, V_{ij}^{\boldsymbol{\sigma}} > 0\}| & \geq & \delta n & \forall i \in [n] \end{array}$$

- For every row/column, a (fixed) linear proportion κn of entries can be removed from $V_n^{\boldsymbol{\sigma}}$ while keeping the remaining matrix irreducible.

Master equations

Theorem

Let V_n be irreducible and consider the following system with unknown the two $n \times 1$ vectors $\boldsymbol{q} = (q_i)$ and $\tilde{\boldsymbol{q}} = (\tilde{q}_i)$:

$$\begin{cases} q_i & = & \frac{(V_n^\mathsf{T}\boldsymbol{q})_i}{s^2+(V_n^\mathsf{T}\boldsymbol{q})_i(V_n\tilde{\boldsymbol{q}})_i} & 1 \le i \le n \\ \tilde{q}_i & = & \frac{(V_n\tilde{\boldsymbol{q}})_i}{s^2+(V_n^\mathsf{T}\boldsymbol{q})_i(V_n\tilde{\boldsymbol{q}})_i} & 1 \le i \le n \\ \sum_{i=1}^n q_i & = & \sum_{i=1}^n \tilde{q}_i & \end{cases}$$

Denote by $\rho(V_n)$ the spectral radius of V_n and by $\vec{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{q} \\ \tilde{\boldsymbol{q}} \end{pmatrix}$. Then this system admits a unique solution satisfying

$$\begin{cases} \vec{\boldsymbol{q}}(s) \succ 0 & \text{for } s \in (0, \sqrt{\rho(V_n)}), \\ \vec{\boldsymbol{q}}(s) = 0 & \text{else} \end{cases}$$

One can define the probability measure on $\mathbb{C}$ by

$$\boxed{\mu_n^{\det}\{z\,,\ |z| \le s\} = 1 - \frac{1}{n}\langle \tilde{\boldsymbol{q}}(s), V_n \boldsymbol{q}(s)\rangle}$$

Example 1: the circular law

Constant variance profile

If $\sigma_{ij}^2 = \sigma^2$ is constant, then the $2n$ master equations merge into a single one:

$$\boxed{q = \frac{\sigma^2 q}{s^2 + \sigma^4 q^2}}$$

and we fortunately recover the **circular law**.

Doubly stochastic variance profile

If V_n is doubly stochastique, then the $2n$ master equations merge into a single one: $\boxed{q = \frac{q}{s^2 + q^2}}$ (circular law). **Hint:** try the equations with

$$\boldsymbol{q} = q \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{q}} = q \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{where} \quad q = \frac{q}{s^2 + q^2}$$

Example 2: separable variance profile

Separable variance profile

Let

$$\mathbb{E}|Y_{ij}|^2 = \frac{1}{n} d_i \tilde{d}_j \ , \quad d_i,\, \tilde{d}_j > 0$$

- Then the $2n$ master equations merge into a single one

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} \frac{d_i \tilde{d}_i}{s^2 + d_i \tilde{d}_i \boldsymbol{u_n}(s)} = 1} \quad \text{and} \quad \mu_n^{\det}\{|z| \le s\} = 1 - \boldsymbol{u_n}(s)$$

In this case,

$$\boxed{\mu_n^Y \ \overset{\mathcal{P}}{\sim}\ \mu_n^{\det}}$$

Example 3: sampled variance profile

Theorem (Cook, Hachem, N., Renfrew)

Let $\sigma : [0,1]^2 \to (0,\infty)$ and consider the **sampled variance profile**:

$$\sigma_{ij}(n) = \sigma\left(\frac{i}{n}, \frac{j}{n}\right)$$

Then $\boxed{\mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \mu_\infty}$ in probability, where

$$\mu_\infty\{|z| \le s\} = 1 - \int_{[0,1]^2} q_\infty(x,s)\,\widetilde{q}_\infty(y,s)\,\sigma^2(x,y)\,dx\,dy$$

with

$$\begin{cases} q_\infty(x,s) = \dfrac{\int_0^1 \sigma^2(y,x) q_\infty(y,s)\,dy}{s^2 + \int_0^1 \sigma^2(y,x) q_\infty(y,s)\,dy \int_0^1 \sigma^2(x,y)\widetilde{q}_\infty(y,s)\,dy}\,, \\ \widetilde{q}_\infty(x,s) = \dfrac{\int_0^1 \sigma^2(x,y)\widetilde{q}_\infty(y,s)\,dy}{s^2 + \int_0^1 \sigma^2(y,x) q_\infty(y,s)\,dy \int_0^1 \sigma^2(x,y)\widetilde{q}_\infty(y,s)\,dy}\,. \end{cases}$$

Example 4

Limiting measure with mass point at zero

Let

$$V_n = \frac{1}{3n}\begin{pmatrix} 0 & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & 0 & 0 \\ \mathbf{1}_{n\times n} & 0 & 0 \end{pmatrix}\ , \qquad \rho^* := \sqrt{\rho(V_n)} = \frac{\sqrt[4]{2}}{\sqrt{3}}$$

Then

$$\boxed{\mu_\infty(dz) = \frac{1}{3}\delta_0(dz) + \frac{12}{\pi}\frac{|z|^2}{\sqrt{1+36|z|^4}}\mathbf{1}_{\{|z|\le\rho^*\}}\ell(dz)}$$

and

$$\boxed{\mu_n^Y \xrightarrow[n\to\infty]{\mathcal{D}} \mu_\infty} \qquad \text{(in probability)}$$

Remark

Under condition

$$\sigma_{ij} \ge \sigma_* > 0\ ,$$

it is proved by Alt, Erdös and Krüger that $\mu_n^{\det}$ admits a density

$$\varphi_n(|z|) > 0 \qquad \text{for} \qquad |z| < \sqrt{\rho(V_n)}\ .$$

Not the case here.

Example 4 - continued

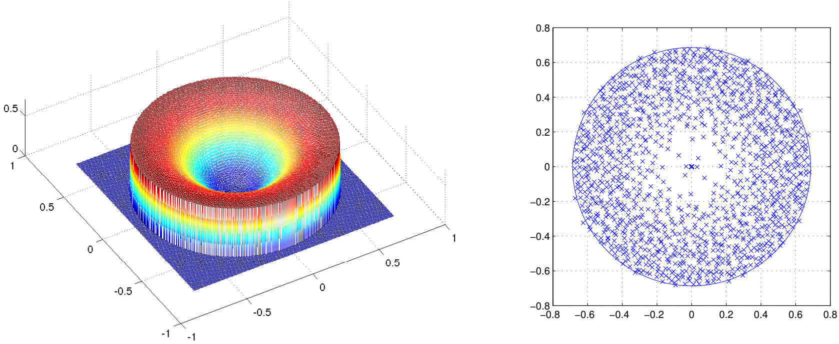

Figure: Density and sampled eigenvalues

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Girko's hermitization trick

Logarithmic potential

$$U_\mu(z) = -\int_{\mathbb{C}} \log|\lambda - z|\mu(d\lambda) \qquad \Rightarrow \qquad \boxed{\mu = -\frac{1}{2\pi}\Delta U_\mu}$$

Hermitization

Let $\nu_n^{Y,z}$ be the spectral distribution of the **Hermitian** model

$$\begin{pmatrix} 0 & Y_n - z \\ (Y_n - z)^* & 0 \end{pmatrix}$$

Then $\nu_n^{Y,z}$ is the symmetrized empirical measure of the singular values of $Y - zI$ and

$$\boxed{-\int_{\mathbb{C}} \log|\lambda - z|\mu_n^Y(d\lambda) = -\int_{\mathbb{R}} \log|t|\, \nu_n^{Y,z}(\,dt)}$$

Deterministic equivalents for the hermitized model

One can prove that there exists a family of deterministic probability measures ν_n^z such that

$$\nu_n^{Y,z} \sim \nu_n^z$$

Each ν_n^z is defined via (its Stieltjes transform from) 2n **Schwinger-Dyson equations**:

$$\begin{cases} p_i &= \dfrac{(V_n^{\mathsf{T}} \boldsymbol{p})_i + \eta}{|z|^2 - ((V_n \tilde{\boldsymbol{p}})_i + \eta)((V_n^{\mathsf{T}} \boldsymbol{p})_i + \eta)} \\[2em] \tilde{p}_i &= \dfrac{(V_n \tilde{\boldsymbol{p}})_i + \eta}{|z|^2 - ((V_n \tilde{\boldsymbol{p}})_i + \eta)((V_n^{\mathsf{T}} \boldsymbol{p})_i + \eta)} \end{cases}, \qquad \eta \in \mathbb{C}^+$$

The key issue: singularity of the logarithm

We study

$$\int_{\mathbb{R}} \log |t| \ \nu_n^{Y,z}(dt) - \int_{\mathbb{R}} \log |t| \nu_n^z(dt)$$

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We study

$$\underbrace{\int_{\mathbb{R}} \log|t| \; \nu_n^{Y,z}(dt)}_{\text{random}} - \underbrace{\int_{\mathbb{R}} \log|t| \nu_n^{z}(dt)}_{\text{deterministic}}$$

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- Probabilistic problem:

$$\int_{\mathbb{R}} \log|t|\ \nu_n^{Y,z}(dt) = \frac{1}{n}\sum_i \log|s_i(Y_n - zI)|$$

$\Rightarrow$ need to **control the smallest singular value** (Cook, '16)

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- Deterministic problem: define (and control!)

$$\int_{\mathbb{R}} \log|t| \ \nu_n^z(dt)$$

$\Rightarrow$ need for estimates such as **Wegner estimates**.

Where do the master equations come from?

Schwinger-Dyson equations ..

$$\begin{cases} p_i(\eta) &= \dfrac{(V_n^\top \boldsymbol{p})_i + \eta}{s^2 - ((V_n \tilde{\boldsymbol{p}})_i + \eta)((V_n^\top \boldsymbol{p})_i + \eta)} \\ \tilde{p}_i(\eta) &= \dfrac{(V_n \tilde{\boldsymbol{p}})_i + \eta}{s^2 - ((V_n \tilde{\boldsymbol{p}})_i + \eta)((V_n^\top \boldsymbol{p})_i + \eta)} \end{cases}$$

.. evaluated along the imaginary axis $(\boldsymbol{\eta} = \boldsymbol{it})$..

$$r_i(t) = \operatorname{im} p_i(\boldsymbol{it}) \ , \qquad \begin{cases} r_i &= \dfrac{(V_n^\top \boldsymbol{r})_i + t}{s^2 + [(V_n^\top \boldsymbol{r})_i + t][(V_n \tilde{\boldsymbol{r}})_i + t]} \\ \tilde{r}_i &= \dfrac{(V_n \tilde{\boldsymbol{r}})_i + t}{s^2 + [(V_n^\top \boldsymbol{r})_i + t][(V_n \tilde{\boldsymbol{r}})_i + t]} \end{cases}$$

.. then one lets $t \downarrow 0$

$$q_i = \lim_{t \downarrow 0} r_i(t) \, , \qquad \begin{cases} q_i &= \dfrac{(V_n^\top \boldsymbol{q})_i}{s^2 + (V_n^\top \boldsymbol{q})_i (V_n \tilde{\boldsymbol{q}})_i} \\ \tilde{q}_i &= \dfrac{(V_n \tilde{\boldsymbol{q}})_i}{s^2 + (V_n^\top \boldsymbol{q})_i (V_n \tilde{\boldsymbol{q}})_i} \end{cases}$$

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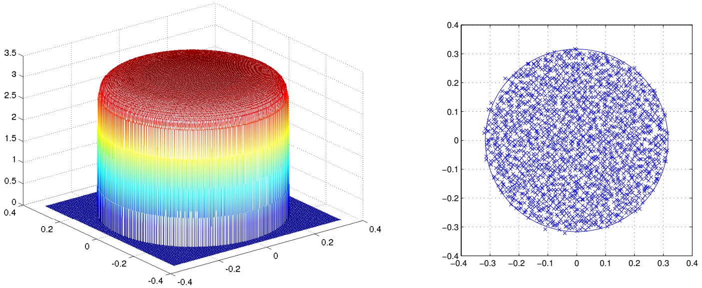

Figure: Density and sampled eigenvalued for the variance profile V_n^1

$$V_n^1: \quad \sigma_{ij}^2 = \sigma^2\left(\frac{i}{n}, \frac{j}{n}\right) \quad \text{with} \quad \sigma^2(x,y) = \mathbf{1}_{\{|x-y|\le 1/20\}}\ .$$

Example 2: modified band variance profile

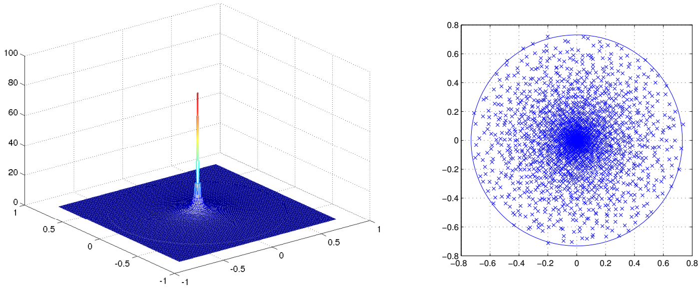

Figure: Density and sampled eigenvalued for the variance profile V_n^2

$$V_n^2 : \quad \sigma_{ij}^2 = \sigma^2\left(\frac{i}{n}, \frac{j}{n}\right) \quad \text{with} \quad \sigma^2(x,y) = (x+y)^2 \mathbf{1}_{\{|x-y| \le 1/10\}} \ .$$

Reference

- Limiting spectral distribution for non-hermitian random matrices with a variance profile. N. Cook, W. Hachem, J. Najim, D. Renfrew (arxiv)