Numéro de la feuille d'examen
à reporter ci-dessous :
N°

Ex 6.1.

$$E[S_t | S_1] \overset{?}{=} E[S_t | B_1]$$ (rmq du TD $\sigma(S_1) = \sigma(B_1)$)

$$S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma(B_t - B_1) + \sigma B_1\right)$$

$$= S_1 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-1) + \underbrace{\sigma(B_t - B_1)}_{\perp\!\!\!\perp\ S_1}\right)$$

donc mesurabilité + $\perp\!\!\!\perp$

$$E[S_t | S_1] = S_1 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-1)\right) \times \underbrace{E[\exp(\sigma(B_t - B_1))]}$$

$$= \Phi_{B_t - B_1}(-i\sigma)$$

$$= \exp\left(\frac{\sigma^2}{2}(t-1)\right)$$

d'où $E[S_t | S_1] = S_1 \exp(\mu(t-1))$

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exo 6.2

$$S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

Comme indiqué en TD $E[S_t | S_T] = E[S_t | B_T]$

on a vu $E[B_t | B_T] = \frac{t}{T} B_T$ à l'exo 5.4

donc on écrit $B_t = \underbrace{B_t - \frac{t}{T} B_T}_{\perp\!\!\!\perp} + \frac{t}{T} B_T$

d'où $S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\left(B_t - \frac{t}{T}B_T\right) + \sigma B_T \frac{t}{T}\right)$

or $S_T^{t/T} = \left[\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B_T\right)\right]^{t/T}$

$$= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_T \frac{t}{T}\right)$$

donc $S_t = S_T^{t/T} \exp\left(\sigma\left(B_t - \frac{t}{T}B_T\right)\right)$.

D'où, par $\perp\!\!\!\perp$ & modularité,

$$E[S_t | S_T] = S_T^{t/T} \, E\left[\exp\left(\sigma\left(B_t - \frac{t}{T}B_T\right)\right)\right]$$

$$= \Phi_{B_t - \frac{t}{T}B_T}(-i\sigma).$$

Rappel

$Var\left(B_t - \frac{t}{T}B_T\right)$

$= t + \frac{t^2}{T^2}T - \frac{2t^2}{T} = t\left(1 - \frac{t}{T}\right)$

$$= \exp\left(\frac{\sigma^2 Var\left(B_t - \frac{t}{T}B_T\right)}{2}\right)$$

$$= \exp\left(\frac{\sigma^2 t}{2}\left(1 - \frac{t}{T}\right)\right).$$

d'où $\boxed{E[S_t | S_T]} = \exp\left(\frac{\sigma^2 t}{2}\left(1 - \frac{t}{T}\right)\right) S_T^{t/T}$

exo 6.3

Même principe pour $E[S_t S_T | S_n]$

comme $E[B_t | B_n] = B_n$ $\qquad B_t - B_n \perp B_n$

$$S_t S_T = \exp\left((\mu - \frac{\sigma^2}{2})(t+T) + \sigma(B_t - B_n) + \sigma(B_T - B_n) + 2\sigma B_n \right)$$

$$= S_n^2 \exp\left((\mu - \frac{\sigma^2}{2})(t+T-2n) + \sigma(B_t - B_n + B_T - B_n) \right)$$

donc $E[S_t S_T | S_n] = S_n^2 \exp\left((\mu - \frac{\sigma^2}{2})(t+T-2n) \right)$

par modularité et indépendance

$$E\Big[\underbrace{\exp\left(\sigma(B_t - B_n) + \sigma(B_T - B_n)\right)}_{A} \Big]$$

Reste à calculer la dernière espérance

vu $T > t$ $\quad B_T - B_n = \underbrace{B_T - B_t}_{} + \underbrace{B_t - B_n}_{}$

$\perp$.

donc $E[A] = \underbrace{E[\exp(2\sigma(B_t - B_n))]}_{\exp\left(\frac{4\sigma^2(t-n)}{2}\right)} \underbrace{E[\exp(\sigma(B_T - B_t))]}_{\exp\left(\frac{\sigma^2(T-t)}{2}\right)}$

Bilan : $E[S_t S_T | S_n] = S_n^2 \exp\left(\mu(t+T-2n)\right)$

$$\times \exp\left(2\sigma^2(t-n) - \frac{\sigma^2}{2} 2(t-n)\right)$$

$$\times \exp\left(\frac{\sigma^2}{2}(T-t) - \frac{\sigma^2}{2}(T-t)\right).$$

$$= S_n^2 \exp\left(\mu(t+T-2n) + \sigma^2(t-n)\right).$$