# The Euler characteristic of a surface from its Fourier analysis in one direction

Nguyen Viet Dang

In this paper, we prove that we can recover the genus of a closed compact surface $S$ in $\mathbb{R}^3$ from the restriction to a generic line of the Fourier transform of the canonical measure carried by $S$. We also show that the restriction on some line in Minkowski space of the solution of a linear wave equation whose Cauchy data comes from the canonical measure carried by $S$, allows to recover the Euler characteristic of $S$.

## 1. Introduction

Let us start with a surface $S$ in $\mathbb{R}^3$. To this surface $S$, we can always associate a natural measure just by integrating test functions on $S$. Indeed, the surface carried measure $\mu$ is the distribution defined as

$$(1.1) \qquad \mu(\varphi) = \int_S \varphi d\sigma$$

where $d\sigma$ is the canonical area element on $S$ induced by the Euclidean metric of $\mathbb{R}^3$ ([21, p. 334], [20, p. 321]) and $\varphi$ a test function. If we assume that $S$ is compact, then we find that the measure $\mu$ can be viewed as a *compactly supported distribution* on $\mathbb{R}^3$, $\mu$ is thus a tempered distribution and therefore it has a well defined Fourier transform denoted by $\widehat{\mu}$.

In harmonic analysis, we are interested in the analytical properties of the Fourier transform $\widehat{\mu}$ of measures $\mu$ which are supported on submanifolds of some given vector space. For instance, in a very recent paper [1], using the concept of wave front set and resolution of singularities, Aizenbud and Drinfeld give a proof that the Fourier transform of an algebraic measure is smooth in some open dense set. More classically, one would like to study the asymptotic behaviour of $\widehat{\mu}$ for large momenta. In the simple case where $\mu$ is the measure carried by a plane $H$ in a vector space $V$, the Fourier transform $\widehat{\mu}$ is a measure carried by the dual plane $H^\perp$ in Fourier space $V^*$ [13, Thm. 7.1.25 p. 173]. Moreover, if the measure $\mu$ is compactly supported, then

1263

we know by Paley–Wiener that its Fourier transform $\widehat{\mu}$ should be a *bounded* function on $\mathbb{R}^3$. Therefore in both cases, for any test function $\varphi$, classical theorems in distribution theory only yield that the Fourier transform $\widehat{\mu\varphi}$ is **bounded**.

However, if we assume that $\mu$ is carried by a *compact* hypersurface $S$ of dimension $n$ with *non vanishing Gauss curvature*, then a celebrated result of Stein (see [20, Theorem 1 p. 322] and also [13, Thm. 7.7.14]) gives finer decay properties on $\widehat{\mu}$. Indeed, he proves that $|\widehat{\mu}| \leqslant C(1 + |\xi|)^{-\frac{n}{2}}$ for some constant $C$ which depends on the volume of $S$ and the Gauss curvature. Stein's result shows the interplay between geometry and analysis, since a simple assumption on the Gauss curvature of $S$ gives sharper decay properties on $\widehat{\mu}$ than a simple application of the Paley–Wiener theorem.

In this paper, we explore the relationship between the Fourier transform $\widehat{\mu}$ of the surface carried measure $\mu$ and the topology of the surface itself. In the same spirit as in the papers [14, 15], we use microlocal analysis and Morse theory in the study of $\widehat{\mu}$. Throughout the paper, the surface $S$ will always be assumed to be smooth and oriented. We state in an informal way the main result of this note:

**Theorem 1.1.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. For a generic line $\ell \subset \mathbb{R}^3$, we can recover the Euler characteristic of the surface $S$ from the restriction $\widehat{\mu}|_\ell$.*

The space of unoriented lines in $\mathbb{R}^3$ is canonically identified with the projective space $\mathbb{RP}^2$ and generic means that the theorem holds true for an open dense set of lines in $\mathbb{RP}^2$. We refer the reader to Theorem 2.2 which explains in *what sense we recover* $\chi(S)$ and gives a precised version of our main result.

Our main result might look surprising because if we knew the full Fourier transform $\widehat{\mu}$, then it would be easy to reconstruct $\mu$ hence $S$ from $\widehat{\mu}$ by Fourier inversion. But to recover the topology of $S$, it is enough to consider the partial information of the restriction of $\widehat{\mu}$ to a line $\ell$. Actually, all the information we need is contained in the asymptotic expansion of $\widehat{\mu}(\xi)$ for large $|\xi|$. In a way, our result is reminiscent of Weyl's tube formula where the asymptotic expansion in $\varepsilon$ of the volume of the tube $S_\varepsilon$ of "thickness" $\varepsilon$ around a surface $S$ gives geometrical data on $S$: the volume and the Euler characteristic of $S$.

As a byproduct of our main theorem, we derive similar results for more general integral transforms. First, we connect our result with the Radon

transform in the spirit of [5, section 5.3]. We denote by $\mathcal{R}\mu$ the Radon transform of $\mu$.

**Theorem 1.2.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. For a generic line $\ell \subset \mathbb{R}^3$, we can recover the Euler characteristic of the surface $S$ from the restriction of $\mathcal{R}\mu$ on $\ell$.*

In the sequel, a Morse function $\psi$ is called *excellent* if its critical values are distinct.

**Theorem 1.3.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. If $\psi \in C^\infty(\mathbb{R}^3)$ is such that its restriction on $S$ is Morse excellent then we can recover the Euler characteristic of $S$ from the map $\lambda \longmapsto \mu\left(e^{i\lambda\psi}\right)$.*

And finally, let us state informally an application of this theorem to the following inverse problem:

**Theorem 1.4.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right) u = 0$ with Cauchy data $u(0) = 0$, $\partial_t u(0) = \mu$. For $x$ in some open dense subset of $\mathbb{R}^3$, set $\ell(x)$ to be the line $\{x\} \times \mathbb{R} \subset \mathbb{R}^{3+1}$, then:*

- *the restriction $u_{\ell(x)}$ is a compactly supported distribution of $t$,*
- *we can recover the Euler characteristic of $S$ from the restriction $u_{\ell(x)}$.*

We give precise statements in Proposition 4.1 and Theorem 4.2 which explain in what sense we recover $\chi(S)$.

In appendix, we gather several useful results in Morse theory and transversality theory which are used in the paper.

**The general principle underlying our results.** The main idea of our note is to think of the stationary phase principle as an analytic version of Lagrangian intersection. The first observation is that a surface carried measure $\mu$ is the simplest example of Lagrangian distribution (in the sense of Maslov Hörmander) with wave front set the conormal $N^*(S)$ of the surface $S$. To probe the wave front set of a distribution, a natural idea is to calculate its Fourier transform or more generally to pair $\mu$ with an oscillatory

Nguyen Viet Dang

function of the form $e^{i\lambda\psi}$ where $d\psi \neq 0$. We can think of this operation as a plane wave analysis of $\mu$. By the stationary phase principle, the main contributions to the asymptotics of $\mu(e^{i\lambda\psi})$ when $\lambda \to +\infty$ come from the points where the graph of $d\psi$ meets the wave front set of $\mu$ which is the conormal $N^*(S)$. Therefore our strategy is to extract topological information from the Lagrangian intersection (Graph of $d\psi$) $\cap N^*(S)$ by the stationary phase principle. This is strongly related to the work of Kashiwara [15, p. 194] and Fu [11] where the Euler characteristic of subanalytic sets (Kashiwara gives an index formula in the context of *constructible sheaves*) is expressed in terms of Lagrangian intersection.

Let us give the central example of the theory.

**Example 1.5.** Let $\psi$ be a Morse function on a manifold $M$ of dimension $n$ and $\Omega$ a test form in $\Omega_c^n(M)$, then the main contributions to the oscillatory integral $\int_M e^{i\lambda\psi}\Omega$ when $\lambda \to +\infty$ come from the critical points of $\psi$, in other words, from the intersection of the Lagrangian graph $d\psi$ and the zero section $\underline{0} \subset T^*M$.

## 2. Proof of Theorem 1.1

### 2.1. Fourier transform, Morse functions and stationary phase

Recall that $S$ is a closed, compact surface embedded in $\mathbb{R}^3$ and $\mu$ denotes the associated surface carried measure. We denote by $d\sigma$ the canonical area element on $S$ induced by the Euclidean metric of $\mathbb{R}^3$. Then the surface carried measure $\mu$ is the distribution defined by the formula:

$$(2.1) \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \mu(\varphi) = \int_S \varphi d\sigma.$$

The Fourier transform $\widehat{\mu}$ reads

$$(2.2) \qquad \widehat{\mu}(\xi) = \int_S e^{-i\xi(x)} d\sigma(x).$$

Since $S$ is compact, $\mu$ is compactly supported thus $\widehat{\mu}$ is a real analytic function by Paley–Wiener theorem. In order to study the asymptotics of $\widehat{\mu}(\lambda\xi)$ for $\lambda$ large, we will think of $\xi \in \mathbb{R}^{3*} \subset C^\infty(\mathbb{R}^3)$ as a linear function on $\mathbb{R}^3$ whose restriction on $S$ is a height function which we denote by $\xi \in C^\infty(S)$.

For every $\xi \in \mathbb{R}^{3*}$, we denote by $[\xi]$ its class in $\mathbb{RP}^2$ and by Theorem 5.2 recalled in Appendix, for an everywhere dense set of $[\xi] \in \mathbb{RP}^2$, the function

$\xi$ is a *Morse function* on $S$. Set $\omega = \frac{\xi}{|\xi|}$, $\lambda = |\xi|$ and note that $\omega$ has the same critical points as $\xi$. The stationary phase principle states that the main contributions to the asymptotics of $\widehat{\mu}(\lambda\omega)$ when $\lambda \to \infty$ come from the *critical points* of the Morse function $\omega$ which are isolated by Theorem 10.4.3 in [7, p. 87]. We denote by Crit $\omega$ the set of critical points of $\omega$. Then in the neighborhood of every $x \in \mathrm{Crit}\,\omega$, we choose local coordinates $(y^1, y^2)$ on $S$ such that $y^1(x) = y^2(x) = 0$ and $d\sigma(x) = |dy^1 dy^2|$ where $d\sigma$ is the canonical area element on $S$. The method of stationary phase ([9, Lemma (19.4) p.95],[18, p. 124]) yields the asymptotic expansion:

$$(2.3) \qquad \widehat{\mu}(\lambda\omega) \underset{\lambda\to\infty}{\sim} \sum_{x\in\mathrm{Crit}\,\omega} \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} e^{i\frac{\pi}{4}(n_+ - n_-)(x)}$$

$$\times \frac{e^{-i\lambda\omega(x)}}{\sqrt{|\det\omega''(x)|}} \left[1 + \sum_{j=1}^{\infty} b_j(x)\lambda^{-j}\right].$$

We want to comment on the geometric interpretation of each of the terms in the expansion:

- $\frac{n}{2} = 1$ in our case since $S$ is a surface.

- $n_+(x)$ (resp $n_-(x)$) is the number of positive (resp negative) eigenvalues of the Hessian $-\omega''$ of the Morse function $-\omega$ at the critical point $x$, the number $n_-(x)$ is the *Morse index* of $-\omega$ at $x$. For surfaces, $e^{i\frac{\pi}{4}(n_+ - n_-)(x)}$ can only take the three values $\{i, 1, -i\}$. Observe that

$$n_-(x) = 1 \mod (2) \Leftrightarrow e^{i\frac{\pi}{4}(n_+ - n_-)(x)} = 1$$
$$n_-(x) = 0 \mod (2) \Leftrightarrow e^{i\frac{\pi}{4}(n_+ - n_-)(x)} = \pm i.$$

- In the local chart $(y_1, y_2)$ around $x$, the Hessian $-\omega''(x)$ is non degenerate since $\omega$ is Morse and it also coincides with the second fundamental form of the surface $S$ at $x$. Therefore $|K(x)| = |\det\omega''(x)|$ where $K(x)$ is the *Gauss curvature* of the surface $S$ at $x$ ([19, 3.3.1 p. 67]).

## 2.2. An oscillatory integral whose singular points are the critical values of the height function

We denote by

$$\mathcal{F}_\lambda^{-1} : v \in \mathcal{S}'(\mathbb{R}) \longmapsto \widehat{v}(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda\, e^{i\tau\lambda} v(\lambda)$$

Nguyen Viet Dang

the inverse Fourier transform w.r.t. variable $\lambda$. We define the oscillatory integral $u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)$ on $\mathbb{R}$ and show that $u$ is singular at the critical values of the Morse function $\omega$. Recall that $K(x)$ denotes the Gauss curvature of $S$ at $x$.

**Proposition 2.1.** Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Let $u$ be the distribution

(2.4)
$$u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right).$$

If $\omega \in C^\infty(S)$ is Morse, then:

- the singular support of $u$ is the set of critical values of $\omega$,

- $u$ is a finite sum of oscillatory integrals on $\mathbb{R}$

- each oscillatory integral has polyhomogeneous symbol whose leading term is $\frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|K(x)|}}$ where $x \in \mathrm{Crit}\ \omega$.

*Proof.* By the stationary phase expansion,

$$\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega) = \sum_{x\in\mathrm{Crit}\ \omega} e^{-i\lambda\omega(x)}b(x,\lambda)$$

where, for every $x \in \mathrm{Crit}\ \omega$, $b(x,\lambda)$ is a polyhomogeneous symbol in $\lambda$:

(2.5)
$$b(x;\lambda) \sim \frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|\det\omega''(x)|}}\left[1 + \sum_{j=1}^\infty b_j(x)\lambda^{-j}\right]$$

with principal symbol $\frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|\det\omega''(x)|}}$ since $|K(x)| = |\det\omega''(x)| \neq 0$. Therefore:

(2.6)
$$u(t) = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)$$
$$= \sum_{x\in\mathrm{Crit}\ \omega}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda(t-\omega(x))}b(x;\lambda)d\lambda$$

is a finite sum of oscillatory integrals ([13, Theorem (7.8.2) p. 237]). By the general theory of oscillatory integrals [17, Theorem IX.47 p. 102] [13,

Theorem 8.1.9 p. 260]:

$$(2.7) \qquad WF(u) = \{(t;\tau)|(\omega(x) - t) = 0, \tau \neq 0\}.$$

From [13, Proposition 8.1.3 p. 254], we deduce that the singular support of $u$ is $\{t|(\omega(x) - t) = 0, x \in \text{Crit } \omega\}$ which is exactly the set of critical values of $\omega$. □

We state and prove a precised version of the main theorem 1.1 given in the introduction:

**Theorem 2.2.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Then for an everywhere dense set $\omega \in \mathbb{S}^2$, the distribution $u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)$ can be canonically decomposed as a sum*

$$(2.8) \qquad u = \sum_{x \in Crit\ \omega} a(x)\delta_{\omega(x)} + r$$

*such that*

- *$\forall x \in Crit\ \omega, a(x) \neq 0$*

- *$r$ is an oscillatory integral with symbol of degree $-1$.*

*The Euler characteristic of $S$ satisfies the identity:*

$$(2.9) \qquad \chi(S) = \sum_{x \in Crit\ \omega} -\frac{a(x)^2}{|a(x)|^2}.$$

*Proof.* By Lemma 5.3 proved in appendix, for an everywhere dense set $\omega \in \mathbb{S}^2$, the corresponding function $\omega \in C^\infty(S)$ is an excellent Morse function which means that if $(x_1, x_2)$ are distinct critical points of $\omega$ then $\omega(x_1) \neq \omega(x_2)$.

In particular, $\omega$ is Morse therefore we can use the results of Proposition 2.1. From the asymptotic expansion (2.5) of the symbol $b$ and the equation of $u$ (2.6), we find that:

$$(2.10) \qquad u = \sum_{x \in \text{Crit } \omega} \frac{e^{i\frac{\pi}{4}(n_+ - n_-)(x)}}{\sqrt{|\det \omega''(x)|}}\delta_{\omega(x)} + r,$$

where $r \in \mathcal{D}'(\mathbb{R})$ is an oscillatory integral whose asymptotic symbol has leading term $\sum_{x \in \text{Crit } \omega} a(x)b_{-1}(x)\lambda^{-1}$ for $a(x) = \frac{e^{i\frac{\pi}{4}(n_+ - n_-)(x)}}{\sqrt{|\det \omega''(x)|}}$. We rewrite the above

formula in a simpler form:

$$(2.11) \qquad u = \sum_{x \in \mathrm{Crit}\ \omega} a(x)\delta_{\omega(x)} + r$$

where $a(x) = \frac{e^{i\frac{\pi}{4}(n_+ - n_-)(x)}}{\sqrt{|\det \omega''(x)|}}$.

Our goal is to express $\chi(S)$ in terms of the coefficients $a(x), x \in \mathrm{Crit}\ \omega$. We recall the definition of the Morse counting polynomial $\mathcal{M}_\omega(T)$ for a given Morse function $\omega$ ([10, Definition C.4 p. 228] ):

$$(2.12) \qquad \mathcal{M}_\omega(T) = \sum_{x \in \mathrm{Crit}\ \omega} T^{n_-(x)}.$$

Observe that

$$n_-(x) = 0 \mod (2) \Leftrightarrow a(x) \in i\mathbb{R}$$
$$n_-(x) = 1 \mod (2) \Leftrightarrow a(x) \in \mathbb{R}$$
$$\Rightarrow -\frac{a(x)^2}{|a(x)|^2} = (-1)^{n_-(x)}.$$

This implies that:

$$(2.13) \qquad \mathcal{M}_\omega(-1) = \sum_{x \in \mathrm{Crit}\ \omega} (-1)^{n_-(x)} = \sum_{x \in \mathrm{Crit}\ \omega} -\frac{a(x)^2}{|a(x)|^2}.$$

To conclude, we use the well known result $\chi(S) = \mathcal{M}_\omega(-1)$ which is a consequence of the Morse inequalities ([10, Theorem C.3 p. 228] and [16, Thm. 5.2 p. 29]) . $\qquad \square$

Let $\mathcal{R}$ be the Radon transform. From the identity relating the Fourier transform and the Radon transform (see [5, p. 19]), we find a relationship between the distribution $u$ of Theorem 2.2 and $\mathcal{R}\mu$:

$$\mathcal{R}\mu(\omega,\tau) = \mathcal{F}_\lambda^{-1}\left(\widehat{\mu}(\lambda\omega)\right)(\tau) \implies u(\tau) = \frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega,\tau)$$
$$\implies \frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega,\tau) = \sum_{x \in \mathrm{Crit}\ \omega} a(x)\delta_{\omega(x)} + r.$$

This immediately proves a precised version of Theorem 1.2 and gives some informations on the Radon transform of the measure $\mu$:

**Theorem 2.3.** *Under the assumptions of Theorem 2.2, let $\mathcal{R}\mu$ be the Radon transform of $\mu$. Then for an everywhere dense set of $\omega \in \mathbb{S}^2$,*

$$(2.14) \qquad \frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega, \tau) = \sum_{x \in Crit\ \omega} a(x)\delta_{\omega(x)} + r$$

*where*

$$\chi(S) = \sum_{x \in Crit\ \omega} -\frac{a(x)^2}{|a(x)|^2}.$$

**2.2.1. Remark.** If we are given a Lagrangian distribution $u \in \mathcal{D}'(\mathbb{R})$ with singular point $t_0$ and $u$ can be written as a sum $u = a\delta_{t_0} + r$ where $a \in \mathbb{C}$ and $r$ is a Lagrangian distribution with asymptotic symbol of degree $-1$, then we can recover $a$ by scaling around the point $t_0$. Indeed, a straightforward calculation yields

$$(2.15) \qquad \lim_{\lambda \to 0} \lambda u(\lambda(. - t_0)) = a\delta_{t_0}$$

where the limit is understood in the sense of distributions. For any test function $\varphi$ which is equal to 1 in a sufficiently small neighborhood of $t_0$, we find that $\lim_{\lambda \to 0} \langle \lambda t(\lambda(. - t_0)), \varphi \rangle = a$.

### 2.3. The example of the sphere

We show how our proof works in the case of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. First, note that any height function restricted on $\mathbb{S}^2$ is Morse excellent and has exactly two critical points. The Fourier transform of $\mu$ is given by the exact formula: $\widehat{\mu}(\xi) = 4\pi \frac{\sin(|\xi|)}{|\xi|}$ therefore

$$\frac{\lambda}{2\pi}\widehat{\mu}(\lambda\omega) = 2\sin(\lambda)$$

$$\implies \mathcal{F}^{-1}\left(\frac{\lambda}{2\pi}\widehat{\mu}(\lambda\omega)\right) = \mathcal{F}^{-1}\left(2\sin(\lambda)\right)$$

$$= \mathcal{F}^{-1}\left(\frac{e^{i\lambda} - e^{-i\lambda}}{i}\right) = i(\delta_{+1} - \delta_{-1}).$$

Finally, the identity 2.9 allows to recover the well known result $\chi(\mathbb{S}^2) = -i^2 - (-i)^2 = 2$.

## 3. Plane wave analysis and topology

Let us recall the statement of Theorem 1.3 before we give a proof:

*Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. If $\psi \in C^\infty(\mathbb{R}^3)$ is such that its restriction on $S$ is Morse excellent then we can recover the Euler characteristic of $S$ from the map $\lambda \longmapsto \mu\left(e^{i\lambda\psi}\right)$.*

The idea to consider oscillatory integrals of the form $\mu\left(e^{i\lambda\psi}\right)$ comes from the coordinate invariant definition of wave front set due to Gabor [4, 12] then corrected by Duistermaat [8, Proposition 1.3.2]. To prove Theorem 1.3, we just repeat the proof of Theorem 2.2 applied to the distribution $u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\mu\left(e^{i\lambda\psi}\right)\right)$ where we replace $\omega$ by $\psi$.

## 4. The wave equation, propagation of singularities and topology

Let us recall the statement of Theorem 1.4:

*Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right)u = 0$ with Cauchy data $u(0) = 0, \partial_t u(0) = \mu$. For $x$ in some open dense subset of $\mathbb{R}^3$, set $\ell(x)$ to be the line $\{x\} \times \mathbb{R} \subset \mathbb{R}^{3+1}$, then:*

1) *the restriction $u_{\ell(x)}$ is a compactly supported distribution of $t$,*

2) *we can recover the Euler characteristic of $S$ from the restriction $u_{\ell(x)}$.*

Let us prove a precised version of claim (1):

**Proposition 4.1.** Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Consider the solution $u$ of the wave equation $\Box u = 0$ with Cauchy data $u(0) = 0, \partial_t u(0) = \mu$. For all $x \in \mathbb{R}^3 \setminus S$:

- $t \longmapsto u(t, x)$ is a compactly supported distribution of $t$

- the singular support of $u(., x)$ is a subset of $\{t \in \mathbb{R} | \exists y \in S \text{ s.t. } |x - y| = |t|, y - x \perp T_y S\}$.

*Proof.* Recall that $\ell_x = \{(t, x) | t \in \mathbb{R}^3\}$ and denote by $N^*(\ell_x) \subset T^*\mathbb{R}^{3+1}$ its conormal bundle. We want to prove that one can restrict the distribution $u$

on $\ell_x$. First, $\Box u = 0$ implies that the wave front set of $u$ lies in the characteristic set of $\Box$ ([13, Theorem 8.3.1]):

$$WF(u) \subset \text{Char } \Box = \{\tau^2 - |\xi|^2\} \implies WF(u) \cap N^*(\ell_x) = \emptyset$$

which means that one can pull–back the distribution $u$ by the embedding $i : \ell_x \hookrightarrow \mathbb{R}^{3+1}$ (see [13, Theorem 8.2.4]). The restriction $i^*u$ is thus well defined, it is compactly supported since the Cauchy data $(0, \mu)$ is compactly supported and by finite propagation speed property for the wave equation.

Secondly, we calculate the wave front set of the restriction $i^*u$. Elements of the cotangent space $T^*\mathbb{R}^{3+1}$ are denoted by $(t, x; \tau, \xi)$ and $\pi$ is the projection $T^*\mathbb{R}^{3+1} \mapsto \mathbb{R}^{3+1}$. Since $u$ is solution of $\Box u = 0$ with Cauchy data $(0, \mu)$, $u$ is given by the representation formula ([3, equation (2.21) p. 12], [2, Theorem 5.3 p. 67], [6, p. 3])

$$(4.1) \qquad u(t, x) = \frac{1}{4\pi t} \int_{\mathbb{R}^3} \delta(|t| - |x - y|)\mu(y)dy.$$

We want to calculate $WF(u)$ using the integral formula (4.1). By finite propagation speed, the condition $x \in \mathbb{R}^3 \setminus S$ ensures that $i^*u = 0$ in some neighborhood of $t = 0$ which means we do not have to consider the contribution of $t = 0$ to $WF(\delta(|t| - |x - y|))$. Denote by $P$ the projection $P : (t, y) \in \mathbb{R}^{3+1} \mapsto t \in \mathbb{R}$, since $WF(\delta(|t| - |x|)) = \{(t, x; \lambda t, -\lambda x)||t| = |x|, \lambda \in \mathbb{R} \setminus \{0\}\} \cup T_0^*\mathbb{R}^{3+1}$, the calculus of wave front set yields:

$$\begin{aligned} WF(u) &\subset P_*WF(\delta(|t| - |x - y|)\mu(y)) \\ &\subset \{(t; \lambda t)|\exists(y; \eta) \in N^*(S), |t| = |x - y|, \lambda(x - y) = \eta\} \\ &= \{(\pm|x - y|; \tau)|(x - y) \perp T_yS, \tau \in \mathbb{R} \setminus \{0\}\} \\ \implies \text{ss } u &= \pi(WF(i^*u)) \\ &\subset \{\pm|x - y||(x - y) \perp T_yS\}. \end{aligned}$$

$\square$

Let us prove that one can recover $\chi(S)$ from $u(., x) \in \mathcal{D}'(\mathbb{R})$ concluding the proof of Theorem 1.4.

**Theorem 4.2.** *Let $S$ be a closed compact surface embedded in $\mathbb{R}^3$ and $\mu$ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right) u = 0$ with Cauchy data $u(0) = 0$,*

$\partial_t u(0) = \mu$. *Then for $x$ in some open dense subset of $\mathbb{R}^3$, we have the canonical decomposition:*

$$(4.2) \qquad -2i\,(t\partial_t + 1)\,u(t,x) = \sum_{y \in Crit\ L_x} a(y)\delta_{|y-x|}(t) + r(t)$$

*where $L_x : y \in S \mapsto |y - x|$ is Morse excellent, $r$ is a finite sum of oscillatory integrals with symbol of degree $-1$.*

$$(4.3) \qquad \chi(S) = \sum_{y \in Crit\ L_x} -\frac{a(y)^2}{|a(y)|^2}.$$

*Proof.* We again use the representation formula for $u$:

$$u(t,x) = \frac{1}{4\pi t} \int_{\mathbb{R}^3} \delta(|t| - |x - y|)\mu(y)dy.$$

Recall that $x \in \mathbb{R}^3 \setminus S$ implies $u(.,x) = 0$ in some neighborhood of $t = 0$. Therefore, for $t \geqslant 0$:

$$u(t,x) = \frac{1}{8\pi^2 t} \int_{\mathbb{R}} d\lambda e^{it\lambda} \int_{\mathbb{R}^3} \mu(y)e^{-i\lambda|x-y|}dy$$

$$\implies -2i\,(t\partial_t + 1)\,u(t,x) = \mathcal{F}^{-1}\left(\frac{\lambda}{2\pi} \int_{\mathbb{R}^3} \mu(y)e^{-i\lambda|x-y|}dy\right).$$

By Lemma 5.5, for $x$ in some open dense set in $\mathbb{R}^3$, the function $L_x : y \in S \mapsto |y - x|$ is Morse excellent. Therefore, repeating the proof of Theorem 1.1 with $L_x$ instead of $-\omega$, we find that

$$-2i\,(t\partial_t + 1)\,u(t,x) = \sum_{y \in \mathrm{Crit}\ L_x} a(y)\delta_{|y-x|}(t) + r(t)$$

where $r$ is a finite sum of oscillatory integrals with symbol of degree $-1$. Finally $\chi(S) = \sum_{y \in \mathrm{Crit}\ L_x} -\frac{a(y)^2}{|a(y)|^2}$.                  $\square$

## 5. Appendix

In this section, we gather several useful results in Morse theory.

### 5.1. Almost all height functions are Morse

Let $S$ be an embedded surface in $\mathbb{R}^3$. For every $x \in S$, we denote by $T_x S$ the tangent plane to $S$ at $x$.

**Definition 5.1.** The map $x \in S \mapsto n(x) \in \mathbb{S}^2$ where $n(x)$ is the oriented unit normal vector to $T_x S$ is called the Gauss map. It induces canonically a *projective* Gauss map denoted by $[n] := x \in S \longmapsto [n](x) \in \mathbb{RP}^2$.

The next theorem [7, Thm. 11.2.2 p. 94] characterizes all height functions which are Morse functions in terms of the Gauss map:

**Theorem 5.2.** *Let $S$ be an embedded surface in $\mathbb{R}^3$. The height function $\xi \in C^\infty(S) := x \in S \mapsto \xi(x)$ is a Morse function precisely when $[\xi] \in \mathbb{RP}^2$ is a regular value of the projective Gauss map $[n]$. It follows that for an* open *everywhere dense set of $[\xi] \in \mathbb{RP}^2$, the height function $\xi \in C^\infty(S)$ is a Morse function.*

### 5.2. Almost all Morse height functions are excellent Morse functions

We refine Theorem 5.2 and show that for generic $\xi \in \mathbb{RP}^2$, the height function $\xi$ is an excellent Morse function i.e. all critical values of $\xi$ are distinct.

**Lemma 5.3.** *Let $S$ be a compact embedded surface in $\mathbb{R}^3$, for an* open *everywhere dense set of $[\xi] \in \mathbb{RP}^2$, the height function $\xi \in C^\infty(S)$ is an excellent Morse function.*

*Proof.* Let $V$ be the set of regular values of the projective Gauss map $[n]$ in $\mathbb{RP}^2$. By Sard's Theorem $V$ is open dense in $\mathbb{RP}^2$ and

$$[\xi] \in V \Leftrightarrow \text{ the height function } \xi \text{ is Morse.}$$

We give next a characterization of Morse height functions which are not Morse excellent. For all $[\xi] \in V$, there is a neighborhood $\Omega$ of $[\xi]$ such that the preimage $[n]^{-1}(\Omega)$ is a disjoint union of open sets $(U_1, \ldots, U_k) \subset S^k$ and each $U_i$ is sent diffeomorphically to $\Omega$ by the Gauss map. Therefore there is a collection of maps

$$(x_1, \ldots, x_k) := [\xi] \in \Omega \subset \mathbb{RP}^2 \longmapsto (x_1([\xi]), \ldots, x_k([\xi])) \in S^k$$

such that $\forall i, [n](x_i([\xi])) = [\xi]$. We claim that for $[\xi] \in \Omega$:

$\xi$ is not Morse excellent $\Leftrightarrow \xi.(x_i([\xi]) - x_j([\xi])) = 0$ for some $1 \le i < j \le k$.

From the fact that

$$d_\xi \xi.(x_i([\xi]) - x_j([\xi])) = \langle (x_i([\xi]) - x_j([\xi])), . \rangle \ne 0$$

Nguyen Viet Dang

we deduce that the set of $[\xi]$ such that the height function $\xi$ is not Morse excellent is a finite union of submanifolds in $\Omega$. It has thus **empty interior** which proves that Morse excellent $\xi$ are open dense.                    $\square$

### 5.3. The distance function to almost every point is Morse

The height function is not the only way to produce Morse functions on $S$. Recall we considered the distance function to $x$, $L_x := y \in S \longmapsto |x - y|$. The set of points $x \in \mathbb{R}^3$ where $L_x^2 \in C^\infty(S)$ fails to be a Morse function is called the set of *focal points*. By the result in [7, Section 11.3 p. 95] and [16, Corollary 6.2 p. 33], the set of *focal points* has null measure in $\mathbb{R}^3$. If $\psi \geqslant 0$ is a non negative Morse function on $S$ with **only positive critical values** then

$$d_x \psi(c) = 0 \implies d_x^2 \sqrt{\psi}(c) = \frac{d_x^2 \psi(c)}{2\sqrt{\psi}} \neq 0.$$

This shows that if $x \notin S$ then $L_x^2$ is a Morse function iff $L_x$ is Morse. Thus:

**Proposition 5.4.** For an *open everywhere dense set* of points $x \in \mathbb{R}^3$, the distance function

$$L_x := y \in S \longmapsto |x - y|$$

is Morse.

### 5.4. The distance function to almost every point is Morse excellent

The next Lemma is needed for the proof of Theorem 1.4.

**Lemma 5.5.** *Let $S$ be a closed compact embedded surface in $\mathbb{R}^3$. For generic $x \in \mathbb{R}^3$, the function $L_x \in C^\infty(S)$ is an excellent Morse function.*

*Proof.* Note that if $x \notin S$ and $L_x^2$ is Morse excellent then so is $L_x$. Therefore, it suffices to prove that $L_x^2$ is Morse excellent for generic $x$. For every $(y, x) \in S \times \mathbb{R}^3$, let use denote by $g_x(y)$ the function $L_x(y)^2$.

For given $x_0$ s.t. $g_{x_0}$ is Morse, we show there is a neighborhood $U$ of $x_0$ such that the critical points of $L_x$ depend smoothly on $x \in U$. Let us call $y_1(x_0), \ldots, y_k(x_0)$ the isolated critical points of $g_{x_0}$. For all $l \in \{1, \ldots, k\}$, $y_l(x_0)$ is a non degenerate critical point of $g_x$ i.e. the Hessian $d_y^2 g_x$ is invertible. Therefore, we can use the implicit function theorem to express the

critical points $(y_i)_{1 \leq i \leq k}$ of $g_x$ as functions of $x \in U$ in such a way that

$$(5.1) \qquad \forall x \in U, \forall i \in \{1, \ldots, k\}, d_y g_x(y_i(x)) = 0.$$

Hence

$L_x^2$ not Morse excellent $\Leftrightarrow g_x(y_i(x)) = g_x(y_j(x))$ for some $1 \leq i < j \leq k$.

Observe that the subset $\Sigma_{ij} = \{x \in U | g_x(y_i(x)) = g_x(y_j(x))\}$ is a surface in $U$ since

$$\partial_x g_x(y_i(x)) - \partial_x g_x(y_j(x)) = 2 \langle ., y_i(x) - y_j(x) \rangle \neq 0,$$

therefore the set of $x$ such that $L_x^2$ fails to be Morse excellent has empty interior in $U$ which proves the claim. $\qquad \square$

## Acknowledgements

## References

[1] A. Aizenbud and V. Drinfeld, *The wave front set of the Fourier transform of algebraic measures*, `arXiv:1212.3630`, (2014).

[2] S. Alinhac, *Hyperbolic Partial Differential Equations*, Universitext, Springer, (2009).

[3] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, American Mathematical Society, third edition, (1971).

[4] C. Brouder, N. V. Dang, and F. Hélein, *Boundedness and continuity of the fundamental operations on distributions having a specified wave front set (with a counter example by Semyon Alesker)*, `arXiv: 1409.7662`, (2014).

[5] C. Brouder, N. V. Dang, and F. Hélein, *A smooth introduction to the wavefront set*, `arXiv:1404.1778`.

[6] C. D.Sogge, *Lectures on Non-Linear Wave equations*, International Press, second edition, (2008).

[7] B. Dubrovin, A. Fomenko, and S. Novikov, *Modern Geometry II– Methods and Applications*, Springer, (1985).

1278 Nguyen Viet Dang

[8] J. J. Duistermaat, *Fourier Integral Operators*, Birkhäuser, Boston, (1996).

[9] G. Eskin, *Lectures on Linear Partial Differential Equations*, Vol. 123 of Graduate Studies in Mathematics, Amer. Math. Soc., Providence, (2011).

[10] M. Farber, *Topology of Closed One-forms*, Vol. 108 of Mathematical surveys and monographs, AMS, (2004).

[11] J. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

[12] A. Gabor, *Remarks on the wave front of a distribution*, Trans. Am. Math. Soc. **170** (1972), 239–44.

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis*, Springer Verlag, Berlin, second edition, (1990).

[14] J.L. Brylinski, A. Dubson, and M. Kashiwara, *Formule de l'indice pour modules holonômes et obstruction d'Euler locale*, C. R. Acad. Sci. Paris Sér. I **293** (1981), 573–576.

[15] M. Kashiwara, *Index theorem for constructible sheaves*, Astérisque **130** (1985), 193–209.

[16] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, (1969).

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II Fourier Analysis, Self-adjointness*, Academic Press, New York, (1975).

[18] S. Bates and A. Weinstein, *Lectures on the Geometry of Quantization*, Vol. 8, AMS, Berkeley mathematics lecture notes, edition (1998).

[19] S. P. Novikov and I. A. Taimanov, *Modern Geometric Structures and Fields*, Vol. 71 of Graduate Studies in Mathematics, AMS, (2006).

[20] E. M. Stein, *Beijing Lectures in Harmonic Analysis*, Annals of Mathematics Studies, Princeton University Press, (1986).

[21] E. M. Stein and R. Shakarchi, *Functional Analysis, An Introduction to Further Topics in Analysis*, Princeton University Press, (2011).

Laboratoire Paul Painlevé (U.M.R. CNRS 8524)
UFR de Mathématiques, Université de Lille 1
59 655 Villeneuve d'Ascq Cédex, France
*E-mail address*: Nguyen-Viet.Dang@math.univ-lille1.fr