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# Renormalization of quantum field theory on Riemannian manifolds

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In this paper, we provide a simple pedagogical proof of the existence of covariant renormalizations in Euclidean perturbative quantum field theory on closed Riemannian manifolds, following the Epstein–Glaser philosophy. We rely on a local method that allows us to extend a distribution defined on an open set $\Omega \subseteq M$ to the whole manifold $M$.

*Keywords*: Renormalization; distributions.

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## 1. Introduction

Renormalization in the Epstein–Glaser sense has played a fundamental role in the construction of perturbative quantum field theories on curved space times. Our aim in this paper is to present a pedagogical and new proof of the existence of covariant renormalization of Euclidean perturbative quantum field theories (pQFT) on closed Riemannian manifolds that is simple, and based on extension of distributions. The advantage of the Riemannian setting is that the propagators are only singular on the diagonals hence we do not need involved methods of microlocal analysis to construct the renormalization. The structure of the paper is first to describe a class of distributions having some moderate growth properties that generalize the example $x^{-1}\Theta(x)$ discussed below and contain the singular Feynman amplitudes encountered in quantum field theory. Then, we construct some analytic tools which allow to extend these distributions as in the above example. We finally use these tools to give a short proof of renormalizability of pQFT on closed Riemannian manifolds in the sense of Epstein–Glaser, extending previous results [36, 37] of the Nikolov and collaborators on flat space. Our approach builds on the works of [5, 6, 14, 17, 19, 20, 38, 39, 44].

### 1.1. *Statement of the main theorem*

#### 1.1.1. *Preliminary definitions*

In Minkowski pQFT, we are interested in making sense of *time-ordered correlation functions* of Wick powers of free fields denoted by

$$\langle \mathcal{T}({:}\phi^{i_1}{:}(x_1)\dots{:}\phi^{i_n}{:}(x_n))\rangle. \tag{1.1}$$

These are objects living on the configuration space $M^n$ that can be expressed formally, using the Feynman rules, as linear combinations of products of the form

$$\prod_{1\le i<j\le n} G(x_i,x_j)^{n_{ij}}, \tag{1.2}$$

where $n_{ij}\in\mathbb{N}$ and $G$ is the Green function, that will be recalled below for the Euclidean case. A product (1.2) is called *Feynman amplitude* and it is depicted pictorially by a graph with $n$ labeled vertices $\{1,\dots,n\}$, where the vertices $i$ and $j$ are connected by $n_{ij}$ unoriented lines. In principle, since the product of distributions is not always well-defined, the previous product (1.2) only makes sense as a formal expression or as a smooth function defined on $M^n$ outside of all the diagonals. In the latter case, the aim of pQFT could be reexpressed as trying to find a distribution extending the mentioned smooth function defined outside of all diagonals and satisfying certain properties to be explained below.

To illustrate the problem of extension of distributions, let us start with a simple example which is discussed in [40, Example 9, p. 140], and actually goes back to Hadamard. Denote by $\Theta$ the Heaviside function (i.e. the indicator function of $\mathbb{R}_{\ge 0}$) and consider the function $x^{-1}\Theta(x)$, viewed as a distribution on $\mathcal{D}'(\mathbb{R}\backslash\{0\})$. The linear map

$$\varphi\mapsto\int_0^\infty dx\frac{\varphi(x)}{x} \tag{1.3}$$

is clearly ill-defined for $\varphi\in\mathcal{D}(\mathbb{R})$ if $\varphi(0)\ne 0$ since the integral $\int_0^\infty dx/x$ diverges. However, the integral $\int_0^\infty dx x^{-1}\varphi(x)$ converges if $\varphi(0)=0$ and an elementary estimate shows that $x^{-1}\Theta(x)$ defines a linear functional on the ideal $x\mathcal{D}(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$ formed by functions vanishing at 0. Note that the following expression

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 dx\frac{(\varphi(x)-\varphi(0))}{x}+\int_1^\infty dx\frac{\varphi(x)}{x} \tag{1.4}$$

converges, for all $\varphi\in\mathcal{D}(\mathbb{R})$. One thus defines an extension of $x^{-1}\Theta(x)$ by

$$x_+^{-1}=\lim_{\varepsilon\to 0}\int_\varepsilon^\infty dx x^{-1}+\log(\varepsilon)\delta, \tag{1.5}$$

where we subtracted the distribution $\log(\varepsilon)\delta$ supported at 0, which becomes singular when $\varepsilon\to 0$, and it is called a *local counterterm*. The distributional extension $x_+^{-1} \in \mathcal{D}'(\mathbb{R})$, called the *Hadamard finite part*, extends the linear functional $x^{-1}\Theta(x) \in (x\mathcal{D}(\mathbb{R}))'$. This example shows the most elementary situation where we can extend a distribution by an *additive renormalization*.

Going back to pQFT, we will work with the Euclidean formulation, i.e. where one uses *Schwinger functions* instead of the time-ordered correlation functions (1.1). In this case we consider a compact Riemannian manifold $(M, g)$ and let $-\Delta_g$ be the corresponding Laplace–Beltrami operator. The Laplace operator has a discrete spectrum $\sigma(-\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty\}$. We will denote by $e_\lambda$ the corresponding eigenfunction to $\lambda \in \sigma(-\Delta_g)$, i.e. solutions of the equation $-\Delta_g e_\lambda = \lambda e_\lambda$. Let us recall the definition of the associated Green function.

**Definition 1.1.** The series

$$\sum_{\lambda\in\sigma(-\Delta_g)\setminus\{0\}} \lambda^{-1} e_\lambda(x) \boxtimes e_\lambda(y) \tag{1.6}$$

converges to a distribution $G$ in $\mathcal{D}'(M \times M)$. Furthermore, $G$ defines a fundamental solution of the Laplace operator $-\Delta_g$, i.e. if $(u, f) \in C^\infty(M)^2$ is a solution of the elliptic equation $\Delta u = f$, then $u(x) = \int_M G(x, y) f(y) dv + k$ for some constant $k$, where $dv$ is the Riemannian volume and $G$ is symmetric with respect to permutation of the variables. We remark that $G$ is a smooth function outside of the diagonal.

### 1.1.2. *Renormalization maps*

In order to encompass all products of the form (1.2), we will consider a slightly more general index set for the variables appearing in them.

**Definition 1.2.** Let $(M, g)$ be a Riemannian manifold. Given any finite subset $I \subseteq \mathbb{N}$ of the positive integers, we denote by $M^I$ the configuration space of points labeled by $I$ and for any subset $J \subseteq I$, $D_J$ is the subset of $M^I$ given by $\{(x_i)_{i\in I} \mid x_j = x_k \text{ for some } (j, k) \in J^2, j \ne k\}$. As usual, if $I = \{1, \ldots, n\}$, we shall denote $M^I$ simply by $M^n$. Define $\mathcal{O}(M^I)$ to be the vector subspace of the space of smooth functions on $M^I \setminus D_I$ generated by

$$\left\{ \prod_{(i<j)\in I^2} G(x_i, x_j)^{n_{ij}} : n_{ij} \in \mathbb{N}_0 \right\}. \tag{1.7}$$

We will now briefly explain the following notation that we will use in this paper. Assume we have a linear map $\mathcal{R} : E \to \mathcal{D}'(M)$, where $E$ is a vector space and $M$ is a smooth manifold. For any open subset $U \subseteq M$, let $i_U : U \hookrightarrow M$ denote the inclusion map. By $\mathcal{R}|_U$, we mean the operator $i_U^* \mathcal{R} : E \to \mathcal{D}'(U)$ obtained as the composition of $\mathcal{R}$ and the pull-back by $i_U$, i.e. taking the restriction of the image of $\mathcal{R}$ to the open subset $U$.

Now, following the recent work [36] by Nikolov, Stora and Todorov, we can give an elegant definition of renormalization scheme as follows.

**Definition 1.3.** Let $(M, g)$ be a Riemannian manifold. A *renormalization scheme* is a sequence of (not necessarily continuous) linear maps $\mathcal{R}_{M^I}[g] : \mathcal{O}(M^I) \to \mathcal{D}'(M^I)$, called *renormalization maps*, indexed by finite subsets $I$ of $\mathbb{N}$ satisfying the following system of *functional equations*:

(i) Given any $t \in \mathcal{O}(M^I)$ and $\varphi \in \mathcal{D}(M^I \backslash D_I)$, then

$$\langle \mathcal{R}_{M^I}[g](t), \varphi\rangle = \langle t, \varphi\rangle. \tag{1.8}$$

This condition expresses the fact that $\mathcal{R}_{M^I}[g](t)$ is a *distributional extension* of $t \in C^\infty(M^I \backslash D_I)$.

(ii) Given any pair of disjoint finite subsets $I', I'' \subseteq \mathbb{N}$ and a Feynman amplitude $G_I = \prod_{i<j\in I^2} G^{n_{ij}}(x_i, x_j)$ of the form given in (1.7) with $I' \sqcup I''$, we have

$$\mathcal{R}_{M^I}(G_I)|_{C_{I',I''}} = \big(\mathcal{R}_{M^{I'}}(G_{I'}) \boxtimes \mathcal{R}_{M^{I''}}(G_{I''})\big) G_{I',I''}|_{C_{I',I''}},$$

where $G_{I'}, G_{I''}$ are defined as $G_I$, $G_{I',I''} = \prod_{(i'<i'')\in I'\times I''} G^{n_{ij}}(x_i, x_j)$ and

$$C_{I',I''} = \big\{(x_i)_{i\in I} \in M^I : x_{i'} \neq x_{i''} \text{ for all } (i', i'') \in I' \times I''\big\}.$$

This equation states that our renormalization map $\mathcal{R}_{M^I}[g]$ factorizes on some regions of the configuration space $M^I$ and translates the fact that renormalization must preserve the locality property.

We are also interested in imposing the following *covariance condition* on the construction of the renormalization scheme with respect to the Riemannian metric $g$. It means that it only depends on the metric and not the chosen coordinates.

(iii) Given any diffeomorphism $\Phi : N \to M$ of closed manifolds, any Riemannian structure $g$ on $M$, any finite set $I \subseteq \mathbb{N}$ and any $t \in \mathcal{O}(M^I)$, then

$$\mathcal{R}_{N^I}[\Phi^* g]\big((\Phi^I)^* t\big) = (\Phi^I)^*\big(\mathcal{R}_{M^I}[g](t)\big), \tag{1.9}$$

where $\Phi^I : N^I \to M^I$ is the diagonal map induced by $\Phi$.

**Remark 1.4.** By choosing a set of isomorphism representatives $\mathcal{M}$ in the groupoid category of closed Riemannian manifolds provided with isometries as morphisms, we see that, in order to satisfy the covariance condition (iii) in the previous definition, it suffices to construct the renormalization maps $\{\mathcal{R}_{M^I}[g]\}_I$ satisfying (i) and (ii) and such that $\mathcal{R}_{M^I}[g]$ is equivariant under the action of the isometry group of $(M, g)$, for each representative $M \in \mathcal{M}$.

#### 1.1.3. *The main result of the paper: Renormalization as a problem of extensions of distributions*

One main ingredient of a Euclidean pQFT on some Riemannian manifold $(M, g)$ is to find some solution $\{\mathcal{R}_{M^I}[g]\}_I$ to the above system of functional equations. The main result of our paper, namely Theorem 6.5, gives the existence of such renormalization maps on a closed Riemannian manifold, based on the nice work [36].

**Theorem 1.5 (Main Theorem).** *Let $(M, g)$ be a smooth compact Riemannian manifold without boundary and $G$ be the Green function of $-\Delta_g$. Then, there exists a solution $\{\mathcal{R}_{M^I}[g]\}_I$ to the system of functional equations of Definition* 1.3 *that is equivariant under the action of the isometry group of $(M, g)$.*

1.1.4. *Comparison to related work*

To our knowledge, one of the first rigorous results on the perturbative renormalization of the $\phi^4$ theory on curved Riemannian manifolds was given by Kopper and Müller (see [27]) and it is based on some implementation of the Wilson–Polchinsky equations to derive the renormalization group flow of the coupling constants. In his book [9] (see also [10]), Costello gives a different approach to the first problem. First, from any action functional of the form $S(\phi) = \int_M \phi\Delta_g\phi + I_{int}(\phi)$, where $\Delta_g$ is the Laplace–Beltrami operator and the interaction part $I_{int}$ is at least cubic in $\phi$, he defines a notion of effective field theory via the effective action

$$\Gamma_\varepsilon(\chi) = \hbar \log\left(\int d\mu_{G_\varepsilon}(\phi)e^{\frac{iS(\phi+\chi)}{\hbar}}\right),$$

where $d\mu_{G_\varepsilon}$ is the Gaussian measure whose covariance is a regularized propagator $G_\varepsilon$ with $G_\varepsilon \to G$ as $\varepsilon \to 0$. He then proves that starting from any local action functional $S$, there is a local action functional $S_\varepsilon^{CT}$ so that the limit

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(\chi) = \hbar \log\left(\int d\mu_{G_\varepsilon}(\phi)e^{\frac{i(S(\phi+\chi)+S_\varepsilon^{CT}(\phi+\chi))}{\hbar}}\right)$$

exists for every power of $\hbar$ (see [9, Theorems 9.3.1 and 10.1.1]). The key point is that $S_\varepsilon^{CT}$ might contain infinitely many counterterms and that the limit can always be defined even for theories that are not renormalizable in the classical sense.

For quantum fields on curved Lorentzian spacetimes, a proof of the renormalizability was first achieved by Brunetti and Fredenhagen in [5], and by Hollands and Wald in [19, 20]. They rely on the Epstein–Glaser approach, which reformulates renormalization as a problem of extension of distributions satisfying physical constraints such as causality. Recently, this method was revisited in the elegant paper [36], which discusses Epstein–Glaser renormalization in flat Minkowski space. Costello's approach is similar to the above methods because they both deal with Feynman amplitudes in position space and make sense for all quantum field theories, even those that are not renormalizable in the classical sense.

Our goal in this paper is to give a simple existence proof of the renormalizability of quantum field theories on arbitrary closed Riemannian manifolds, following the Epstein–Glaser philosophy. It thus gives an alternative approach to the one by Costello. To reach our goal, we will need to revisit some methods in analysis originally developed by Whitney in [47], and which were in turn improved by Malgrange and Łojasiewicz, to compare these techniques with the approach by scaling of Meyer in [33] and the first author in [11]. We will finally apply them to our renormalization problem.

In the mathematical literature, the idea to consider extendible distributions really goes back to Łojasiewicz (see [28]), whereas tempered functions already appear in the work [29, 30] of Malgrange. However, the first general definition of a tempered distribution on any open set $U$ in some manifold $M$ is due to M. Kashiwara: a distribution is tempered if it is extendible to $\overline{U}$ (see [22, Lemma 3.2, p. 332], or also [7]). By our Theorem 4.1, this will in turn imply that these distributions are in $\mathcal{T}_{M\setminus\partial U}$, i.e. they have moderate growth along $\partial U$. The previously mentioned work by Kashiwara was further extended in [16, 25, 26]. On the other hand, tempered functions and distributions were also recently studied in the context of real algebraic geometry in [1, 7] with applications to representation theory. A different approach to the extension problem in terms of scaling was developed by Meyer in his book [33]. His purpose was to study the singular behavior at given points of irregular functions with applications to multifractal analysis (see [23]).

## 2. The General Problem of Extension of Distributions

We recall that, given a smooth manifold $M$, $C^\infty(M)$ (also denoted by $\mathcal{E}(M)$) has a unique structure of Fréchet algebra (see [34, Theorem 14.2]), which can be described as follows. Let $\{K_\ell\}_{\ell\in\mathbb{N}_0}$ be a countable collection of compact subsets of $M$ such that $M = \cup_{\ell\in\mathbb{N}_0} K_\ell^\circ$ and $K_\ell$ is included in a chart $(U_{i_\ell}, \phi_{i_\ell})$ of the atlas of $M$. For $\ell, m \in \mathbb{N}_0$, define

$$p_{\ell,m}(f) = \sup_{x\in\phi_{i_\ell}(K_\ell)} \sup_{\bar{\alpha}\in\mathbb{N}^n_{0,\leq m}} \left| 2^m \frac{\partial^\alpha (f\circ\phi_{i_\ell}^{-1})}{\partial x^\alpha}(x)\right|, \tag{2.1}$$

where $\mathbb{N}^n_{0,\leq m}$ is the subset of $\mathbb{N}^n_0$ formed by the elements $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ such that $|\bar{\alpha}| = \alpha_1 + \cdots + \alpha_n \leq m$, and $f \in C^\infty(M)$. This family of seminorms induces a structure of Fréchet algebra on $C^\infty(M)$ (see [31, IV.4.(2)]). A similar construction tells us that $C^m(M)$ (also denoted by $\mathcal{E}_m(M)$) is a Fréchet algebra, for any $m \in \mathbb{N}_0$. Given a compact subset $K \subseteq M$, we will denote by $\mathcal{D}_K(M)$ the subspace of the LCS $\mathcal{E}(M)$ formed by the smooth functions whose compact support is included in $K$. Let $\mathcal{D}(M)$ be the vector subspace of $\mathcal{E}(M)$ formed by all smooth functions on $M$ of compact support, and its usual locally convex topology, for which $\mathcal{D}(M)$ is an (LF)-space. If $\Omega \subseteq M$ is an open subset, $\mathcal{D}(\Omega)$ will denote the subset of $\mathcal{D}(M)$ formed by the smooth functions whose compact support is included in $\Omega$. Moreover, given a closed subset $K \subseteq M$, we will denote by $\mathcal{D}_K(\Omega)$ the vector subspace of $\mathcal{D}(\Omega)$ formed by the smooth functions whose compact support is included in $K$.

Since we will treat the case of Riemannian manifolds, there is a canonical identification of LCS between $C^\infty(M)$ and the space of 1-densities $\mathrm{Vol}(M)$, by means of the Riemannian density of $M$, and the same is true if the concerned objects have compact support. As a consequence, we can (and will) consider a distribution on a Riemannian manifold $M$ to be a continuous linear functional of $\mathcal{D}(M)$. We will denote them by $\mathcal{D}'(M)$. We refer the reader to [21, Chap. 6], for details. Given a compact subset $K \subseteq M$, we will denote by $\mathcal{D}'_K(M)$ the vector subspace of $\mathcal{D}'(M)$

formed by distributions whose support is included in $K$. The vector subspace of $\mathcal{D}'(M)$ formed by all distributions of compact support is canonically identified with $\mathcal{E}'(M)$. We also remark that the dual spaces considered previously are in principle provided with the strong topology, unless otherwise stated.

### 2.1. *An abstract characterization of the extension problem and a brief summary of the results*

In order to deal with the requirement (i) in Definition 1.3, we first investigate the following problem which has a simple formulation. Let $M$ be a smooth manifold and $\Omega \subseteq M$ be an open subset. A distribution $t \in \mathcal{D}'(\Omega)$ is *extendible* to $M$ if and only if it belongs to the image of the restriction map

$$\mathcal{D}'(M) \to \mathcal{D}'(\Omega). \tag{2.2}$$

As this map is not surjective, the previous extension problem of distributions is tantamount to explicitly determining the image of (2.2), that we are going to denote by $\mathcal{T}(\Omega)$. It is a LCS with subspace topology of that of $\mathcal{D}'(\Omega)$. Since (2.2) is clearly continuous, its kernel $\mathcal{D}'_{M\setminus\Omega}(M)$ is a closed subspace of $\mathcal{D}'(M)$. Moreover, $\mathcal{D}'_{M\setminus\Omega}(M)$ is the space formed by all distributions $t \in \mathcal{D}'(M)$ satisfying that $\mathrm{supp}(t) \subseteq M\setminus\Omega$, so we get a sequence of LCS

$$0 \to \mathcal{D}'_{M\setminus\Omega}(M) \to \mathcal{D}'(M) \to \mathcal{T}(\Omega) \to 0 \tag{2.3}$$

such that the underlying short sequence of vector spaces is exact. By the First Isomorphism theorem, we see that there is a bijective continuous linear map from $\mathcal{D}'(M)/\mathcal{D}'_{M\setminus\Omega}(M)$ onto the subspace $\mathcal{T}(\Omega)$ of $\mathcal{D}'(\Omega)$ formed by the extendible distributions. We remark that the previous map is not in general a topological isomorphism, since the mapping (2.2) is not necessarily closed.

Even though extendible distributions do not form a sheaf (cf. Remark 3.2), they satisfy the following nice property, due to Łojasiewicz in the case $M$ is the Euclidean space (see [28, Sec. 5, Proposition 1, p. 96]), and whose proof applies *verbatim* to this more general situation.

**Lemma 2.1.** *Let $\Omega \subseteq M$ be an open set of a smooth manifold $M$, and let $t \in \mathcal{D}'(\Omega)$ be a distribution. Then, $t$ is extendible to $M$ if and only if there is an open covering $\{\Omega_i\}_{i\in I}$ of $M$ such that $t|_{\Omega_i\cap\Omega}$ is extendible to $M$, for all $i \in I$. One may even assume that $\Omega_i$ is relatively compact, for all $i \in I$.*

We will introduce in Sec. 3.1 a natural growth condition on $t \in \mathcal{D}'(\Omega)$ that measures the singular behavior of $t$ near the boundary $\partial\Omega$ and that addresses the previous issue: if $t$ satisfies the referred growth condition, then there exists a distribution $\bar{t} \in \mathcal{D}'(M)$ such that the restriction of $\bar{t}$ to $\Omega$ coincides with $t$. Moreover, we will explicitly construct in Sec. 4.1 a linear map $\mathcal{P}_\Omega : \mathcal{T}(\Omega) \to \mathcal{D}'(M)$ such that for all $t \in \mathcal{T}(\Omega)$, $\mathcal{P}_\Omega(t)|_\Omega = t$, and eventually give explicit formulas for $\mathcal{P}_\Omega$. We will discuss the different possibilities for extension maps $\mathcal{P}_\Omega$ in case $M = \mathbb{R}^n$ which is the local case.

Our approach in the present paper combines the more traditional one in the mathematical physics literature where one tries to extend a distribution on $M\backslash X$ to $M$, where $X$ is a closed submanifold and where the singularities of the distributions are measured in terms of the scaling degree by means of Euler vector fields (see [12]), and a more general approach where distributions are extended along closed subsets $X$ and the singular behavior is measured by the distance function to $X$. Note that in general the notion of scaling in the transverse directions to $X$ is not even well defined, which is not the case for the notion of moderate growth. Another advantage of the framework presented in our paper is its great flexibility, since we can extend directly Feynman amplitudes on the complement of all the diagonals in the configuration spaces, which thus involve stratified sets and not submanifolds.

### 2.2. *Some ideals associated to the extension of distributions*

#### 2.2.1. *Taylor decomposition*

Let $X \subseteq M$ be a closed subset of $M$ and $m \in \mathbb{N}_0$. Denote by $\mathcal{I}_X^{m+1}(M)$ the closed ideal of $C^m(M)$ formed by the functions satisfying that all their derivatives of order less than or equal to $m$ vanish at any point of $X$. Then we have a short exact sequence of Fréchet spaces

$$0 \to \mathcal{I}_X^{m+1}(M) \overset{\iota_m}{\to} C^m(M) \to \mathcal{E}^m(X) \to 0, \tag{2.4}$$

where $\mathcal{E}^m(X)$ is precisely the Banach space of Whitney jets on $X$ (see [30, Definition 2.3, p. 3]). This short exact sequence has even a splitting of Fréchet spaces (see [30, p. 10] or [2, Theorem 2.3, p. 146]), where we recall that a short exact sequence of Fréchet spaces means that the sequence of underlying vector spaces is exact (see [32, p. 70]). Since (2.4) is an exact sequence of Fréchet spaces, the dual sequence of vector spaces

$$0 \to \mathcal{E}^m(X)' \to C^m(M)' \overset{\iota_m'}{\to} \mathcal{I}_X^{m+1}(M)' \to 0 \tag{2.5}$$

is exact (see [32, Proposition 26.4, p. 308]).

**Definition 2.2.** Let $X$ be a closed subset of $M$. A *Taylor decomposition of* $C^m(M)$ *along* $X$ is a continuous projector $\Pi : C^m(M) \to C^m(M)$ with image $\mathcal{I}_X^{m+1}(M)$. Equivalently, a Taylor decomposition of $C^m(M)$ along $X$ is given by a (continuous) splitting of (2.4).

**Remark 2.3.** The reader can think of the Taylor decomposition of $C^m(M)$ as a way to decompose a $C^m$ function as a sum of a *Taylor remainder* in $\mathcal{I}_X^{m+1}(M)$, which vanishes at order $m$ on $X$, and a *Taylor polynomial*, which is some function in a fixed complement space of $\mathcal{I}_X^{m+1}(M)$ in $C^m(M)$ given by the kernel of $\Pi$. For example, if $X = \{x\}$ is given by a single point in $U \subseteq \mathbb{R}^n$, $\mathcal{E}^m(X)$ is isomorphic to the space $\mathbb{R}_m[X_1, ..., X_n]$ of abstract polynomials of degree less than or equal to $m$ in $n$ variables. In this case, we can choose the projector $\Pi : C^m(U) \to C^m(U)$ such that $\Pi(f)$ is the usual Taylor polynomial of $f$ at $x$ of degree $m$.

We will use the following proposition for classifying the possible extensions of an extendible distribution.

**Proposition 2.4.** *Let* $M = \mathbb{R}^n$ *and* $X \subseteq M$ *be a closed subset. Then, given any* $m \in \mathbb{N}_0$, *there is a canonical bijection between*

(i) *the space of Taylor decompositions of* $C^m(M)$ *along* $X$;
(ii) *the collection of closed subspaces* $B$ *of* $C^m(M)$ *such that* $C^m(M) = \mathcal{I}_X^{m+1}(M) \oplus B$;
(iii) *the space of continuous linear maps* $\mathcal{R}$ *from* $\mathcal{I}_X^{m+1}(M)'$ *to* $\mathcal{E}'_m(M)$ *such that* $\iota'_m \circ \mathcal{R}$ *is the identity map of* $\mathcal{I}_X^{m+1}(M)'$, *where* $\mathcal{I}_X^{m+1}(M)'$ *and* $\mathcal{E}'_m(M)$ *are provided with the weak*$^\star$ *topology.*

*Moreover, any of these spaces is nonempty.*

**Proof.** The equivalence between conditions (i) and (ii) follows directly from the Open mapping theorem for Fréchet spaces (see [32, Theorem 24.30]), whereas the equivalence between conditions (i) and (iii) follows from the Bipolar theorem (see [32, Theorem 22.13]). Finally, the nonemptiness is a consequence of the Whitney extension theorem (see [30, p. 10] or [2, Theorem 2.3, p. 146]). □

A continuous linear map $\mathcal{R}$ from $\mathcal{I}_X^{m+1}(M)'$ to $\mathcal{E}'_m(M)$ such that $\iota'_m \circ \mathcal{R}$ is the identity map of $\mathcal{I}_X^{m+1}(M)'$, where $\mathcal{I}_X^{m+1}(M)'$ and $\mathcal{E}'_m(M)$ are provided with the weak$^\star$ topology, will be called a *renormalization map* of order $m$.

Let $\mathcal{I}_X^\infty(M)$ be the closed ideal of $C^\infty(M)$ formed by all functions whose derivatives of all orders vanish at every point of $X$. This is a nuclear Fréchet space since it is a closed subspace of the nuclear Fréchet space $C^\infty(M)$. We then define the Fréchet space $\mathcal{E}(X)$ as the quotient of $C^\infty(M)$ by $\mathcal{I}_X^\infty(M)$, i.e. we have the short exact sequence of Fréchet spaces

$$0 \to \mathcal{I}_X^\infty(M) \to C^\infty(M) \to \mathcal{E}(X) \to 0. \tag{2.6}$$

One can think of the space $\mathcal{E}(X)$ as some sort of $\infty$-jets in "the transverse directions" to $X$.

### 2.2.2. *An abstract characterization of the extendible distributions of compact support*

We first remark that the strong dual of $\mathcal{E}(X)$ is canonically isomorphic to the closed subspace $(\mathcal{I}_X^\infty(M))^\perp$ of the strong dual of $C^\infty(M)$ given by the continuous functionals that vanish on $\mathcal{I}_X^\infty(M)$ (see [32, Lemma 23.31]). Moreover, $(\mathcal{I}_X^\infty(M))^\perp$ coincides with the subspace $C^\infty(M)'_X$ of $C^\infty(M)'$ given by the distributions with compact support included in $X$ (the inclusion $(\mathcal{I}_X^\infty(M))^\perp \subseteq C^\infty(M)'_X$ is trivial, whereas the other contention follows from [21, Theorem 2.3.3]). Hence, by taking the strong dual of the sequence (2.6) and taking into account the previous comments, we obtain the short sequence of (DNF) spaces (see [7, Appendix A], for a nice short

AQ: We have inserted brackets around 2.6 to show it's an equation. OK?

exposition)

$$0 \to C^\infty(M)'_X \to C^\infty(M)' \to \mathcal{I}^\infty_X(M)' \to 0. \tag{2.7}$$

We remark that the previous short sequence is exact for the underlying structures of vector spaces (see [32, Proposition 26.4]). Hence, by the First Isomorphism theorem, we conclude that there is a bijective continuous linear map from $C^\infty(M)'/C^\infty(M)'_X \simeq (\mathcal{I}^\infty_X(M))'$ onto $(\mathcal{I}^\infty_X(M))'$. Furthermore, since $C^\infty(M)$ is a Fréchet–Schwartz space, [32, Proposition 26.24], implies that this map is a topological isomorphism. If the manifold $M$ is compact, then there is a morphism from the short exact sequence (2.7) of (DNF) to (2.3) such that the first two maps are topological isomorphisms but the third map (from $\mathcal{I}^\infty_X(M)'$ to $\mathcal{T}(\Omega)$) is only bijective and continuous.

**Remark 2.5.** When $X$ is a submanifold of $\mathbb{R}^n$, it is interesting to think of $\mathcal{E}(X)$ as smooth functions *restricted to the formal neighborhood of* $X$. We can think of the *formal neighborhood of* $X$ as the topological dual of $\mathcal{E}(X)$ which is nothing but the space of distributions $\mathcal{E}'_X(\mathbb{R}^n)$ with compact support contained in $X$.

### 2.2.3. *An explicit construction of* $\Pi$ *for diagonals*

The aim of this subsubsection is to explicitly construct a set of renormalization maps that satisfy a certain covariance condition with respect to the choice of the Riemannian metric $g$ on a manifold $M$. Therefore, we are led to construct a projection map $\Pi[M, g]$ in the particular case where the closed subset is the small diagonal $d_n = \{x_1 = \cdots = x_n\}$ of the configuration space $M^n$, for every $n \in \mathbb{N}$, such that $\Pi[M, g]$ is covariant with respect to the Riemannian manifold $(M, g)$, i.e. $\Pi$ naturally induces a functor on the (groupoid) category of closed Riemannian manifolds provided with isometric maps (see (2.12)). Pick a Riemannian metric $g$ on $M$ and consider the $(n-1)$th fiber product $E_n(M) = TM \times_M \cdots \times_M TM \to M$. It is a vector bundle over $M$ whose fiber over $x \in M$ is $(T_xM)^{n-1}$. An element of the bundle $E_n(M)$ will be denoted by $(x; v_2, \ldots, v_n)$ where $x$ lives on the base and $v_2, \ldots, v_n$ are in $T_xM$. Using the metric $g$, for every $x \in M$, we can define an exponential map $\exp_x : U_x \subseteq T_xM \to M$, which is a local diffeomorphism on a neighborhood $U_x$ of $0 \in T_xM$. We thus define a map

$$\mathcal{E}_n : (x, v_2, \ldots, v_n) \in \mathcal{U} \mapsto \big(x, \exp_x(v_2), \ldots, \exp_x(v_n)\big) \in M^n, \tag{2.8}$$

which is a diffeomorphism on some neighborhood $\mathcal{U} \subseteq E_n(M)$ of the zero section.

On the other hand, consider the commutative Lie group $\mathbb{R}_{>0}$ for the usual product of the real numbers, and the action $\sigma$ of $\mathbb{R}_{>0}$ on $E_n(M)$ given by scaling in the fibers, i.e. $\sigma(\lambda, (x; v)) = (x; \lambda v) \in E_n(M)$, where $\lambda \in \mathbb{R}_{>0}$ and $(x; v) \in E_n(M)$. Hence, for every $(x; v) \in E_n(M)$, $\sigma_{(x;v)} : \mathbb{R}_{>0} \to E_n(M)$ is smooth and one defines the vector field $\rho : E_n(M) \to TE_n(M)$, called the *Euler vector field* [12], by

$$\rho(x; v) = \left.\frac{d\sigma_{(x;v)}(\lambda)}{d\lambda}\right|_{\lambda=1}.$$

It is clear that $\rho$ is complete and its global flow $\Phi_\rho$ sends $(t,(x;v)) \in \mathbb{R} \times E_n(M)$ to $\sigma(e^t,(x;v))$. Consider the subalgebra $\mathcal{A}$ of $C^\infty(E_n(M))$ given by all the smooth functions $f$ that are polynomial on the fibers of $E_n(M)$, i.e. $f|_{E_n(M)_x} : E_n(M)_x \to \mathbb{R}$ is a polynomial function, for all $x \in M$. Since the map $\sigma(\lambda,-)$ gives an action of $\mathbb{R}_{>0}$ on $\mathcal{A}$ by automorphisms of algebras via $f \mapsto f \circ \sigma(\lambda^{-1},-)$, it induces an action of the corresponding Lie algebra $\mathbb{R}$ on $\mathcal{A}$ by derivations. In particular, $\rho$ acts by derivations on $\mathcal{A}$. The next lemma shows that this action has spectrum included in $\mathbb{N}_0$, and its spectral decomposition is given by the Taylor expansion.

**Lemma 2.6 (Spectral Projectors).** *There is a decomposition $\mathcal{A} = \oplus_{k\in\mathbb{N}_0}\mathcal{A}_k$ such that $\mathcal{A}_k$ is the eigenspace of $\rho$ associated with the eigenvalue $k \in \mathbb{N}_0$ and a sequence of spectral projectors $\{\Pi_k\}_{k\in\mathbb{N}_0}$, where $\Pi_k : C^\infty(E_n(M)) \to \mathcal{A}_k$ such that, given any $f \in C^\infty(E_n(M))$ and any $N \in \mathbb{N}_0$,*

$$f - \sum_{k=0}^{N} \Pi_k(f) \in \mathcal{I}_{\underline{0}}^{N+1}(E_n(M)),$$

*where $\mathcal{I}_{\underline{0}}^{N+1}(E_n(M))$ is the ideal of functions all of whose derivatives of order less than or equal to $N$ vanish along the zero section $\underline{0} \subseteq E_n(M)$.*

Note that the projectors $\Pi_k$ are algebraic analogues of spectral projectors appearing in [13], where the difference is that the Euler vector field $\rho$ has critical set equal to a submanifold instead of singular points for Morse gradients and the discussion here is only local.

**Proof.** Let $e^{t\rho} = \Phi_\rho(t)$ denote the one parameter group of diffeomorphisms generated by the Euler field $\rho$. For every $k \in \mathbb{N}_0$, we define the projector $\Pi_k$ by

$$\Pi_k(f) = \frac{1}{k!}\left(\frac{d}{d\lambda}\right)^k (e^{-\log(\lambda)\rho*} f)\Big|_{\lambda=1}. \tag{2.9}$$

Observe that by its definition, $\Pi_k$ is global and intrinsic. Also by definition it is clear that $\rho\Pi_k = k\Pi_k$. Now we will consider the action of $\Pi_k$ in some local trivialization of the bundle $E_n(M)$ to prove that the remainder $f - \sum_{k=0}^N \Pi_k(f)$ really vanishes at order $N$ along the zero section of $E_n(M)$. Recall that $E_n(M)$ is an Euclidean bundle whose metric depends only on the metric $g$ since $E_n(M)$ is a fiber product of $(TM, g)$ viewed as an Euclidean bundle. Over some contractible open subset $U$, the bundle $E_n(M)|_U$ admits some orthonormal moving coframe $(h_x^i)_{i=1}^{(n-1)d}$, for $x \in U$. For any chart $\Phi : U \to \Omega \in \mathbb{R}^d$ the map $(x;v) \in E_n(M)|_U \mapsto (\Phi(x), h_x^i(v)) \in \Omega \times \mathbb{R}^{(n-1)d}$ trivializes the bundle over $U$ and $(h^i)_i$ can be thought of as linear coordinates in the fibers. Then the vector field $\rho$ reads $\sum_i h^i\partial_{h^i}$ in this trivialization and the result follows from the usual Taylor expansion in the variables $(h^i)_i$. Hence by some slight notation abuse for $f \in C^\infty(E_n(M)|_U)$, we can write in the above trivialization

$$f(x,h) = \sum_{|\alpha|\le N} \frac{h^\alpha}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial h^\alpha}(x,0) + O\big(|h|^{N+1}\big)$$

and we thus find the explicit formula for the spectral projector

$$\Pi_k = \sum_{|\alpha|=k} \frac{|h^\alpha\rangle}{\alpha!} \langle \partial_h^\alpha \delta_0(h)|, \tag{2.10}$$

where

$$\Pi_k(f) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial h^\alpha}(x, 0) \tag{2.11}$$

is homogeneous of degree $k$ with respect to scaling, i.e. $\rho\Pi_k = k\Pi_k$. Hence, given a contractible open subset $U$ as before, every $f$ compactly supported function $f$ at $x \in U$ has a Taylor expansion

$$f - \sum_{k=0}^{N} \Pi_k(f) \in \mathcal{I}_{\underline{0}}^{N+1}(E_n(M)|_U).$$

The result for $f$ defined on the whole manifold $M$ follows from the fact that $\rho$ is globally defined on $E_n(M)$ and by a classical argument using partitions of unity. □

By the above construction we also obtain the following result.

**Corollary 2.7.** *The projectors $\{\Pi_k\}_{k\in\mathbb{N}_0}$ constructed above only depend on the metric $g$.*

**Proposition 2.8.** *Let $(M, g)$ be a closed Riemannian manifold of dimension $d$, and let $n \in \mathbb{N}$. For every $m \in \mathbb{N}_0$, there is a projector $\Pi_{\leq m}[M, g] : C^\infty(M^n) \to C^\infty(M^n)$ such that $\mathrm{Im}(\Pi_{\leq m}[M, g]) \subseteq \mathcal{I}_{d_n}^{m+1}(M^n)$. Moreover, the construction of $\Pi_{\leq m}[M, g]$ satisfies that*

$$\Phi^*\big(\Pi_{\leq m}[M, g]\varphi\big) = \Pi_{\leq m}[N, g'](\Phi^*\varphi), \tag{2.12}$$

*for every $\varphi \in C^\infty(M^n)$ and every diffeomorphism $\Phi : (M, g) \to (N, g')$, where $\Phi^*\varphi \in C^\infty(N^n)$ is the obviously induced map.*

**Proof.** Assume that the injectivity radius of $M$ is greater than $\rho > 0$ (see [24, Definition 1.4.6]). Let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth function such that $\chi = 1$ if $|t| \leq \rho^2/4$ and $\chi = 0$ if $|t| \geq \rho^2$. We denote by $\delta : M \times M \to \mathbb{R}_{\geq 0}$ the distance function on $M \times M$ induced by the metric $g$, which is smooth on $\delta^{-1}[0, \rho)$. On configuration space $M^n$, set $\delta_n(x_1, \ldots, x_n) = \delta^2(x_1, x_2) + \cdots + \delta^2(x_1, x_n)$. Then, set:

$$\Pi_{\leq m}[M, g](\varphi) = \chi(\delta_n)(\mathcal{E}_n)_* \left(\mathcal{E}_n^*\left(\chi(\delta_n)\varphi\right) - \sum_{k=0}^{m} \Pi_k(\mathcal{E}_n^*\left(\chi(\delta_n)\varphi\right))\right) + \big(1 - \chi^2(\delta_n)\big)\varphi.$$

It only depends on the metric $g$ and the choice of test function $\chi$, but not on the chosen coordinates on $M$ or $M^n$. □

## 3. Distributions of Moderate Growth

### 3.1. *Generalities*

We introduce now one of the main notions of this work.

**Definition 3.1.** Let $M$ be a smooth manifold and let $\Omega \subseteq M$ be an open subset. Set $X = M\backslash\Omega$. Pick any Riemannian metric $g$ on $M$ and let $d$ be the distance function on $M$ induced by $g$. A distribution $t \in \mathcal{D}'(\Omega)$ has *moderate growth* (along $X$) if for every compact set $K$ included in $M$, there are finite seminorms $p_{\ell_1,m_1}, \ldots, p_{\ell_N,m_N}$ and a pair of constants $C, s \in \mathbb{R}_{\geq 0}$ such that

$$|t(\varphi)| \leq C(1 + d(\mathrm{supp}(\varphi), X)^{-s}) \sup_{1\leq i\leq N} p_{\ell_i,m_i}(\varphi), \tag{3.1}$$

for all $\varphi \in \mathcal{D}(\Omega)$ with support included in $K$. We denote by $\mathcal{T}(\Omega)$ the set of distributions in $\mathcal{D}'(\Omega)$ with moderate growth.

**Remark 3.2.** Note that the mapping $\Omega \mapsto \mathcal{T}(\Omega)$ clearly forms a separated presheaf on $M$. We remark however that it is not necessarily a sheaf. Moreover, taking into account that all metrics on $M$ are locally equivalent, we see that $\mathcal{T}(\Omega)$ is in fact independent of the choice of Riemannian metric $g$ of $M$, so $\mathcal{T}(\Omega)$ is well-defined.

On the other hand, assume there is $\bar{t} \in \mathcal{D}'(M)$ and set $t = \bar{t}|_\Omega$. Then, (3.1) is clearly satisfied with $s = 0$, so $t$ is of moderate growth.

The next result follows directly from Leibniz's rule and a standard manipulation of upper bounds.

**Lemma 3.3.** *Let $M$ be a smooth manifold and let $\Omega \subseteq M$ be an open subset. If $t \in \mathcal{D}'(\Omega)$ is a distribution of moderate growth along $M\backslash\Omega$ and $f \in C^\infty(\Omega)$ is a smooth function, then the distribution $ft \in \mathcal{D}'(\Omega)$ also has moderate growth along $M\backslash\Omega$.*

### 3.2. *The local case*

We will consider the following special situation for distributions (of moderate growth). All along this subsection $M \subseteq \mathbb{R}^n$ will denote an open subset, $X \subseteq \mathbb{R}^n$ will be a compact subset included in $M$ and $\Omega = M\backslash X$. Set $\mathcal{I}_X(M)$ to be the subset of $\mathcal{E}(M)$ formed by all smooth functions $\varphi$ satisfying that

$$\mathrm{supp}(\varphi) \cap X = \emptyset. \tag{3.2}$$

Note that $\mathcal{I}_X(M)$ canonically includes $\mathcal{D}(\Omega)$. The aim of this subsection is to provide an equivalent but simpler description of a distribution $t \in \mathcal{D}'(\Omega)$ of moderate growth along $X$ having compact support (see Proposition 3.6). In this case, we define $\|\varphi\|_m^Y = \sup_{x\in Y,|\alpha|\le m} |\partial_x^\alpha \varphi(x)|$, for any subset $Y \subseteq \Omega$ and any smooth function defined on $\Omega$.

We first note that, by precisely the same argument as the one used to prove that the continuous dual of $\mathcal{E}(M)$ coincides with the vector subspace of $\mathcal{D}'(M)$ formed by the distributions of compact support, we have the following result.

**Fact 3.4.** Let $t \in \mathcal{D}'(\Omega)$ be a distribution with $\overline{\mathrm{supp}(t)}$ compact in $M$. Then, $t$ has moderate growth along $X$ if and only if there are finite $(C, s, m) \in \mathbb{R}^2_{\ge 0} \times \mathbb{N}_0$ such that

$$|t(\varphi)| \le C\big(1 + d\big(\mathrm{supp}(\varphi), X\big)^{-s}\big)\|\varphi\|_m^\Omega, \tag{3.3}$$

for all $\varphi \in \mathcal{I}_X(M)$.

Given $m \in \mathbb{N}_0 \cup \{\infty\}$, we recall that $\mathcal{I}_X^{m+1}(M)$ is the closed ideal of $C^m(M)$ formed by all functions whose derivatives of order (strictly) less than $m+1$ vanish at every point of $X$. It has the subspace topology of $C^m(M)$.

We will need the following technical result.

**Lemma 3.5.** *Let $Y \subseteq \mathbb{R}^n$ be a compact subset and let $(d, m) \in \mathbb{N}_0^2$ be two nonnegative integers. Then, there is a family of functions $\chi_\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R}_{\ge 0})$ parametrized by $\lambda \in (0, 1]$ satisfying that $\chi_\lambda = 1$ if $d(x, Y) \le \lambda/8$, $\chi_\lambda = 0$ if $d(x, Y) \ge \lambda$, and such that there exists a constant $\tilde{C} \ge 0$ satisfying that*

$$\|\chi_\lambda \varphi\|_m^K \le \tilde{C}\lambda^d \|\varphi\|_{m+d}^{K\cap\{d(x,Y)\le\lambda\}}, \tag{3.4}$$

*for all $K \subseteq \mathbb{R}^n$ compact, $\lambda \in (0, 1]$ and $\varphi \in \mathcal{I}_Y^{m+d+1}(\mathbb{R}^n)$, where the constant $\tilde{C}$ does not depend on $\varphi$ nor $\lambda$.*

**Proof.** Choose $\phi \in C^\infty(\mathbb{R}^n, \mathbb{R}_{\ge 0})$ such that $\int_{\mathbb{R}^n} \phi = 1$, and $\phi = 0$ if $|x| \ge 3/8$. Then, set $\phi_\lambda(x) = \lambda^{-n}\phi(\lambda^{-1}x)$, for all $x \in \mathbb{R}^n$, and let $\alpha_\lambda$ be the characteristic function of the set

$$\left\{x \in \mathbb{R}^n \,|\, d(x, Y) \le \frac{\lambda}{2}\right\}.$$

Define $\chi_\lambda$ to be the convolution product $\phi_\lambda * \alpha_\lambda$. Hence $\chi_\lambda(x) = 1$ if $d(x, Y) \le \lambda/8$, and it equals 0 if $d(x, Y) \ge \lambda$. By Leibniz's rule one has

$$\partial^\alpha(\chi_\lambda\varphi)(x) = \sum_{|k|\le|\alpha|} \binom{\alpha}{k} \partial^k\chi_\lambda(x)\partial^{\alpha-k}\varphi(x),$$

for every $\alpha$ such that $|\alpha| \le m$. It suffices to estimate each term $\partial^k\chi_\lambda\partial^{\alpha-k}\varphi(x)$ of the above sum, where $|k| \le |\alpha|$ and $x \in K$. For any such multi-index $k$, there is $C_k > 0$ such that $|\partial^k\chi_\lambda(x)| \le C_k/\lambda^{|k|}$ for all $x \in \mathbb{R}^n\backslash Y$, and $\mathrm{supp}(\partial^k\chi_\lambda) \subseteq \{x \in \mathbb{R}^n \,|\, d(x, Y) \le \lambda\}$. Therefore, for all $\varphi \in \mathcal{I}_Y^{m+d+1}(\mathbb{R}^n)$, $x \in \mathrm{supp}(\partial^k\chi_\lambda\partial^{\alpha-k}\varphi)$, and

$y \in Y$ such that $d(x, Y) = |x - y|$, we find that $\partial^{\alpha-k}\varphi \in \mathcal{I}^{|k|+d+1}$ since it vanishes at $y$ with order at least $|k| + d$. As a consequence,

$$\partial^{\alpha-k}\varphi(x) = \sum_{|\beta|=|k|+d} (x-y)^{\beta} R_{\beta}(x),$$

where the right-hand side is just the integral remainder in Taylor's expansion of $\partial^{\alpha-k}\varphi$ around $y$. It only depends on the jet of $\varphi$ of order less than or equal to $m + d$. Hence,

$$\big|\partial^k \chi_\lambda \partial^{\alpha-k}\varphi(x)\big| \leq \frac{C_k}{\lambda^{|k|}} \sum_{|\beta|=|k|+d+1} \big|(x-y)^{\beta} R_{\beta}(x)\big|.$$

Since $R_\beta$ only depends on the jet of $\varphi$ of order less than or equal to $m + d$, we see that

$$\big|\partial^k \chi_\lambda \partial^{\alpha-k}\varphi(x)\big| \leq C_k \lambda^d \sup_{\substack{x\in K,\\ d(x,Y)\leq\lambda}} \sum_{|\beta|=|k|+d} \big|R_{\beta}(x)\big|,$$

for all $x \in K$, and the conclusion easily follows. □

We provide now the main result of this subsection.

**Proposition 3.6.** *Let $t \in \mathcal{D}'(\Omega)$ be a distribution having compact support (included in $\Omega$). Then, $t$ has moderate growth along $X$ if and only if there are constants $C \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{N}_0$ such that*

$$\big|t(\varphi)\big| \leq C\|\varphi\|_m^{\Omega}, \tag{3.5}$$

*for all $\varphi \in \mathcal{I}_X(M)$.*

**Proof.** By Fact 3.4, $t$ has moderate growth along $X$ if and only if there exists $(C, s, m) \in \mathbb{R}^2_{\geq 0} \times \mathbb{N}_0$ such that

$$\big|t(\varphi)\big| \leq C\big(1 + d\big(\mathrm{supp}(\varphi), X\big)^{-s}\big)\|\varphi\|_m^{\Omega}, \tag{3.6}$$

for all $\varphi \in \mathcal{I}_X(M)$.

If $s = 0$, then there is nothing to prove. It remains to treat the case $s > 0$, which we suppose from now on. Since $t$ has compact support, consider a smooth function $f$ of compact support such that $f(x) = 1$ for all $x$ in a neighborhood of $\mathrm{supp}(t)$. As $t(f\varphi) = t(\varphi)$, we may (and will) assume that $\varphi$ has compact support. Our idea is to absorb the divergence in (3.6) by a dyadic decomposition, as follows. Let $\{\chi_\lambda\}_{\lambda\in(0,1]}$ be the family of maps constructed in Lemma 3.5 for $Y = X$. Given any $\varphi \in \mathcal{D}(\Omega) \cap \mathcal{I}_X(M)$, there exists $N \in \mathbb{N}$ such that $\chi_{2^{-N}}\varphi = 0$. In consequence, $t(\varphi) = t((1 - \chi_{2^{-N}})\varphi)$, and, in particular,

$$t(\varphi) = \sum_{j=0}^{N-1} t\big((\chi_{2^{-j}} - \chi_{2^{-j-1}})\varphi\big) + t\big((1 - \chi_1)\varphi\big).$$

We easily estimate $t((1-\chi_1)\varphi)$ by $|t((1-\chi_1)\varphi)| \leq C\|\varphi\|_m^{\Omega}$, for all $\varphi \in C^\infty(\mathbb{R}^n)$, and for some constant $C$, since the support of $1-\chi_1$ does not meet $X$. Choose $d \in \mathbb{N}$ such that $d-s>0$. Then,

$$\begin{aligned}\big|t(\chi_1\varphi)\big| &\leq \sum_{j=0}^{N-1} \big|t\big((\chi_{2^{-j}}-\chi_{2^{-j-1}})\varphi\big)\big| \\ &\leq C\sum_{j=0}^{N-1}\big(1+d\big(\mathrm{supp}\big(\varphi(\chi_{2^{-j}}-\chi_{2^{-j-1}})\big),X\big)^{-s}\big)\|(\chi_{2^{-j}}-\chi_{2^{-j-1}})\varphi\|_m^{\Omega}, \\ &\leq C\sum_{j=0}^{N-1}(1+2^{s(j+4)})(2^{-jd}+2^{-(j+1)d})\tilde{C}\|\varphi\|_{m+d}^{\Omega} \leq C'\|\varphi\|_{m+d}^{\Omega},\end{aligned}$$

where we have used the moderate growth property on the second inequality and Lemma 3.5 in the third, and

$$C' = \tilde{C}C(1+2^{-d})\sum_{j=0}^{\infty}2^{-jd}(1+2^{(j+4)s}) < +\infty,$$

which is a convergent series, since $d-s>0$, and it is independent of $N$ and $\varphi$. Hence, we have proved that there exists $C' \in \mathbb{R}_{\geq 0}$ and $m'$ such that

$$\big|t(\varphi)\big| \leq C'\|\varphi\|_{m'}^{\Omega}$$

for all $\varphi \in \mathcal{D}(\Omega)$, where $m' = m+d$ and $d$ is any integer such that $d>s$. The proposition is thus proved. □

We will also need the following result.

**Lemma 3.7.** *Assume $M=\mathbb{R}^n$. Let $t \in \mathcal{D}'(\Omega)$ be a distribution having compact support (included in $\Omega$). If $t$ has moderate growth along $X$, then there is a nonnegative integer $m \in \mathbb{N}_0$ such that $t$ has a unique continuous extension $t_m \in (\mathcal{I}_X^{m+1}(M))'$ given by*

$$t_m(\varphi) = \lim_{\lambda\to 0}\lim_{\varepsilon\to 0} t\big((1-\chi_\lambda)\phi_\varepsilon * \varphi\big), \tag{3.7}$$

*where $\varphi \in \mathcal{I}_X^{m+1}(M)$, $\{\chi_\lambda\}_{\lambda\in(0,1]}$ is the family of cut-off functions defined in Lemma 3.5, and $\phi_\varepsilon$ is any mollifier. Furthermore, if $\varphi \in \mathcal{I}_X^{m+1}(M)\cap\mathcal{E}(M)$, then*

$$t_m(\varphi) = \lim_{\lambda\to 0} t\big((1-\chi_\lambda)\varphi\big). \tag{3.8}$$

**Proof.** Let $m \in \mathbb{N}_0$ be the nonnegative integer given by Proposition 3.6. It suffices to prove that $\mathcal{I}_X^{m+1}(M)$ is the closure in $\mathcal{E}^m(M)$ of the space $\mathcal{I}_X(M)$ of smooth functions whose support does not meet $X$. Let $\phi_\varepsilon$ be a smooth mollifier. By a classical regularization argument, we have $\lim_{\varepsilon\to 0}(1-\chi_\lambda)\phi_\varepsilon * \varphi = (1-\chi_\lambda)\varphi$ in

$\mathcal{E}^m(M)$, for all $\varphi \in \mathcal{E}^m(M)$. Moreover, $\lim_{\lambda\to 0}(1-\chi_\lambda)\varphi \to \varphi$ in $\mathcal{I}_X^{m+1}(M)$. Indeed, by Lemma 3.5 (see [30, p. 11]), we have

$$\|\chi_\lambda\varphi\|_m^K \leq \tilde{C}\|\varphi\|_m^{K\cap\left\{d(x,\Omega^c)\leq\lambda\right\}} \to 0,$$

for all $\varphi \in \mathcal{I}_X^{m+1}(M)$ and all compact subsets $K \subseteq \mathbb{R}^n$, when $\lambda \to 0$. Hence, $\varphi = \lim_{\lambda\to 0}(1-\chi_\lambda)\varphi$ with respect to the topology induced by that of $\mathcal{E}^m(M)$. This proves the claim. □

**Remark 3.8.** Taking into account that any distribution of compact support in an open subset $M$ of $\mathbb{R}^n$ can be canonically regarded as a distribution of compact support in the whole space by an extension by zero, it is clear that Fact 3.4 and Proposition 3.6 also hold if one replaces $\varphi \in \mathcal{I}_X(M)$ by $\varphi \in \mathcal{I}_X(\mathbb{R}^n)$. Analogously, (3.8) of Lemma 3.7 also holds if one replaces $\varphi \in \mathcal{I}_X^{m+1}(\mathbb{R}^n)$ by $\varphi \in \mathcal{I}_X^{m+1}(M)$.

## 4. The Main Result: Extendible Distributions have Moderate Growth

### 4.1. *The statement*

We will now present the first main result of this paper, mentioned in Sec. 2.1.

**Theorem 4.1.** *Let $M$ be a smooth manifold and $\Omega$ be an open subset of $M$. Set $X = M\backslash\Omega$. Then, the following are equivalent*:

(i) $t \in \mathcal{D}'(\Omega)$ *is extendible to* $M$;
(ii) $t \in \mathcal{D}'(\Omega)$ *has moderate growth*;
(iii) *there is a family of smooth functions* $\{\beta_\lambda\}_{\lambda\in(0,1]} \in C^\infty(M)^{(0,1]}$ *and a family of neighborhood $U_\lambda$ of $X$ in $M$ such that*

   (a) $(\beta_\lambda)|_{U_\lambda} \equiv 0$, *for all* $\lambda \in (0,1]$;
   (b) $\lim_{\lambda\to 0}\beta_\lambda(x) = 1$, *for all* $x \in \Omega$;

   *and a family of distributions* $\{c_\lambda\}_{\lambda\in(0,1]} \in \mathcal{D}'(M)^{(0,1]}$ *with support in $X$ such that the limit*

$$\lim_{\lambda\to 0}(t\beta_\lambda - c_\lambda) \tag{4.1}$$

   *exists in $\mathcal{D}'(M)$ and defines an extension of $t$, where we remark that $t\beta_\lambda$ is naturally regarded as a distribution in $\mathcal{D}'(M)$ by* (a).

**Proof.** It clear that (iii) implies (i), and (i) implies (ii) by Remark 3.2. It only remains to prove that (ii) implies (iii). This will be done in Sec. 4.2. □

Our moderate growth condition is weaker than the hypothesis of [22, Lemma 3.3]. Theorem 4.1 can also be viewed as a generalization of [33, Theorem 2.1, p. 48], and [5, Theorem 5.2, p. 645], which only treat the extension problem in the case of a point. Condition (iii) in the above theorem is a generalization of Hadamard's definition of finite parts of distributions. This is beautifully explained in Meyer's book [33] (see p. 45), and it also linked with the appearance of local counterterms in the renormalization of Feynman amplitudes in pQFT. After proving this theorem, we will use it in the proof of Theorem 5.3, which states that the product of distributions in $\mathcal{D}'(M)$ with functions which are tempered in $\Omega$ (see Definition 5.1 for the algebra $\mathcal{M}(\Omega)$ of tempered functions) is renormalizable. This also implies that the space of extendible distributions (or, equivalently, of distributions in $\mathcal{T}(\Omega)$) is a module over $\mathcal{M}(\Omega)$ (see Theorem 5.4).

**Remark 4.2.** Note that the map sending $t \in \mathcal{T}(\Omega)$ to $\bar{t} \in \mathcal{D}'(M)$ given by (4.1) is linear. We will denote it by $P_\Omega$. Let $G$ be any compact group acting on $M$ such that the action preserves $\Omega$ and $M\backslash\Omega$. As, a consequence, the short exact sequence (2.3) is of $G$-modules. By using the standard Weyl's unitarian trick (see [46, §5]), we also obtain a $G$-equivariant section $P_\Omega^G : \mathcal{T}(\Omega) \to \mathcal{D}'(M)$ of $\mathcal{D}'(M) \to \mathcal{T}(\Omega)$. Indeed, setting $P_\Omega^G = (\int_G g \cdot P_\Omega dg)/(\int_G dg)$, where $dg$ is an invariant Haar measure on $G$, we obtain the purported $G$-equivariant section.

### 4.2. *Proof of Theorem* 4.1

We will first prove a restricted version of Theorem 4.1, given by taking the manifold $M$ to be an open subset of $\mathbb{R}^n$.

**Proposition 4.3.** *Let $M$ be an open subset of $\mathbb{R}^n$, which is regarded as a manifold, and let $t \in \mathcal{D}'(\Omega)$ be a distribution of compact support. Then, statements* (i)–(iii) *in Theorem* 4.1 *are equivalent.*

**Proof.** As explained in the proof of Theorem 4.1, the only nontrivial implication is (ii) $\Rightarrow$ (iii). Since any distribution of compact support in an open subset of $\mathbb{R}^n$ can be canonically extended by zero to a distribution of compact support in $\mathbb{R}^n$, we will assume without loss of generality that $M = \mathbb{R}^n$. Let $m \in \mathbb{N}_0$ be the nonnegative integer given by Proposition 3.6, $\{\chi_\lambda\}_{\lambda\in(0,1]}$ be the family of smooth functions considered in Lemma 3.7 for $Y = X$, and $\phi_\epsilon$ be a mollifier. Set $\beta_\lambda = 1 - \chi_\lambda$. Note that $\beta_\lambda$ satisfies the conditions stated in (iii) of Theorem 4.1. By Lemma 3.7, $t$ has a unique continuous extension $t_m \in \mathcal{I}_X^{m+1}(M)'$ given by

$$t_m(\varphi) = \lim_{\lambda\to 0} \lim_{\varepsilon\to 0} t((1 - \chi_\lambda)\phi_\varepsilon * \varphi), \tag{4.2}$$

where $\varphi \in \mathcal{I}_X^{m+1}(M)$.

As recalled in Proposition 2.4, the short exact sequence (2.4) has a continuous splitting, so there is a continuous retraction $I_m : C^m(\mathbb{R}^n) \to \mathcal{I}_X^{m+1}(\mathbb{R}^n)$ of the inclusion $\mathcal{I}_X^{m+1}(\mathbb{R}^n) \to C^m(\mathbb{R}^n)$. Set $B = \operatorname{Ker}(I_m)$ and $P_m : C^m(\mathbb{R}^n) \to B$ be the continuous linear map given by $P_m = \mathrm{id}_{C^m(\mathbb{R}^n)} - I_m$. For any $\varphi \in C^m(\mathbb{R}^n)$, we now define

$$\begin{aligned}\bar{t}_m(\varphi) &= \lim_{\lambda\to 0}\lim_{\varepsilon\to 0} t\big((1-\chi_\lambda)\phi_\varepsilon * I_m(\varphi)\big)\\ &= \lim_{\lambda\to 0}\lim_{\varepsilon\to 0} t\big((1-\chi_\lambda)\phi_\varepsilon * \varphi\big) - \lim_{\lambda\to 0}\lim_{\varepsilon\to 0} t\big((1-\chi_\lambda)\phi_\varepsilon * P_m(\varphi)\big). \qquad (4.3)\end{aligned}$$

Set

$$c_\lambda(\varphi) = \lim_{\varepsilon\to 0} t\big((1-\chi_\lambda)\phi_\varepsilon * P_m(\varphi)\big),$$

for all $\varphi \in C^m(\mathbb{R}^n)$. This defines a family of distributions $\{c_\lambda\}_{\lambda\in(0,1]}$ of compact support included in $X$. It is now clear that (4.3) is tantamount to (4.1), and the proposition follows. □

**Proof of Theorem 4.1 from Proposition 4.3.** Choose a locally finite cover of $M$ by relatively compact open charts $\{(U_i,\phi_i)\}_{i\in I}$ and a subordinated smooth partition of unity $\{\varphi_i\}_{i\in I}$, where $K_i = \mathrm{supp}(\varphi_i)$ is a compact subset of $U_i$. Define $V_i = \phi_i(U_i)$ and $Y_i = \phi_i(X\cap K_i)$. Then $V_i$ is an open subset of $\mathbb{R}^n$, $Y_i$ is a compact subset of $V_i$, and $t_i = (\phi_i)_*(t\varphi_i) \in \mathcal{D}'(V_i\backslash Y_i)$ is a distribution of moderate growth along $Y_i$. By Proposition 4.3, for each $i\in I$, there exists a family of smooth functions $\{\beta_{i,\lambda}\}_{\lambda\in(0,1]} \in C^\infty(V_i)^{(0,1]}$ and a family of neighborhood $U_{i,\lambda}$ of $Y_i$ in $V_i$ such that

(a) $(\beta_{i,\lambda})|_{U_{i,\lambda}} \equiv 0$, for all $\lambda\in(0,1]$;
(b) $\lim_{\lambda\to 0}\beta_{i,\lambda}(x) = 1$, for all $x\in V_i$;

and a family of distributions $\{c_{i,\lambda}\}_{\lambda\in(0,1]} \in \mathcal{D}'(V_i)^{(0,1]}$ with support in $Y_i$ such that the limit

$$\lim_{\lambda\to 0}\big(t_i\beta_{i,\lambda} - c_{i,\lambda}\big) \qquad (4.4)$$

exists in $\mathcal{D}'(V_i)$ and defines an extension $\bar{t}_i$ of $t_i$. Define

$$\beta_\lambda = \sum_{i\in I}\varphi_i(\beta_{i,\lambda}\circ\phi_i) \in C^\infty(M)$$

and

$$c_\lambda = \sum_{i\in I}(\phi_i^{-1})_*(c_{i,\lambda}) \in \mathcal{E}'(M).$$

We recall that the last sum is well defined for it is locally finite and each summand is a distribution of compact support, so it is canonically extended by zero to a distribution of compact support in $M$. Moreover, the support of $c_\lambda$ is included in $X$, for each summand satisfies that condition. Then, (iii) is satisfied, and the theorem is proved. □

**Remark 4.4.** The divergences of the first term in the third member of (4.3) come from the fact that $\varphi \notin \mathcal{I}_X^{m+1}(\mathbb{R}^n)$. However, these divergences are local in the sense they can be subtracted by the counterterm given by the last term of (4.3), which becomes singular when $\lambda \to 0$, and only depend on the restriction to $X$ of the $m$-jets of $\varphi$. Indeed, the fact that $\varphi$ vanishes near $X$ implies that, if $\varphi \in \mathcal{I}_X^{m+1}(\mathbb{R}^n)$, then $P_m\varphi = 0$. We remark that the family of distributions $\{c_\lambda\}_\lambda$ are exactly the counterterms that appear in the renormalization procedure in QFT.

### 4.3. *The ambiguity group*

Define the *ambiguity group* $G_m$ of order $m \in \mathbb{N}_0$ as the collection of linear, continuous, bijective maps from $C^m(\mathbb{R}^n)$ to itself preserving $\mathcal{I}_X^{m+1}(\mathbb{R}^n)$. Note that $g \in G_m$ implies $g^{-1}$ is continuous by the Open mapping theorem, so $G_m$ is a group. Let $\mathcal{R}$ be the renormalization map corresponding to a retraction $I_m : C^m(\mathbb{R}^n) \to \mathcal{I}_X^{m+1}(\mathbb{R}^n)$ of the inclusion $\mathcal{I}_X^{m+1}(\mathbb{R}^n) \to C^m(\mathbb{R}^n)$. In other words, $\mathcal{R}$ is the continuous dual of $I_m$. The group $G_m$ naturally acts on the space of renormalization maps. Indeed, given $g \in G_m$, $t \in \mathcal{I}_X^{m+1}(\mathbb{R}^n)'$ and $\varphi \in C^m(\mathbb{R}^n)$, define $(g.\mathcal{R})(t)(\varphi) = \mathcal{R}(t)(g(\varphi)) = t(I_m \circ g(\varphi))$.

## 5. Renormalized Products

### 5.1. *Generalities*

As explained in the introduction, in pQFT we need to renormalize products of Green functions. Therefore we usually need to control the behavior of products of distributions with smooth functions that are singular along some closed sets.

**Definition 5.1.** Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. A function $f \in C^\infty(\Omega)$ is said to be *tempered* if for every compact $K \subseteq \mathbb{R}^n$ and every $m \in \mathbb{N}_0$, there exist $C$ and $s$ in $\mathbb{R}_{\geq 0}$ such that

$$\sup_{|\alpha|\leq m} \big|\partial^\alpha f(x)\big| \leq C\big(1 + d(x, \Omega^c)^{-s}\big), \tag{5.1}$$

for all $x \in K \cap \Omega$. The set of all tempered functions on $\Omega$ will be denoted by $\mathcal{M}(\Omega, \mathbb{R}^n) \subseteq C^\infty(\Omega)$.

Note that tempered functions form a subalgebra of $C^\infty(\Omega)$ by Leibniz's rule. It is immediate that this definition can be generalized to any open subset $\Omega$ of a smooth manifold $M$.

**Proposition 5.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. Consider $t \in \mathcal{T}(\Omega)$ and $f \in C^\infty(\Omega)$ satisfying the following conditions*:

(a) *there exists $(C, s_1) \in \mathbb{R}^2_{\geq 0}$ such that*

$$\big|t(\varphi)\big| \leq C(1 + d(\mathrm{supp}(\varphi), \Omega^c)^{-s_1})\|\varphi\|_m^K,$$

*for all $\varphi \in \mathcal{D}(\Omega)$*;

(b) *there exists $(C_m, s_2) \in \mathbb{R}^2_{\geq 0}$ such that*

$$\sup_{|\alpha|\leq m} \big|\partial^\alpha f(x)\big| \leq C_m\big(1 + d(x, \Omega^c)^{-s_2}\big),$$

*for all $x \in K \cap \Omega$.*

*Then, there is $C' > 0$ such that*

$$\left|ft(\varphi)\right| \leq C'(1 + d(\mathrm{supp}(\varphi), \Omega^c)^{-(s_1+s_2)})\|\varphi\|_m^K, \tag{5.2}$$

*for all $\varphi \in \mathcal{D}(\Omega)$.*

**Proof.** The claim follows from the inequalities

$$\begin{aligned}\left|ft(\varphi)\right| &\leq C(1 + d(\mathrm{supp}(\varphi), \Omega^c)^{-s_1})\|f\varphi\|_m^K \\ &\leq CC_m 2^{mn}(1 + d(\mathrm{supp}(\varphi), X)^{-s_1})(1 + d(\mathrm{supp}(\varphi), \Omega^c)^{-s_2})\|\varphi\|_m^K \\ &\leq \underbrace{4CC_m 2^{mn}}_{C'}(1 + d(\mathrm{supp}(\varphi), \Omega^c)^{-(s_1+s_2)})\|\varphi\|_m^K,\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$. □

**Theorem 5.3.** *Let $M$ be a manifold and $\Omega \subseteq M$ be an open subset. For all $f \in \mathcal{M}(\Omega)$ and all $t \in \mathcal{D}'(M)$, there exists a distribution $\mathcal{R}(ft) \in \mathcal{D}'(M)$ which coincides with the regular product $ft$ in $\Omega$.*

**Proof.** By a classical argument on partitions of unity (as the one used in the proof of Theorem 4.1), we may reduce to the case where $\Omega$ is an open subset of a relatively compact open set $M \subseteq \mathbb{R}^n$. Moreover, we may even assume that $f \in \mathcal{M}(\Omega)$ and $t \in \mathcal{D}'(\Omega)$ is a distribution of compact support included in $\Omega$, so it canonically extends to $t \in \mathcal{E}'(\mathbb{R}^n)$. By Proposition 4.3, it suffices to prove that $ft$ has moderate growth, which is a consequence of the previous proposition. □

**Example.** Our result shares some similarities with [33, Theorems 4.2 and 4.3, pp. 83–85], where Meyer renormalizes the product of distributions $S_\gamma t$ at a point $x_0 \in \mathbb{R}^n$, where $S_\gamma(x) = \mathrm{fp}\,|x - x_0|^\gamma$ is the Hadamard finite part of $|x - x_0|^\gamma$, $t$ is some kind of weakly homogeneous distribution of degree $s$ at $x_0$ and $s + \gamma \in \mathbb{R}\backslash\{-n - m : m \in \mathbb{N}_0\}$. He shows that the renormalized product $S_\gamma t$ is locally weakly homogeneous of degree $s + \gamma$ at $x_0$.

Proposition 4.3 gives the following direct consequence of Theorem 5.3.

**Corollary 5.4.** *$\mathcal{T}(\Omega)$ is a $\mathcal{M}(\Omega)$-module.*

This was also proved by Malgrange (see [29, Proposition 1, p. 4]).

### 5.2. *Gluing properties*

The following property plays a central role in our approach to renormalization *à la* Epstein–Glaser and it allows to avoid the use of partitions of unity.

**Definition 5.5.** Let $X$ and $Y$ be two closed sets of an open set $U$ of the Euclidean space $\mathbb{R}^n$. They are said to be *regularly situated* (*in* $U$) if given any $x_0 \in X \cap Y$ there exist a neighborhood $W$ of $x_0$ and constants $C > 0$ and $m \in \mathbb{N}$ such that

$$d(x, X) + d(x, Y) \geq Cd(x, X \cap Y)^m, \tag{5.3}$$

for all $x \in W$.

More generally, two closed sets of a manifold $M$ of dimension $n$ are called *regularly situated* if there is an atlas $\{(U_i, \phi_i)\}_{i \in I}$ of $M$ such that $\overline{\phi_i(X \cap U_i)}$ and $\overline{\phi_i(Y \cap U_i)}$ are regularly situated in $\mathbb{R}^n$, for all $i \in I$.

Finally, we will say that a finite family $\{V_j\}_{j \in J}$ of open sets of a manifold $M$ are *regularly good* if for all nonempty subsets $J', J'' \subseteq J$ such that $J' \cap J'' = \emptyset$, $\partial(\cup_{j \in J'} V_j) \cup \partial(\cup_{j \in J' \cup J''} V_j)$ and $\partial(\cup_{j \in J''} V_j) \cup \partial(\cup_{j \in J' \cup J''} V_j)$ are regularly situated.

The following result is due to Łojasiewicz in the case of bounded open sets in the Euclidean space (see [28, Sec. 5, Proposition 6, p. 98]).

**Proposition 5.6.** *Let $U$ and $V$ be two regularly good open subsets of a manifold $M$ of dimension $n$, i.e. such that $X = \partial U \cup \partial(U \cup V)$ and $Y = \partial V \cup \partial(U \cup V)$ are regularly situated. Then the short sequence of vector spaces*

$$0 \to \mathcal{T}(U \cup V) \xrightarrow{\iota} \mathcal{T}(U) \oplus \mathcal{T}(V) \xrightarrow{p} \mathcal{T}(U \cap V) \to 0$$

*is exact, where $\iota(u) = (u|_U, u|_V)$ and $p(v, w) = v|_{U \cap V} - w|_{U \cap V}$, for all $u \in \mathcal{T}(U \cup V)$, $v \in \mathcal{T}(U)$ and $w \in \mathcal{T}(V)$.*

**Proof.** Let $(U_i, \phi_i)_{i \in I}$ be a locally finite atlas of $M$ such that $\overline{\phi_i(X \cap U_i)}$ and $\overline{\phi_i(Y \cap U_i)}$ are regularly situated in $\mathbb{R}^n$, for all $i \in I$. By Lemma 2.1, it suffices to show that $t|_{U_i} \in \mathcal{T}((U \cup V) \cap U_i)$, for all $i \in I$. Hence, by replacing $U$ by $\phi_i(U \cap U_i)$, $V$ by $\phi_i(V \cap U_i)$ and $t|_{U_i}$ by $\phi_i^*(t|_{U_i})$, we might assume that $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $t$ is a distribution on an open set of $\mathbb{R}^n$ including $U$ and $V$. The definition of $X$ and $Y$ being regularly situated is clearly equivalent to the definition that $\partial U$ and $\partial V$ are regularly separated by $\partial(U \cup V)$ (for the definition, see [28, Sec. 3, p. 91]). By [28, Sec. 5, Proposition 6, p. 98], $t \in \mathcal{T}(U \cup V)$, and the proposition follows. □

We will now recall a result showing that the regularly situated hypothesis is fairly general. For the definition of semianalytic and subanalytic sets of a real analytic manifold, we refer the reader to [3, Definitions 2.1 and 3.1], respectively. We only remark that any semianalytic set is clearly subanalytic, any finite intersection and finite union of a subanalytic sets is again subanalytic, as well as the complement and the closure of any subanalytic set.

The local version of the next result, where $M$ is an open subset of $\mathbb{R}^n$, can be found in [3, Corollary 6.7]. The general version follows from observing that Definition 5.5 is of local nature.

**Proposition 5.7.** *Let $M$ be an analytic manifold, and let $X$ and $Y$ be two closed subanalytic subsets of $M$. Then, $X$ and $Y$ are regularly situated.*

## 6. Renormalization of Feynman Amplitudes in Euclidean Quantum Field Theories: The Proof of Theorem 1.5

### 6.1. *Feynman amplitudes are tempered*

We will give in this section the main application of our extension techniques: the proof of Theorem 1.5. Our approach to renormalization follows the philosophy of Brunetti and Fredenhagen in [4–6], and Nikolov, Stora and Todorov in [36], which goes back to the papers [14, 15]. It is essentially based on the concept of extension of distributions. However, we will use the nice formalism of *renormalization maps* of Nikolov (see [36, 37]) which is closest in spirit to the present paper. In what follows, we will always assume that $(M, g)$ is a smooth $d$-dimensional Riemannian manifold with Riemannian metric $g$. We denote by $\Delta_g$ the Laplace–Beltrami operator corresponding to $g$, and we consider the Green function $G \in \mathcal{D}'(M \times M)$ of the operator $\Delta_g + m^2$, for $m \in \mathbb{R}_{\geq 0}$. $G$ is the Schwartz kernel of the operator inverse of $\Delta_g + m^2$ (see [43, Appendix 1]), which always exists when $M$ is compact and $m^2 \notin \text{Spec}(\Delta_g)$. In the noncompact case, the existence and uniqueness for the Green function usually depends on the global properties of $\Delta_g$ and $(M, g)$. For instance, if $(M, g)$ has *bounded geometry* in the sense of [8, p. 33], and [41] (see also [43, Definition 1.1, Appendix 1], and [42, Definition 1.1, p. 3]), then under some conditions of spectral theoretic nature on $\Delta_g + m^2$ (see [43, Appendix 1]), the operator inverse $(\Delta_g + m^2)^{-1} : L^p(M) \to L^p(M)$ exists for $p \in (1, +\infty)$, and its Schwartz kernel is $G$.

In any case, assuming that $G$ exists, we have the following well-known result about the asymptotics of $G$ near the diagonal.

**Lemma 6.1.** *Let $(M, g)$ be a smooth Riemannian manifold and $\Delta_g$ the corresponding Laplace operator. If $G \in \mathcal{D}'(M \times M)$ is the fundamental solution of $\Delta_g + m^2$, then $G$ is tempered in $M^2 \backslash D_2$, where $D_2 \subseteq M \times M$ denotes the diagonal.*

**Proof.** This follows from the estimate in [45, Proposition 2.2(2.5)], applied to the Green function $G$, which is the Schwartz kernel of an elliptic pseudodifferential operator of degree $-2$, for $G$ is a parametrix of the Laplace–Beltrami operator $\Delta_g + m^2$. □

### 6.2. *Basic definitions on configuration spaces*

We recall that for every finite subset $I \subseteq \mathbb{N}$ and any open subset $U \subseteq M$, we define the configuration space $U^I = \{(x_i)_{i \in I} \,|\, x_i \in U, \ \forall\, i \in I\}$ of $|I|$ particles in $U$ labeled by the subset $I \subseteq \mathbb{N}$. In the sequel, we will distinguish two types of diagonals in $U^I$: the *big diagonal* $D_I = \{(x_i)_{i \in I} \,|\, \exists\, (i \neq j) \in I^2, x_i = x_j\}$, which represents configurations where at least two particles collide, and the *small diagonal* $d_I = \{(x_i)_{i \in I} \,|\, \forall\, (i, j) \in I^2, x_i = x_j\}$, where all particles in $U^I$ collapse over the same element. For every pair of elements $i, j \in I$ such that $i \neq j$, set $d^I_{\{i,j\}}$ to be the subset $\{x_i = x_j\}$ of the configuration space $M^I$. For simplicity, the configuration space

$M^{\{1,\dots,n\}}$ and the corresponding *big* and *small* diagonals $D_{\{1,\dots,n\}}$ and $d_{\{1,\dots,n\}}$, as well as the set $d^{\{1,\dots,n\}}_{\{i,j\}}$ will be denoted by $M^n$, $D_n$, $d_n$, and $d^n_{\{i,j\}}$, respectively. For any finite subset $I \subseteq \mathbb{N}$, a *Feynman amplitude* will denote any element of the form $\prod_{i<j\in I^2} G(x_i,x_j)^{n_{ij}} \in C^\infty(M^I\backslash D_I), n_{ij}\in\mathbb{N}_0$.

### 6.3. *The vector subspace $\mathcal{O}(D_I, .)$ generated by Feynman amplitudes*

As explained in Sec. 1.1.2, in QFT, the extension of Feynman amplitudes to the whole configuration space should satisfy some consistency conditions in order to be compatible with the fundamental requirement of locality.

Recall that for any open subset $\Omega \subseteq M^I$, we denote by $\mathcal{M}(\Omega\backslash D_I)$ the algebra of tempered functions in $\Omega\backslash D_I$. We introduce the vector space $\mathcal{O}(D_I,\Omega) \subseteq C^\infty(\Omega\backslash D_I)$ generated by the Feynman amplitudes, i.e.

$$\mathcal{O}(D_I,\Omega) = \left\langle \left\{ \prod_{i<j\in I^2} G(x_i,x_j)^{n_{ij}} : n_{ij}\in\mathbb{N}_0 \right\} \right\rangle. \tag{6.1}$$

By Lemma 6.1, $\mathcal{O}(D_I,\Omega) \subseteq \mathcal{M}(\Omega\backslash D_I)$.

### 6.4. *Axioms for renormalization maps: Factorization property as a consequence of locality*

We will now present a slightly different but equivalent form of the notion of renormalization maps given in Definition 1.3(i) and (ii). We remark that these axioms are simplified versions of those appearing in [36, Sec. 5, pp. 33–35].

**Definition 6.2.** A collection of linear maps $\{\mathcal{R}_{\Omega\subseteq M^I}\}_{\Omega,I} : \mathcal{O}(D_I,\Omega) \to \mathcal{D}'(\Omega)$, where $I$ runs over the finite subsets of $\mathbb{N}$ and $\Omega$ runs over the open subsets of $M^I$, is called a *renormalization scheme* if the following conditions are satisfied.

(i) For any finite set $I \subseteq \mathbb{N}$ and any open set $\Omega \subseteq M^I$, $\mathcal{R}_{\Omega\subseteq M^I}(t)|_{\Omega\backslash D_I} = t$ for all $t \in \mathcal{O}(D_I,\Omega)$;

(ii) For every pair of open subsets $\Omega_1 \subseteq \Omega_2 \subseteq M^I$, we require that
$$\langle \mathcal{R}_{\Omega_2\subseteq M^I}(f), \varphi\rangle = \langle \mathcal{R}_{\Omega_1\subseteq M^I}(f), \varphi\rangle,$$
for all $f \in \mathcal{O}(D_I,\Omega_2)$ and $\varphi \in \mathcal{D}(\Omega_1)$;

(iii) The renormalization maps satisfy the *factorization property*, given as follows. Given any pair of disjoint finite subsets $I', I'' \subseteq \mathbb{N}$, and open set $\Omega \subseteq M^I$ and a Feynman amplitude $G_I = \prod_{i<j\in I^2} G^{n_{ij}}(x_i,x_j) \in \mathcal{O}(D_I,\Omega)$ with $I' \sqcup I''$, we have
$$\mathcal{R}_\Omega(G_I)|_{\Omega_{I',I''}} = \big(\mathcal{R}_{M^{I'}}(G_{I'}) \boxtimes \mathcal{R}_{M^{I''}}(G_{I''})\big) G_{I',I''}|_{\Omega_{I',I''}},$$
where $G_{I'}, G_{I''}$ are defined as $G_I$, $G_{I',I''} = \prod_{(i'<i'')\in I'\times I''} G^{n_{ij}}(x_i,x_j)$ and
$$\Omega_{I',I''} = \big\{(x_i)_{i\in I} \in \Omega : x_{i'} \neq x_{i''},\ \text{for all } (i',i'') \in I'\times I''\big\}.$$

The most important condition is the factorization property (iii), which is imposed in [36, Eq. (2.2), p. 5]. We recall that, as usual, the renormalization map $\mathcal{R}_{\Omega\subseteq M^I}$ with $\Omega = M^I$ is typically denoted just by $\mathcal{R}_{M^I}$.

### 6.5. *The main idea on how to define Renormalization maps*

In order to define $\mathcal{R}$ on $M^I$, for every Feynman amplitude $t \in \mathcal{O}(D_I, M^I)$, it suffices to define $\mathcal{R}_{\Omega_i\subseteq M^I}$ for a finite open cover $\{\Omega_i\}_i$ of $M^I\backslash D_I$ satisfying that the open sets $\{\Omega_i\}_i$ are regularly situated and such the maps $\mathcal{R}_{\Omega_i\subseteq M^I}$ coincide on the overlaps $\Omega_i \cap \Omega_j$ and each $\mathcal{R}_{\Omega_i\subseteq M^I}(t)$ has moderate growth in $\mathcal{T}(\Omega_i)$. Indeed, by the gluing property for distributions with moderate growth given in Proposition 5.6, the various sections $\{\mathcal{R}_{\Omega_i\subseteq M^I}(t)\}_i$ glue together to define an element $\mathcal{R}_{M^I\backslash D_I}(t) \in \mathcal{T}(M^I\backslash D_I)$.

### 6.6. *Covering lemma*

We now state a key result in the sequel. Its first part is due to Popineau and Stora (see [36, Lemma 2.2, p. 6] and also [39, 44]).

**Lemma 6.3.** *Let $M$ be a smooth manifold of dimension $d$. For any nonempty subset $I \subsetneq \{1,\dots,n\}$, let $C_I = \{(x_1,\dots,x_n)\,|\,\forall i \in I, \forall j \notin I, x_i \neq x_j\} \subseteq M^n$. Note that $C_I$ is the complement of $\cup_{i\in I,j\notin I} d^n_{\{i,j\}}$ in $M^n$. Then,*

$$\bigcup_I C_I = M^n\backslash d_n, \tag{6.2}$$

*where $I$ runs over all nonempty strict subsets of $\{1,\dots,n\}$. Moreover, the family $\{C_I\}_I$ is regularly good.*

**Proof.** Note first that, if $(x_1,\dots,x_n) \notin d_n$, then at least two points $x_i$ and $x_j$ differ for $(i,j) \in \{1,\dots,n\}^2$. In consequence, $(x_1,\dots,x_n) \in C_I$, for $I = \{j \in \{1,\dots,n\} : x_j = x_i\}$, which in turn implies that (6.2) holds.

We will now prove that the finite collection of open subsets $\{C_I\}_I$ is regularly good, i.e. given $\{I_{j'} : j \in J'\}$ and $\{I_{j''} : j \in J''\}$ be two nonempty and disjoint families of nonempty strict subsets of $\{1,\dots,n\}$, $X = \partial(\cup_{j'\in J'} C_{I_{j'}}) \cup \partial(\cup_{j\in J'\cup J''} C_{I_j})$ and $Y = \partial(\cup_{j''\in J''} C_{I_{j''}}) \cup \partial(\cup_{j\in J'\cup J''} C_{I_j})$ are regularly situated. By [18, Proposition 8], the smooth manifold $M$ admits a compatible analytic structure, which then induces an analytic structure on the cartesian power $M^n$ of $M$. Furthermore, any diagonal $d^n_{\{i,j\}}$ inside $M^n$ is a closed real analytic subset, which in turn implies that $C_I$ is a semianalytic set of $M^n$, so *a fortiori* subanalytic. By the preservation of the subanalyticity property under finite unions, finite intersections, complements and closures, we conclude that $X$ and $Y$ are also subanalytic, so regularly situated, by Proposition 5.7. The statement is thus proved. □

### 6.7. *Recursive property of the renormalization maps*

The following result is proved in [36, Lemmas 2.2 and 2.3, p. 6].

If $t = \prod_{1\leq i<j\leq n} G(x_i,x_j)^{n_{ij}}$ is a Feynman amplitude and $I \subsetneq \{1,\ldots,n\}$ is a nonempty subset, we introduce the following elements

$$t_I = \prod_{\substack{i,j\in I\\ i<j}} G(x_i,x_j)^{n_{ij}}, \quad t_{I^c} = \prod_{\substack{i,j\in I^c\\ i<j}} G(x_i,x_j)^{n_{ij}},$$
$$t_{I,I^c} = \prod_{(i,j)\in I\times I^c} G(x_i,x_j)^{n_{ij}}. \tag{6.3}$$

**Lemma 6.4.** *Let $n \in \mathbb{N}$ and let $\{\mathcal{R}_{\Omega\subseteq M^I}\}_{\Omega,I}$ be a collection of renormalization maps defined for all $I \subseteq \mathbb{N}$ such that $|I| < n$ and satisfying the axioms of Definition* 6.2. *Consider the open cover $\{C_I\}_I$ defined in Lemma* 6.3 *and a Feynman amplitude $t = \prod_{1\leq i<j\leq n} G(x_i,x_j)^{n_{ij}}$. Then, given two nonempty subsets $I, J \subsetneq \{1,\ldots,n\}$, we have the identity*

$$\mathcal{R}_{M^I}(t_I)\mathcal{R}_{M^{I^c}}(t_{I^c})t_{I,I^c}|_{C_I\cap C_J} = \mathcal{R}_{M^J}(t_J)\mathcal{R}_{M^{J^c}}(t_{J^c})t_{J,J^c}|_{C_I\cap C_J} \tag{6.4}$$

*on the open set $C_I \cap C_J$, which in turn implies that*

$$\mathcal{R}_{C_I\subseteq M^n\setminus d_n}|_{C_I\cap C_J} = \mathcal{R}_{C_J\subseteq M^n\setminus d_n}|_{C_I\cap C_J}. \tag{6.5}$$

*As a consequence, the renormalization map $\mathcal{R}_{M^n\setminus d_n\subseteq M^n}$ exists and it is uniquely determined by the renormalizations maps $\mathcal{R}_{M^I}$ for all $|I| < n$.*

**Proof.** See [36, pp. 6–7], for a detailed proof. □

The previous result clearly generalizes to any subset $L$ of $\mathbb{N}$ having $n$ elements, but we have stated it for the case $L = \{1,\ldots,n\}$ for simplicity. Note also that the above Lemma does not ascertain the existence of the renormalization map $\mathcal{R}_{M^n}$.

### 6.8. *The existence theorem for renormalization maps*: *The proof of Theorem* 1.5

We finally provide the following short proof of the existence of renormalization maps on general closed Riemannian manifolds.

**Theorem 6.5.** *Let $(M,g)$ be a closed Riemannian manifold, $\Delta_g$ be the corresponding Laplace operator, and $G$ be the Green function of $\Delta_g + m^2$, where $m \geq 0$. We recall that for any configuration space $M^I$, where $I$ is a finite subset of $\mathbb{N}$, and any open subset $\Omega \subseteq M^I$, $\mathcal{O}(D_I,\Omega) \subseteq \mathcal{M}(D_I,\Omega)$ is the vector space generated by the Feynman amplitudes $\prod_{i<j\in I^2} G(x_i,x_j)^{n_{ij}}$, $n_{ij} \in \mathbb{N}_0$.*

*Then, there exists a collection of renormalization maps $\{\mathcal{R}_{\Omega\subseteq M^I}\}_{\Omega,I}$, where $I$ runs over the finite subsets of $\mathbb{N}$ and $\Omega$ runs over the open subsets of $M^I$ which satisfies the three axioms of Definition* 6.2. *They can even be constructed so that they satisfy the covariance condition* (iii) *in Sec.* 1.1.2.

**Proof.** We proceed by induction on the number $n \in \mathbb{N}$ of elements of the configuration space. Now assume that all renormalization maps $\{\mathcal{R}_{\Omega\subseteq M^I}\}_{\Omega,I}$ for $|I| \leq n-1$ are constructed and satisfy the list of axioms of Definition 6.2, as well as the covariance condition. It suffices to show that $\mathcal{R}_{\Omega\subseteq M^I}$ exists for all finite subsets $I \subseteq \mathbb{N}$ satisfying that $|I| = n$ and all open subsets $\Omega \subseteq M^I$, and it fulfills the covariance condition. By Definition 6.2(ii), it suffices to prove the previous statement for $\mathcal{R}_{M^I}$ and all finite subsets $I \subseteq \mathbb{N}$ satisfying that $|I| = n$. For simplicity, we will only deal with the case $\mathcal{R}_{M^n}$, but the same argument holds in general.

For $n = 2$, the renormalization map $\mathcal{R}_{M^2} : \mathcal{O}(D_2, M^2) \to \mathcal{D}'(M^2)$ exists since propagators are tempered along diagonals by Lemma 6.1 and their powers can be renormalized by Theorem 5.3. For $n > 2$ and any generic Feynman amplitude $t = \prod_{1\leq i<j\leq n} G(x_i, x_j)^{n_{ij}} \in \mathcal{O}(D_n, M^n)$, Lemmas 6.3 and 6.4 tell us that $\mathcal{R}_{M^n\setminus d_n}(\prod_{1\leq i<j\leq n} G(x_i, x_j)^{n_{ij}})$ exists and it is unique. Recall that we can write

$$\mathcal{R}_{C_I}(t) = \underbrace{\mathcal{R}_{M^I}(t_I)}_{\in\mathcal{D}'(M^I)}\underbrace{\mathcal{R}_{M^{I^c}}(t_{I^c})}_{\in\mathcal{D}'(M^{I^c})}\underbrace{t_{I,I^c}}_{\in\mathcal{M}(\partial C_I, M^n)}, \tag{6.6}$$

where we use the notation of (6.3). The product $\mathcal{R}_{M^I}(t_I)\mathcal{R}_{M^{I^c}}(t_{I^c})$ belongs to $\mathcal{D}'(M^n)$ and the product $t_{I,I^c} = \prod_{(i,j)\in I\times I^c} G^{n_{ij}}(x_i, x_j)$ is tempered in $C_I$. It follows from Theorem 5.3 that the distribution

$$\mathcal{R}_{C_I}\left(\prod_{1\leq i<j\leq n} G(x_i, x_j)^{n_{ij}}\right) = \underbrace{\prod_{(i,j)\in I\times I^c} G(x_i, x_j)^{n_{ij}}}_{\in\mathcal{M}(\partial C_I, M^n)}\underbrace{\mathcal{R}_{M^I}(G_I)\mathcal{R}_{M^{I^c}}(G_{I^c})}_{\in\mathcal{D}'(M^n)}$$

in $\mathcal{D}'(C_I)$ has moderate growth in $C_I$, so for every $C_I$, $\mathcal{R}_{M^n\setminus d_n}(t)|_{C_I} \in \mathcal{T}(C_I)$. Since the open sets $C_I$ are regularly good by Lemma 6.4, Proposition 5.6 tells us that $\mathcal{R}_{M^n\setminus d_n}(t) \in \mathcal{T}(\cup C_I) = \mathcal{T}(M^n\setminus d_n)$, so $\mathcal{R}_{M^n\setminus d_n}(t)$ is extendible. Note that, for $n = 2$, $\mathcal{R}_{M^n\setminus d_n}$ clearly satisfies the covariance axiom, for the Feynman amplitudes clearly do. Moreover, for $n > 2$, the inductive hypothesis and the explicit expression (6.6) of $\mathcal{R}_{M^n\setminus d_n}$ in terms of the renormalization maps $\{\mathcal{R}_{\Omega\subseteq M^I}\}_{\Omega,I}$ for $|I| \leq n-1$ imply that $\mathcal{R}_{M^n\setminus d_n}$ also satisfies the covariance axiom.

We now set $\mathcal{R}_{M^n}(t)$ to be any extension of $\mathcal{R}_{M^n\setminus d_n}(t)$ that is equivariant with respect to the action of the group of isometries of $(M, g)$. Indeed, since the isometry group $\mathrm{Iso}(M, g)$ of any closed Riemannian manifold is compact ($\mathrm{Iso}(M, g)$ is a Lie group by [35, Theorem 9], whereas the Arzelà–Ascoli theorem shows that it is compact if $M$ is so), Remark 4.2 tells us that $\mathcal{R}_{M^n}(t) = \mathcal{P}^{\mathrm{Iso}(M,g)}_{M^n\setminus d_n}(\mathcal{R}_{M^n\setminus d_n}(t))$ does the job. Alternatively, the existence of such $\mathcal{R}_{M^n}(t)$ also follows from Proposition 2.8. In any case, since the extension $\mathcal{R}_{M^n}t$ of $\mathcal{R}_{M^n\setminus d_n}t$ is compatible with the action of the group of isometries of $(M, g)$, the former also satisfies the covariance axiom, as explained in Remark 1.4. The theorem is thus proved. □

An important remark is that the sequence of renormalization maps constructed in the above proof is not unique and has infinitely many degrees of freedom at each step of the induction since we can choose many possible extensions for the distribution $\mathcal{R}_{M^n \setminus d_n}(\prod_{1\leq i<j\leq n} G(x_i, x_j)^{n_{ij}})$ and these are precisely controlled by the ambiguity group considered in Sec. 4.3. Moreover, they are related to the renormalization ambiguities which are encountered in renormalization of pQFT on curved spacetimes.

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