SPECTRAL ANALYSIS OF MORSE-SMALE GRADIENT FLOWS

NGUYEN VIET DANG AND GABRIEL RIVIÈRE

Abstract. On a smooth, compact and oriented manifold without boundary, we give a complete description of the correlation function of a Morse-Smale gradient flow satisfying a certain nonresonance assumption. This is done by analyzing precisely the spectrum of the generator of such a flow acting on certain anisotropic spaces of currents. In particular, we prove that this dynamical spectrum is given by linear combinations with integer coefficients of the Lyapunov exponents at the critical points of the Morse function. Via this spectral analysis and in analogy with Hodge-de Rham theory, we give an interpretation of the Morse complex as the image of the de Rham complex under the spectral projector on the kernel of the generator of the flow. This allows us to recover classical results from differential topology such as the Morse inequalities and Poincaré duality.

1. Introduction

Consider a smooth ($\mathcal{C}^\infty$) flow $(\varphi^t)_{t\in\mathbb{R}}$ acting on a smooth, compact, oriented manifold M which has no boundary and which is of dimension $n \geq 1$. A natural question to raise is whether the limit

$$\lim_{t\to+\infty} \varphi^{-t*}(\psi)$$

exists for any smooth function ψ defined on M. This is of course very unlikely to happen in general, and a natural setting where one may expect some convergence is the class of dynamical systems with hyperbolic behaviour and for a nice enough reference measure. For instance, if φ^t is a topologically transitive Anosov flow [1] and if we study the weak limit with respect to a so-called Gibbs measure, it is known from the works of Bowen, Ruelle and Sinai that such a limit exists and is equal to the average of ψ with respect to the Gibbs measure[1] [67, 13]. If one is able to show that this equilibrium state exists, a second natural question to raise is: can one describe the fluctuations? For instance, what is the rate of convergence to this state?

These problems are naturally related to the study of the operator generating the flow:

$$\mathcal{L} : \psi \in \mathcal{C}^\infty(M) \mapsto -\frac{d}{dt}\left(\varphi^{-t*}(\psi)\right)|_{t=0} \in \mathcal{C}^\infty(M).$$

Note that, by duality, this operator acts on the space of distributions $\mathcal{D}'(M)$. In recent years, many progresses have been made in the study of such operators acting on suitable Banach spaces of distributions when the flow φ^t enjoys the Anosov property. In [55], Liverani defined Banach spaces of distributions with "anisotropic Hölder regularity" for

[1]Recall that a well-known example is the Liouville measure for the geodesic flow on a negatively curved manifold.

which he could make a precise spectral analysis of $\mathcal{L}$. For instance, he proved the existence of a *spectral gap for contact Anosov flows* from which one can deduce that, for every $t \geq 0$ and for every ψ_1, ψ_2 in $\mathcal{C}^\infty(M)$,

$$C_{\psi_1,\psi_2}(t) := \int_M \varphi^{-t*}(\psi_1)\psi_2 d\text{vol}_g = \int_M \psi_1 d\text{vol}_g \int_M \psi_2 d\text{vol}_g + \mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda t}), \tag{1}$$

where $\Lambda > 0$ is some fixed positive constant related to the spectral gap of $\mathcal{L}$ and where vol_g is the Riemannian volume. His construction was inspired by similar results for diffeomorphisms [11] and by a proof of Dolgopyat which holds in the 2-dimensional case [25]. Introducing Banach spaces inside $\mathcal{D}'(M)$ contrasts with earlier approaches to these questions where symbolic coding of Anosov flows was used to describe the weak convergence of $\varphi^{-t*}(\psi)$. For more general Anosov flows, Butterley and Liverani also showed how this direct approach allows to make a meromorphic extension for the Laplace transform of the correlation function $C_{\psi_1,\psi_2}(t)$ to the entire half plane [19]. This extended earlier works of Pollicott [62] and Ruelle [64] which were also based on the use of symbolic dynamics. Such poles (and their corresponding eigenstates) describe in some sense the fine structure of the long time dynamics and are often called *Pollicott-Ruelle resonances*. Pushing further this direct approach [40], Giulietti, Liverani and Pollicott extended this spectral analysis to anisotropic spaces of currents and they proved that, for any smooth Anosov flow, the Ruelle zeta function has a meromorphic extension to $\mathbb{C}$. In the case of Anosov geodesic flows satisfying certain pinching assumptions, they also showed that (1) also holds for the Bowen-Margulis measure (and not only with respect to the Riemannian volume).

In parallel to this approach via spaces of anisotropic Hölder distributions, it was observed that the spectral analysis of Anosov flows can in fact be understood as a semiclassical problem which fits naturally in the theory of semiclassical resonances [48, 29]. Building on earlier works for Anosov diffeomorphisms by Baladi-Tsujii [6, 7] and Faure-Roy-Sjöstrand [31] involving microlocal tools, this kind of approach to Pollicott-Ruelle resonances was initiated for Anosov flows by Tsujii [73, 74], Faure-Sjöstrand [32] and Faure-Tsujii [34, 33]. Thanks to this microlocal approach, several new results or alternative proofs were obtained in the last few years: upper bounds on the counting function in the high frequency limit near the imaginary axis [32, 23], explicit bounds on the size of the spectral gap [73, 74, 61, 33], meromorphic extension of the Ruelle zeta function [28], band structure of the resonances [34, 33, 26]. We refer to the survey article of Gouëzel [42] for a recent account on these progresses.

Regarding the important steps made in the Anosov case, it is natural to understand to what extent these methods can be adapted to more general dynamical systems satisfying weaker chaotic features. A natural extension to consider is the class of hyperbolic flows with discontinuities. Results in this direction were obtained recently by Baladi-Liverani [5] and Baladi-Demers-Liverani [4]. For instance, they proved exponential decay of correlations for the 2-dimensional finite horizon Sinai billiard flow [4]. We also refer to [71] for results on open billiard flows and to [24] for billiard maps. Another natural direction is to consider the case of open systems which was recently studied by Dyatlov and Guillarmou via microlocal techniques [27] – see also [2, 35] for similar problems concerning expanding maps. Their framework is in fact related to the so-called Axiom A flows [70, 13]. In particular, they show that Pollicott-Ruelle resonances can be defined *locally* on a small neighborhood of any basic set of a given Axiom A flow with no critical points – see also [64, 6, 7, 41] for earlier results in the case of Axiom A diffeomorphisms.

1.1. **Morse-Smale gradient flows and a question from Bowen's notebook.** Given a Morse function f and a Riemannian metric g, one can define a gradient flow φ_f^t and hence give the simplest example of an Axiom A flow in the sense of Smale [70, p. 803]. Such dynamical systems of course do not satisfy the Anosov property and hyperbolicity is in some sense concentrated at the critical points of the Morse function. In his notebook [14, Problem 1], Bowen raised the following natural question concerning these flows : *To what extent does the gradient flow near a critical point depend on the metric ?*

In the present article, we aim at extending the microlocal approach of Faure-Sjöstrand-Tsujii for this kind of dynamical systems. As a consequence, it will allow us to give some insights on that question as we will obtain a rather complete description of the asymptotics of the *global* "correlation function" for generic gradient flows. More precisely, we can in some sense track the dependence on the metric in the terms of our asymptotic expansion. For instance, our analysis will among other things show that all the decay rates will be given by integer linear combinations of the Lyapunov exponents which depend only on the "germ" of the metric at the critical points of f – see equation (3) below.

Near the critical points of the Morse function, this kind of dynamical system has in fact a very simple structure and it behaves like the gradient flow induced by a hyperbolic matrix acting on $\mathbb{R}^n$ thanks to the Grobman-Hartman Theorem [44]. We note that hyperbolic linear models on $\mathbb{R}^n$ were already analyzed via microlocal techniques by Faure and Tsujii in [34, Ch. 3-4] where it also served as a local picture for their global model. Here we will make use of similar ideas and one of the main task of the present article will be to understand how these local models can be patched together in our dynamical framework in order to give a *global description of the dynamics*. As we shall see, it will allow us to give a rather precise description of the spectrum of $\mathcal{L}$ and of the long time dynamics of the flow φ_f^t acting on smooth differential forms of any degree.

1.2. **From dynamics to differential topology.** Recall that gradient flows associated with a Morse function are also interesting because of their deep connections with differential topology which first appeared in the pioneering works of Thom [72] and Smale [68, 69]. Our microlocal approach to the properties of gradient flows will allow among other things to give new (spectral) interpretations of some results of Laudenbach [53, 54] and Harvey-Lawson [45, 46] and to recover some classical facts from differential topology such as the finiteness of the Betti numbers, the Morse inequalities and Poincaré duality. The works of Witten [76], Helffer-Sjöstrand [47], Bismut-Zhang [10, 77], Burghelea et al. [17, 16, 18] developped the relationship between Morse theory, Hodge theory and Ray-Singer analytic torsion. For an introduction to this spectral approach to Morse theory, we refer the reader to the survey of Henniart [49]. To compare with our results, recall that Witten's approach applies for instance to the operator $d_s = d + si_V$ where $s > 0$, d is the usual coboundary operator and i_V is the contraction operator induced by the vector field V of φ^t. More precisely, he studied the *elliptic* Hamiltonian $(d_s+d_s^*)^2$ [76, Sect. 3] and gave, in the limit $s\to+\infty$, a topological interpretation of its kernel in terms of the zero set of V. Here, instead of $(d_s+d_s^*)^2$, we will study the operator $s\mathcal{L}=d_s^2$ in the specific case of Morse-Smale vector fields. We will show that, even if $\mathcal{L}$ is not elliptic, its spectrum on convenient Sobolev spaces has a very nice structure which also carries some topological informations on the manifold. For a more detailed exposition on these relations between dynamical systems, topology and spectral theory, we refer to the classical survey article of Bott [12].

2. Statement of the main results

2.1. **Dynamical framework.** We fix f to be a smooth ($\mathcal{C}^\infty$) Morse function, meaning that f has only finitely many critical points and that these points are non degenerate. We denote by $\text{Crit}(f)$ the set of critical points. For simplicity, we shall always assume that f is excellent in the sense that, given $a\neq b$ in $\text{Crit}(f)$, one has $f(a)\neq f(b)$. If we consider a smooth Riemannian metric g on M, we can define a vector field V_f as follows

$$\forall(x,v)\in TM,\ d_xf(v)=\langle V_f(x),v\rangle_{g(x)}. \tag{2}$$

This vector field generates a complete flow on M [54, Ch. 6] that we denote by φ_f^t. Given any point a in $\text{Crit}(f)$, we can define its stable (resp. unstable) manifold, i.e.

$$W^{s/u}(a):=\left\{x\in M:\ \lim_{t\to+/-\infty}\varphi_f^t(x)=a\right\}.$$

One can show that $W^s(a)$ (resp. $W^u(a)$) is an embedded submanifold in M of dimension $0\leq r\leq n$ (resp. $n-r$) where r is the index of the critical point [75]. Note also that $W^u(a)\cap W^s(a)=\{a\}$. A remarkable property of these submanifolds is that *they form a partition of the manifold* M [72], i.e.

$$M=\bigcup_{a\in\text{Crit}(f)}W^s(a),\ \text{and}\ \forall a\neq b,\ W^s(a)\cap W^s(b)=\emptyset.$$

The above property also holds true for the unstable manifolds. This partition in stable (and unstable) leaves will play a central role in our analysis. Among these Morse gradient flows, Smale introduced a particular family of flows [68]. Namely, given any a and b in $\text{Crit}(f)$, he required that $W^s(a)$ *and* $W^u(b)$ *intersect transversally* whenever they intersect. This assumption also turns out to be a crucial ingredient to make our proofs work. We will use the terminology **Morse-Smale** for any gradient flow enjoying the above properties. Finally, for any point a in $\text{Crit}(f)$, we define $L_f(a)$ as the only matrix satisfying

$$\forall\xi,\eta\in T_aM,\ d_a^2f(\xi,\eta)=g_a(L_f(a)\xi,\eta). \tag{3}$$

As a is a nondegenerate critical point, $L_f(a)$ is symmetric with respect to g_a and invertible. Its eigenvalues are called the **Lyapunov exponents** at the point a and we write them as

$$\chi_1(a)\leq\ldots\leq\chi_r(a)<0<\chi_{r+1}(a)\leq\ldots\leq\chi_n(a),$$

where r is the index of the critical point a. All along the article, we will often make the assumption that (f,g) is a *smooth Morse pair* inducing a Morse-Smale gradient flow. By smooth Morse pair, we roughly mean that there is a smooth linearizing chart for V_f near any critical point. By the Sternberg-Chen Theorem [20], this is for instance satisfied when, for every critical point a, the Lyapunov exponents $(\chi_j(a))_{1\leq j\leq n}$ are rationally independent – see paragraph 3.5 for more details.

Remark 2.1. Let us fix some conventions. We will denote by $\mathbb{N}^*$ the set of positive integers $\{1,2,\ldots\}$ while $\mathbb{N}$ will be the set of nonnegative integers $\{0,1,2,\ldots\}$. We will use $\alpha = (\alpha_1,\ldots,\alpha_n)$ for a multi-index in $\mathbb{N}^n$. Given any critical point a of f, we denote by $|\chi(a)|$ the vector $(|\chi_1(a)|,\ldots,|\chi_n(a)|)$. For any $0\leq k\leq n$, $\Omega^k(M)$ will be the space of smooth differential forms of degree k and $\mathcal{D}'^{,k}(M)$ will be the topological dual of $\Omega^{n-k}(M)$, i.e. the space of currents of degree k (or of dimension $n-k$). For an introduction to the theory of currents, we refer to [63, 66].

2.2. **Correlation function.** The main concern of the article will be to perform a spectral analysis of the operator $\mathcal{L}_{V_f}$ acting on appropriate spaces of currents. As an application of our analysis, we will prove the following result on the asymptotic behaviour of $\varphi_f^{-t*}(\psi)$:

Theorem 2.2. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let $0\leq k\leq n$.*

Then, for every a in $\mathrm{Crit}(f)$ *and for every α in $\mathbb{N}^n$, there exists a continuous linear operator,*

$$\pi_{a,k}^{(\alpha)}:\Omega^k(M)\rightarrow\mathcal{D}'^{,k}(M),$$

such that for every $\Lambda>0$, for every ψ_1 in $\Omega^k(M)$, for every ψ_2 in $\Omega^{n-k}(M)$, and for every $t\geq 0$,

$$\int_M \varphi_f^{-t*}(\psi_1)\wedge\psi_2 = \sum_{a\in\mathrm{Crit}(f)}\ \sum_{\alpha\in\mathbb{N}^n:\alpha.|\chi(a)|<\Lambda} e^{-t\alpha.|\chi(a)|}\int_M \pi_{a,k}^{(\alpha)}(\psi_1)\wedge\psi_2 + \mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda t})$$

where $\alpha.|\chi(a)| = \sum_{i=1}^n \alpha_i|\chi_i(a)|$. Moreover, for every a in $\mathrm{Crit}(f)$ *and for every α in $\mathbb{N}^n$, one has*[2]

- *for every ψ_1 in $\Omega^k(M)$, the support of $\pi_{a,k}^{(\alpha)}(\psi_1)$ is contained in $\overline{W^u(a)}$,*
- $0\leq \mathrm{rk}(\pi_{a,k}^{(\alpha)})\leq 2^n$,
- $\mathrm{rk}(\pi_{a,k}^{(0)})=\delta_{k,r}$ *where r is the index of a,*
- $\mathrm{rk}(\pi_{a,k}^{(\alpha)})=\frac{n!}{k!(n-k)!}$ *for every $\alpha\in(\mathbb{N}^*)^n$.*

This Theorem gives us an asymptotic expansion at any order of the *correlation function*[3] associated with a Morse-Smale gradient flow and it is a consequence of Proposition 7.8 which is in fact slightly more precise. As we shall see in this Proposition, we will also provide a more or less explicit expression of the operator $\pi_{a,k}^{(\alpha)}$ near the critical point a. These operators of course depend on the choice of the Riemannian metric used to define

[2]We will in fact give a (rather combinatorial) explicit expression of $\mathrm{rk}(\pi_{a,k}^{(\alpha)})$ in the proof – see Remark 7.6.

[3]Here we make a small abuse of terminology as correlation functions are usually concerned with invariant measures.

the gradient flow as well as the Lyapunov exponents appearing in the above asymptotic expansion. In that sense, this result gives a partial answer to the question mentionned in the introduction concerning the dependence of gradient flows on the underlying metric [14, Problem 1].

Each term appearing in the sum looks also very much like the expansion obtained by Faure and Tsujii in the case of linear models acting on $\mathbb{R}^n$ [34, Ch. 3-4]. Here, one of the main difficulty will be to understand how these local models can be glued together in order to obtain a result valid on the whole manifold. We also mention that similar expansions appear via techniques from complex analysis in the case of analytic expanding circle maps arising from finite Blaschke products [8, 9]. Even if our flows are in some sense degenerate Axiom A flows, this result is also closely related to the recent results of Dyatlov and Guillarmou on Pollicott-Ruelle resonances for open systems [27]. Maybe the main difference with this reference is that Theorem 2.2 holds *globally* on M and not only in a neighborhood of the critical points[4]. If we consider the time 1 map of the flow, the induced diffeomorphism $h = \varphi_f^{-1}$ is probably one of the simplest example of an Axiom A (but not Anosov) diffeomorphism [70], even if it is a kind of "trivial" example as all the basic sets are reduced to fixed points. Resonances of general Axiom A diffeomorphisms were studied by Ruelle in [65] via methods of symbolic dynamics while the direct functional approach of these dynamical systems was developped by Baladi–Tsujii [6, 7] and by Gouëzel–Liverani [41]. As for the case of Axiom A flows treated in [27], these results focused on the dynamics near a convenient basic set. In our case, it would correspond to restrict ourselves to ψ_1 and ψ_2 supported in a neighborhood of a fixed critical point. Here, due to the simple structure of the diffeomorphism, the asymptotic expansion of the correlation function can be made in the entire complex plane without restrictions on ψ_1 and ψ_2 and with discrete poles on the real axis which are completely determined by the Lyapunov exponents on the basic sets of the diffeomorphism (i.e. the critical points of f). Moreover, the corresponding resonant states can also be made explicit.

We made a kind of (global) nonresonance assumption in the statement of Theorem 2.2. This assumption ensures that the generator of the flow does not have any Jordan blocks. Regarding this point, we should mention that, in [41, p.436], Gouëzel and Liverani raised the question of the existence of Jordan blocks for *nontopologically transitive* Axiom A diffeomorphisms. As far as we know, the above Theorem gives the first example of such diffeomorphisms where *all* the Jordan blocks in the spectrum are trivial. In [36], Frenkel, Losev and Nekrasov were lead to similar problems in the context of quantum field theory. In particular, as a by product of their analysis, they obtain the complete asymptotic for the correlation function of the flow associated with the height function on the 2-sphere endowed with its canonical metric. In that case, the Lyapunov exponents are all equal to ± 1 (hence resonant) and they prove that there are indeed infinitely many polynomial factors (hence non trivial Jordan blocks) in the asymptotic expansion. In the general case (including the case of [36]), we still obtain an asymptotic expansion for the correlation function which may involve polynomial factors in t except for the peripheral eigenvalue

[4]We also note that [27] made the assumption that the vector field does not vanish.

$\lambda = 0$. We refer to paragraph 5.3 for a complete expression in the general setting. We just mention here the *leading term of the asymptotics* in the general case:

Theorem 2.3. *Let (f, g) be a smooth Morse pair generating a Morse-Smale gradient flow φ_f^t. Let $0 \leq k \leq n$.*

Then, for every a in $\text{Crit}(f)$ of index k, there exist

- *U_a in $\mathcal{D}'^{,k}(M)$ whose support is equal to $\overline{W^u(a)}$,*
- *S_a in $\mathcal{D}'^{,n-k}(M)$ whose support is equal to $\overline{W^s(a)}$,*

such that, for every

$$0 < \Lambda < \min \left\{ |\chi_j(b)| : \ 1 \leq j \leq n, \ b \in \text{Crit}(f) \right\},$$

for every (ψ_1, ψ_2) in $\Omega^k(M) \times \Omega^{n-k}(M)$, and for every $t \geq 0$,

$$\int_M \varphi_f^{-t*}(\psi_1) \wedge \psi_2 = \sum_{a \in \text{Crit}(f): \ \text{ind}(a) = k} \int_M S_a \wedge \psi_1 \int_M U_a \wedge \psi_2 + \mathcal{O}_{\psi_1, \psi_2}(e^{-\Lambda t}).$$

Except for the remainder term, this result was first proved by Harvey and Lawson via techniques from geometric measure theory and under slightly more restrictive assumptions on the gradient flows [45, 46]. This Theorem follows from Proposition 6.9 which is again slightly more precise. To our knowledge, the currents U_a and S_a appearing in the leading term of the asymptotic expansion were first constructed by Laudenbach [53, 54] in the case of a "locally flat metric adapted to the Morse coordinates" – see paragraph 3.5 for the definition. In the following, we shall refer to them as Laudenbach's currents. The difficulty is that the submanifolds $W^u(a)$ and $W^s(a)$ are not a priori properly embedded and one has to justify that the currents of integration are well defined. Precisely, one can integrate on $W^u(a)$ a differential form ψ whose support is included in a compact part of $W^u(a)$ but integration of a general form whose support may intersect the boundary needs to be justified. This can solved by analyzing the mass of the currents near the boundary of the unstable (resp. stable manifold) and this requires a careful description of the structure of the boundary of $W^u(a)$ [53, 54]. Even if it is in a different manner, similar difficulties involving the boundary will of course occur at some point in our analysis and we shall deal with this problem via dynamical techniques following the works of Smale [68] and Weber [75] – see for instance Lemmas 3.6 and 3.8.

After properly defining the spectral framework of our problem, we will recover the existence of these currents as a consequence of our spectral analysis. They correspond to the kernel of the operator $\mathcal{L}_{V_f}$ acting on appropriate anisotropic spaces of currents. The advantage of this approach is that it allows to treat more general families of gradient flows and that it sheds a new (spectral) light on these natural dynamical objects. A difficulty may be that it relies on microlocal techniques which are maybe not as well-known as the geometric measure theory used by Harvey and Lawson in [45, 46] to give an interpretation of these currents as a limit of the correlation function under the assumption of finite volume – see also [58] for generalizations of this result. In any case, Theorem 2.2 generalizes this type of result in the sense that it does not only give the existence of the limit but also a rate of convergence to this equilibrium state and the full asymptotic expansion as $t \to +\infty$.

2.3. **Topological interpretation of the leading term.** One of the main application of this dynamical approach to Morse theory is that the partition of the manifold into unstable components has beautiful topological implications [72, 68, 46, 54]. In section 8, we will explain how to recover some classical results from differential topology (e.g. finiteness of Betti numbers, Poincaré duality, Morse inequalities) via our spectral approach and via some analogies with Hodge-de Rham theory. For that purpose, we can set, for every $0 \leq k \leq n$,

$$C^k(f) := \text{span}\left\{U_a : \text{ind}(a) = k\right\}.$$

In analogy with Hodge-de Rham theory where one uses the formula $\Delta = d \circ d^* + d^* \circ d$, we can write the Cartan formula

$$\mathcal{L}_{V_f} = d \circ i_{V_f} + i_{V_f} \circ d,$$

where d is the coboundary operator and i_{V_f} is the contraction by the vector field V_f. We will verify that $(C^*(f), d)$ induces a cohomological complex while $(C^*(f), i_{V_f})$ induces a homological complex. The first complex is known as the Morse complex (also sometimes called the Thom-Smale-Witten complex) and we will call the second one the **Morse-Koszul** complex. Using our analysis of Morse-Smale gradient flows, we will give a purely spectral proof of the following results:

Theorem 2.4. *Let (f, g) be a smooth Morse pair generating a Morse-Smale gradient flow φ_f^t. Then, the following holds.*

(1) *The maps*

$$\mathbb{P}^{(k)} : \psi \in \Omega^k(M) \mapsto \sum_{a \in \text{Crit}(f):\ \text{ind}(a)=k} \left(\int_M S_a \wedge \psi\right) U_a \in C^k(f)$$

induce a quasi–isomorphism between the cohomology of the de Rham complex $(\Omega^*(M), d)$ and the cohomology of the Morse complex $(C^*(f), d)$.

(2) *The Euler characteristic of M is equal to the Euler characteristic of the Morse-Koszul complex $(C^*(f), i_{V_f})$.*

The first part of the Theorem is due to Laudenbach in the case of a "locally flat metric adapted to the Morse coordinates" [53, 54]. It recovers the classical fact that the de Rham complex $(\Omega^*(M), d)$ is quasi-isomorphic to the Morse complex. From this, it is classical to deduce the finiteness of the Betti numbers and the so-called Morse inequalities – see section 8 for more details. The second part seems new and it shows that the Morse-Koszul complex has nice topological content. As we shall see, the dimension of the homology of degree k of the Morse-Koszul complex is equal to the number of critical points of index k for every $0 \leq k \leq n$. On the other hand, we will verify that the dimension of the homology of degree k of the classical Koszul complex $(\Omega^*(M), i_{V_f})$ is equal to 0 for every $1 \leq k \leq n$ while the 0-th homology has dimension $|\,\text{Crit}(f)|$. This may sound a little bit surprising as it shows that the Morse-Koszul complex $(C^*(f), i_{V_f})$ is not quasi-isomorphic to the classical

Koszul complex $(\Omega^*(M), i_{V_f})$ via the collection of maps $\mathbb{P}^k, 0 \leqslant k \leqslant n$. This is due to the fact that, contrary to the coboundary operator d, i_{V_f} is sensitive to the regularity of the elements of the complex.

2.4. **About the proof: spectral analysis of $\mathcal{L}_{V_f}$.** As was already alluded, understanding the asymptotic properties of φ_f^{-t*} is related to the spectral analysis of its generator which is nothing else but the Lie derivative $\mathcal{L}_{V_f}$ along the vector field V_f. One of the main difficulty one encounters when trying to describe this spectrum is to find good Banach spaces containing $\Omega^*(M)$ and where $\mathcal{L}_{V_f}$ has nice spectral properties such as discrete spectrum. We already explained that important progresses were made recently concerning this problem especially in the case of Anosov vector fields. Among the type of spaces that can be considered, we can very roughly distinguish two types: (1) anisotropic Hölder spaces like in [55], (2) anisotropic Sobolev spaces obtained via microlocal techniques like in [32]. Here, we will in fact closely follow the construction of Faure-Sjöstrand in [32] (see also [28, 33] in the case of currents) and explain how to adapt it to our dynamical framework. One of the main issue we have to deal with is the asymptotic behaviour of the Hamiltonian lift of φ_f^t. In particular, we have to verify that the attractor and the repeller of the normalized Hamiltonian flow are compact subsets – see Lemmas 3.6 and 3.8. This is one of the first place where we will strongly use our extra assumptions on the flow, namely the Smale transversality and the (smooth) linearizing property near every critical point.

After setting properly this dynamical framework and its asymptotic properties, we can closely follow the construction from [32] which requires minor (but necessary) modifications and which will be described in section 4 – see also appendices A and B. Given any $\Lambda > 0$ and any $0 \leq k \leq n$, this procedure allows us to construct an anisotropic Sobolev space $\mathcal{H}_k^{m_\Lambda}(M) \subset \mathcal{D}'^{,k}(M)$ such that

$$-\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^{m_\Lambda}(M) \to \mathcal{H}_k^{m_\Lambda}(M),$$

and such that the operator has only discrete spectrum with finite multiplicity in the half plane $\{\mathrm{Re}(z) > -\Lambda\}$. According to [32, Th. 1.5], these values are independent of the choice of our anisotropic space. These complex numbers are called the Pollicott-Ruelle resonances of $-\mathcal{L}_{V_f}^{(k)}$ [62, 64], and they correspond to the poles of the meromorphic extension of $(-\mathcal{L}_{V_f}^{(k)} - z)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$ to the complex plane. We denote these poles by $\mathcal{R}_k(f, g)$.

In the case where (f, g) is a smooth Morse pair inducing a Morse-Smale gradient flow, we will obtain several results on their structure that we will now describe:

(1) Any element in $\mathcal{R}_k(f, g)$ is contained in $(-\infty, 0]$ and is a linear combination with integer coefficients of the Lyapunov exponents at a fixed critical point a (Proposition 5.1).
(2) Up to its multiplicity, we can determine exactly which linear combination appears in the spectrum (Proposition 5.1).

(3) The dimension of the kernel is equal to the number of critical points of index k, and we have an expression for any element of the basis in a neighborhood of any critical point of index a (Propositions 6.7 and 6.8).
(4) If all the Lyapunov exponents are rationally independent, we can determine the multiplicity of every element in $\mathcal{R}_k(f,g)$ and the local expression of the eigenmodes near the associated critical point (Propositions 7.5 and 7.7).
(5) The algebraic multiplicity of an eigenvalue is always equal to its geometric multiplicity (Propositions 6.1 and 7.2).
(6) In particular, we can determine Weyl asymptotics in terms of the Lyapunov exponents (Proposition 7.9) and we recover the classical Lefschetz trace formula (Proposition 7.10).

The combination of all these results allows to prove Theorem 2.2 and Theorem 2.3– see Propositions 7.8 and 6.9. The proofs of these different spectral results will heavily rely on the construction of the spaces $\mathcal{H}_k^{m_\Lambda}(M)$. In particular, we will deduce from that construction that we have a very strong information on the wave front set of our eigenmodes [50], namely it is contained in the unstable direction of the gradient flow. Combining this a priori information to the dynamical structure of our flow and to results from distribution theory and microlocal analysis, we will be able to reconstruct in some sense a lot of informations on the Pollicott-Ruelle resonances and their corresponding eigenmodes. As a by-product of our spectral analysis (see paragraph 6.4.5), we observe that our description of the spectrum of $\mathcal{L}_{V_f}^{(k)}$ implicitely shows the following result:

Corollary 2.5. *Let (f,g) be a smooth Morse pair generating a Morse-Smale gradient flow φ_f^t and let $0 \leq k \leq n$. Then, for every ψ in $\Omega^k(M)$, the cohomological equation*

$$\mathcal{L}_{V_f}^{(k)} u = \psi$$

admits a solution in $\mathcal{H}_k^{m_\Lambda}(M)$ if and only if, for every critical point a of index k, one has

$$\langle S_a, \psi \rangle = 0.$$

From the construction, one knows that the anisotropic Sobolev space $\mathcal{H}_k^{m_\Lambda}(M)$ is contained in the standard Sobolev space $\mathcal{H}_k^{-N}(M)$ for some large positive constant N. In the case of a strongly non-resonant Morse-Smale gradient flow, the argument of paragraph 6.4.5 could be adapted to find criteria under which the cohomological equation $(\mathcal{L}_{V_f}^{(k)} + z)u = \psi$ admits solution in $\mathcal{H}_k^{-N}(M)$ for $N > 0$ large enough depending on the real part of z.

2.5. **Witten's approach to Morse theory and supersymmetric quantum mechanics.** In [76], Witten recovered the classical Morse inequalities by studying the operator $d_\hbar := \hbar e^{-\frac{f}{\hbar}} d e^{\frac{f}{\hbar}}$ and its adjoint $d_\hbar^*$ in the semiclassical limit $\hbar \to 0$. More precisely, he obtained this result by considering the bottom of the spectrum of the twisted Laplacian $d_\hbar^* d_\hbar + d_\hbar d_\hbar^*$. Even if the operator $\mathcal{L}_{V_f}$ is of different nature, we would like to emphasize the following common point with Witten's approach. Like in [32, 28], the strategy is to consider the action of $\mathcal{L}_{V_f}^{(k)}$ on certain anisotropic Sobolev spaces $\mathcal{H}_k^{m_\Lambda}(M)$. As we shall see in

Appendix B, this is equivalent to conjugating the operator by a certain elliptic pseudodifferential operator $\mathrm{Op}(\mathbf{A}^{(k)}_{m_\Lambda})$ with variable order and to consider the action of the conjugated operator on the Hilbert space $L^2(M, \Lambda^k(T^*M))$. Compared with Witten, we deform the operator $\mathcal{L}_{V_f}$ (instead of d and d^*) by conjugating it by $\mathrm{Op}(\mathbf{A}^{(k)}_{m_\Lambda})$ (instead of $e^{\frac{f}{\hbar}}$). Following the works of Faure-Sjöstrand, the effect of this conjugation is to unveil some dynamical resonances of the operator $\mathcal{L}^{(k)}_{V_f}$, and, like in Witten's approach, this dynamical spectrum carries some topological informations related to the Morse complex – see section 8 for more details.

Recall that Witten's theory was inspired by supersymmetric (SUSY) quantum mechanics. In some sense, this framework allows to compare our approach with Hodge theory (and also with Witten's work). This unifying point of view is discussed from the physical perspective in inspiring works of Losev et al. [36, 37, 52, 56, 57]. In order to make this parallel between Hodge theory and the present work, we observe that the ($N = 1$) supersymmetric quantum mechanical system associated with our problem is given by:

- a $\mathbb{Z}_2$-graded state space $\mathcal{H}^{m_\Lambda}(M) = \mathcal{H}^{m_\Lambda}_+(M) \oplus \mathcal{H}^{m_\Lambda}_-(M)$ of anisotropic Sobolev currents with bosonic (resp fermionic) states $\mathcal{H}^{m_\Lambda}_+(M)$ (resp $\mathcal{H}^{m_\Lambda}_-(M)$) of even (resp odd) degree.
- a fermion operator $(-1)^F$ whose eigenspaces are $(\mathcal{H}^{m_\Lambda}_+(M), \mathcal{H}^{m_\Lambda}_-(M))$ with eigenvalue ± 1 depending on the parity of the state,
- two operators of symmetry, i.e. $Q_1 = d + i_{V_f}$ and $Q_2 = i(d - i_{V_f})$ which satisfies $Q_1Q_2 + Q_2Q_1 = 0$ (i_{V_f} plays the role of d^* in Hodge theory),
- a Hamiltonian which is given by the Lie derivative $\mathcal{L}_{V_f}$ and which verifies $\mathcal{L}_{V_f} = Q_1^2 = Q_2^2 = di_{V_f} + i_{V_f}d$ from Cartan's formula.

As we shall see, the currents U_a appearing in Corollary 2.3 can be interpreted as *Lagrangian states* whose wave front set lies in conormals of unstable manifolds. From the point of view of supersymmetry, they span the *vacuum* $C^*(f)$ (generated by eigenstates of minimal energy) of our quantum theory.

2.6. **Organization of the article.** In section 3, we gather some crucial dynamical preliminaries and introduce some notations that will be used all along the article. In section 4, we make use of our dynamical assumptions in order to construct anisotropic spaces of currents which are adapted to our problem. As our construction is very close to the one in [32], we mostly focus on the differences, namely the construction of the escape function whose detailed proof is postponed to appendix A. In section 5, we make use of the regularity properties of the eigenmodes to determine exactly the Pollicott-Ruelle spectrum (up to its multiplicities). In section 6, we study the case of the eigenvalue 0. Section 7 is devoted to describe the local structure of the eigenmodes under some nonresonance assumptions. From that, we deduce some results on the multiplicity of the eigenvalues. We explain in section 8 how to deduce some classical results of differential topology from the results obtained in the previous sections. In appendix A, we give the complete proof of the construction of the escape function. Appendix B is devoted to a brief reminder of [32] concerning the proof of Proposition 4.2. Finally, appendix C collects some facts on asymptotic expansions that we use at several stages of our work.

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3. Morse-Smale gradient flows

In this section, we briefly collect some facts on the dynamical properties of Morse-Smale gradient flows. The main new results of this section are Lemmas 3.6 and 3.8 which are related to earlier works of Weber [75] on the space of connecting orbits for gradient flows. These two lemmas are the crucial ingredients to develop the machinery of anisotropic Sobolev spaces of Faure and Sjöstrand [32]. We also fix some conventions that we will use all along the article. For the well-known results, we follow the lines of [54, 75] and we refer to these references for a more detailed exposition. Recall that, in all the article, M denotes a smooth ($\mathcal{C}^\infty$), oriented, compact manifold without boundary.

3.1. **Gradient flows.** Let $f : M \to \mathbb{R}$ be a smooth function on M. If we fix a Riemannian metric g on M (compatible with our orientation), then we can define the corresponding gradient vector field as follows:

$$\forall x \in M,\ \langle \text{grad} f(x), .\rangle_{g(x)} := df(x).$$

In local coordinates, this can we written as

$$V_f(x) := \text{grad} f(x) = \sum_{i,j=1}^n g^{ij}(x)\partial_{x_j} f \partial_{x_j},$$

where $(g^{ij}(x))_{1\leq i,j\leq n}$ is the induced Riemannian metric on T_x^*M. Under our geometric assumptions (compactness of the manifold), one knows that the gradient vector field induces a complete flow that we denote by

$$\varphi_f^t : M \to M.$$

If it does not create any particular confusion, we will sometimes use the convention $x(t) = \varphi_f^t(x_0)$ for a fixed x_0 in M. Note that, for any integral curve $t \mapsto x(t)$ of the gradient vector field, one has

$$\forall t_1, t_2 \in \mathbb{R},\ f(x(t_2)) - f(x(t_1)) = \int_{t_1}^{t_2} \|\text{grad} f(x(t))\|^2_{g(x(t))} dt. \tag{4}$$

In other words, f can only increase along the flow lines. Suppose now that f is a **Morse function**. We denote by $\mathrm{Crit} f$ its critical points. For the sake of simplicity, we will also assume our Morse function f to be **excellent** which means that all critical values are distinct. Recall that such functions are dense in the topological space $\mathcal{C}^\infty(M,\mathbb{R})$. The Morse Lemma tells us

Lemma 3.1 (Morse Lemma)**.** *Let f be a Morse function on a Riemannian manifold (M,g). Then, near any critical point a, there is a system of coordinates $(z_i)_i$ such that the point a is given by $z=0$, and such that*

$$f(x) = f(a) - \frac{z_1^2}{2} - \ldots - \frac{z_r^2}{2} + \frac{z_{r+1}^2}{2} + \ldots + \frac{z_n^2}{2},$$

*for some $0 \leq r \leq n$. The integer r is called the **index** of the critical point a. We will either denote it by $r(a)$ or* $\mathrm{ind}(a)$.

An important property of the dynamical system $\varphi_f^t : M \to M$ is that, for any given x_0 in M, there exist two points x_- and x_+ in $\mathrm{Crit} f$ such that

$$\lim_{t\to\pm\infty} \varphi_f^t(x_0) = x_\pm. \tag{5}$$

3.2. **Stable and unstable manifolds.** Let a be a critical point of f. The stable (resp. unstable) manifold $W^s(a)$ (resp. $W^u(a)$) is defined as the set of points x in M satisfying $\varphi_f^t(x) \to a$ as $t \to +\infty$ (resp. $t \to -\infty$). From [75, Th. 2.7], one knows that $W^s(a)$ (resp. $W^u(a)$) is a *smooth submanifold* of dimension $r(a)$ (resp. $n - r(a)$) where $0 \leq r(a) \leq n$ is the index of the critical point a. Note that, for more general vector fields with an hyperbolic point, the stable (resp. unstable) manifold is a priori only injectively immersed in M. The fact that we consider a gradient flow allows to show that it is also embedded [75, Th. 2.7], even if it is not a priori properly embedded.

We will say that the gradient flow φ_f^t satisfies the **Morse-Smale assumption** if for every pair of critical points (a,b), the submanifolds $W^s(a)$ and $W^u(b)$ are transversal. Note that, in the case where $a=b$, the intersection of the tangent spaces is in fact reduced to $\{0\}$. When $a \neq b$ and $W^s(a) \cap W^u(b)$ is non empty, then this assumption ensures that $r(b) < r(a)$.

Let us now fix some conventions. Given any x_0 in M, there exists a unique pair of critical points (x_-, x_+) such that x_0 belongs to $W^u(x_-) \cap W^s(x_+)$. We define

$$E^u(x_0) := T_{x_0}W^u(x_-) \text{ and } E^s(x_0) := T_{x_0}W^s(x_+).$$

Note that, whenever x_0 is not a critical point of f, the intersection of these two subspaces is not reduced to 0 as they both contains the direction of the flow. From our transversality assumption, one has $T_{x_0}M = E^u(x_0) + E^s(x_0)$. We refer to paragraph 3.3 for a more detailed description of these subspaces at the critical points of f. We can also introduce the dual spaces $E_u^*(x_0)$ and $E_s^*(x_0)$ which are defined as the annihilators of these unstable and stable spaces, i.e. $E_{u/s}^*(x_0)(E^{u/s}(x_0)) = 0$. From the Morse-Smale transversality assumption, one can verify that, for any x_0 in M,

$$E_u^*(x_0) \cap E_s^*(x_0) = \{0\}. \tag{6}$$

3.3. **Lyapunov exponents.** Given every point x in M, we define $L_f(x)$ as the unique matrix satisfying

$$\forall \xi, \eta \in T_xM,\ \langle L_f(x)\xi, \eta\rangle_{g(x)} = d_x^2 f(\xi, \eta).$$

Let a be an element in $\text{Crit}(f)$. The matrix $L_f(a)$ corresponds to the linearization of V_f at the point a. It can be shown [75, Lemma 2.5] that

$$\forall t \in \mathbb{R},\ d\varphi_f^t(a) = \exp(tL_f(a)).$$

Moreover, from the definition and from the Morse assumption, one can verify that $L_f(a)$ is an invertible matrix which is symmetric with respect to the Riemannian metric g. In particular, it is diagonalizable and we denote by $(\chi_j(a))_{j=1,\ldots,n}$ its eigenvalues. These nonzero real numbers are called the **Lyapunov exponents** of the system. They depend both on f and on the metric g. We will always suppose that $\chi_i(a) < 0$ for $1 \leq i \leq r$ and $\chi_i(a) > 0$ for $r + 1 \leq i \leq n$. Moreover, there exists a basis of eigenvectors which is orthonormal with respect to the metric g. According to [75, Th. 2.7], the stable (resp. unstable) space is in fact equal to the direct sum of eigenspaces corresponding to the negative (resp. positive) eigenvalues of $L_f(a)$.

3.4. **Lift to the cotangent space.** We will now explain how one can lift this gradient flow to the cotangent space T^*M. We associate to the vector field V_f an Hamiltonian function H_f which can be written as follows:

$$\forall (x, \xi) \in T^*M,\ H_f(x, \xi) := \xi\left(V_f(x)\right).$$

This Hamiltonian function also induces an Hamiltonian flow that we denote by $\Phi_f^t : T^*M \to T^*M$. We note that, by construction,

$$\Phi_f^t(x, \xi) := \left(\varphi_f^t(x), \left(d\varphi_f^t(x)^T\right)^{-1}\xi\right),$$

and that *this flow induces a diffeomorphism between* $T^*M - \{0\}$ *and* $T^*M - \{0\}$. When it does not lead to any confusion, we will also write $\Phi_f^t(x, \xi) = (x(t), \xi(t))$. We note that this flow induces a smooth flow on the unit cotangent bundle S^*M, i.e.

$$\forall t \in \mathbb{R},\ \forall (x, \xi) \in S^*M,\ \tilde{\Phi}_f^t(x, \xi) = \left(\varphi_f^t(x), \frac{\left(d\varphi_f^t(x)^T\right)^{-1}\xi}{\left\|\left(d\varphi_f^t(x)^T\right)^{-1}\xi\right\|}\right).$$

We denote by $\tilde{X}_{H_f}$ the induced smooth vector field on S^*M.

3.5. **Adapted coordinates.** We will say in the following that (f, g) is a **smooth Morse pair** if, given any critical point a of f, one can find an open neighborhood V_a of a and a system of *smooth* (meaning $\mathcal{C}^\infty$) local coordinate charts $(z_j)_{j=1,\ldots,n} = (x, y)$ such that, in this coordinate chart, the vector field V_f reads

$$V_f := \sum_{j=1}^n \chi_j(a) z_j \partial_{z_j} = -\sum_{j=1}^r |\chi_j(a)| x_j \partial_{x_j} + \sum_{j=r+1}^n |\chi_j(a)| y_j \partial_{y_j}. \tag{7}$$

The key point for us is that this change of coordinates is smooth which will allow us to take as many derivatives as we want in the following sections where we aim at using microlocal techniques. Requiring that there exists a smooth change of coordinates for which the gradient vector field can be linearized may a priori look as a strong assumption. We will briefly discuss below two situations where this assumption is satisfied, the second one being in some sense rather general.

When this assumption is satisfied, we shall say that we have an **adapted system of coordinates**. We note that the function f may not have a nice expression in these coordinates, meaning that f may a priori not have a Morse-type expression. In such a coordinate chart, the gradient flow reads

$$\begin{aligned}\varphi_f^t(z) &= (e^{t\chi_1(a)}z_1,\ldots,e^{t\chi_n(a)}z_n)\\ &= (e^{-|\chi_1(a)|t}x_1,\ldots,e^{-|\chi_r(a)|t}x_r,e^{|\chi_{r+1}(a)|t}y_{r+1},\ldots,e^{|\chi_n(a)|t}y_n).\end{aligned}$$

Let us fix some conventions that we will use in the following. For every critical point a, we denote the change of coordinates by

$$\kappa_a : w\in V_a\subset M\rightarrow (x,y)\in W_a=(-\delta_a,\delta_a)^n\subset\mathbb{R}^n,$$

where $\delta_a>0$ is some small enough parameter.

Remark 3.2. Let $(U,\kappa)=(u^i)$ be local coordinates on M. Whenever the chart is of class $\mathcal{C}^1$, one can lift in a canonical way these coordinates into coordinates (u^i,v_j) on the cotangent space T^*M by using $(\mathcal{K},T^*U)$, where $\mathcal{K}(x,\xi)=(\kappa(x),(d\kappa(x)^T)^{-1}\xi)$. When we make a change of coordinates $(\tilde{u}^i,\tilde{v}_j)$, one can verify that $\tilde{v}$ is the image of v under a linear transformation (depending only on the coordinate charts (u^i) and $(\tilde{u}^i)$).

We can also write the expression of the Hamiltonian flow in the corresponding adapted coordinate chart near a critical point a. In such a chart, one can write

$$H_f(z,\zeta)=\sum_{j=1}^n\chi_j(a)z_j\zeta_j=-\sum_{j=1}^r|\chi_j(a)|x_j\xi_j+\sum_{j=r+1}^n|\chi_j(a)|y_j\eta_j.$$

In particular, the map Φ_f^t can be written in this local coordinate chart as

$$\Phi_f^t(z,\zeta)=(e^{\chi_1(a)t}z_1,\ldots e^{\chi_n(a)t}z_n;e^{-\chi_1(a)t}\zeta_1,\ldots e^{-\chi_n(a)t}\zeta_n).$$

3.5.1. *Locally flat metrics.* The vector field V_f is a priori not linear in the chart of the Morse Lemma. In fact, there might be no Morse chart in which $g_{ij}(0)$ is diagonal and the vector field linear. We will call the metric g **locally flat with respect** to f if, for any critical point of f, there exists a *smooth Morse chart* (z_i) in which the vector field V_f has the linear form

$$V_f(z)=\sum_{j=1}^n\chi_j(a)z_j\partial_{z_j}.$$

Such flows are also sometimes refered as tame flows. This type of locally flat metrics appears for instance in [45, 54]. It is shown in [45] that, given any Morse function, one can find an adapted metric g such that the Morse-Smale transversality assumption is satisfied. Moreover, they proved that this property is more or less generic among such metrics.

3.5.2. *Sternberg-Chen Theorem.* We would like to justify that asking for a smooth change of coordinates which linarizes the gradient flow is in some sense generic. For that purpose, we just recall Sternberg-Chen's Theorem on the linearization of vector fields near hyperbolic critical points [20] (see also [59, Th. 9, p.50]):

Theorem 3.3 (Sternberg-Chen)**.** *Let $X(x)=\sum_j a_j(x)\partial_{x_j}$ be a smooth vector field defined in a neighborhood of 0 in $\mathbb{R}^n$. Suppose that $X(x)=0$. Denote by (χ_j) the eigenvalues of $L:=(\partial_{x_k}a_j(0))_{k,j}$. Suppose that the eigenvalues satisfy the **non resonant assumption**,*

$$\forall\ k_1,\dots,k_n\in\mathbb{Z}\ s.t\ k_1,\dots k_n\geq 2,\ \forall\ 1\leq i\leq n,\ \chi_i\neq\sum_{j=1}^n k_j\chi_j.$$

Then, there exists a smooth diffeomorphism h which is defined in a neighborhood of 0 such that

$$X\circ h(x)=dh\circ(Lx.\partial_x).$$

The classical Grobman-Hartman Theorem [44] ensures the existence of a conjugating homeomorphism. The crucial point of the Sternberg-Chen Theorem is that the conjugating map is smooth provided some non resonance assumption is made. Applying this Theorem locally near the critical points of f allows to show the existence of a smooth and adapted system of coordinates. Note that this non-resonant assumption on the eigenvalues is for instance satisfied if, for every a in $\text{Crit}(f)$, the Lyapunov exponents $(\chi_j(a))_{j=1,\dots,n}$ are rationally independent. In section 7, we will in fact make the assumption that *all* the Lyapunov exponents are rationally independent.

3.6. **Attractor and repeller of the Hamiltonian flow.** We now introduce the following subsets of T^*M:

$$\Gamma_+=\bigcup_{x\in M}E_s^*(x),\ \Gamma_-=\bigcup_{x\in M}E_u^*(x),\ \text{and}\ \Gamma=M\times\{0\}.$$

We have then

Lemma 3.4. *Suppose that (f,g) is a smooth Morse pair which generates a Morse-Smale gradient flow φ_f^t. One has, for every (x,ξ) in T^*M with $\xi\neq 0$,*

$$(x,\xi)\in\Gamma_\pm\implies\lim_{t\to\pm\infty}\|\xi(t)\|_{x(t)}=0,$$

and

$$(x,\xi)\notin\Gamma_\pm\implies\lim_{t\to\pm\infty}\|\xi(t)\|_{x(t)}=+\infty,$$

where $(x(t),\xi(t))=\Phi_f^t(x,\xi)$.

This Lemma tells us that the trapped set of the Hamiltonian flow is reduced to the zero section of T^*M. The proof of this Lemma will crucially use the fact that we made the Morse-Smale assumption and that we have a smooth (at least $\mathcal{C}^1$) change of coordinates which linearizes the vector field.

Proof. We only consider the case where $t\to+\infty$ (the other case can be obtained by replacing f by $-f$). Let (x,ξ) be an element in T^*M with $\xi\neq 0$. There exists a critical point x_+ of f such that $\lim_{t\to+\infty}\varphi_f^t(x)=x_+$. In particular, for $t>0$ large enough, $\varphi_f^t(x)$ belongs to the adapted chart around x_+ which was defined in paragraph 3.5. Up to a translation of time, one can write that, in this system of adapted coordinates and for every $t\geq 0$

$$\Phi_f^t(x,\xi)=(e^{-|\chi_1|t}x_1,\ldots,e^{-|\chi_r|t}x_r,0,\ldots,0,e^{|\chi_1|t}\xi_1,\ldots,e^{|\chi_r|t}\xi_r,e^{-|\chi_{r+1}|t}\eta_{r+1},\ldots,e^{-|\chi_n|t}\eta_n).$$

Hence, as all the norms can be made uniformly equivalent to the Euclidean norm in a small neighborhood of x_+, one can find two positive constants $0<C_1<C_2$ such that

$$C_1\left(\sum_{j=1}^r e^{2|\chi_j|t}\xi_j^2+\sum_{j=r+1}^n e^{-2|\chi_j|t}\eta_j^2\right)\leq\|\xi(t)\|_{x(t)}^2\leq C_2\left(\sum_{j=1}^r e^{2|\chi_j|t}\xi_j^2+\sum_{j=r+1}^n e^{-2|\chi_j|t}\eta_j^2\right).$$

The fact that (x,ξ) belongs to Γ_+ is exactly equivalent to the fact that $\xi_1=\ldots=\xi_r=0$ from which one can easily conclude the expected property. □

Introduce now the two following disjoint subsets of S^*M:

$$\Sigma_u:=S^*M\cap\Gamma_+,\ \text{and}\ \Sigma_s:=S^*M\cap\Gamma_-.$$

Then, one has:

Lemma 3.5. *Suppose that (f,g) is a smooth Morse pair which generates a Morse-Smale gradient flow φ_f^t. One has*

$$\forall(x,\xi)\in S^*M-\Sigma_s,\ \lim_{t\to-\infty}d_{S^*M}\left(\tilde{\Phi}_f^t(x,\xi),\Sigma_u\right)=0,\tag{8}$$

and

$$\forall(x,\xi)\in S^*M-\Sigma_u,\ \lim_{t\to+\infty}d_{S^*M}\left(\tilde{\Phi}_f^t(x,\xi),\Sigma_s\right)=0.\tag{9}$$

This lemma tells us that Σ_u and Σ_s are in a certain weak sense repeller and attractor of the flow $\tilde{\Phi}_f^t$. A stronger version of this fact will be given in Lemma 3.8.

Proof. We proceed as in the proof of Lemma 3.4, and we just treat the case where $t\to+\infty$. Let (x,ξ) be an element in $S^*M-\Sigma_u$. In other words, (x,ξ) does not belong Γ_+. Using the notations of the proof of Lemma 3.4, it means that

$$\Phi_f^t(x,\xi)=(e^{-|\chi_1|t}x_1,\ldots,e^{-|\chi_r|t}x_r,0,\ldots,0,e^{|\chi_1|t}\xi_1,\ldots,e^{|\chi_r|t}\xi_r,e^{-|\chi_{r+1}|t}\eta_{r+1},\ldots,e^{-|\chi_n|t}\eta_n),$$

with $\xi_j\neq 0$ for some $1\leq j\leq r$. By letting $t\to+\infty$, we find that that any accumulation point of $\tilde{\Phi}_f^t(x,\xi)$ is of the form $(0,\ldots,0,\tilde{\xi}_1,\ldots,\tilde{\xi}_r,0,\ldots,0)$. Equivalently, every accumulation point belongs to Σ_s. □

3.7. **Compactness.** In order to make the machinery of anisotropic Sobolev space work, it will first be important for us that Σ_u and Σ_s are compact subsets of S^*M. This assumption is verified for gradient flow satisfying the Morse-Smale assumption:

Lemma 3.6. *Suppose that (f, g) is a smooth Morse pair which induces a gradient flow with the Morse-Smale property. Then, the subsets Σ_u and Σ_s are compact in S^*M.*

This property combined with Lemma 3.8 will be crucial in our construction of anisotropic Sobolev spaces. In particular, they are necessary to prove Lemma 2.1 from [32] which is at the heart of this construction. We note that the proof of Lemma 3.6 requires both the Morse-Smale assumption and the existence of a (at least $\mathcal{C}^1$) linearizing chart for the flow.

Remark 3.7. Even if this Lemma sounds natural, the proof is a little bit subtle and it is related to the so-called Whitney regularity condition [60, Ch. 7] – see also the appendix of [54] for related results in the case of locally flat metrics. Here, we are aiming at weaker results than in these references and we shall give a proof of our Lemma which is based on purely "dynamical arguments". Our argument is in fact very close to the proof of the compactness of the space of connecting orbits of Weber in [75, Th. 3.8] – see also [68] for earlier related results of Smale. In this reference, it was proved that the space of connecting orbits between two critical points a and b is "compact up to broken orbits". It means that, for a fixed sequence $(x_p)_{p\geq 1}$ in $W^u(a)\cap W^s(b)$, there exists (up to extraction) a sequence of critical points $a = b_0, b_1, \dots, b_l = b$ and a finite sequence of points u_m in $W^u(b_m)\cap W^s(b_{m+1})$ such that

$$\forall \epsilon_1 > 0, \exists p_0, \forall p \geq p_0,\ d\left(\mathcal{O}(x_p), \cup_{0\leq m\leq l-1}\mathcal{O}(u_m)\right) < \epsilon_1,$$

where $\mathcal{O}(x)$ denotes the orbit of x under the flow φ_f^t. The key "dynamical argument" in the proof of Weber was to use the Grobman-Hartman linearization Theorem around the critical points of f. Here, we want to prove something slightly stronger in the sense that we will have to keep track of the behaviour of the cotangent vectors in the phase space S^*M and not only of the points in the position space M. For that purpose, we will crucially makes use of the fact that we have a smooth (at least $\mathcal{C}^1$) chart where the vector field can be linearized. In some sense, our compactness statement on S^*M requires the Sternberg-Chen's Theorem while Weber's compactness statement on M only required the Grobman-Hartman's Theorem.

Proof. We only treat the case of Σ_s as the case of Σ_u can be obtained by replacing f by $-f$. In order to prove compactness, we will just prove that Σ_s is closed (as S^*M is compact). Consider (z_m, ζ_m) a converging subsequence in Σ_s. We denote by (z_∞, ζ_∞) its accumulation point which belongs to S^*M. We will proceed by contradiction and suppose that (z_∞, ζ_∞) does not belong to Σ_s. Without loss of generality, we can suppose that, for every $m \geq 1$, z_m belongs to the unstable manifold $W^u(a)$ of some critical point a. Observe also that z_∞ belongs to $W^u(b_1)$ for some b_1 in $\text{Crit}(f)$. If $a = b_1$, we can apply the backward flow Φ^{-T} for T large enough to the sequence $(z_m; \zeta_m)_m$ and its limit (z_∞, ζ_∞) so that we may assume that the sequence and its limit are in the linearized chart near a and go directly to the final step of our proof.

Suppose now that $a \neq b_1$. In that case, we will show that there exists a converging subsequence $(z_m^{(1)}, \zeta_m^{(1)})$ in $\Sigma_s \cap W^u(a)$ such that the accumulation point $(z_\infty^{(1)}, \zeta_\infty^{(1)})$ belongs to $W^s(b_1) - \{b_1\}$ but not to Σ_s. For that purpose, we note that $z_\infty \in W^u(b_1) \implies \lim_{T\to+\infty} \varphi^{-T}(z_\infty) = b_1$. Hence by continuity of $\varphi_f^{-T} : M \mapsto M$, applying the flow $\tilde{\Phi}_f^{-T}$ for some large $T > 0$ and up to another extraction, we can suppose that, for every $m \geq 1$, z_m belongs to the neighborhood of b_1 with adapted coordinates as defined in paragraph 3.5. In other words, we can write z_m as a point $(x_{m,1}, \dots, x_{m,r}, y_{m,r+1}, \dots, y_{m,n})$ such that,

$$\sum_{j=1}^{r} |x_{m,j}| + \sum_{j=r+1}^{n} |y_{m,j}| \leq \frac{\delta_{b_1}}{2},$$

where $\delta_{b_1} > 0$ is the size of the chart of adapted coordinates defined in paragraph 3.5. As $a \neq b_1$, we also know that, for every $m \geq 1$, there exists $1 \leq j \leq r$ such that $x_{m,j} \neq 0$. Otherwise the sequence x_m would belong to $W^u(b_1)$ which is given in local coordinates near b_1 by the equations $\{x_j = 0, 1 \leqslant j \leqslant r\}$. In particular, for every $m \geq 1$, there exists some $T_m > 0$ such that

$$z_m^{(1)} = \varphi_f^{-T_m}(z_m) = (e^{T_m} x_{m,1}, \dots, e^{T_m} x_{m,r}, e^{-T_m} y_{m,r+1}, \dots, e^{T_m} y_{m,n}) = (x_{m,1}^{(1)}, \dots, x_{m,r}^{(1)}, y_{m,r+1}^{(1)}, \dots, y_{m,n}^{(1)})$$

verifies

$$\exists 1 \leq j \leq r \text{ s.t. } |x_{m,j}^{(1)}| \geq \frac{\delta_{b_1}}{2}.$$

One can verify that any accumulation point $z_\infty^{(1)}$ belongs to $W^s(b_1) - \{b_1\}$. Up to this point, we did not use our assumption on ζ_∞ and we did not track what happens for the covector ζ_m under application of the flow $\tilde{\Phi}_f^{-T_m}$. From our assumption that (z_∞, ζ_∞) does not belong to Σ_s and from property (8), we find that $d\left(\tilde{\Phi}_f^{-T}(z_\infty, \zeta_\infty), \Sigma_u\right) \to 0$ when $T \to +\infty$. Fix now $p \geq 1$. Since Σ_u is locally given by $\{\xi_1 = \dots = \xi_r = 0\}$, we observe that if we pick $T > 0$ larger in the first step (in a way that depends on $p \geq 1$) and up to another extraction, we suppose that one can find $m(p) \geq 1$ such that $\zeta_{m(p)}$ belongs to a small conic neighborhood of size $1/p$ of $\{\xi_1 = \ldots = \xi_r = 0\}$. If the conic neighborhood is chosen small enough (i.e. if $p \geq 1$ is large enough), then it is stable under the action of $\tilde{\Phi}_f^{-T_m}$. Hence, if we set $(z_m^{(1)}, \zeta_m^{(1)}) = \tilde{\Phi}_f^{-T_m}(z_m, \zeta_m)$, then, for every p large enough, $\zeta_{m(p)}^{(1)}$ still belongs to a conic neighborhood of the stable direction of size $1/p$. Note that, as in the proof of Lemma 3.4, this property follows from the expression we have for the Hamiltonian flow Φ_f^t in the adapted system of coordinates.

Up to another extraction, we can thus suppose that the limit $(z_\infty^{(1)}, \zeta_\infty^{(1)})$ satisfies

$$z_\infty^{(1)} \in W^s(b_1) - \{b_1\}, \text{ and } \zeta_\infty^{(1)} \in E_s^*(z_\infty^{(1)}),$$

which means in fact that $(z_\infty^{(1)}, \zeta_\infty^{(1)}) \in \Sigma_u$. In particular, as $E_u^* \cap E_s^* = \{0\}$ from the Morse-Smale assumption, we can deduce that $(z_\infty^{(1)}, \zeta_\infty^{(1)})$ does not belong to Σ_s which implies the expected property.

Now, there exists another critical point b_2 such that $z_\infty^{(1)}$ belongs to $W^u(b_2)$. If $b_2 \neq a$, we can apply the same procedure one more time. As there are only finitely many critical points, this iteration should terminate with a convergent sequence $(z_m^{(l)}, \zeta_m^{(l)})_m$ in $\Sigma_s \cap S^*_{W^u(a)}M$ such that its limit $(z_\infty^{(l)}, \zeta_\infty^{(l)}) \in S^*_{W^u(a)}M$ does not belong to Σ_s by construction. We can iterate the flow $\tilde{\Phi}_f^t$ in backward times once again. Thus, we can suppose that $(z_\infty^{(l)}, \zeta_\infty^{(l)})$ belongs to the adapted chart around a. As $W^u(a)$ is a smoothly embedded manifold around the point a, we can finally get the contradiction as any sequence in Σ_s belonging to this small neighborhood should then converge to a point in Σ_s.

□

3.8. **Invariant neighborhoods.** We conclude this dynamical section with the following Lemma which states that Σ_u and Σ_s are repeller and attractor in a slightly stronger sense than in Lemma 3.5.

Lemma 3.8. *Let (f, g) be a smooth Morse pair that induces a Morse-Smale gradient flow. Let $\epsilon > 0$. Then, there exists an open neighborhood V^s of Σ_s which is of size $\leq \epsilon$ and such that*

$$\forall t \geq 0,\ \tilde{\Phi}_f^t(V^s) \subset V^s.$$

The same holds for Σ_u in backward times.

One more time, the proof makes use of the existence of a (at least $\mathcal{C}^1$) linearizing chart for the flow. We also make use of the fact that the critical values of f are distinct.

Proof. Again, we only treat the case of Σ_s (the case of Σ_u can be obtained by replacing f by $-f$). Recall that we assumed our Morse function f to be excellent, meaning that all its critical values are distinct. Thus, we can define the following **total order** relation between critical points. We say that $a < b$ if $f(a) < f(b)$. This relation allows to order the critical points as $a_1 < a_2 < \cdots < a_K$.

The proof of this lemma requires one more time a delicate analysis of the flow. We construct the neighborhood in a progressive manner. First, we build a small neighborhood of the projection of Σ_s on M which is equal to $\cup_{1 \leq j \leq K : r(a_j) \neq 0} W^u(a_j)$. Then, we adjust the construction to be able to lift this open set into a small open neighborhood of Σ_s inside S^*M. In order to construct the neighborhood in M, we fix, for every $1 \leq j \leq K$ for which a_j has positive Morse index (i.e. $r(a_j) > 0$), the following open neighborhood of a_j inside M:

$$R(a_j, \epsilon_j, \epsilon_j') := \{(x, y) : \forall j,\ |x_j| < \epsilon_j',\ |y_j| < \epsilon_j\},$$

where $\epsilon_j, \epsilon_j' > 0$ are small enough parameters to ensure that this defines an ϵ neighborhood of a_j. We will now adjust the values of these parameters to construct a neighborhood of the projection of Σ_s which is invariant by φ_f^t for every $t \geq 0$. For that purpose, we proceed by induction starting from the largest values of f.

First, we observe that every point whose trajectory enters $R(a_j, \epsilon_j, \epsilon_j')$ will either stay in this open set for every $t \geq 0$, or escape this open set (in a maybe arbitrarily large time) by crossing the following subset of M:

$$F(a_j, \epsilon_j, \epsilon_j') := \{(x, y) : \forall j,\ |x_j| < \epsilon_j',\ \exists j,\ |y_j| = \epsilon_j\}$$

which is one of the face of the boundary of $R(a_j,\epsilon_j,\epsilon_j')$. We will inductively construct from the maximum of f a system of open neighborhoods $R(a_j,\epsilon_j,\epsilon_j')_j, r(a_j)>0$ such that for every face $F(a_j,\epsilon_j,\epsilon_j')$ of $R(a_j,\epsilon_j,\epsilon_j')$, there exists a finite time $T_j>0$, such that for every $x\in F(a_j,\epsilon_j,\epsilon_j')$, the trajectory $t\mapsto\varphi_f^t(x)$ meets $\cup_{j<i}R(a_i,\epsilon_i,\epsilon_i')$ for some $t\in(0,T_j)$. For $j=K$, one can verify that the neighborhood is invariant by the flow in positive time provided that we pick $\epsilon_K=\epsilon_K'>0$ small enough to ensure that we are in the neighborhood of adapted coordinates defined in paragraph 3.5. Suppose now that we have fixed the values of ϵ_i and ϵ_i' for every $i>j$ with $r(a_i)\neq 0$ and that $r(a_j)\neq 0$. We will explain how to fix the value of ϵ_j and ϵ_j'. We claim that the forward trajectory of every point inside $F(a_j,\epsilon_j,\epsilon_j')$ will reach

$$\bigcup_{i>j:r(a_i)\neq 0} R(a_i,\epsilon_i,\epsilon_i')$$

in a finite time $0<t<T_j$ where T_j depends only on ϵ_i,ϵ_i' with $i>j$ satisfying $r(a_i)\neq 0$ and on ϵ_j. In particular, this time can be made uniform in ϵ_j'. Assume by contradiction that, for every $m>0$ and for every $T>0$, there exists $x_{m,T}$ in $F(a_j,\epsilon_j,1/m)$ such that the orbit $t\in[0,T]\mapsto\varphi_f^t(x_{m,T})$ does not meet the subset $\bigcup_{i>j:r(a_i)\neq 0}R(a_i,\epsilon_i,\epsilon_i')$. We fix $T>0$, and, by compactness, one can extract a subsequence such that $x_{m,T}\to x_{\infty,T}$ as $m\to+\infty$ where $x_{\infty,T}$ belongs to $W^u(a_j)$ is at distance $>\mathcal{O}(\epsilon_j)$ of a_j. We now extract another subsequence (as $T\to+\infty$) and we obtain a point $x_\infty\neq a_j$ in $W^u(a_j)$ that would not reach $\bigcup_{i>j:r(a_i)\neq 0}R(a_i,\epsilon_i,\epsilon_i')$ in finite time. This contradicts the fact that $\lim_{t\to+\infty}\varphi_f^t(x_\infty)$ is equal to a_i for some $i>j$ satisfying $r(a_i)\neq 0$.

Recall now that the distance between two trajectories can grow at most exponentially under the flow [78, Lemma 11.11]. Hence, if we choose $\epsilon_j'>0$ small enough, we can ensure that, the forward trajectory of every point inside $F(a_j,\epsilon_j,\epsilon_j')$ will remain ϵ close to $W^u(a_j)$ up the finite time $t\leq T_j$ where it will enter one of the neighborhood $R(a_i,\epsilon_i,\epsilon_i')$ with $i>j$ and $r(a_i)>0$. This construction defines a family of open neighborhood of the critical points a_j of index >0 whose forward trajectory under the flow will remain within a distance $\epsilon>0$ of $\cup_{1\leq j\leq K:r(a_j)\neq 0}W^u(a_j)$ which is exactly the projection of Σ_s on M. Then, we set

$$\mathcal{N}:=\bigcup_{t\geq 0}\bigcup_{1\leq j\leq K:r(a_j)\neq 0}\varphi_f^t(R(a_j,\epsilon_j,\epsilon_j')).$$

By construction, this set is invariant by φ_f^t. Moreover, it defines a neighborhood of $\cup_{1\leq j\leq K:r(a_j)\neq 0}W^u(a_j)$ which is of size $\leq\epsilon$.

It now remains to verify that we can lift this neighborhood into a neighborhood of size ϵ of Σ_s. For that purpose, we rely on the fact that, our smooth system of coordinate chart allows to linearize also the Hamiltonian flow Φ_f^t. Hence, we fix another positive parameter $\epsilon_j''>0$ and we consider above each neighborhood $R(a_j,\epsilon_j,\epsilon_j')$ an open neighborhood $\tilde{R}(a_j,\epsilon_j,\epsilon_j',\epsilon_j'')$ in S^*M made of unit covectors which are within a distance $<\epsilon_j''$ of $\xi_1=\ldots=\xi_r=0$. For every fixed choice of $\epsilon_j>0$ and $\epsilon_j''>0$, we can use the compactness of Σ_s to fix $\epsilon_j'>0$ small enough to ensure that this defines indeed a neighborhood of size $<\epsilon$ of $\Sigma_s\cap S^*R(a_j,\epsilon_j,\epsilon_j')$. Using the fact that the distance between two trajectories can grow at most exponentially under the flow $\tilde{\Phi}_f^t$, we can argue by induction as in the case of M. More precisely, at each step of the induction, we can fix $\epsilon_j'' > 0$ small enough in a way that depends only on the values of ϵ_j and of $\epsilon_i^{(*)}$ with $i > j$ and $r(a_i) > 0$ and such that

$$\tilde{\mathcal{N}} := \bigcup_{t\geq 0} \bigcup_{1\leq j\leq K: r(a_j)\neq 0} \tilde{\Phi}_f^t(\tilde{R}(a_j, \epsilon_j, \epsilon_j', \epsilon_j''))$$

defines a forward invariant open neighborhood of Σ_s of size $< \epsilon$. □

4. Spectral properties of the transfer operator acting on currents

In [32], Faure and Sjöstrand constructed some anisotropic Sobolev spaces adapted to the spectral study of transfer operators for Anosov flows – see also [31] for the case of Anosov diffeomorphisms. The extension of this microlocal approach to anisotropic spaces of currents was done by Dyatlov and Zworski in [28, Sect. 3] by a slightly different approach. The definition of these spaces is based on the construction of a nice escape function adapted to the dynamics of the flow, namely a function that is strictly decreasing along the flow. This type of construction appears in fact naturally in the study of semiclassical resonances – see e.g. the appendix of [38]. In our framework, a natural candidate to pick regarding (4) is the opposite of the function f. However, near the critical points of f, the derivative along the flow vanishes, and we have to find an appropriate candidate for this part of phase space. This can be done by using the hyperbolicity of the flow at these points and mimicking the construction of [32].

This section is organized as follows. First, we state the existence of a nice escape function enjoying the dynamical features of [32, 28]. This allows us to define some Sobolev spaces of anisotropic currents following these references. Finally, we recall the spectral properties of $-\mathcal{L}_{V_f}^{(k)}$ acting on these spaces. The main difference with the above references is the construction of the escape function which requires modifications compared with the setting from [32, Lemma 2.1] where the authors made use of the Anosov property. Lemmas 3.6 and 3.8 will in fact ensure that the construction of Faure and Sjöstrand can be extended to our framework.

From this point on, we will always assume that (f, g) is a **smooth Morse pair generating a Morse-Smale gradient flow**.

4.1. Construction of anisotropic Sobolev spaces.

4.1.1. *Escape function.* The key ingredient in the construction of [32] is the following Lemma which will allow us to define appropriate Sobolev spaces where the operator $\mathcal{L}_{V_f}$ has nice spectral properties.

Lemma 4.1 (Escape function). *Let $N_0, N_1 > 4\|f\|_{\mathcal{C}^0}$ be two elements in $\mathbb{R}$. Then, there exist $c_0 > 0$ (depending on (M, g) but not on N_0 and N_1) and a smooth function $m(x, \xi)$ in $\mathcal{C}^\infty(T^*M)$ with bounded derivatives and which*

- *takes values in $[-2N_0, 2N_1]$,*
- *is 0 homogeneous for $\|\xi\|_x \geq 1$,*

- *is* $\leq -\frac{N_0}{2}$ *on a conic neighborhood of* Γ_- *(for* $\|\xi\|_x \geq 1$*),*
- *is* $\geq \frac{N_1}{2}$ *on a conic neighborhood of* Γ_+ *(for* $\|\xi\|_x \geq 1$*),*

and such that there exists $R_0 > 0$ *for which the escape function*

$$G_m(x,\xi) := m(x,\xi)\log(1+\|\xi\|_x^2)$$

verifies, for every (x,ξ) *in* T^*M *with* $\|\xi\|_x \geq R_0$*,*

$$X_{H_f}.(G_m)(x,\xi) \leq -C_N := -c_0 \min\{N_0, N_1\}.$$

Now that we have settled the dynamical framework precisely in section 3, the construction of the function G_m closely follows the one from [32]. For the sake of exposition, we postpone the detailed proof of this result to appendix A, and we just mention the key ingredients: (1) f is strictly decreasing along the flow, (2) there exists a $\mathcal{C}^1$ chart of adapted coordinates (see paragraph 3.5), (3) the attractor and repeller of the Hamiltonian flow (Lemmas 3.6 and 3.8) are compact. Lemma 4.1 is in fact the only step in the construction of the anisotropic Sobolev space where one uses the dynamical properties of the flow under consideration.

4.1.2. *Anisotropic Sobolev spaces.* Let us now define the corresponding anisotropic Sobolev spaces. We fix $N_0, N_1 > 4\|f\|_{\mathcal{C}^0}$ large and we set

$$A_m(x,\xi) := \exp G_m(x,\xi), \tag{10}$$

where $G_m(x,\xi)$ is given by Lemma 4.1. Following paragraph 1.1.2 in [32], one can define the following anisotropic Sobolev space

$$\mathcal{H}^m(M) := \mathrm{Op}(A_m)^{-1}(L^2(M)),$$

where $\mathrm{Op}(A_m)$ is a pseudodifferential operator[5] with principal symbol A_m.

We now briefly collect some facts concerning these spaces and we refer to [31, Sect. 3.2] for more properties of these spaces. The space $\mathcal{H}^m(M)$ is endowed with a Hilbert structure induced by the Hilbert structure on $L^2(M)$. The space

$$\mathcal{H}^{-m}(M) = \mathrm{Op}(A_m)L^2(M)$$

is the topological dual of $\mathcal{H}^m(M)$. The anisotropic Sobolev space $\mathcal{H}^m(M)$ is a reflexive space. Finally, one has

$$C^\infty(M) \subset \mathcal{H}^m(M) \subset \mathcal{D}'(M),$$

where the injections are continuous.

[5]Note that this requires to deal with symbols of variable orders whose symbolic calculus was described in Appendix A of [31]. This can be done as the symbol $m(x,\xi)$ belongs to the standard class of symbols $S^0(T^*M)$.

4.1.3. *Anisotropic Sobolev spaces of currents.* Let $0 \leq k \leq n$. We consider the vector bundle $\Lambda^k(T^*M)$ of exterior k forms. We define $\mathbf{A}_m^{(k)}(x,\xi) := A_m(x,\xi)\mathbf{Id}$ belonging to $\text{Hom}(\Lambda^k(T^*M))$. We fix the inner product $\langle , \rangle_{g^*}^{(k)}$ on $\Lambda^k(T^*M)$ which is induced by the metric g on M.

This allows to define the Hilbert space $L^2(M,\Lambda^k(T^*M))$ and to introduce an anisotropic Sobolev space of currents by setting

$$\mathcal{H}_k^m(M) = \text{Op}(\mathbf{A}_m^{(k)})^{-1}L^2(M,\Lambda^k(T^*M)),$$

where $\text{Op}(\mathbf{A}_m^{(k)})$ is a pseudodifferential operator with principal symbol $\mathbf{A}_m^{(k)}$. We refer to [28, App. C.1] for a brief reminder of pseudodifferential operators with values in vector bundles. In particular, adapting the proof of [31, Cor. 4] to the vector bundle valued framework, one can verify that $\mathbf{A}_m^{(k)}$ is an elliptic symbol, and thus $\text{Op}(\mathbf{A}_m^{(k)})$ can be chosen to be invertible. Mimicking the proofs of [31], we can deduce some properties of these spaces of currents. First of all, they are endowed with a Hilbert structure inherited from the L^2-structure on M. The space

$$\mathcal{H}_k^m(M)' = \text{Op}(\mathbf{A}_m^{(k)})L^2(M,\Lambda^k(T^*M))$$

is the topological dual of $\mathcal{H}_k^m(M)$ which is in fact reflexive. We also note that the space $\mathcal{H}_k^m(M)$ can be identified with $H^m(M)\otimes_{\mathcal{C}^\infty(M)}\Omega^k(M)$. Finally, one has

$$\Omega^k(M) \subset \mathcal{H}_k^m(M) \subset \mathcal{D}'^{,k}(M),$$

where the injections are continuous.

4.2. **Identifying the dual.** Recall that the Hodge star operator is the unique isomorphism $\star_k : \Lambda^k(T^*M) \to \Lambda^{n-k}(T^*M)$ such that, for every ψ_1 in $\Omega^k(M)$ and ψ_2 in $\Omega^{n-k}(M)$,

$$\int_M \psi_1 \wedge \psi_2 = \int_M \langle \psi_1, \star_k^{-1}\psi_2\rangle_{g^*(x)}^{(k)}\omega_g(x),$$

where $\langle ., .\rangle_{g^*(x)}^{(k)}$ is the induced Riemannian metric on $\Lambda^k(T^*M)$. In particular, $\star_k$ induces an isomorphism from $\mathcal{H}_k^m(M)'$ to $\mathcal{H}_{n-k}^{-m}(M)$, whose Hilbert structure is given by the scalar product

$$(\psi_1,\psi_2) \in \mathcal{H}_{n-k}^{-m}(M)^2 \mapsto \langle \star_k^{-1}\psi_1, \star_k^{-1}\psi_2\rangle_{\mathcal{H}_k^m(M)'}.$$

Thus, the topological dual of $\mathcal{H}_k^m(M)$ can be identified with $\mathcal{H}_{n-k}^{-m}(M)$, where, for every ψ_1 in $\Omega^k(M)$ and ψ_2 in $\Omega^{n-k}(M)$, one has the following duality relation:

$$\langle \psi_1,\psi_2\rangle_{\mathcal{H}_k^m\times\mathcal{H}_{n-k}^{-m}} = \int_M \psi_1\wedge\psi_2 = \langle \text{Op}(\mathbf{A}_m^{(k)})\psi_1, \text{Op}(\mathbf{A}_m^{(k)})^{-1}\star_k^{-1}\overline{\psi_2}\rangle_{L^2} = \langle \psi_1, \star_k^{-1}\psi_2\rangle_{\mathcal{H}_k^m\times(\mathcal{H}_k^m)'}.$$

4.3. **Discrete spectrum.** The main result on the spectral properties of $-\mathcal{L}_{V_f}^{(k)}$ acting on these anisotropic spaces is the following Proposition:

Proposition 4.2 (Discrete spectrum)**.** *The operator $-\mathcal{L}_{V_f}^{(k)}$ defines a maximal closed unbounded operator on $\mathcal{H}_k^m(M)$,*

$$-\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M),$$

with domain given by $\mathcal{D}(-\mathcal{L}_{V_f}^{(k)}) := \{u \in \mathcal{H}_k^m(M) : -\mathcal{L}_{V_f}^{(k)} u \in \mathcal{H}_k^m(M)\}$*. It coincides with the closure of* $-\mathcal{L}_{V_f}^{(k)} : \Omega^k(M) \to \Omega^k(M)$ *in the graph norm for operators. Moreover, there exists a constant* C_0 *in* $\mathbb{R}$ *(that depends on the choice of the order function* $m(x,\xi)$*) such that* $-\mathcal{L}_{V_f}^{(k)}$ *has empty spectrum for* $\mathrm{Re}(z) > C_0$*. Finally, the operator*

$$-\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M),$$

has a discrete spectrum with finite multiplicity in the domain

$$\mathrm{Re}(z) > -C_N + C,$$

where $C > 0$ *depends only the choice of the metric (and the local coordinate charts) and* $C_N > 0$ *is the constant from Lemma 4.1.*

The second part on the discrete spectrum is obtained by showing that the operator is $(-\mathcal{L}_{V_f}^{(k)} - z)$ is a Fredholm operator of index 0 depending analytically on z in the corresponding half plane [48, 78]. In the case of Anosov flows, the proof of this result was given by Faure-Sjöstrand in [32, Sect. 3] for $k = 0$ while the extension to the case of currents was done by Dyatlov-Zworski in [28, Sect. 3]. Note that the proofs in both references are of slightly different nature but they both crucially rely on the properties of the escape function used to define the anisotropic space $\mathcal{H}_k^m(M)$. Up to some adaptations to deal with the case of currents, we can in fact follow the proof of [32] for the extension to the case of currents – see appendix B for a brief account on the proof of Faure and Sjöstrand.

We now list some properties of this spectrum. As in [32, Th. 1.5], we can show that the eigenvalues (counted with their algebraic multiplicity) and the eigenspaces of $-\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M)$ are in fact independent of the choice of escape function. For every $0 \leq k \leq n$, we call the eigenvalues the **Pollicott-Ruelle resonances of index** k. For later use, we will write

$$\mathcal{R}_k(f,g) := \{\text{Pollicott-Ruelle resonances of index } k\} \subset \mathbb{C}.$$

In other words, these complex numbers are the poles of the meromorphic extension of the resolvent

$$\left(-\mathcal{L}_{V_f}^{(k)} - z\right)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M).$$

Finally, we note that, by duality, the same spectral properties holds for the dual operator

$$(-\mathcal{L}_{V_f}^{(k)})^\dagger = -\mathcal{L}_{V_{-f}}^{(n-k)} : \mathcal{H}_{n-k}^{-m}(M) \to \mathcal{H}_{n-k}^{-m}(M). \tag{11}$$

5. Computing the spectrum

In sections 5 to 7, we aim at describing the eigenvalues and the eigenmodes in great details. In order to explain the idea in a simpler manner, we start with the simpler question of determining the Pollicott-Ruelle resonances of $-\mathcal{L}_{V_f}^{(k)}$ without describing their corresponding eigenmodes and/or their algebraic multiplicity. In sections 6 and 7, we will refine the arguments presented here in order to obtain a complete description of the spectrum. Yet, in order to emphasize the main ideas of the proof, we start with this simpler question, and the main result of this section reads:

Proposition 5.1. *Suppose that* (f, g) *is a smooth Morse pair inducing a Morse-Smale gradient flow. Then, one has, for every* $0 \leq k \leq n$,

$$\mathcal{R}_k(f, g) = \mathcal{I}_k := \bigcup_{a \in \text{Crit}(f)} \mathcal{I}_k(a),$$

where, for every a *in* $\text{Crit}(f)$ *of index* $0 \leq r \leq n$,

$$\mathcal{I}_k(a) := \bigcup_{I \subset \{1,\ldots,r\},\ J \subset \{r+1,\ldots,n\} : |J|-|I|=k-r} \left\{ -\sum_{j=1}^{n} \alpha_j |\chi_j(a)| :\ \forall j \in I \cup J,\ \alpha_j \geq 1 \right\}. \tag{12}$$

In other words, *up to the multiplicities*, we are able to determine exactly the Pollicott-Ruelle spectrum. Building on the construction of section 4, we shall now give the proof of this Proposition. We emphasize that the existence of a *smooth* linearizing chart is crucial in our proof. With lower regularity (say $\mathcal{C}^1$), we would not a priori obtain such a precise description. We proceed in several steps. First, we start with a simple propagation Lemma which will be used at several stages of our proof. Then, we use our a priori information on the regularity of the resonant states to show the inclusion $\mathcal{R}_k(f, g) \subset \mathcal{I}_k$. Finally, by considering appropriate test functions, we obtain the converse inclusion $\mathcal{I}_k \subset \mathcal{R}_k(f, g)$.

5.1. **Propagation Lemma.** In our proof, we intend to use a simple propagation Lemma that we will now prove.

Lemma 5.2 (Propagation Lemma)**.** *Let* $0 \leq k \leq n$, *let* z *in* $\mathbb{C}$ *and let* $u \in \mathcal{D}'^{,k}(M)$ *be a solution of* $\mathcal{L}_{V_f}^{(k)} u = zu$. *If* $u|_U = 0$ *where* $U \subset M$ *is some open subset then* u *vanishes on the larger open subset* $\bigcup_{t \in \mathbb{R}} \varphi_f^t(U)$.

Proof. We shall establish the result by a duality argument. First, we note that, for every ψ in $\Omega^{n-k}(M)$,

$$\frac{d}{dt} \left\langle \varphi_f^{t*} u, \psi \right\rangle = \left\langle \varphi_f^{t*} \mathcal{L}_{V_f}^{(k)} u, \psi \right\rangle = z \left\langle \varphi_f^{t*} u, \psi \right\rangle.$$

Hence, solving the ODE yields

$$\varphi_f^{t*} u = e^{tz} u,\ \forall t \in \mathbb{R}.$$

Assume now $x \in \bigcup_{t \in \mathbb{R}} \varphi_f^t(U)$. It means that there is some $t_0 \in \mathbb{R}$ such that $x \in \varphi_f^{t_0}(U)$ which is an open subset of M. Let ψ be an element in $\Omega^{n-k}(\varphi_f^{t_0}(U))$. We have the identity

$$\langle u, \psi \rangle = \langle (\varphi_f^{t_0})^* u, (\varphi_f^{-t_0})_* \psi \rangle = \langle e^{t_0 z} u, (\varphi_f^{-t_0})_* \psi \rangle = 0$$

since supp $\left((\varphi_f^{-t_0})_* \psi \right) \subset U$ and $u|_U = 0$. □

5.2. **Proving $\mathcal{R}_k(f,g)\subset\mathcal{I}_k$.** We let $0\leq k\leq n$. Let $z=\lambda+i\gamma$ be an element in $\mathcal{R}_k(f,g)$. We will prove that z belongs to $\mathcal{I}_k$ via a kind of induction argument on the set of critical points. For that purpose, we order this set $a_1<a_2<\ldots<a_K$ by increasing order as in the proof of Lemma 3.8. In order to clarify the main idea, we will explain first the argument in the case of critical points of index 0, i.e. minima of f. Then, we will treat the general case using the pull-back Theorem of Hörmander [50].

5.2.1. *Smoothness near minima.* From the definition of $\mathcal{R}_k(f,g)$, one knows that there exists u in $\mathcal{H}_k^m(M)$ such that $\mathcal{L}_{V_f}^{(k)}u=-zu$. From [32, Th. 1.5], we know that the spectrum and the generalized eigenvalues are intrinsic, i.e. they do not depend on the choice of the order function $m(x,\xi)$. Thus, for any given N, we know from Sobolev injections that, if we pick N_1 large enough in the definition of m, u is of class $\mathcal{C}^N$ in a small neighborhood of any minimum. The neighborhood may depend on the choice of N_1. Yet, by a propagation argument, this remains true on the image of this neighborhood under the flow φ_f^t. In other words, u is smooth in the neighborhood of any local minimum of f, i.e. close to every point a of index $r(a)=0$. We have then the following alternative

- either u identically vanishes in a neighborhood of every point of index 0,
- or there exists w_0 in a small neighborhood of a point a of index 0 such that w_0 belongs to the neighborhood of adapted coordinates of paragraph 3.5 and $u(w_0)\neq 0$.

Suppose that we are in the second case. Following the proof of the propagation Lemma, we write that, for every $t\geq 0$,

$$u\circ\varphi_f^{-t}(w_0)=e^{(\lambda+i\gamma)t}u(w_0).$$

Equivalently, if we set $t=-\ln s$ for $0<s\leq 1$, one has

$$u\circ\varphi_f^{\ln s}(w_0)=s^{-(\lambda+i\gamma)}u(w_0). \tag{13}$$

Recall that, if we integrate the flow of $\varphi_f^{\ln s}$ in the adapted coordinates $w=(y_i)_{1\leq i\leq n}$ of paragraph 3.5, we get

$$\varphi_f^{\ln s}(w)=(s^{\chi_i(a)}y_i)_{i=1,\ldots,n}.$$

Hence, when we look at equality (13), the flow acts on the $(y_i)_{1\leq i\leq n}$ variables via scalings with **different speeds** depending on the various Lyapunov exponents. We would now like to conclude that equality (13) combined with $u(w_0)\neq 0$ implies that z belongs to $\mathcal{I}_k(a)$.

For that purpose, we will use Taylor expansion of u in both even and odd variables as u is a **smooth form** near a. Indeed, we think of exterior forms $u\in\Omega^k(M)$ as super functions, i.e. as elements

$$u(y_1,\ldots,y_n;dy_1,\ldots,dy_n)$$

of the Grassmann algebra $C^\infty(M)[dy_1,\ldots,dy_n]$ over the algebra $C^\infty(M)$. Hence, as u is smooth, it admits a Taylor expansion in both **even** $(y_1,\ldots,y_n)$ and **odd** variables $(dy_1,\ldots,dy_n)$. The Taylor formula reads :

$$u(y;dy)\sim\sum_{(\alpha,\beta)\leqslant N}u_{\alpha,\beta}y^\alpha dy^\beta$$

where (α, β) are multi–indices and the polynomials are of bounded degree in the odd variables dy. Note that this Taylor expansion in the adapted coordinate charts makes sense because *we are considering smooth change of variables.* We now introduce n scaling variables $(s_1, \ldots, s_n) \in \mathbb{R}^n$. Then, with the conventions of paragraph 3.5, we consider the map $\Phi : (s_j, y_j)_{1\leq j\leq n} \in (-1,1)^n \times W_a \mapsto (s_j y_j)_{1\leq j\leq n} \in W_a$ which acts by pull–back as :

$$\Phi^* u(s, x, dx) = u(s_j y_j, s_j dy_j) \tag{14}$$

which implies that $\Phi^* u(s, y, dy)$ is **smooth** in $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$. However our goal is to study the asymptotic behaviour of $u(s^{\chi_j(a)} y_j, s^{\chi_j(a)} dy_j)$ near $s = 0$. Thanks to Lemma C.3, we can deduce that this expression has a polyhomogeneous expansion in s. Combining this observation to equation (13) (with $u(w_0) \neq 0$) and to Lemma C.4, we deduce that

$$\gamma = 0, \text{ and } \lambda = -\sum_{j=1}^{n} \alpha_j \chi_j(a),$$

for some $\alpha_j \in \mathbb{N}$. We also note that, as u belongs to $\Omega^k(M)$, at least k of the α_j must be ≥ 1. This shows that z belongs to $\mathcal{I}_k(a)$ except if u was identically vanishing in a neighborhood of a. Our next step will be to generalize this argument near any critical point of f.

5.2.2. *The general case.* Assume now that, for all $i \leqslant j$, the eigenfunction u vanishes near a_i and assume the germ of u near a_{j+1} is non vanishing (see Lemma 5.3 below). In that case, we would like to prove that z is of the form:

$$z = -\sum_{l=1}^{n} \alpha_l |\chi_l(a_{j+1})|,$$

with some restrictions on the coefficients α_l. For that purpose, we start with the following central observation:

Lemma 5.3. *Let $u \in \mathcal{D}'^{,k}(M)$ be some eigencurrent of $-\mathcal{L}_{V_f}^{(k)}$ acting on $\mathcal{H}_k^m(M)$.*

If u vanishes in some neighborhood of all a_i for $i \leqslant j$, then u restricted to the level $f^{-1}(< f(a_{j+1}))$ vanishes. Moreover, if the germ $u|_{V_{a_{j+1}}} \neq 0$ (for the adapted chart $\kappa_{a_{j+1}} : V_{a_{j+1}} \to W_{a_{j+1}}$ defined in paragraph 3.5), then the germ $u|_{V_{a_{j+1}}}$ is supported in the germ of unstable manifold $W^u(a_{j+1}) \cap V_{a_{j+1}}$.

Remark 5.4. A first consequence of this Lemma is that there is necessarily a critical point a in a neighborhood of which u does not vanish.

Proof. Assume without loss of generality that $f(a_{j+1}) = 0$. The level $f^{-1}(< 0)$ contains only the critical points $\{a_1, \ldots, a_j\}$. Moreover, since the value of the potential f is monotonic along the flow it follows that the level $f^{-1}(< 0)$ is contained in the union of unstable manifolds $\bigcup_{i\leq j} W^u(a_i)$. Hence, by the propagation Lemma, $u|_{\{f^{-1}(<0)\}} = 0$. Now consider some open set V in $V_{a_{j+1}}$ which does not interset $W^u(a_{j+1})$. Using the facts that f is excellent that $W^u(a_{j+1})$ is an embedded submanifold and that f must increase along the flow, one knows that, for every x in V, $x_- = \lim_{t\to-\infty} \varphi_f^t(x)$ belongs to $\{a_1, \ldots, a_j\}$. Using the propagation Lemma one more time, one can then deduce that $u_{|V} = 0$. This is valid for any open set $V \subset V_{a_{j+1}}$ which does not intersect $W^u(a_{j+1})$. In other words, $u_{|U}$ is supported in the germ of unstable manifold $W^u(a_{j+1}) \cap V_{a_{j+1}}$, which concludes the proof of the Lemma. □

We want to proceed as in the first step of the induction. In other words, we let ψ be a smooth test form in $\Omega^{n-k}(M)$ which is compactly supported in $V_{a_{j+1}}$. We can then write, for every s in $(0,1]$,

$$\left\langle (\varphi_f^{\ln s})^* u, \psi \right\rangle = s^{-(\lambda+i\gamma)} \langle u, \psi \rangle.$$

As we made the assumption that u is not identically vanishing in the neighborhood of a_{j+1}, we can choose ψ compactly supported near a_{j+1} such that $\langle u, \psi \rangle \neq 0$. Then, we would like to prove that the left-hand side of the equality admits a polyhomogeneous expansion in s which is indexed by the set $\mathcal{I}_k(a_{j+1})$. Combining this to Lemma C.4, we would then deduce that z is of the expected form. Thus, our last task is to prove that $\left\langle (\varphi_f^{\ln s})^* u, \psi \right\rangle$ admits a polyhomogeneous expansion indexed by $\mathcal{I}_k(a_{j+1})$. For that purpose, we shall work using the local coordinates (x, y) defined in paragraph 3.5. We denote by $\tilde{u}$ the image of u in this chart. From Lemma 5.3, this defines a current which is carried in $(-\delta_a, \delta_a)^n \cap \{x = 0\}$. This can be extended into a current defined on $\tilde{W}_a := \mathbb{R}^{r(a)} \times (-\delta_a, \delta_a)^{n-r(a)}$ by setting $\tilde{u} = 0$ outside $(-\delta_a, \delta_a)^n$. Then, we introduce the following map

$$\Phi : (s_1, \ldots, s_n; x, y) \in (-1,1)^n \times \tilde{W}_a \mapsto (s_i^{-1} x_i, s_j x_j) \in \tilde{W}_a.$$

Note that this is well defined except if $s_i = 0$ for some $1 \leq i \leq r(a)$. We also define the partial maps:

$$\Phi^1 : (s_1, \ldots, s_n; x, y) \in (-1,1)^n \times \tilde{W}_a \mapsto (x_j, s_j y_j) \in \tilde{W}_a,$$

and

$$\Phi^2 : (s_1, \ldots, s_n; x, y) \in (-1,1)^n \times \tilde{W}_a \mapsto (s_j x_j, y_j) \in \tilde{W}_a.$$

Contrary to Φ, these two maps are well defined for $\mathbf{s}$ belonging to the whole set $(-1,1)^n$. Let $\mathbf{s}$ be a point in $(-1,1)^n$ with all entries which are non vanishing. In that case, we can write

$$\langle \Phi(\mathbf{s})^* \tilde{u}, \tilde{\psi} \rangle = \langle \Phi^2(\mathbf{s})_* \Phi^1(\mathbf{s})^* \tilde{u}, \tilde{\psi} \rangle = \langle \Phi^1(\mathbf{s})^* \tilde{u}, \Phi^2(\mathbf{s})^* \tilde{\psi} \rangle. \tag{15}$$

This is valid as long as $s_j \neq 0$ for every $1 \leq j \leq n$. Our next step is to show that this extends as a smooth function on $(-1,1)^n$. From the previous expression, one can observe that the main concern is to be able to study the smoothness of $\Phi^1(\mathbf{s})^* \tilde{u}$ in the variable $(s_j)_j \in (-1,1)^n$. Recall that u is an eigenvector of $-\mathcal{L}_{V_f}^{(k)}$ acting on a certain anisotropic Sobolev space $\mathcal{H}_k^m(M)$. According to [32, Th. 1.5], the eigenmodes are independent of the choice of the order function $m(x, \xi)$ satisfying the assumptions of Lemma 4.1. As in the case of minima, letting $N_1 \to +\infty$ in this Lemma and using Lemma 5.3, one finds that the wave front set $WF(\tilde{u})$ of $\tilde{u}$ satisfies the following

$$WF(\tilde{u}) \subset \left\{ (0, y, \xi, 0) \in T^* \tilde{W}_a : \xi \neq 0 \right\}. \tag{16}$$

We would now like to define the pull-back of $\tilde{u}$ under the map Φ^1, and, for that purpose, we shall apply Hörmander's pullback Theorem [50, Th. 8.2.4] – see also [15]. Hence, we have to compute the normal $N^*_{\Phi^1} \subset T^*\tilde{W}_a$ of the map Φ^1,

$$\begin{aligned} N^*_{\Phi^1} &= \{(x, s_j y_j; \xi, \eta) \text{ such that } (0,0,0) = (\xi,\eta) \circ d_{(\mathbf{s},x,y)}\Phi, (\xi,\eta) \neq (0,0)\} \\ &= \left\{(x, s_j y_j; \xi, \eta) \text{ such that } (0,0,0) = (\xi,\eta) \circ \begin{pmatrix} 0 & 1 & 0 \\ y & 0 & (s_j)_j \end{pmatrix}, (\xi,\eta) \neq (0,0)\right\} \\ &= \left\{(x, s_j y_j; \xi, \eta) \text{ such that } \left(\sum_j y_j \eta^j, \xi, \sum_j s_j \eta^j\right) = (0,0,0), (\xi,\eta) \neq (0,0)\right\} \\ &= \left\{(x, s_j y_j; 0, \eta) \text{ such that } \eta \neq 0, \sum_j s_j \eta^j = \sum_j y_j \eta^j = 0\right\}. \end{aligned}$$

In particular, from (16), $N^*_{\Phi^1} \cap WF(\tilde{u})$ is empty. Hence, we can apply Hörmander's pullback Theorem, i.e. $(\Phi^1)^*\tilde{u}$ is well defined and its wave front set is contained in

$$\begin{aligned} (\Phi^1)^* WF(\tilde{u}) &= \left\{\left(\mathbf{s}, x, y; \sum_j y_j \eta^j, \xi, \sum_j s_j \eta^j\right) \text{ such that } \left(x, \sum_j s_j y_j; \xi, \eta\right) \in WF(\tilde{u})\right\} \\ &\subset \{(\mathbf{s}, 0, y; 0, \xi, 0) \text{ such that } \xi \neq 0\}. \end{aligned}$$

As $\tilde{\psi}$ is a smooth test form, we note that $(\Phi^1)^*\tilde{u} \wedge (\Phi^2)^*\tilde{\psi}$ is a current of degree n on $(-1,1)^n \times \tilde{W}_a$ whose wave front set is included in $(\Phi^1)^*WF(\tilde{u})$. Consider now the push-forward of this current under the map

$$\mathbf{p} : (x, y, \mathbf{s}) \in \tilde{W}_a \times (-1,1)^n \mapsto \mathbf{s} \in (-1,1)^n$$

By the push-forward Theorem [50, 15], the wave front set of the pushforward distribution is included in

$$\mathbf{p}_*\left((\Phi^1)^* WF(\tilde{u})\right) = \left\{(\mathbf{s}; \sigma) \text{ such that } (\mathbf{s}, x, y; \sigma, 0, 0) \in (\Phi^1)^* WF(\tilde{u}),\ \sigma \neq 0\right\} = \emptyset.$$

In other words, the pushforward distribution is a smooth function in the variable $\mathbf{s} \in (-1,1)^n$. In particular, according to (15), $\langle \Phi(\mathbf{s})^*\tilde{u}, \tilde{\psi}\rangle$ has a well–defined Taylor expansion in $\mathbf{s}$ around 0. Then, we can combine Lemma C.3 to the fact that, in our system of adapted coordinates, the reparametrized flow $\varphi_f^{\ln s}$ can be written $(s^{\chi_j(a)} x_j, s^{\chi_j(a)} y_j)$. From that, we deduce the expected property, i.e. $\left\langle (\varphi_f^{\ln s})^* u, \psi \right\rangle$ admits a polyhomogeneous expansion indexed by $(\alpha.|\chi(a)|)_{\alpha \in \mathbb{N}^n}$. In order to conclude the proof, we should observe that $u(x, y, dx, dy)$ is of degree k and $\psi(x, y, dx, dy)$ of degree $n-k$. This forces that some of the α_j do not vanish when we express z as a combination of the Lyapunov exponents, i.e. z must in fact belong to the set $\mathcal{I}_k(a_{j+1})$. This concludes the proof of the first inclusion, i.e. $\mathcal{R}_k(f,g) \subset \mathcal{I}_k$.

Remark 5.5. We shall use this kind of arguments several times in the following. We observe that we have just been able to prove that $\langle \varphi_f^{\ln s *} u, \psi\rangle$ has a polyhomogeneous expansion indexed by the set $\mathcal{I}_k(a_{j+1})$ and that our proof only made use of the facts that the support of u near a_{j+1} was included in $W^u(a_{j+1})$ and that its wave front is included in E_u^* near a_{j+1}.

5.3. **Asymptotic expansion of the correlation function.** Before proving the converse inclusion, let us draw some consequences of the fact that $\mathcal{R}_k(f,g)\subset\mathcal{I}_k$ following the lines of [31]. From (50) in the appendix, we know that $(\varphi_f^{-t})^*$ generates also a strongly continuous semi-group from $\mathcal{H}_k^m(M)$ to $\mathcal{H}_k^m(M)$ for every $1\leq k\leq n$ whose norm is bounded by e^{tC_0}. Fix now $\Lambda>0$. Suppose without loss of generality that $-\Lambda$ does not belong to $\mathcal{I}_k$. From Proposition 4.2, we now observe that, for every $\Lambda>0$, one can find a weight function $m(x,\xi)$ such that the operator

$$-\mathcal{L}_{V_f}^{(k)}:\mathcal{H}_k^m(M)\rightarrow\mathcal{H}_k^m(M)$$

has only discrete spectrum with finite multiplicity in the half plane $\mathrm{Re}(\lambda)>-\Lambda$. Moreover, from the fact that $\mathcal{R}_k(f,g)\subset\mathcal{I}_k$, the operator *has only finitely many eigenvalues in this region* which are real and nonpositive. We denote by $-\lambda_i^{(k)}$ the eigenvalues of $-\mathcal{L}_{V_f}^{(k)}$ (counted with their algebraic multiplicities). Note that each eigenvalue may a priori be associated with a Jordan block of size $d_i^{(k)}\geq 1$. Following [48, App. A], we fix a Jordan path in $\mathbb{C}$ which separates the eigenvalues in the half plane $\mathrm{Re}(\lambda)>-\Lambda$ from the rest of the spectrum. Then, according to this reference, the *spectral projector* associated with this finite part of the spectrum can be written as

$$\Pi_\Lambda^{(k)}:=\int_\gamma(-\mathcal{L}_{V_f}^{(k)}-z)^{-1}dz.$$

We can then split the operator $-\mathcal{L}_{V_f}^{(k)}$ as follows:

$$-\mathcal{L}_{V_f}^{(k)}:=\Pi_\Lambda^{(k)}\circ(-\mathcal{L}_{V_f}^{(k)})\circ\Pi_\Lambda^{(k)}+(\mathrm{Id}-\Pi_\Lambda^{(k)})\circ(-\mathcal{L}_{V_f}^{(k)})\circ(\mathrm{Id}-\Pi_\Lambda^{(k)}).$$

According to [30, p. 244-246], the spectrum of the operator $(\mathrm{Id}-\Pi_\Lambda^{(k)})\circ(-\mathcal{L}_{V_f}^{(k)})\circ(\mathrm{Id}-\Pi_\Lambda^{(k)})$ is contained in the half plane $\mathrm{Re}(\lambda)<-\Lambda$ while the finite rank part can be written as

$$\Pi_\Lambda^{(k)}\circ(-\mathcal{L}_{V_f}^{(k)})\circ\Pi_\Lambda^{(k)}=\sum_{i:\lambda_i^{(k)}\leq\Lambda}\left(\sum_{l=1}^{d_i^{(k)}}-\lambda_i^{(k)}|u_{i,l}^{(k)}\rangle\langle v_{i,l}^{(k)}|+\sum_{l=1}^{d_i^{(k)}-1}|u_{i,l}^{(k)}\rangle\langle v_{i,l+1}^{(k)}|\right)\tag{17}$$

where

- $(u_{i,l}^{(k)})_{\lambda_i^{(k)}\leq\Lambda,l=1,\dots d_i^{(k)}}$ belongs to $\mathcal{H}_k^m(M)\subset\mathcal{D}'^{,k}(M)$,
- $(v_{i,l}^{(k)})_{\lambda_i^{(k)}\leq\Lambda,l=1,\dots d_i^{(k)}}$ belongs to $\mathcal{H}_{n-k}^{-m}(M)\subset\mathcal{D}'^{,n-k}(M)$,
- $|u\rangle\langle v|:\psi\in\mathcal{H}_k^m(M)\mapsto\langle v,\psi\rangle u\in\mathcal{H}_k^m(M)$

Recall from [32, Th. 1.5] that these "generalized eigendistributions" are intrinsic and that they do not depend on the choice of the order function m. We also note that the vectors $v_*^{(k)}$ give rise to a Jordan basis for the spectral decomposition of the dual operator acting on $\mathcal{H}_{n-k}^{-m}(M)$. We now want to relate this spectral decomposition to the correlation function from the introduction :

Proposition 5.6. *Let $0 \leq k \leq n$. Then, for every $i \geq 0$, there is an integer $d_i^{(k)} \geq 1$ s.t. for any $\Lambda > 0$, for every $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$ and for every $t \geq 0$,*

$$\langle (\varphi_f^{-t})^* \psi_1, \psi_2 \rangle = \sum_{i:\lambda_i^{(k)} < \Lambda} e^{-\lambda_i^{(k)} t} \sum_{l=0}^{d_i^{(k)}-1} \frac{t^l}{l!} \sum_{j=l+1}^{d_i^{(k)}} \langle u_{i,j}^{(k)}, \psi_2 \rangle \langle v_{i,j+l}^{(k)}, \psi_1 \rangle + \mathcal{O}_{\psi_1,\psi_2,\Lambda}(e^{-\Lambda t}). \tag{18}$$

In fact, the result also holds for any ψ_1 in $\mathcal{H}_k^m(M)$ provided the parameters (N_0, N_1) involved in the definition of m are large enough.

Note that the sum is finite and that all the quantities involved in the sum are independent of the choice of the order function m. This expression gives us an asymptotic expansion for the correlation function at any order of precision. As was already explained, all the $\lambda_i^{(k)}$ appearing in the sum belong to the set $-\mathcal{I}_k \subset \mathbb{R}_+$. The rest of the article is devoted to a more precise understanding of the terms appearing in this asymptotic expansion. Yet, before that, let us prove this Proposition.

Proof. Fix $q \geq 1$. We first follow the arguments of [31, Th. 1] applied to the hyperbolic diffeomorphism $\varphi_q := \varphi_f^{-\frac{1}{q}}$ rather than to the generator $-\mathcal{L}_{V_f}^{(k)}$. Precisely, following this reference, we can verify that the order function m_{N_0,N_1} from Lemma A.1 satisfies the assumptions of [31, Lemma 2]. Then, following almost verbatim [31, section 3.2], we can deduce that the transfer operator

$$\varphi_q^* : \psi \in \mathcal{H}_k^m(M) \to \varphi_f^{-\frac{1}{q}*} \psi \in \mathcal{H}_k^m(M)$$

defines a bounded operator on the anisotropic space $\mathcal{H}_k^m(M)$ which can decomposed as

$$\varphi_q^* = \hat{r}_{m,q} + \hat{c}_{m,q}, \tag{19}$$

where $\hat{c}_{m,q}$ is a compact operator and the remainder $\hat{r}_{m,q}$ has small operator norm bounded as : $\|\hat{r}_{m,q}\| \leq e^{\frac{C - \frac{C_N}{q}}{3}}$ (for some uniform C that may be slightly larger than the one from Proposition 4.2). Note that, for every $q \in \mathbb{N}$, we can make $\|\hat{r}_{m,q}\|$ arbitrarily small by choosing N large enough. The proof follows similar lines as for the definition of the spectrum of $-\mathcal{L}_{V_f}^{(k)}$ except that we deal with the propagator at discrete times instead of the generator. Again, we can verify that the spectrum is intrinsic, i.e. independent of the choice of order function. This is because the eigenvalues and associated spectral projectors correspond to the poles and residues of a *discrete resolvent* defined as an operator from $\Omega^k(M)$ to $\mathcal{D}'^{,k}(M)$ as follows : consider the series $\sum_{l=0}^{+\infty} e^{-lz} \varphi_q^{l*}$. Then, by the direct bound :

$$\| \sum_{l=0}^{+\infty} e^{-lz} \varphi_q^{l*} \psi \|_{\mathcal{H}_k^m(M)} \leqslant \sum_{l=0}^{+\infty} e^{-l\mathrm{Re}(z)} \|\varphi_q^*\|^l \|\psi\|_{\mathcal{H}_k^m(M)},$$

we deduce that, for $\mathrm{Re}(z)$ large enough, the series $\sum_{l=0}^{+\infty} e^{-lz} \varphi_q^{l*} \psi$ converges absolutely in $\mathcal{H}_k^m(M)$ for every test form $\psi \in \Omega^k(M)$. Hence, by the continuous injections $\Omega^k(M) \hookrightarrow$

$\mathcal{H}_k^{m_{N_0,N_1}}(M) \hookrightarrow \mathcal{D}'^{,k}(M)$, the identity

$$(\mathrm{Id} - e^{-z}\varphi_q^*)^{-1} := \sum_{l=0}^{+\infty} e^{-lz}\varphi_q^{l*} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$$

holds true for $\mathrm{Re}(z)$ large enough. A consequence of the decomposition (19) is that the resolvent of φ_q^*

$$(\lambda - \varphi_q^*)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$$

has a meromorphic extension from $|\lambda| > e^{C_0}$ to $\lambda \in \mathbb{C}$ with poles of finite multiplicity which correspond to the eigenvalues of the operator φ_q^* [31, Corollary 1]. In other words, $(\mathrm{Id} - e^{-z}\varphi_q^*)^{-1} : \Omega^k(M) \to \mathcal{D}'^{,k}(M)$ has a meromorphic extension from $\mathrm{Re}(z) > C_0$ (with $C_0 > 0$ large enough) to $z \in \mathbb{C}$ with poles of finite multiplicity. Denote by $\tilde{\pi}_{\lambda,q}^{(k)}$ the spectral projector of φ_q^* associated to the eigenvalue λ which is obtained from the contour integral formula :

$$\tilde{\pi}_{\lambda,q}^{(k)} = \frac{1}{2i\pi}\int_\gamma \left(\mu - \varphi_q^*\right)^{-1} d\mu$$

where γ is a small circle around λ. This corresponds to the residues of the discrete resolvent at $e^z = \lambda$. As φ_q^* commutes with $-\mathcal{L}_{V_f}^{(k)}$, we can deduce that the range of $\tilde{\pi}_{\lambda,q}^{(k)}$ is preserved by $-\mathcal{L}_{V_f}^{(k)}$. In particular, any eigenvalue z_0 of $-\mathcal{L}_{V_f}^{(k)}$ on that space must verify $e^{\frac{z_0}{q}} = \lambda$. As we know that any resonance of $-\mathcal{L}_{V_f}^{(k)}$ is real, we can deduce that the poles of $(\mathrm{Id} - e^{-z/q}\varphi_q^*)^{-1}$ belong to $\mathcal{R}_k(f,g) \subset \mathbb{R}$ modulo $2i\pi\mathbb{Z}$. Take now z_0 in $\mathcal{R}_k(f,g)$. We want to show that

$$\tilde{\pi}_{e^{z_0},1}^{(k)} = \pi_{z_0}^{(k)}, \tag{20}$$

where $\pi_{z_0}^{(k)}$ is the spectral projector of $-\mathcal{L}_{V_f}^{(k)}$ associated to the eigenvalue z_0. Equivalently, the spectral projectors are the same for both problems. Once it will be done, the proposition is just a consequence of decomposition (19) for $q = 1$ when t is an integer as a consequence of [31, Corollary 1]. When t is a positive real number, we can conclude by writing $\varphi^{-t} = \varphi^{-[t]}\varphi^{-t+[t]}$ and by using (50) from the appendix.

In order to show (20), we first observe that $\tilde{\pi}_{e^{z_0/q},q}^{(k)} = \tilde{\pi}_{e^{z_0},1}^{(k)}$ and we decompose the resolvent $(z + \mathcal{L}_{V_f}^{(k)})^{-1}$ as follows:

$$(z + \mathcal{L}_{V_f}^{(k)})^{-1} = \sum_{l=0}^{+\infty} e^{-\frac{z}{q}}\varphi_q^* \int_0^{\frac{1}{q}} e^{-zt}\varphi_f^{-t*}dt = (\mathrm{Id} - e^{-\frac{z}{q}}\varphi_q^*)^{-1}\int_0^{\frac{1}{q}} e^{-zt}\varphi_f^{-t*}dt.$$

For $\mathrm{Re}(z)$ large enough, this expression makes sense viewed as an operator from $\Omega^k(M)$ to $\mathcal{D}'^{,k}(M)$. We have seen that it can be meromorphically continued to $\mathbb{C}$ by using the fact that we have built a proper spectral framework and that we may pick N_0 and N_1 arbitrarily large. Consider now a small contour γ around z_0 containing no other elements of $\mathcal{R}_k(f,g)$. Integrating over this contour tells us that, for every $q \geq 1$:

$$\pi_{z_0}^{(k)} = \tilde{\pi}_{e^{z_0/q},q}^{(k)} q \int_0^{\frac{1}{q}} e^{-z_0 t} \varphi_f^{-t*} dt = \tilde{\pi}_{e^{z_0},1}^{(k)} \int_0^1 e^{-t\frac{z_0}{q}} \varphi_f^{-\frac{t}{q}*} dt.$$

As an operator on $\Omega^k(M)$, we can observe that $\int_0^1 e^{-t\frac{z_0}{q}} \varphi_f^{-\frac{t}{q}*} dt$ converges to the identity as $q \to +\infty$. Hence, $\pi_{z_0}^{(k)} = \tilde{\pi}_{e^{z_0},1}^{(k)}$ as expected. □

5.4. **Proving $\mathcal{I}_k \subset \mathcal{R}_k(f,g)$.** Note that the formula we obtained for the correlation function in (18) shows that the function $s \in (0,1] \mapsto \langle \varphi^{\ln(s)*}\psi_1, \psi_2\rangle$ admits an asymptotic expansion in $(\ln s)^l s^\lambda$ where $-\lambda$ belongs to the spectrum of $\mathcal{L}_{V_f}^{(k)}$ acting on the anisotropic space $\mathcal{H}_k^m(M)$. We also observe that this asymptotic expansion is valid for ψ_1 in $\mathcal{H}_k^m(M)$ and ψ_2 in $\Omega^{n-k}(M)$.

This implies the following **criteria**: *If there is some pair $(\psi_1,\psi_2) \in \mathcal{H}_k^m(M) \times \Omega^{n-k}(M)$ such that s^λ shows up in the asymptotic expansion of $\langle \psi_1, \varphi^{\ln(s)*}\psi_2\rangle$ near $s=0$ then λ belongs to the spectrum of $\mathcal{L}_{V_f}^{(k)}$.* Hence, for every fixed $0 \leq k \leq n$ and every $-\lambda \in \mathcal{I}_k$, we would show that $\lambda \in \mathcal{R}_k(f,g)$ if we could find some pair $(\psi_1,\psi_2) \in \mathcal{H}_k^m(M) \times \Omega^{n-k}(M)$ such that $\langle \varphi^{\log(s)*}\psi_1, \psi_2\rangle$ has a polyhomogeneous expansion in s with s^λ appearing in the asymptotic expansion.

For that purpose, we fix a critical point a of index r and $0 \leq k \leq n$. We work with the adapted coordinates (x,y) introduced in paragraph 3.5. We let $0 \leq \theta_1(x) \leq 1$ (resp. $\theta_2(y)$) be a smooth function which is compactly supported in $(-\delta_a,\delta_a)^r$ (resp. $(-\delta_a,\delta_a)^{n-r}$) which is equal to 1 in a neighborhood of 0. We fix a multi-index $(\alpha,\beta) = (\alpha_1,\ldots,\alpha_r,\beta_{r+1},\ldots\beta_n)$, and two sets

$$I \subset \{1,\ldots,r\}, \text{ and } J \subset \{r+1,\ldots,n\},$$

such that $|J|-|I| = k-r$. In these cooordinates, we define

$$\psi_1(x,y,dx,dy) = \theta_1(x)\theta_2(y)\delta_0^{(\alpha)}(x)y^\beta \left(\wedge_{i\notin I} dx_i\right) \wedge \left(\wedge_{i\in J} dy_i\right),$$

and

$$\psi_2(x,y,dx,dy) = \theta_1(x)\theta_2(y) \left(\wedge_{i\in I} dx_i\right) \wedge \left(\wedge_{i\notin J} dy_i\right).$$

Note that ψ_1 (more precisely its image in $\mathcal{D}'^{,k}(M)$) belongs to the anisotropic Sobolev space $\mathcal{H}_k^m(M)$ provided that we choose N_0 large enough in the definition of the order function $m(x,\xi)$. We can then write, for every $0 < s \leq 1$,

$$\langle (\varphi_f^{\ln s})^*\psi_1, \psi_2\rangle = \pm s^{\lambda_{\alpha,\beta,I,J}} \left\langle \delta_0(x)y^\beta, \theta_1(x)\theta_2(y)(\partial^\alpha\theta_1)((s^{|\chi_i(a)|}x_i)_i)\theta_2((s^{|\chi_i(a)|}y_i)_i)\right\rangle. \tag{21}$$

where

$$\lambda_{\alpha,\beta,I,J} := \sum_{i=1}^r \alpha_i|\chi_i(a)| + \sum_{i=r+1}^n \beta_i|\chi_i(a)| + \sum_{1\leq i\leq r: i\in I} |\chi_i(a)| + \sum_{r+1\leq i\leq n: i\in J} |\chi_i(a)|.$$

Writing the Taylor expansion of (21), one finds that

$$\langle (\varphi_f^{\ln s})^*\psi_1, \psi_2\rangle = \pm s^{\lambda_{\alpha,\beta,I,J}} \theta_1(0)\theta_2(0) \left\langle \delta_0^{(\alpha)}(x)y^\beta, \theta_1(x)\theta_2(y)\right\rangle (1+o(1)).$$

Hence, if we choose θ_1 and θ_2 such that the leading term is not zero in this asymptotic expansion, we find that $-\lambda_{\alpha,\beta,I,J}$ belongs to $\mathcal{R}_k(f,g)$. We shall come back to this construction in section 7 where we will construct a family of linearly independent eigenmodes for each eigenspace.

6. Description of the kernel

In the previous section, we found the Pollicott-Ruelle spectrum of any degree $0 \leq k \leq n$. Yet, we did not discuss the multiplicity or the existence of Jordan blocks in the spectrum. We will now consider this question in sections 6 and 7. The argument presented in these two sections is a refinement of the argument of section 5 and it relies on a Theorem of Schwartz [66] which describes the distributions carried by a smooth submanifold inside M. The general case is slightly involved combinatorially and it requires some nonresonance assumption. Hence, we will first expose the proof under the simplifying assumption that $\lambda = 0$. The case $\lambda \neq 0$ will follow similar lines (up to the resonance assumption) and it will be discussed in section 7. Precisely, we will show that there is no Jordan blocks in the kernel (Proposition 6.1) and that there exists a "canonical" basis of the kernel which is carried by the closure of the unstable manifolds $W^u(a)$ (Proposition 6.7 and 6.8).

6.1. Jordan blocks.

Let us first show the absence of Jordan blocks in the kernel:

Proposition 6.1. *Suppose that (f,g) is a smooth Morse pair which induces a Morse-Smale gradient flow. Let $0 \leq k \leq n$. Then, when acting on a convenient*[6] *anisotropic space $\mathcal{H}_k^m(M)$, one has*

$$\operatorname{Ker}(\mathcal{L}_{V_f}^{(k)}) = \operatorname{Ker}((\mathcal{L}_{V_f}^{(k)})^2).$$

In other words, there is no Jordan blocks in the kernel.

We start with the following Lemma:

Lemma 6.2. *Let $1 \leq j \leq K$. Then, there exists an open neighborhood V_{a_j} of a_j such that, for every $i < j$ with $r(a_i) \geq r(a_j)$, one has*

$$W^u(a_i) \cap V_{a_j} = \emptyset.$$

Proof. Let $1 \leq j \leq K$. Let $i < j$ satisfying $r(a_i) \geq r(a_j)$. The closure of $W^u(a_i)$ is a compact subset. In order to prove this Lemma, we suppose by contradiction that a_j belongs to $\overline{W^u(a_i)}$. It means that there exists a sequence $(x_p)_{p\geq 1}$ in $W^u(a_i)$ such that x_p converges to a_j as $p \to +\infty$. Without loss of generality, we can suppose that there exists a unique b in $\operatorname{Crit}(f)$ such that, for every $p \geq 1$, $x_p \in W^s(b)$. Using the conventions of Remark 3.7 for our fixed sequence $(x_p)_{p\geq 1}$, there exists a sequence of critical points $a_i = b_0, b_1, \ldots, b_l = b$ and a finite sequence of points u_m in $W^u(b_m) \cap W^s(b_{m+1})$ such that

$$\forall \epsilon_1 > 0, \exists p_0, \forall p \geq p_0,\ d\left(\mathcal{O}(x_p), \cup_{0\leq m\leq l-1}\mathcal{O}(b_m)\right) < \epsilon_1,$$

[6] It means that there is a discrete spectrum for $\operatorname{Re}(\lambda) < C$ if $C > 0$.

where $\mathcal{O}(x)$ denotes the orbit of x under the flow φ_f^t. In particular, as $x_p \to a_j$, this implies that $a_j = b_m$ for some $0 < m \leq l$. By the Morse-Smale transversality assumption, this implies that $r(a_j) > r(a_i) = r(b_0)$ which gives the contradiction. □

We can now give the proof of Proposition 6.1. Suppose by contradiction that there exists a Jordan block associated to the eigenvalue 0 for a certain degree of currents k. Then, it means that there exists $u_0 \neq 0$ and $u_1 \neq 0$ in our anisotropic Sobolev space of currents $\mathcal{H}_k^m(M)$ such that

$$\mathcal{L}_{V_f}^{(k)} u_0 = 0 \text{ and } \mathcal{L}_{V_f}^{(k)} u_1 = u_0.$$

Integrating these expressions, we find that, for all t in $\mathbb{R}$,

$$(\varphi_f^t)^* u_0 = u_0 \text{ and } (\varphi_f^t)^* u_1 = u_1 + t u_0.$$

As in our computation of the spectrum, we let $t = \ln s$ with $0 < s \leq 1$,

$$(\varphi_f^{\ln s})^* u_1 - u_1 = (\ln s) u_0. \tag{22}$$

As above, we order our critical points $a_1 < a_2 < \ldots < a_K$ using the fact that the critical values of f are distinct.

We now use this Lemma to get the expected contradiction. We fix $j \geq 0$. We suppose that u_0 is vanishing in a neighborhood of any critical point $(a_i)_{i \leq j}$ and that it does not vanish in a neighborhood of a_{j+1}. According to Lemma 5.3, we can deduce that $\text{supp}(u_0) \cap V_{a_{j+1}}$ is included in $W^u(a_{j+1})$. Arguing as in paragraph 5.2.2 (i.e. via the pull-back Theorem of Hörmander), we can verify that $\langle (\varphi_f^{\ln s})^* u_0, \psi \rangle = \langle u_0, \psi \rangle$ has a *bounded* asymptotic expansion in s for ψ a smooth test form compactly supported in $V_{a_{j+1}}$. Moreover, we can choose ψ such that the right hand side of the equality does not vanish. Hence, the leading order of this expansion must be of degree 0. This implies that a_{j+1} is a critical point of index $r(a_{j+1}) = k$.

We would now like to prove that, near a_{j+1}, u_1 is also supported in $W^u(a_{j+1})$. We fix V an open subset of $V_{a_{j+1}}$ which does not intersect $W^u(a_{j+1})$. From Lemma 6.2, one knows that, for every x in V, there exists $i \leq j$ such that $a_i = \lim_{t \to -\infty} \varphi_f^t(x)$ and $r(a_i) < r(a_{j+1})$. Hence, we would conclude that $\text{supp}(u_1) \cap V_{a_{j+1}}$ is included in $W^u(a_{j+1})$ if we could show that, for every $i \leq j$ with $r(a_i) < k$, u_1 identically vanishes in an open neighborhood of a_i.

Let $i_0 \leq j$ be an index with $r(a_{i_0}) < k$. Then, either $\text{supp}(u_1) \cap V_{a_{i_0}}$ is included in $W^u(a_{i_0})$, or, as $\mathcal{L}_{V_f} u_1 = u_0 = 0$ on $f^{-1}(< f(a_{j+1}))$, we can deduce by propagation that there exists a critical point a of smaller index such that u_1 does not vanish in a neighborhood of a. Hence, without loss of generality, we can suppose that $\text{supp}(u_1) \cap V_{a_{i_0}}$ is included in $W^u(a_{i_0})$. As $\mathcal{L}_{V_f} u_1 = 0$ in $V_{a_{i_0}}$, we can argue one more time as in paragraph 5.2.2. From that, we deduce that

$$\langle (\varphi_f^{\ln s})^* u_1, \psi \rangle = \langle u_1, \psi \rangle$$

has a bounded asymptotic expansion in s for every choice of ψ compactly supported in $V_{a_{i_0}}$. Using the fact that $r(a_{i_0}) < k$, we conclude that the left hand side must go to 0 as $s \to 0^+$. Thus, one has $\langle u_1, \psi \rangle = 0$ as expected from which we deduce that $\text{supp}(u_1) \cap V_{a_{j+1}}$ is included in $W^u(a_{j+1})$.

Thanks to the fact that $\text{supp}(u_1) \cap V_{a_{j+1}}$ is included in $W^u(a_{j+1})$ and to the fact that u_1 belongs to our family of anisotropic spaces, we can argue one more time as in paragraph 5.2.2. We find then that $\langle (\varphi_f^{\ln s})^* u_1, \psi\rangle$ has a bounded asymptotic expansion as $s \to 0^+$ for any smooth test function ψ supported near a_{j+1}. Using then that u_1 verifies equation (22), we can finally conclude that $\langle u_0, \psi\rangle = 0$ for every ψ supported near 0 which gives the contradiction to the fact that there exists a nontrivial Jordan block in the kernel.

Finally, let us conclude with the following fact which has been implicitely proved following the lines of paragraph 5.2.2:

Proposition 6.3. *Suppose that (f, g) is a smooth Morse pair which induces a Morse-Smale gradient flow. Let $u \neq 0$ be an element of $\mathcal{H}_k^m(M)$ satisfying $\mathcal{L}_{V_f}^{(k)} u = 0$. Let a be the critical point of f satisfying the following properties:*

- *u does not vanish in any neighborhood of a,*
- *for every a' in $\text{Crit}(f)$ satisfying $f(a') < f(a)$, u identically vanishes near a'.*

Then, the Morse index of a equals k.

6.2. **Exponential convergence of the correlation function.** Before continuing our description of the kernel, let us draw a simple consequence of the fact that there is no Jordan blocks in the kernel of $\mathcal{L}_{V_f}^{(k)}$. Using the spectral expansion (18), one has, for every $\psi_1 \in \Omega^k(M)$ and every ψ_2 in $\Omega^{n-k}(M)$,

$$\forall t \geq 0,\ \langle (\varphi_f^{-t})^* \psi_1, \psi_2\rangle = \sum_{i:\lambda_i^{(k)}=0} \langle u_{i,1}^{(k)}, \psi_2\rangle \langle v_{i,1}^{(k)}, \psi_1\rangle + \mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda_+ t}), \tag{23}$$

for some $\Lambda_+ > 0$ which is independent of ψ_1 and ψ_2. We now aim at describing more precisely the elements in the kernel. Note that this asymptotic expansion is valid when (f, g) is a smooth Morse pair inducing a Morse-Smale gradient flow.

Remark 6.4. To prove this property, we could probably use less regularity of the linearizing chart. In fact, our construction of the order function m only makes use of the fact that the linearizing chart is $\mathcal{C}^1$. Smoothness is only important if we want to determine exactly all the elements inside $\mathcal{R}_k(f, g)$.

6.3. **Background material on currents.** In order to describe the elements of the kernel, we start with some background material on the theory of currents which will also be useful in section 7. By a Theorem of Schwartz [66, Th. 37 p. 102] whose adaptation to the case of currents is straightforward, we first recall that :

Theorem 6.5 (Schwartz)**.** *Let u be a current of degree k supported by a smooth submanifold X embedded in M. Suppose that in a small neighbohood of $x \in X$, one has a system of coordinate functions $(x_i, y_j)_{1\leqslant i\leqslant r, r+1\leqslant j\leqslant n}$ where the coordinates $(x_i)_{1\leqslant i\leqslant r}$ are transversal coordinates of X, i.e. the submanifold X is given by the equations $\{x_i = 0, 1 \leqslant i \leqslant r\}$. Then the current u reads locally as a finite sum :*

$$u(x, y) = \sum_{\alpha, |I|+|J|=k} u_{\alpha,I,J}(y)\partial_x^\alpha \delta_{\{0\}}^{\mathbb{R}^r}(x) dx^I \wedge dy^J \tag{24}$$

where (α, I, J) are multi–indices, the $u_{\alpha,I,J}$ are distributions in $\mathcal{D}'(\mathbb{R}^{n-r})$.

If we denote by N^*X the conormal cycle of X, we also have the following property [21, Lemma 9.2]:

Lemma 6.6. *Suppose that the assumptions of the previous Theorem hold and use the same notations. If $WF(u) \subset N^*(X)$, then the current u reads*

$$u(x, y) = \sum_{\alpha,|I|+|J|=k} u_{\alpha,I,J}(y)\partial_x^\alpha \delta_{\{0\}}^{\mathbb{R}^r}(x)dx^I \wedge dy^J \tag{25}$$

where the $u_{\alpha,I,J}$ are smooth functions in $C^\infty(\mathbb{R}^{n-r})$.

6.4. **Description of the eigenmodes.** Let $0 \leq k \leq n$. In this paragraph, we will construct a "canonical" basis for the kernel of the operator $\mathcal{L}_{V_f}^{(k)}$ acting on the anisotropic space $\mathcal{H}_k^m(M)$. We proceed in three steps. First, we show that, near the "smallest" critical point a of index k, an element in the kernel must be proportional to the current of integration $[W^u(a)]$ on the unstable manifold. Then, we prove that the germ of current near to 0 can be extended into a current carried by $\overline{W^u(a)}$ that we denote by U_a. Finally, we show that these currents form indeed a basis of the kernel.

The construction of these currents U_a is known to be a delicate task and, to our knowledge, it was first made by Laudenbach under some tameness assumption on the Morse-Smale gradient flow [53, 54]. Here, we will show how our microlocal machinery allows to give an alternative proof of this construction for quite general Morse-Smale gradient flows.

6.4.1. *Local form near the critical point.* Let $u \neq 0$ be an element in $\mathcal{H}_k^m(M)$ such that $\mathcal{L}_{V_f}^{(k)} u = 0$. As above, we order the critical points $a_1 < \ldots a_K$ according to the values of f. We denote by j the index such that, for every $i < j$, u vanishes in a neighborhood of a_i and such that u does not vanish near a_j. Recall from Lemma 5.3 that such a j exists and that $\text{supp}(u) \cap V_{a_j}$ is included in $W^u(a_j)$ for some small enough neighborhood V_{a_j} of a_j. We have also shown in Proposition 6.3 that a_j must be of index k.

Using the above results, we deduce that, in the adapted coordinates of paragraph 3.5, the current u reads as a finite sum :

$$u(x, y) = \sum_{\alpha,|I|+|J|=k} u_{\alpha,I,J}(y)\partial_x^\alpha \delta_{\{0\}}^{\mathbb{R}^k}(x)dx^I \wedge dy^J \tag{26}$$

where the $u_{\alpha,I,J}$ are smooth functions in $C^\infty(\mathbb{R}^{n-k})$. The fact that $u_{\alpha,I,J}$ is smooth follows from Lemma 6.6 and from the fact that we can pick N_1 arbitrarly large in the definition of the order function m used to define our anisotropic Sobolev space. Let now ψ be a smooth test form of degree $n-k$ supported near a_j. Using the fact that u satisfies $\mathcal{L}_{V_f}^{(k)} u = 0$, one has, for every $0 < s \leq 1$,

$$\langle u, \psi \rangle = \langle \varphi_f^{\ln s *} u, \psi \rangle.$$

Plugging the expression of the flow in these coordinates into the explicit form of u given by equation (26), one finds that u must be of the following form

$$u(x, y) = u_0(y)\delta_{\{0\}}^{\mathbb{R}^k}(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k,$$

where u_0 is a smooth function. Using one more time the invariance by the flow φ_f^t, one can deduce that u_0 is **constant** in a neighborhood of the origin. Using the result of Corollary D.4 in appendix D of [22], we recognize that $\delta_{\{0\}}^{\mathbb{R}^k}(x)dx_1 \wedge \ldots \wedge dx_k$ is the integral formula for the current of integration on the germ of submanifold $W^u(a_j) = \{x_i = 0, 1 \leqslant i \leqslant k\}$. Hence, our last statement means that locally near a_j, $u = c[W^u(a_j)]$ is a multiple of the current of integration on $W^u(a_j)$.

6.4.2. *Extension of the local form to M.* Our next step will be to prove that the current of integration on the germ of unstable submanifold near any critical point a of index r can be extended in a natural manner to M.

For that purpose, we still work with the adapted coordinates (x, y) introduced in paragraph 3.5. According to [22, Cor. D.4], the current of integration on $W^u(a)$ is well defined near a and it can be written locally as

$$[W^u(a)] = \delta_0^{\mathbb{R}^r}(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_r.$$

Using the conventions of paragraph 3.5, we let $0 \leq \theta \leq 1$ be a smooth cutoff function which is equal to 1 on $(-\delta_a/2, \delta_a/2)^n$ and to 0 outside $(-\delta_a, \delta_a)^n$. Thanks to this cutoff function, we define the following element of $\mathcal{D}'^{,k}(M)$:

$$\tilde{U}_a := \theta(x, y)\delta_0^{\mathbb{R}^r}(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_r.$$

By a straightforward calculation, $\tilde{U}_a$ satisfies $\mathcal{L}_{V_f}\tilde{U}_a = 0$ on $(-\delta_a/4, \delta_a/4)^n$. One can also verify that, for every order function satisfying the properties of Lemma 4.1, $\tilde{U}_a$ belongs to the space $\mathcal{H}_r^m(M)$. Recall that $(\varphi_f^{-t})^*$ generates a strongly continuous semigroup associated to the operator $-\mathcal{L}_{V_f}^{(r)}$ acting on $\mathcal{H}_r^m(M)$. As we have shown that there is no Jordan block in the kernel and that $-\mathcal{L}_{V_f}^{(r)}$ has a spectral gap, we can deduce that, in $\mathcal{H}_r^m(M)$ (hence in $\mathcal{D}'^{,r}(M)$),

$$\lim_{t\to+\infty} \varphi_f^{-t*}(\tilde{U}_a) = \sum_{i:\lambda_i^{(r)}=0} \langle \tilde{U}_a, v_{i,0}^{(r)} \rangle u_{i,0}^{(r)}. \tag{27}$$

We denote by U_a this limit current which satisfies $\mathcal{L}_{V_f}^{(r)}(U_a) = 0$. Let us now verify that U_a is an extension of $\tilde{U}_a|_{(-\delta_a/4,\delta_a/4)^n}$. For that purpose, we write the double inclusion

$$Y := \{(0, y) : \forall r + 1 \leq j \leq n, \ |y_j| \leqslant \delta_a/4\} \subset X := \text{supp }(\tilde{U}_a) \subset W^u(a).$$

Set also $X_m = \varphi_f^{mt_0}(X)$, and $Y_m = \varphi_f^{mt_0}(Y)$ for some fixed $t_0 > 0$. Then, we have the following sequences of inclusions, for all $m \in \mathbb{N}$:

$$\begin{aligned} Y_m \subset X_m \subset W^u(a), \\ Y_m \subset Y_{m+1}, \\ \text{supp } \varphi_f^{-mt_0*}\left(\tilde{U}_a\right) = X_m \subset \overline{W^u(a)}. \end{aligned}$$

These inclusions are simple consequences of the inclusion ($Y \subset X \subset W^u(a)$) and of the fact that $W^u(a)$ is stable by application of the one-parameter group $(\varphi_f^t)_t$. In particular, from (27), we have

$$\text{supp}(U_a) \subset \overline{W^u(a)}. \tag{28}$$

In the local chart $(-\delta_a/4, \delta_a/4)^n$ around a, we can write

$$(\varphi_f^{-t_0})^*(\tilde{U}_a)(x, y, dx, dy) = \theta\left((e^{-t_0\chi_j(a)}x_j)_j, (e^{-t_0\chi_j(a)}y_j)_j\right)\delta_0^{\mathbb{R}^r}(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_r.$$

As θ is constant and equal to 1 on $(-\delta_a/2, \delta_a/2)^n$, we have, provided that we pick $t_0 > 0$ small enough,

$$(\varphi_f^{-t_0})^*(\tilde{U}_a)(x, y, dx, dy) = \theta(x, y)\,\delta_0^{\mathbb{R}^r}(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_r = \tilde{U}_a(x, y, dx, dy).$$

To summarize, we have

$$[W^u(a)]|_{(-\delta_a/4,\delta_a/4)^n} = \tilde{U}_a|_{(-\delta_a/4,\delta_a/4)^n} = \left((\varphi_f^{-t_0})^*(\tilde{U}_a)\right)|_{(-\delta_a/4,\delta_a/4)^n}.$$

From this, we can infer that, for every $m \geq 0$,

$$\left(\varphi_f^{-mt_0*}\tilde{U}_a\right)|_{\varphi_f^{mt_0}((-\delta_a/4,\delta_a/4)^n)} = \left(\varphi_f^{-(m+1)t_0*}\tilde{U}_a\right)|_{\varphi_f^{mt_0}((-\delta_a/4,\delta_a/4)^n)}.$$

Recall that the support of both currents is included in X_{m+1}. The above equality tells us that they coincide on the subset Y_m. From that, we can deduce that, for every $m \geq 0$,

$$\left(\varphi_f^{-mt_0*}\tilde{U}_a\right)|_{(-\delta_a/4,\delta_a/4)^n} = \left(\varphi_f^{-(m+1)t_0*}\tilde{U}_a\right)|_{(-\delta_a/4,\delta_a/4)^n}.$$

Hence, for every $m \geq 1$, one has

$$\left(\varphi_f^{-(m+1)t_0*}\tilde{U}_a\right)|_{(-\delta_a/4,\delta_a/4)^n} = \tilde{U}_a|_{(-\delta_a/4,\delta_a/4)^n},$$

which, by letting $m \to +\infty$ implies that

$$[W^u(a)]|_{(-\delta_a/4,\delta_a/4)^n} = U_a|_{(-\delta_a/4,\delta_a/4)^n}.$$

To summarize, we have proved the following result:

Proposition 6.7. *Let (f, g) be a smooth Morse pair which induces a Morse-Smale gradient flow. Let a be a critical point of index r and let $0 \leq \theta \leq 1$ be a smooth cutoff function which is compactly supported in a small enough neighborhood V_a of a, and equal to 1 in an open neighborhood of a.*

Then, there exists an open neighborhood $\tilde{V}_a \subset V_a$ of a such that the current

$$U_a := \sum_{i:\lambda_i^{(r)}=0} \langle \theta[W^u(a)], v_{i,1}^{(r)} \rangle u_{i,1}^{(r)}$$

satisfies

- $U_a|_{\tilde{V}_a} = [W^u(a)]|_{\tilde{V}_a}$,
- $\text{supp}(U_a) = \overline{W^u(a)}$,
- $\mathcal{L}_{V_f}^{(r)}(U_a) = 0$.

We call these currents **Laudenbach's currents of degree** r.

6.4.3. *The generation property.* Even if the currents depend a priori on the choice of cutoff function, the following Proposition shows that they are in some sense "canonical":

Proposition 6.8. *Let $0 \leq k \leq n$. The family of currents*

$$\{U_a : \ a \in \text{Crit}(f) \ \text{and} \ \text{ind}(a) = k\}$$

forms a basis of the kernel of the operator

$$\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M).$$

Proof. Let us first show that this family of currents is linearly independent. For that purpose, we suppose that

$$\sum_{a \in \text{Crit}(f):\text{ind}(a)=k} \alpha_a U_a = 0.$$

Let a be the "smallest" point of index k, in the sense that, for every other point $a' \neq a$ of index k, $f(a') > f(a)$. We pick ψ a smooth form which is compactly supported near a and such that $\langle [W^u(a)], \psi \rangle \neq 0$. As the support of U_b is contained in $\overline{W^u(b)}$ for any critical point b of index k, we can deduce (provided that the support of ψ is small enough) that

$$0 = \sum_{a \in \text{crit}(f):\text{ind}(b)=k} \alpha_b \langle U_b, \psi \rangle = \alpha_a \langle U_a, \psi \rangle = \alpha_a \langle [W^u(a)], \psi \rangle.$$

From this, we deduce that $\alpha_a = 0$. By induction, we can conclude that the family contains only linearly independent currents.

It remains to verify that this family generates all the kernel. Let $u \neq 0$ be an element in $\mathcal{H}_k^m(M)$ in the kernel of $\mathcal{L}_{V_f}^{(k)}$. From the argument of paragraph 6.4.1, we know that u must be equal to $c_a[W^u(a)]$ in a neighborhood of a where a is the "smallest" critical point where u does not identically vanish and where c_a is a fixed constant. Set now

$$u_1 = u - c_a U_a.$$

We know that u_1 belongs to $\mathcal{H}_k^m(M)$ and that it satisfies $\mathcal{L}_{V_f}^{(k)} u_1 = 0$. Morever, by construction, we know that u_1 vanishes identically near every critical point a' with $a' \leq a$. Repeating the process a finite number of times, we finally get that

$$u = \sum_{a \in \text{Crit}(f):\text{ind}(a)=k} c_a U_a,$$

for some c_a in $\mathbb{R}$.

□

6.4.4. *Back to the correlations.* We now turn back to the correlation function. For that purpose, we fix $\Lambda > 0$ which is strictly smaller than $\min\{|\chi_j(a)| : 1 \leq j \leq n, a \in \text{Crit}(f)\}$. As in paragraph 5.3, we write the spectral decomposition associated with the spectral projector of the eigenvalue $\lambda = 0$. We have shown that there is no Jordan blocks for this eigenvalue (Prop. 6.1), and that we can choose a basis of eigenmodes $(U_a)_a$ indexed by the critical points of index k. Moreover, all the elements in this basis can be chosen in such a way that the support of U_a is equal to $\overline{W^u(a)}$. We denote by S_a the corresponding dual basis. Proceeding as in paragraph 5.3, we have then

$$\forall\psi_1\in\mathcal{H}_k^m(M),\ \forall\psi_2\in\mathcal{H}_{n-k}^{-m}(M),\ \langle\varphi_f^{-t*}\psi_1,\psi_2\rangle=\sum_{a:\text{ind}(a)=k}\langle U_a,\psi_2\rangle\langle S_a,\psi_1\rangle+\mathcal{O}_{\psi_1,\psi_2}(e^{-\Lambda t}),$$

for every $t>0$. Applying the arguments of the previous paragraphs to the operator $\mathcal{L}_{V_{-f}}^{(n-k)}$ acting on the anisotropic space $\mathcal{H}_{n-k}^{-m}(M)$, we can construct a basis of the kernel that we denote by $(\overline{S}_a)_a$ indexed by the critical points of index k. Mimicking the above procedure, we can impose that $\overline{S}_a$ has support contained in $\overline{W^s(a)}$ and that $\overline{S}_a$ coincides with $[W^s(a)]$ in a neighborhood of the critical point a. In particular, as $\overline{W^s(a)}\cap\overline{W^u(a)}=\{a\}$, we can use our local adapted coordinates near a to find that $\langle\overline{S}_a,U_a\rangle=1$. Consider now $a'\neq a$ of index k. If we are able to show that $\langle\overline{S}_a,U_{a'}\rangle=0$ for every such a', then we will have that $\overline{S}_a=S_a$. To prove this, we just need to observe that $\overline{W^s(a)}\cap\overline{W^u(a')}=\emptyset$. In fact, according to Remark 3.7 applied to f and $-f$, we find that, if x belongs to $\overline{W^s(a)}\cap\overline{W^u(a')}$, then $\text{ind}(x_-)\geq k$ and $\text{ind}(x_+)\leq k$, where $x\in W^u(x_-)\cap W^s(x_+)$. In other words, from the Morse-Smale assumption, $x_-=x_+$. From Lemma 6.2, we would then have $a=a'$ which gives the contradiction.

To summarize, we have shown:

Proposition 6.9. *Let (f,g) be a smooth Morse pair which induces a Morse-Smale gradient flow. Let $\Lambda>0$ be such that*

$$\Lambda<\min\{|\chi_j(a)|:1\leq j\leq n,a\in\text{Crit}(f)\}.$$

Then, for every $0\leq k\leq n$, and, for every $a\in\text{Crit}(f)$ of index k, there exists

$$U_a\in\mathcal{H}_k^m(M)\ \textit{and}\ S_a\in\mathcal{H}_{n-k}^{-m}(M)$$

whose support are respectively equal to $\overline{W^u(a)}$ and $\overline{W^s(a)}$ and satisfying

$$\forall t\geq 0,\ \varphi_f^{-t*}=\sum_{a:\text{ind}(a)=k}|U_a\rangle\langle S_a|+\mathcal{O}_{\mathcal{H}_k^m\to\mathcal{H}_k^m}(e^{-\Lambda t}).$$

This implies in particular Theorem 2.3 from the introduction. We also note that we recover Theorem 3.3 from [46] via our spectral analysis of gradient flows. Our result is slightly stronger in two ways: (1) it holds for quite general families of Morse-Smale gradient flows (not necessarily associated with a locally flat metric); (2) it gives a rate of convergence in the asymptotic. This kind of decomposition of the transfer operator into a product of stable and unstable distributions is also at the heart of the spectral analysis of Anosov vector fields performed by Faure and Tsujii in [34, 33]. As we shall see in section 7, we can in fact generalize this expansion at any order.

6.4.5. *Solving the cohomohological equation $\mathcal{L}_{V_f}^{(k)}u=\psi$.* Note that we can use our spectral decomposition to prove Corollary 2.5. In fact, given any $\psi\in\Omega^k(M)$, one can write

$$\psi=\mathbb{P}^{(k)}(\psi)+(\text{Id}-\mathbb{P}^{(k)})(\psi),$$

for $\mathbb{P}^{(k)}$ associated with the eigenvalue 0 – see e.g. paragraph 8.4.2 below. As we can solve $\mathcal{L}_{V_f}^{(k)}(u) = (\mathrm{Id} - \mathbb{P}^{(k)})(\psi)$ from the spectral properties of $\mathcal{L}_{V_f}^{(k)}$, the problem

$$\mathcal{L}_{V_f}^{(k)}(u) = \psi$$

admits a solution in $\mathcal{H}_k^m(M)$ if and only if

$$\mathcal{L}_{V_f}^{(k)}(u) = \mathbb{P}^{(k)}(\psi) = \sum_{a \in \mathrm{Crit}(f):\mathrm{ind}(a)=k} \langle S_a, \psi \rangle U_a$$

has a solution in $\mathcal{H}_k^m(M)$. As the U_a forms a basis of the kernel of $\mathcal{L}_{V_f}^{(k)}$ which does not contain any Jordan block, this is equivalent to the fact that, for every a of index k, $\langle S_a, \psi \rangle = 0$.

7. The case $\lambda \neq 0$

In this section, we describe some properties of the eigenmodes when $\lambda \neq 0$. Except for the use of the nonresonance assumption, the proof is the same as in the case $\lambda = 0$ but the arguments are somewhat more involved due to the possible multiplicity at every critical point. Precisely, we will one more time show that there exists a "canonical" basis of eigenmodes which are carried by the closure of the unstable manifolds $W^u(a)$ (Propositions 7.5 and 7.7) and that there are no Jordan blocks (Proposition 7.2). In all this section, we make the assumption that the Lyapunov exponents

$$\{\chi_j(a) : 1 \leq j \leq n,\ a \in \mathrm{Crit}(f)\}$$

are *rationally independent*. From paragraph 3.5, we know that this implies in particular that (f, g) is a smooth Morse pair. Hence we can apply the results from section 5. We will crucially use this nonresonance assumption to ensure the absence of Jordan blocks. From this, we will be able to construct a more or less explicit basis of eigenmodes.

Remark 7.1. This (global) nonresonance assumption may sound a little bit surprising but it seems to us that removing this assumption may be a subtle issue. In fact, our proof works under the following assumptions:

- (f, g) is a smooth Morse pair generating a Morse-Smale gradient flow,
- If there exists $\alpha \neq 0$ and $\alpha' \neq 0$ in $\mathbb{N}^n$ such that

$$\sum_{j=1}^n \alpha_j |\chi_j(a)| = \sum_{j=1}^n \alpha'_j |\chi_j(a')|,$$

  then $a = a'$.

7.1. **Jordan blocks.** As for the kernel, we start by proving the absence of Jordan blocks. In other words, the algebraic multiplicity is always equal to the geometric multiplicity.

Proposition 7.2. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let $0 \leq k \leq n$. Then, when acting on a convenient anisotropic space $\mathcal{H}_k^m(M)$, one has, for every $\lambda \in \mathcal{R}_k(f,g)$,*

$$\mathrm{Ker}((\mathcal{L}_{V_f}^{(k)} + \lambda)) = \mathrm{Ker}((\mathcal{L}_{V_f}^{(k)} + \lambda)^2).$$

Proof. Once again, the proof is very similar to the case of the kernel. We only treat the case $\lambda \neq 0$. Suppose by contradiction that there exists a Jordan block associated to the eigenvalue $\lambda > 0$ for a certain degree k. Once again, it means that there exists $u_0 \neq 0$ and $u_1 \neq 0$ in our anisotropic Sobolev space of currents $\mathcal{H}_k^m(M)$ such that

$$\mathcal{L}_{V_f}^{(k)} u_0 = \lambda u_0 \text{ and } \mathcal{L}_{V_f}^{(k)} u_1 = \lambda u_1 + u_0.$$

Integrating these expressions, we find that, for all t in $\mathbb{R}_-$,

$$(\varphi_f^t)^* u_0 = e^{\lambda t} u_0 \text{ and } (\varphi_f^t)^* u_1 = e^{\lambda t}\left(u_1 + t u_0\right).$$

As in our computation of the spectrum, we let $t = \ln s$ with $0 < s \leq 1$,

$$(\varphi_f^{\ln s})^* u_1 - s^{\lambda} u_1 = s^{\lambda} (\ln s) u_0. \tag{29}$$

Following the proof of paragraph 5.2.2, we denote by $j+1$ the index point such that, for every $i \leq j$, u_0 vanishes in a neighborhood of a_i and such that u_0 does not vanish near a_{j+1}. This implies that $\mathrm{supp}(u_0) \cap V_{a_{j+1}}$ is included in $W^u(a_{j+1})$ and that λ is of the following form:

$$\lambda = \sum_{i=1}^{n} \alpha_i |\chi_i(a_{j+1})|.$$

As $\mathcal{L}_{V_f}^{(k)} u_1 = \lambda u_1 + u_0$, we know that $\mathcal{L}_{V_f}^{(k)} u_1 = \lambda u_1$ on the open set $f^{-1}(< f(a_{j+1}))$. Suppose now that there exists $i_0 \leq j$ such that u_1 does not identically vanish near a_{i_0}. Without loss of generality, we may suppose that i_0 is minimal. In such a neighborhood, one has $\mathcal{L}_{V_f}^{(k)} u_1 = \lambda u_1$ as u_0 vanishes near a_{i_0}. Arguing as in paragraph 5.2.2 one more time, we would find that

$$\lambda = \sum_{i=1}^{n} \alpha_i |\chi_i(a_{i_0})|.$$

As we supposed that $\lambda \neq 0$ and that the Lyapunov exponents are rationally independent, this would lead to a contradiction. Hence, u_1 vanishes near any critical point a_i with $i \leq j$. As u_0 is locally supported on $W^u(a_{j+1})$, we know that u_1 still satisfies the eigenvalue equation $\mathcal{L}_{V_f}^{(k)} u_1 = \lambda u_1$ near a_{j+1} and outside $W^u(a_{j+1})$. By propagation, we deduce that u_1 is locally supported on $W^u(a_{j+1})$. According to Remark 5.5, we are then able to infer that $\langle (\varphi_f^{\ln s})^* u_1, \psi \rangle$ has a (bounded) polyhomogeneous expansion in s as $s \to 0^+$ for every smooth test form ψ supported near a_{j+1}. From our assumption on j, one can find ψ such that $\langle u_0, \psi \rangle \neq 0$ which gives the expected contradiction when we write (29). □

We conclude this paragraph by mentioning the following analogue of Proposition 6.3 which we used above and which was implicitely proved in paragraph 5.2.2:

Proposition 7.3. *Suppose that (f, g) is a smooth Morse pair generating a Morse-Smale gradient flow. Let $u \neq 0$ be an element of $\mathcal{H}_k^m(M)$ satisfying $\mathcal{L}_{V_f}^{(k)} u = \lambda u$. Let a be the critical point of f satisfying the following properties:*

- *u does not vanish in any neighborhood of a,*
- *for every a' in $\text{Crit}(f)$ satisfying $f(a') < f(a)$, u identically vanishes near a'.*

Then, $-\lambda$ belongs to the index set $\mathcal{I}_k(a)$.

Note that this part of the proof is based on the proof of paragraph 5.2.2 and it does not require the rational independence of the Lyapunov exponents.

7.2. **Description of the eigenmodes.** We now use similar arguments as at was done in section 6 in order to construct a basis of every eigenspace. Compared with the case $\lambda = 0$, there will be some combinatorial complications due to the possible multiplicity of the eigenvalue at a critical point a. In this paragraph, we fix $0 \leq k \leq n$ and λ an eigenvalue of the operator

$$\mathcal{L}_{V_f}^{(k)} : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M).$$

Recall from Proposition 5.1 that λ must be of the form

$$\lambda = \sum_{j \in I \cup J} (\alpha_j + 1)|\chi_j(a)| + \sum_{j \in (I \cup J)^c} \alpha_j |\chi_j(a)|,$$

where

- a is a critical point of index r,
- for every $1 \leq j \leq n$, α_j is a nonnegative integer,
- $I \subset \{1, \ldots, r\}$ and $J \subset \{r+1, \ldots, n\}$ such that $|J| - |I| = k - r$.

We shall now proceed as in the case of the kernel. First, we determine the local form of an eigenmode near critical points. Then, we show how to extend these local models into currents defined on M. Finally, we show that these currents form a basis of the eigenspace associated to λ.

Remark 7.4. The case $\lambda = 0$ was already treated in the previous paragraph. Thus, we will always suppose $\lambda \neq 0$ in this paragraph.

7.2.1. *Local form near the "smallest" critical point.* Let $u \neq 0$ be an element in $\mathcal{H}_k^m(M)$ such that $\mathcal{L}_{V_f}^{(k)} u = \lambda u$. As before, we denote by j the index such that, for every $i < j$, u vanishes in a neighborhood of a_i and such that u does not vanish near a_j. Recall that $\text{supp}(u) \cap V_{a_j}$ is included in $W^u(a_j)$ for some small enough neighborhood V_{a_j} of a_j. Thanks to Proposition 7.3, $-\lambda$ belongs to $\mathcal{I}_k(a_j)$. In order to alleviate notations, we will write $a_j = a$ in the following.

Using Schwartz's Theorem and Lemma 6.6 one more time, we deduce that, in the adapted coordinates of paragraph 3.5, the current u reads as a finite sum :

$$u(x, y, dx, dy) = \sum_{\alpha', |I'| + |J'| = k} u_{\alpha', I', J'}(y) \partial_x^{\alpha'} \delta_{\{0\}}^{\mathbb{R}^r}(x) dx^{I'} \wedge dy^{J'} \tag{30}$$

where the $u_{\alpha',I',J'}$ are smooth functions in $C^\infty(\mathbb{R}^{n-r})$. A direct calculation shows us that, in a small enough neighborhood of a, one has, for every $0 < s \leq 1$,

$$(\varphi_f^{\ln s*}u)(x,y,dx,dy) = \sum_{\alpha',|I'|+|J'|=k} u_{\alpha',I',J'}((s^{\chi_j(a)}y_j)_j)\partial_x^{\alpha'}\delta_{\{0\}}^{\mathbb{R}^r}(x)s^{\tilde{\lambda}_{I',J',\alpha'}}dx^{I'}\wedge dy^{J'},$$

where

$$\tilde{\lambda}_{I',J',\alpha'} := \sum_{j=1}^r(\alpha_j'+1)|\chi_j(a)| - \sum_{j\in I'}|\chi_j(a)| + \sum_{j\in J'}|\chi_j(a)|.$$

On the other hand, as u satisfies $\mathcal{L}_{V_f}^{(k)}u = \lambda u$, we know that, for every smooth test form ψ of degree $n-k$ and for every $0 < s \leq 1$,

$$\langle \varphi_f^{\ln s*}u, \psi\rangle = s^\lambda\langle u,\psi\rangle.$$

Combining this equality to the local form of u, we find

$$s^\lambda\langle u,\psi\rangle = \sum_{\alpha',|I'|+|J'|=k} s^{\tilde{\lambda}_{I',J',\alpha'}}\left\langle \partial_x^{\alpha'}\delta_{\{0\}}^{\mathbb{R}^r}(x), u_{\alpha',I',J'}((s^{\chi_j(a)}y_j)_j)dx^{I'}\wedge dy^{J'}\wedge\psi(x,y,dx,dy)\right\rangle.$$

Write now the Taylor expansion of $u_{\alpha',I',J'}$ (which is $\mathcal{C}^\infty$). From that, we find that

$$u_{\alpha',I',J'}(y) = c_{\alpha',I',J'}y_{r+1}^{\alpha_{r+1}'}\dots y_n^{\alpha_n'},$$

where $c_{\alpha',I',J'}$ is some fixed constant, α_j' belongs to $\mathbb{N}$ for every $r+1\leq j\leq n$ and

$$\tilde{\lambda}_{I',J',\alpha'} + \sum_{j=r+1}^n \alpha_j'|\chi_j(a)| = \lambda.$$

Equivalently, one has

$$\lambda = \sum_{j=1}^r(\alpha_j'+1)|\chi_j(a)| + \sum_{j=r+1}^n\alpha_j'|\chi_j(a)| - \sum_{j\in I'}|\chi_j(a)| + \sum_{j\in J'}|\chi_j(a)|.$$

To summarize, this shows that the current u reads in the adapted cooordinates near a:

$$u(x,y,dx,dy) = \sum_{\alpha,I,J:(*)} c_{\alpha,I,J}\,(y\partial_x)^\alpha\,\delta_{\{0\}}^{\mathbb{R}^r}(x)\,(\wedge_{j\notin I}dx_j)\wedge(\wedge_{j\in J}dy_j)\,, \tag{31}$$

where $c_{\alpha,I,J}$ are some fixed constant and where $(*)$ means that (α,I,J) satisfies

- for every $1\leq j\leq n$, $\alpha_j\in\mathbb{N}$,
- $I\subset\{1,\dots,r\}$, $J\subset\{r+1,\dots,n\}$,
- $|J|-|I| = k-r$,
- $\lambda = \sum_{j\in I\cup J}(\alpha_j+1)|\chi_j(a)| + \sum_{j\in(I\cup J)^c}\alpha_j|\chi_j(a)|$,

7.2.2. *Extension of the local form to M.* Mimicking what was done in the case of the kernel, we will now explain how the local form obtained in (31) can be extended into a natural eigencurrent carried by the closure of $W^u(a)$.

For a fixed triple (α, I, J) satisfying the conditions $(*)$, we define

$$\tilde{U}_a^{\alpha,I,J}(x,y,dx,dy) := \theta(x,y)\,(y\partial_x)^\alpha\,\delta_{\{0\}}^{\mathbb{R}^r}(x)\,(\wedge_{j\notin I}dx_j)\wedge(\wedge_{j\in J}dy_j)\,, \tag{32}$$

where θ is the same smooth function as in paragraph 6.4.2. By construction, one can verify that

$$\mathcal{L}_{V_f}^{(k)}\tilde{U}_a^{\alpha,I,J} = \lambda\tilde{U}_a^{\alpha,I,J}$$

on the open neighborhood $(-\delta_a/4, \delta_a/4)^n$. Moreover, this current belong to the anisotropic space $\mathcal{H}_k^m(M)$ provided that we pick N_0 large enough (compared with $|\alpha|$) in the definition of the order function m. Using the conventions of (17), we then set

$$U_a^{\alpha,I,J} = \sum_{i:\lambda_i^{(k)}=\lambda}\langle\tilde{U}_a^{\alpha,I,J}, v_{i,1}^{(k)}\rangle u_{i,1}^{(k)},$$

which obviously satisfies the eigenvalue equation:

$$\mathcal{L}_{V_f}^{(k)}U_a^{\alpha,I,J} = \lambda U_a^{\alpha,I,J}$$

Let us now describe some properties of this current. First, we let ψ be a smooth $n-k$ form carried outside $\overline{W^u(a)}$. For such a form and for every $0 < s \leq 1$, one has $\langle\varphi_f^{-\ln s*}\tilde{U}_a^{\alpha,I,J}, \psi\rangle = 0$. Hence, every term in the asymptotic expansion 18 must vanish. In particular, one has $\langle U_a^{\alpha,I,J}, \psi\rangle = 0$ for every smooth test form supported outside $\overline{W^u(a)}$. Equivalently, one has

$$\operatorname{supp}\left(U_a^{\alpha,I,J}\right) \subset \overline{W^u(a)}.$$

By invariance under the gradient flow, the support is in fact equal to $\overline{W^u(a)}$. Like in the case of the kernel, we would like to verify that $\tilde{U}_a^{\alpha,I,J}$ and $U_a^{\alpha,I,J}$ coincide in a neighborhood of the critical point a. For that purpose, we let $\psi(x,y,dx,dy)$ be a some smooth test form carried in the neighborhood with adapted coordinates. Mimicking the calculation of paragraph 5.4, one finds that, for every $0 < s \leq 1$,

$$\langle\varphi_f^{\ln s*}\tilde{U}_a^{\alpha,I,J}, \psi\rangle = s^\lambda(1+o(1))\left\langle(y\partial_x)^\alpha\,\delta_{\{0\}}^{\mathbb{R}^r}(x)\,(\wedge_{j\notin I}dx_j)\wedge(\wedge_{j\in J}dy_j)\,,\psi\right\rangle.$$

Using one more time the spectral expansion of the correlation function (18) and using the fact that there is *no Jordan blocks*, one can identify the term of order s^λ in the asymptotic. In particular, this implies that

$$\langle U_a^{\alpha,I,J}, \psi\rangle = \sum_{i:\lambda_i^{(k)}=\lambda}\langle\tilde{U}_a^{\alpha,I,J}, v_{i,1}^{(k)}\rangle\langle u_{i,1}^{(k)}, \psi\rangle = \left\langle(y\partial_x)^\alpha\,\delta_{\{0\}}^{\mathbb{R}^r}(x)\,(\wedge_{j\notin I}dx_j)\wedge(\wedge_{j\in J}dy_j)\,,\psi\right\rangle,$$

for every smooth test form ψ compactly supported in a small enough neighborhood of a. To summarize, we have shown the following:

Proposition 7.5. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let a be a critical point of index r, let $0 \leq k \leq n$ and let $0 \leq \theta \leq 1$ be a smooth cutoff function which is compactly supported in a small enough* *neighborhood V_a of a, and equal to 1 in an open neighborhood of a. Let I be a subset of $\{1,\dots,r\}$ and J be a subset of $\{r+1,\dots,n\}$ satisfying $|J|-|I|=k-r$. Let α be an element in $\mathbb{N}^n$. Set $[W^u(a)]_{\alpha,I,J}$ to be the image in the adapted coordinate chart of*

$$(y\partial_x)^\alpha\,\delta^{\mathbb{R}^r}_{\{0\}}(x)\,(\wedge_{j\notin I}dx_j)\wedge(\wedge_{j\in J}dy_j)\,.$$

Then, there exists an open neighborhood $\tilde{V}_a\subset V_a$ of a such that the current

$$U_a^{\alpha,I,J}:=\sum_{i:\lambda_i^{(r)}=\lambda}\langle\theta[W^u(a)]_{\alpha,I,J},v_{i,1}^{(r)}\rangle u_{i,1}^{(r)}$$

satisfies

- $U_a^{\alpha,I,J}|_{\tilde{V}_a}=[W^u(a)]_{\alpha,I,J}|_{\tilde{V}_a}$,
- $\operatorname{supp}(U_a^{\alpha,I,J})=\overline{W^u(a)}$,
- $\mathcal{L}_{V_f}^{(k)}(U_a^{\alpha,I,J})=\lambda U_a^{\alpha,I,J}$ *with*

$$\lambda=\sum_{j\in I\cup J}(\alpha_j+1)|\chi_j(a)|+\sum_{j\in(I\cup J)^c}\alpha_j|\chi_j(a)|.$$

This Proposition gives a family of natural currents generalizing Laudenbach's currents which were only defined for the eigenvalue 0. Note that they are well defined as soon as (f,g) is a smooth Morse pair generating a Morse-Smale gradient flow. Up to the linearization chart, their expression is more or less explicit. For every λ in $\mathcal{I}_k(a)$, we define the "multiplicity" of λ as

$$m_k(\lambda):=|\{(\alpha,I,J)\text{ satisfying }(*)\}|\,, \tag{33}$$

where $(*)$ means that (α,I,J) satisfies

- for every $1\leq j\leq n$, $\alpha_j\in\mathbb{N}$,
- $I\subset\{1,\dots,r\}$, $J\subset\{r+1,\dots,n\}$,
- $|J|-|I|=k-r$,
- $\lambda=\sum_{j\in I\cup J}(\alpha_j+1)|\chi_j(a)|+\sum_{j\in(I\cup J)^c}\alpha_j|\chi_j(a)|$.

Remark 7.6. In order to compute the Weyl's law for our operators, it will be convenient to rewrite things in a slightly different manner. More precisely, for any given α in $\mathbb{N}^n$, we set

$$m_{k,a}(\alpha):=|\{(I\times J\subset\{1,\dots,r\}\times\{r+1,\dots,n\}:\ |J|-|I|=k-r,\text{ and }\forall j\in I\cup J,\ \alpha_j\geq 1\}|\,,$$

where r is the index of a. With these conventions and thanks to the rational independence, any $\alpha.|\chi(a)|$ appears with multiplicity $m_{k,a}(\alpha)$ in $\mathcal{R}_k(f,g)$.

7.2.3. *The generation Theorem.* We conclude this section by showing that the currents we have just constructed generate a basis of $\operatorname{Ker}(\mathcal{L}_{V_f}^{(k)}+\lambda)$, i.e.

Proposition 7.7. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let $0\leq k\leq n$ and let $\lambda\neq 0$ be an element in $\mathcal{R}_k(f,g)$. The* *family of currents*

$$\left\{U_a^{\alpha,I,J} : \sum_{j\in I\cup J}(\alpha_j+1)|\chi_j(a)| + \sum_{j\in(I\cup J)^c}\alpha_j|\chi_j(a)| = -\lambda\right\}$$

forms a basis of the kernel of the operator

$$\mathcal{L}_{V_f}^{(k)} + \lambda : \mathcal{H}_k^m(M) \to \mathcal{H}_k^m(M).$$

In particular, the kernel of this operator is of dimension $m_k(\lambda)$.

Proof. The proof is almost the same as in the case $\lambda = 0$, and we briefly adapt it in this context.

Let us first show that it generates the kernel of $\mathcal{L}_{V_f}^{(k)} + \lambda$. This follows from the discussion from paragraph 7.2.1. If we take u in $\mathcal{H}_k^m(M)$ satisfying $\mathcal{L}_{V_f}^{(k)} u = -\lambda u$, then (31) gives us a family of constants $c_{\alpha,I,J}$. We then set $\tilde{u} = u - \sum_{\alpha,I,J} c_{\alpha,I,J} U_a^{\alpha,I,J}$ where a is the smallest critical point where u does vanish in a neighborhood. Note that λ belongs to $\mathcal{I}_k(a)$ from Proposition 7.3. One still has that $\mathcal{L}_{V_f}^{(k)}\tilde{u} = -\lambda\tilde{u}$. From paragraph 7.2.1, we also know that $\tilde{u}$ vanishes near any critical point b satisfying $f(b) \leq f(a)$. Then, combing the rational independence of the Lyapunov exponents with Proposition 7.3, we conclude that $\tilde{u} = 0$.

Let us now briefly verify that these elements are independent. Suppose that one can write

$$\sum_{(\alpha,I,J)\text{satisfying } (*)} \gamma_{\alpha,I,J} U_a^{\alpha,I,J} = 0$$

We write this relation near the critical point a and we use that the germs of current are (from proposition 7.5) linearly independent near this critical point. This implies that $\gamma_{\alpha,I,J} = 0$ for every (α, I, J) associated with a.

□

7.3. **Back to correlations.** Mimicking the arguments of paragraph 6.4.4, we obtain a full asymptotic for the correlation function:

Proposition 7.8. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Let $0 \leq k \leq n$.*

Then, for every a in Crit(f) *and for every α in $\mathbb{N}^n$, there exist*[7]

- $(U_{a,k}^{\alpha,j})_{j=1,\ldots,m_{k,a}(\alpha)}$ *in $\mathcal{D}'^{,k}(M)$ whose supports is equal to $\overline{W^u(a)}$,*
- $(S_{a,n-k}^{\alpha,j})_{j=1,\ldots,m_{k,a}(\alpha)}$ *in $\mathcal{D}'^{,n-k}(M)$ whose supports is equal to $\overline{W^s(a)}$,*

such that, for every $\Lambda > 0$ and for every $t \geq 0$,

$$\varphi_f^{-t*} = \sum_{a\in\text{Crit}(f)} \sum_{\alpha\in\mathbb{N}^n:\alpha.|\chi(a)|<\Lambda} e^{-\alpha.|\chi(a)|t} \sum_{j=1}^{m_{k,a}(\alpha)} |U_{a,k}^{\alpha,j}\rangle\langle S_{a,n-k}^{\alpha,j}| + \mathcal{O}_{\Omega^k\to\mathcal{D}'^{,k}}(e^{-\Lambda t}).$$

In particular, this implies Theorem 2.2 from the introduction.

[7]See Remark 7.6 for the precise definition of $m_{k,a}(\alpha)$.

7.3.1. *The duality map* $U^{\alpha,j}_{a,k} \mapsto S^{\alpha,j}_{a,n-k}$ *as a generalized Fourier transform.* The duality map $U^{\alpha,j}_{a,k} \mapsto S^{\alpha,j}_{a,n-k}$ has in fact a nice interpretation in terms of Fourier transform in the linearizing chart near the critical point a. Recall that the Fourier transform can be defined on the space of tempered currents on $\mathbb{R}^n$ [66, p. 396]. This notion of Fourier transform coincides with the Fourier (sometimes called super Fourier) transform appearing in the theory of Berezin integrals [43, Ch. 7] – see section 5 of [22] for a brief reminder on Berezin integrals. In fact, a tempered distribution $u(z, dz)$ can be identified with an element in the Grassmann algebra $\mathcal{S}'(\mathbb{R}^n)[dz_1, \dots, dz_n]$. Using the conventions of [22], one can define the **Fourier transform** as follows:

$$\forall u \in \mathcal{S}'(\mathbb{R}^n)[dz_1, \dots, dz_n],\ \mathcal{F}(u)(z, dz) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(n|n)}} u(p, P) e^{-i(p.z+P.dz)} dpdP.$$

Thanks to Proposition 7.5, we know exactly the local form of the current $U^{\alpha,j}_{a,k}$ near the critical point a, and we can check that the map $U^{\alpha,j}_{a,k} \mapsto S^{\alpha,j}_{a,n-k}$ is locally near a equal to $\mathcal{F}$ up to some normalizing constant depending only on n and k.

7.4. **Asymptotic formulas.** In order to conclude this section, we will give some nice asymptotic formulas that can be easily derived from our description of the spectrum.

7.4.1. *Weyl asymptotics.* Due to the fact that we obtained an explicit expression for the spectrum of the transfer operator, we can easily obtain some Weyl's formula. More precisely,

Proposition 7.9 (Weyl Law). *Let* $0 \leq k \leq n$ *and let* φ_f^t *be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent.. Then, one has*

$$|\{\lambda \in \mathcal{R}_k(f, g) : |\lambda| \leq \Lambda\}| = \frac{\Lambda^n}{k!(n-k)!} \sum_{a \in \text{Crit}(f)} \frac{1}{\prod_{j=1}^n |\chi_j(a)|} + \mathcal{O}(\Lambda^{n-1}),\ \text{as } \Lambda \to +\infty,$$

where the elements in $\mathcal{R}_k(f, g)$ *are counted with their algebraic multiplicities.*

Proof. From Remark 7.6 and Propositions 7.7 and 7.2, one knows that

$$|\{\lambda \in \mathcal{R}_k(f, g) : |\lambda| \leq \Lambda\}| = \sum_{a \in \text{Crit}(f)} \sum_{\alpha \in \mathbb{N}^n : \alpha.|\chi(a)| \leq \Lambda} m_{k,a}(\alpha).$$

Hence, we can fix a critical point a and compute $\sum_{\alpha \in \mathbb{N}^n : \alpha.|\chi(a)| \leq \Lambda} m_{k,a}(\alpha)$. We write

$$\sum_{\alpha \in \mathbb{N}^n : \alpha.|\chi(a)| \leq \Lambda} m_{k,a}(\alpha) = |\{\alpha \in \mathbb{N}^n; I \subset \{1, \dots, r\}, J \subset \{r+1, \dots, n\} : \alpha.|\chi(a)| \leq \Lambda \text{ and } (**)\}|,$$

where r is the index of a and where $(**)$ means that $|J| - |I| = k - r$ and $\forall j \in I \cup J,\ \alpha_j \geq 1$. We start by fixing a pair (I, J) where $I \subset \{1, \dots, r\}$ and $J \subset \{r+1, \dots, n\}$ subject to the condition $|J| - |I| = k - r$. We then want to compute

$$|\{\alpha \in \mathbb{N}^n : \forall j \in I \cup J,\ \alpha_j \geq 1 \text{ and } \alpha.|\chi(a)| \leq \Lambda\}|.$$

One can verify that

$$|\{\alpha \in \mathbb{N}^n : \forall j \in I \cup J,\ \alpha_j \geq 1 \text{ and } \alpha.|\chi(a)| \leq \Lambda\}| = |\{\alpha \in \mathbb{N}^n : \alpha.|\chi(a)| \leq \Lambda\}| + \mathcal{O}(\Lambda^{n-1}).$$

Then, one has

$$|\{\alpha \in \mathbb{N}^n : \alpha.|\chi(a)| \leq \Lambda\}| = \text{Vol}\left(\{x \in (\mathbb{R}_+)^n : |\chi(a)|.x \leq \Lambda\}\right) + \mathcal{O}(\Lambda^{n-1}),$$

which is the volume of a simplical domain. Hence, one has

$$|\{\alpha \in \mathbb{N}^n : \forall j \in I \cup J,\ \alpha_j \geq 1 \text{ and } \alpha.|\chi(a)| \leq \Lambda\}| = \frac{\Lambda^n}{n!|\prod_{j=1}^n \chi_j(a)|} + \mathcal{O}(\Lambda^{n-1}).$$

This is valid for any $I \subset \{1, \ldots, r\}, J \subset \{r+1, \ldots, n\}$ subject to the condition $|J| - |I| = k - r$. One can remark that the number of such $I \times J$ is equal to the number of $I' \times J \subset \{1, \ldots, r\} \times \{r+1, \ldots, n\}$ subject to the condition $|J| + |I'| = k$. This is exactly equal to $\begin{pmatrix} n \\ k \end{pmatrix}$. This concludes the proof of the Proposition. □

7.4.2. *Trace formulas.* In this paragraph, we discuss briefly some trace formulas related to our problem. For every $0 \leq k \leq n$ and every $\lambda \geq 0$, we set

$$C^k(f, \lambda) := \text{Ker}(\mathcal{L}_{V_f}^{(k)} - \lambda),$$

where we mean the kernel of the operator in an appropriate anisotropic Sobolev space as above. We define then the spaces of even (bosonic) and odd (fermionic) eigenstates:

$$C^{\text{even}}(f, \lambda) := \bigoplus_{k \equiv 0(\text{mod}2)} C^k(f, \lambda), \text{ and } C^{\text{odd}}(f, \lambda) := \bigoplus_{k \equiv 1(\text{mod}2)} C^k(f, \lambda)$$

The **fermion number** operator $(-1)^F$ acts on

$$C(f, \lambda) = C^{\text{even}}(f, \lambda) \oplus C^{\text{odd}}(f, \lambda)$$

with eigenvalue ± 1 depending on the parity of the state. Let now $\theta : \mathbb{R} \to \mathbb{C}$. We define the **super[8]-trace** as follows:

$$\begin{aligned} \text{Str}\left(\theta\left(\mathcal{L}_{V_f}\right)\right) &= \text{Tr}\left((-1)^F \theta\left(\mathcal{L}_{V_f}\right)\right) \\ &= \text{Tr}\left(\theta\left(\mathcal{L}_{V_f}\right)\rceil_{C^{\text{even}}(f,\lambda)}\right) - \text{Tr}\left(\theta\left(\mathcal{L}_{V_f}\right)\rceil_{C^{\text{odd}}(f,\lambda)}\right) \\ &:= \sum_{\lambda \in \cup_{k=0}^n \mathcal{R}_k(f,g)} \theta(\lambda)\left(\dim C^{\text{even}}(f, \lambda) - \dim C^{\text{odd}}(f, \lambda)\right). \end{aligned}$$

This allows to define a notion of super-trace as soon as the last quantity is well-defined. In order to avoid too many complications that would be beyond the scope of this article, we take this as a definition of the trace in our framework. We note that this is related to the notion of flat trace – see e.g. [28, Sect. 2.4].

The operators d and i_{V_f} both commute with $\mathcal{L}_{V_f}$. Hence, $Q = (d + i_{V_f})$ defines an operator

$$Q_\lambda : C^{\text{even}}(f, \lambda) \oplus C^{\text{odd}}(f, \lambda) \mapsto C^{\text{odd}}(f, \lambda) \oplus C^{\text{even}}(f, \lambda).$$

[8]One more time, the prefix super just emphasizes the fact that we are considering functions of odd (dz_i) and even (z_i) variables.

which exchanges chiralities. We observe that, for every $\lambda > 0$, Q_λ is an isomorphism since $Q_\lambda^2 = \mathcal{L}_{V_f} = \lambda \mathrm{Id}$. In particular, for every $\lambda > 0$, one has

$$\dim C^{\text{even}}(f,\lambda) = \dim C^{\text{odd}}(f,\lambda).$$

Combined with Proposition 6.8, this implies

$$\mathrm{Str}\left(\theta\left(\mathcal{L}_{V_f}\right)\right) = \theta(0)\sum_{k=0}^{n}(-1)^k|\{a\in\mathrm{Crit}(f):\mathrm{ind}(a)=k\}| = \theta(0)\sum_{a\in\mathrm{Crit}(f)}(-1)^{\mathrm{ind}(a)}.$$

By the classical Morse inequalities, the right-hand side of this equality is equal to $\theta(0)\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. We shall prove this property in section 8.

Let us now specialize this result when we take $\theta(\lambda) = e^{-\lambda t}\mathbf{1}_{[0,\Lambda]}(\lambda)$ some fixed $\Lambda > 0$. In that case, we recover the following version of the Atiyah–Bott–Lefschetz fixed point Theorem [3]:

Proposition 7.10. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Then, one has, for every $\Lambda > 0$, and for every $t > 0$,*

$$\sum_{k=0}^{n}(-1)^k\,\mathrm{Tr}\left(\Pi_\Lambda^{(k)}\varphi_f^{-t*}\Pi_\Lambda^{(k)}\right) = \sum_{x=\varphi_f^{-t}(x)}\frac{\det(\mathrm{Id}-d_x\varphi_f^{-t})}{|\det(\mathrm{Id}-d_x\varphi_f^{-t})|}, \tag{34}$$

where $\Pi_\Lambda^{(k)}$ is the spectral projector defined in paragraph 5.3 and Tr *is the standard trace.*

In the terminology of [3], the left-hand side of (34) is called the **Lefschetz number** of φ_f^{-t*} (more precisely of $\Pi_\Lambda^{(k)}\varphi_f^{-t*}\Pi_\Lambda^{(k)}$). As was already mentionned, we will verify in the next section that the right-hand side is equal to the Euler characteristic $\chi(M)$ of M. Note that, after integrating the previous equality against $t^{s-1}e^{-zt}$ between 0 and $+\infty$, one can write the following expression for the spectral (super-)zeta function of $\mathcal{L}_{V_f} + z$:

$$\zeta(s,z) := \frac{1}{\Gamma(s)}\sum_{k=0}^{n}(-1)^k\int_0^{+\infty}t^s e^{-zt}\,\mathrm{Tr}\left(\Pi_\Lambda^{(k)}\varphi_f^{-t*}\Pi_\Lambda^{(k)}\right)\frac{dt}{t} = \frac{\chi(M)}{z^s}.$$

Up to the normalizing factors[9] t^s and $\Gamma(s)$, this function looks like a (spectral) analogue of the semiclassical zeta function of Faure and Tsujii [33, Sect. 1]. In particular, at $s = 0$, this is formally equal to $\chi(M)$. If we differentiate this expression with respect to s and evaluate it at 0, we find that $e^{-\partial_s\zeta(0,z)} = z^{\chi(M)}$. Equivalently, the super-determinant of $(\mathcal{L}_{V_f} + z)$ verifies:

Corollary 7.11. *Let φ_f^t be a Morse-Smale gradient flow all of whose Lyapunov exponents are rationally independent. Then, one has, for every $\Lambda > 0$ and for every z in $\mathbb{C}^*$,*

$$\prod_{k=0}^{n}\det\left(\Pi_\Lambda^{(k)}(\mathcal{L}_{V_f}^{(k)} + z)\Pi_\Lambda^{(k)}\right)^{(-1)^k} = z^{\chi(M)}. \tag{35}$$

[9] In some sense, this allows to take into account the fact that, in our dynamical framework, there are fixed points.

where $\Pi_\Lambda^{(k)}$ *is the spectral projector defined in paragraph 5.3 and* det *is the standard determinant.*

8. Topological considerations

Studying Morse functions has deep connections with the topology of the manifold, and we will now describe some topological consequences of our spectral analysis of the operator $\mathcal{L}_{V_f}^{(*)}$. In all this section, we still suppose that (f, g) is a smooth Morse pair inducing a Morse-Smale gradient flow but we do not suppose a priori that the Lyapunov exponents are rationally independent. The results presented here are in fact related to the description of the Morse complex given by Laudenbach in [53, 54] and to the interpretation of Morse theory given by Harvey and Lawson in [46]. The main novelty here is the spectral interpretation of these results in analogy with Hodge-de Rham theory.

8.1. **De Rham cohomology.** We start with a brief reminder on de Rham cohomology [63, 66]. Recall that, for every $k \geq 0$, the coboundary operator d sends any element in $\Omega^k(M)$ to an element in $\Omega^{k+1}(M)$, and that it satisfies $d \circ d = 0$. In particular, one can define a cohomological complex $(\Omega^*(M), d)$ associated with d:

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \ldots \to \Omega^n(M) \to 0.$$

This complex is also called the de Rham complex. An element ω in $\Omega^*(M)$ such that $d\omega = 0$ is called a **cocycle** while an element ω which is equal to $d\alpha$ for some $\alpha \in \Omega^*(M)$ is called a **coboundary**. We define then

$$Z^k(M) = \mathrm{Ker}(d) \cap \Omega^k(M), \text{ and } B^k(M) = \mathrm{Im}(d) \cap \Omega^k(M).$$

Obviously, $B^k(M) \subset Z^k(M)$, and the quotient space $\mathbb{H}^k(M) = Z^k(M)/B^k(M)$ is called the **k-th de Rham cohomology**.

According to [66, p. 344-345], the coboundary operator d can be extended into a map acting on the space of currents. This allows to define another cohomological complex $(\mathcal{D}'^{,*}(M), d)$:

$$0 \to \mathcal{D}'^{,0}(M) \to \mathcal{D}'^{,1}(M) \to \ldots \to \mathcal{D}'^{,n}(M) \to 0,$$

where we recall that $\mathcal{D}'^{,k}(M)$ is the topological dual of $\Omega^{n-k}(M)$. One can similarly define the k-th cohomology of that complex. A remarkable result of de Rham is that these two cohomologies coincide [63, Ch. 4] – see also [66, p.355] for a generalization of this result.

Theorem 8.1 (de Rham)**.** *Let u be an element in $\mathcal{D}'^{,k}(M)$ satisfying $du = 0$.*

1. *There exists ω in $\Omega^k(M)$ such that $u - \omega$ belongs to $\mathrm{Im}(d) \cap \mathcal{D}'^{,k}(M)$.*
2. *If $u = dv$ with (u, v) in $\Omega^k(M) \times \mathcal{D}'^{,k-1}(M)$, then there exists ω in $\Omega^{k-1}(M)$ such that $u = d\omega$.*

Remark 8.2. In the following, we will need something slightly more precise than (i). Namely suppose that u is a cocycle in $\mathcal{H}_k^{m+k}(M)$. We denote by $\Delta_g^{(k)}$ the Laplace-Beltrami operator acting on $L^2(M, \Lambda^k(T^*M))$. We can find a pseudodifferential operator A_k of order -2 such that $u - \Delta_g^{(k)} A_k u$ belongs to $\Omega^k(M)$. As u is a cocycle, one can deduce that $d\Delta_g^{(k)} A_k u \in$

$\Omega^{k+1}(M)$. From the ellipticity of Δ_g, we find that $dA_k u \in \Omega^{k+1}(M)$. This implies that $d^* dA_k u \in \Omega^k(M)$ and thus $u - dd^* A_k(u)$ belongs to $\Omega^k(M)$. As u belongs to $\mathcal{H}_k^{m+k}(M)$, we get a refinement for point (i) in the sense that $u - \omega = dd^* A_k(u)$ belongs to $d(\mathcal{H}_{k-1}^{m+k-1}(M))$.

8.2. **Finite dimensional complexes.** Proving that the k-th cohomology is finite dimensional requires more work – see e.g. [63, Ch. 4]. Before deducing that result from our spectral analysis of $\mathcal{L}_{V_f}$, we start with some general considerations on finite dimensional cohomological complexes. Consider a cohomological complex (C^*, d) associated with the coboundary operator:

$$0 \to C^0 \to C^1 \to \ldots \to C^n \to 0,$$

where for every $0 \leq k \leq n$, C^k is a finite dimensional subspace of $\mathcal{D}'^{,k}(M)$.

Remark 8.3. A famous example of such complexes appears in Hodge theory where one considers $C^k(\Delta) = \operatorname{Ker}(\Delta_g^{(k)})$. From the ellipticity of the Laplace-Beltrami operator on that space, one can deduce the fact that $C^k(\Delta)$ is a finite dimensional space inside $\Omega^k(M)$.

Consider now the complexes induced by the operator $\mathcal{L}_{V_f}$. For that purpose, we pick N_0 and N_1 large enough in the definition of the order function $m(x, \xi)$, and we define

$$C^k(f) := \operatorname{Ker}\left(\mathcal{L}_{V_f}^{(k)}\right)$$

which is a finite dimensional space. Recall one more time from [32, Th. 1.5] that these spaces are intrinsic in the sense that they do not depend on the choice of the order function m. As d commutes with the Lie derivative $\mathcal{L}_{V_f}$, one can verify that, if $\mathcal{L}_{V_f}^{(k)} u = 0$ with u in $\mathcal{H}_k^m(M)$, then $\mathcal{L}_{V_f}^{(k+1)}(du) = 0$ with du belonging to $\mathcal{H}_{k+1}^{m+1}(M)$. Hence, the coboundary operator d induces a *finite dimensional cohomological complex* $(C^*(f), d)$

$$0 \to C^0(f) \to C^1(f) \to \ldots \to C^n(f) \to 0.$$

We can now apply Proposition 6.8 and we find that $\dim(C^k(f))$ is in fact equal to the number $c_k(f)$ of critical point of f which are of index k.

8.3. **Morse type inequalities.** Consider a finite dimensional complex (C^*, d). We briefly recall how to obtain Morse type inequalities in that abstract framework arguing as in [54, Ch. 6]. For that purpose, we define

$$Z^k(C^*) = \operatorname{Ker}(d) \cap C^k, \text{ and } B^k(C^*) = \operatorname{Im}(d) \cap C^k.$$

As above, we define the quotient space (or the k-th cohomology of the complex):

$$\mathbb{H}^k(C^*) := Z^k(C^*)/B^k(C^*).$$

We denote by $\beta_k(C^*) < \infty$ the dimension of that quotient space. We also introduce

$$b_k(C^*) = \dim B^k(C^*), \ c_k(C^*) = \dim C^k, \text{ and } z_k(C^*) = \dim Z^k(C^*).$$

We observe that

$$\beta_k(C^*) = z_k(C^*) - b_k(C^*) \text{ and } c_k(C^*) = b_{k+1}(C^*) + z_k(C^*).$$

We now write that, for every $k \geq 0$,

$$0 \leq b_{k+1}(C^*) = (c_k(C^*) - \beta_k(C^*)) - (c_{k-1}(C^*) - \beta_{k-1}(C^*)) + \dots$$

From this expression, we can deduce the following **Morse type inequalities** associated with the complex (C^*, d):

$$\forall 0 \leq k \leq n,\ \sum_{j=0}^{k} (-1)^{k-j} c_j(C^*) \geq \sum_{j=0}^{k} (-1)^{k-j} \beta_j(C^*), \tag{36}$$

and using $b_{n+1}(C^*) = 0$:

$$\sum_{j=0}^{n} (-1)^{n-j} c_j(C^*) = \sum_{j=0}^{n} (-1)^{n-j} \beta_j(C^*). \tag{37}$$

In the case where we pick $C^* = C^*(f)$, inequalities (36) and (37) are exactly the Morse inequalities for the complex $C^*(f)$ which is nothing else but the Morse complex (also called Thom-Smale-Witten complex).

8.4. **The Morse complex is isomorphic to the de Rham complex.** Let $0 \leq k \leq n$. We would like now to give *a spectral proof* of the fact that the k-th cohomology of the Morse complex is isomorphic to the de Rham cohomology of degree k. A proof of this result based on the theory of currents can be found in [54, Ch. 6] in the case of locally flat metrics. Here, we give an alternative proof of that result based on our spectral analysis of the operator $\mathcal{L}_{V_f}$. The main idea is just that the Morse complex is the limit as $t \to +\infty$ of the de Rham complex evolved under the semi-group φ_f^{-t*}. This observation is also contained in the works of Harvey and Lawson on finite volume flows [45, 46].

8.4.1. *Propagating the de Rham complex in finite time.* For every $t > 0$, we can define a complex $(\varphi_f^{-t*}\Omega^*(M), d)$. This complex is *homotopic* to the de Rham complex $(\Omega^*(M), d)$ via the **cochain homotopy equation**:

$$u - \varphi_f^{-t*} u = dR_t u + R_t du \tag{38}$$

$$R_t := \int_0^t i_{V_f} \circ \varphi_f^{-s*} ds, \tag{39}$$

where i_{V_f} is the contraction operator by the vector field V_f. This formula can be obtained as follows:

$$u - \varphi_f^{-t*} u = \int_0^t \mathcal{L}_{V_f} \circ \varphi_f^{-s*} u ds = \int_0^t (d \circ i_{V_f} + i_{V_f} \circ d) \circ \varphi_f^{-s*} u ds,$$

where the first equality is just the Taylor formula while the second equality follows from Cartan's formula [66, p. 351]: $\mathcal{L}_{V_f}^{(k)} = i_{V_f} \circ d + d \circ i_{V_f}$. In particular, one can verify from these relations that the semi-group φ_f^{-t*} induces an **isomorphism** in cohomology $H(\Omega^*(M), d) \simeq H(\varphi_f^{-t*}\Omega^*(M), d)$. Equivalently, the cohomologies of the two complexes are isomorphic via the map φ_f^{-t*}. In order to prove that the Morse complex $(C^*(f), d)$ is isomorphic (in cohomology) to $(\Omega^*(M), d)$, we now have to let t tends to $+\infty$ in the above homotopy equation. This can be done rigorously thanks to the fact that there 0 is an isolated eigenvalue in the spectrum of $\mathcal{L}_{V_f}$ and thanks to the spectral projector associated with the eigenvalue 0. This will exactly be the content of the next paragraph. Once the limit homotopy equation will be settled, *we will still have to verify that it induces an isomorphism in cohomology* between the de Rham and Morse complex – see paragraph 8.4.3.

Remark 8.4. Note that, in Hodge theory, the strategy is exactly the same except that we replace our semigroup φ_f^{-t*} by the heat flow $e^{-t\Delta_g}$.

8.4.2. *Spectral decomposition.* Let $m(x,\xi)$ be an order function with N_0 and N_1 sufficiently large to ensure that 0 is an isolated eigenvalue with finite algebraic multiplicity (eventually equal to 0). Introduce the *spectral projector* associated with the eigenvalue 0:

$$\mathbb{P}^{(k)} := \int_\gamma \frac{dz}{(z-\mathcal{L}_{V_f}^{(k)})} : \mathcal{H}_k^m(M) \to C_k(f),$$

where γ is a small Jordan path which separates 0 from the rest of the spectrum of $\mathcal{L}_{V_f}^{(l)}$ acting on $\mathcal{H}_l^m(M)$ for every $0 \leq l \leq n$ – see [48, App. A]. This operator commutes with $\mathcal{L}_{V_f}^{(k)}$. According to [30, p.244-246], one knows that

$$\mathcal{L}_{V_f}^{(k)} : \left(\mathrm{Id}_{\mathcal{H}_k^m} - \mathbb{P}^{(k)}\right)\mathcal{H}_k^m(M) \to \left(\mathrm{Id}_{\mathcal{H}_k^m} - \mathbb{P}^{(k)}\right)\mathcal{H}_k^m(M)$$

does not contain 0 in its spectrum. In particular, we can write the following decomposition:

$$\mathrm{Id}_{\mathcal{H}_k^m} = \mathbb{P}^{(k)} + \mathcal{L}_{V_f}^{(k)} \circ \left((\mathcal{L}_{V_f}^{(k)})^{-1} \circ \left(\mathrm{Id}_{\mathcal{H}_k^m} - \mathbb{P}^{(k)}\right)\right).$$

By Cartan's formula [66, p. 351], one knows that $\mathcal{L}_{V_f}^{(k)} = i_{V_f} \circ d + d \circ i_{V_f}$. Hence,

$$\mathrm{Id}_{\mathcal{H}_k^m} = \mathbb{P}^{(k)} + (d \circ i_{V_f} + i_{V_f} \circ d) \circ \left((\mathcal{L}_{V_f}^{(k)})^{-1} \circ \left(\mathrm{Id}_{\mathcal{H}_k^m} - \mathbb{P}^{(k)}\right)\right). \tag{40}$$

Moreover, d commutes with $\mathcal{L}_{V_f}$ hence with $\mathbb{P}^{(k)}$ from the expression of the spectral projector. Hence, for every u in $(\mathrm{Id} - \mathbb{P}^{(k)})\mathcal{H}_k^m(M)$, $du \in (\mathrm{Id} - \mathbb{P}^{(k+1)})\mathcal{H}_{k+1}^{m+1}(M)$, and one has $\mathcal{L}_{V_f}^{-1} \circ du = \mathcal{L}_{V_f}^{-1} \circ d \circ \mathcal{L}_{V_f} \circ \mathcal{L}_{V_f}^{-1}(u) = d \circ \mathcal{L}_{V_f}^{-1}(u)$. From that, we infer that d also commutes with $\mathcal{L}_{V_f}^{-1} \circ \left(\mathrm{Id}_{\mathcal{H}_k^m} - \mathbb{P}^{(k)}\right)$. Combining this last observation with the fact that d commutes with $\mathbb{P}^{(k)}$ and with (40), we finally find that, for every u in $\mathcal{H}_k^m(M)$, one has

$$u = \mathbb{P}^{(k)}(u) + d \circ R_\infty^{(k)}(u) + R_\infty^{(k+1)} \circ d(u), \tag{41}$$

where $R_\infty^{(n+1)} = 0$ and, for every $0 \leq l \leq n$,

$$R_\infty^{(l)} := i_{V_f} \circ (\mathcal{L}_{V_f}^{(l)})^{-1} \circ \left(\mathrm{Id}_{\mathcal{H}_l^m} - \mathbb{P}^{(l)}\right).$$

Remark 8.5. Recall that, from our complete description of the spectrum of $\mathcal{L}_{V_f}$, one has, for every u in $\mathcal{H}_k^m(M)$,

$$\mathbb{P}^{(k)}(u) = \lim_{t\to+\infty} \varphi_f^{-t*}u = \sum_{a\in \mathrm{Crit}(f):\ \mathrm{ind}(a)=k} \langle S_a, u\rangle U_a \in C^k(f).$$

We note that our cochain homotopy equation should be thought as the limit as $t \to +\infty$ of the cochain homotopy equation between the de Rham complex $(\Omega^*(M), d)$ and the propagated complex $(\varphi_f^{-t*}\Omega^*(M), d)$. Up to its spectral interpretation, this type of "limit homotopy equation" already appears in the works of Harvey and Lawson [46, Th. 2.3].

Remark 8.6. In Hodge theory, the analogue of this decomposition would consist in using the spectral projector associated with the eigenvalue 0 of $\Delta_g^{(k)} = dd^* + d^*d$, this projects currents on **harmonic** forms. Using the same convention for the spectral projector associated with 0, we would get

$$u = \mathbb{P}^{(k)}(u) + d \circ R_\infty^{(k)} u + R_\infty^{(k+1)} \circ d(u),$$

where, for every $0 \leq k \leq n$,

$$R_\infty^{(l)} := d^* \circ (\Delta_g^{(l)})^{-1} \circ \left(\mathrm{Id} - \mathbb{P}^{(l)}\right).$$

As we shall see below, this relation is the key step to prove that the de Rham cohomology is isomorphic to harmonic forms.

8.4.3. *Cohomological consequences.* As the coboundary operator d commutes with $\mathcal{L}_{V_f}$, it also commutes with $\mathbb{P}^{(k)}$. In particular, the map $\mathbb{P}^{(k)}$ induces a map from $Z^k(M)$ to $Z^k(C^*(f))$. We will now show (using our spectral approach) that it induces an isomorphism between the quotient spaces:

Proposition 8.7. *Let $0 \leq k \leq n$. The map*

$$\mathbb{P}^{(k)} : \Omega^k(M) \to C^k(f)$$

induces an isomorphism between the vector spaces $\mathbb{H}^k(C^*(f), d)$ and $\mathbb{H}^k(M)$.

From that Proposition, we recover the classical fact that $\mathbb{H}^k(M)$ is *a finite dimensional space* for every $0 \leq k \leq n$. Its dimension is called the k-th Betti number that we will denote by $b_k(M)$. With the notations of paragraph 8.2, we have $b_k(M) = \beta_k(C^*(f))$ for every $0 \leq k \leq n$. In particular, if we apply (36) and (37) in the case of the complex $(C^*(f), d)$, we recover the classical Morse inequalities:

Corollary 8.8 (Morse inequalities)**.** *Let*

$$c_k(f) = |\{a \in \mathrm{Crit}(f) \ \textit{s.t.} \ \mathrm{ind}(a) = k\}|.$$

Then, for all $k \in \{0, \ldots, n\}$, we have:

$$\sum_{j=0}^{k} (-1)^{k-j} c_j(f) \geq \sum_{j=0}^{k} (-1)^{k-j} b_j(M),$$

with equality in the case[10] *$k = n$.*

[10] Recall that in that case, the sum is the Euler characteristic $\chi(M)$ of M.

Proof of Proposition 8.7. Let us start with injectivity. Let u be a cocycle in $\Omega^k(M)$ such that $\mathbb{P}^{(k)}(u) = 0$. We use equality (41), and we find that

$$u = d \circ R_\infty^{(k)}(u),$$

which exactly says that u is a coboundary for the complex $(\mathcal{D}'^{,*}(M), d)$. As u is smooth, we know from de Rham Theorem 8.1 that u is a coboundary in $\Omega^k(M)$.

Let us now consider the surjectivity. Fix u a cocycle in $\text{Ker}(\mathcal{L}_{V_f}^{(k)}$. From Remark 8.2, we know that there exists $\omega \in \Omega^k(M)$ and v in $\mathcal{H}_{k-1}^{m+k-1}(M)$ such that $u - \omega = dv$. Writing the cochain homotopy equation (41) for ω, we find that

$$\omega = \mathbb{P}^{(k)}(\omega) + d \circ R_\infty^{(k)}(\omega).$$

This implies that

$$u = \mathbb{P}^{(k)}(\omega) + d\left(R_\infty^{(k)}(\omega) + v\right).$$

By construction, $R_\infty^{(k)}(\omega)+v$ belongs to $\mathcal{H}_{k-1}^{m+k-1}(M)$. Hence, applying the spectral projector to the previous equality and as d commutes with $\mathbb{P}^{(k)}$ (thanks to the integral expression of the spectral projector), we find that

$$u = \mathbb{P}^{(k)}(\omega) + d \circ \mathbb{P}^{(k)}\left(R_\infty^{(k)}(\omega) + v\right),$$

which proves the surjectivity. □

Remark 8.9. In Hodge theory, the surjectivity is in fact slightly simpler provided we use some results from elliptic theory in a slightly different form. In fact, consider a harmonic form u i.e. such that $\Delta_g u = 0$. From the ellipticity of the operator Δ_g, we know that u is smooth and thus belongs to $\Omega^{(*)}(M)$.

Remark 8.10. We note that it would be tempting to consider the complex associated with a nonzero eigenvalue λ of $\mathcal{L}_{V_f}$. Yet, this complex is trivial from the point of view of cohomology. In fact, if $du = 0$ and $(\mathcal{L}_{V_f} - \lambda)u = 0$ for some $\lambda \neq 0$, then we have that

$$\mathcal{L}_{V_f} u = \lambda u.$$

Combining the Cartan formula to the fact that $du = 0$, we would get

$$d \circ i_{V_f} u = \lambda u.$$

As $\lambda \neq 0$ and as i_{V_f} commutes with $\mathcal{L}_{V_f}$, we find that u is a coboundary of $(C^*(f), d)$. In particular, if we fix some (finite or not) subset A *containing* 0 inside $\bigcup_{0\leq k\leq n} \mathcal{R}_k(f, g)$, then one can define the complex $(C^*(f, A), d)$ such that

$$\forall 0 \leq k \leq n,\ C^k(f, A) := \text{span}\{U_{a,k}^{\alpha,j} :\ a \in \text{Crit}(f),\ \alpha.|\chi(a)| \in A \text{ and } 1 \leq j \leq m_{k,a}(\alpha)\}.$$

From the above observation, one can easily deduce that the complex $(C^*(f, A), d)$ is also isomorphic to the de Rham complex via the map

$$\mathbb{P}_A^{(k)}(\psi) := \sum_{a\in\text{Crit}(f)} \sum_{\alpha\in\mathbb{N}^n:\alpha.|\chi(a)|\in A} \sum_{j=1}^{m_{k,a}(\alpha)} \langle S_{a,k}^{\alpha,j}, \psi\rangle U_{a,k}^{\alpha,j}.$$

8.5. **Poincaré duality and $f \mapsto -f$.** By construction, one knows that the currents $(S_a)_{a\in\text{Crit}(f)}$ is the dual basis to $(U_a)_{a\in\text{Crit}(f)}$ for the duality bracket between $\mathcal{H}_k^m(M)$ and $\mathcal{H}_{n-k}^{-m}(M)$ which coincides (in the case of smooth forms) with the standard duality bracket between $\mathcal{D}'^{,*}(M)$ and $\Omega^{n-*}(M)$. Moreover, from paragraph 6.4.4, it is in fact a basis of the kernel of the operator $\mathcal{L}_{V_{-f}}^{(*)}$ acting on $\mathcal{H}_{n-*}^{-m}(M)$. We set

$$C^{n-k}(-f) := \text{Ker}(\mathcal{L}_{V_{-f}}^{(n-k)}).$$

We can then define the following complex associated with the coboundary operator d:

$$0 \to C^0(-f) \to \ldots \to C^{n-1}(-f) \to C^n(-f) \to 0.$$

As was already explained, the two complexes $(C^*(f), d)$ and $(C^*(-f), d)$ are dual to each other via the duality between $\mathcal{H}_k^m(M)$ and $\mathcal{H}_{n-k}^{-m}(M)$, i.e.

$$\forall (u, v) \in C^k(f) \times C^{n-k}(-f),\ \langle u, v\rangle = \langle u, v\rangle_{\mathcal{H}_k^m(M),\mathcal{H}_{n-k}^{-m}(M)} = \int_M u \wedge v.$$

Introduce now the following **Poincaré isomorphism** between $C^k(f)$ and the dual of $C^{n-k}(-f)$:

$$\mathcal{P}_0^{(k)} : u \in C^k(f) \mapsto \langle u, .\rangle \in C^{n-k}(-f)'.$$

We observe that $\langle u, v\rangle$ does not depend on the cohomology class of u and v. Hence, $\mathcal{P}_0^{(k)}$ induces a linear map between $\mathbb{H}^k(C^*(f), d)$ and $\mathbb{H}^{n-k}(C^*(-f), d)'$. We now follow closely [54, Ch. 6] and verify that this is in fact an isomorphism betweeen the quotient spaces. Suppose that θ is a linear form on $\mathbb{H}^{n-k}(C^*(-f), d)$. This induces a linear form θ on $Z^{n-k}(C^*(-f), d)$ which vanishes on $B^{n-k}(C^*(-f), d)$. By the Hahn-Banach Theorem, we extend this linear form to $C^{n-k}(-f)$. From the duality between $C^k(f)$ and $C^{n-k}(-f)$, there exists a unique u in $C^k(f)$ such that $\theta(v) = \langle u, v\rangle$ for every v in $C^{n-k}(-f)$. As θ vanishes on the image of d, we find that, for every v in $C^{n-k-1}(-f)$, $\langle u, dv\rangle = 0$ from which one can deduce that $du = 0$. This shows surjectivity of the linear map induced by $\mathcal{P}_0^{(k)}$. If we intertwine the role of f and $-f$, we find a linear surjection from $\mathbb{H}^{n-k}(C^*(-f), d)$ to $\mathbb{H}^k(C^*(f), d)'$. This implies that all the spaces have the same dimension. Hence, $\mathcal{P}_0^{(k)}$ induces an isomorphism between $\mathbb{H}^k(C^*(f), d)$ and $\mathbb{H}^{n-k}(C^*(-f), d)'$ for every $0 \leq k \leq n$. Combined with Proposition 8.7 applied to both f and $-f$, this implies the following well known result:

Proposition 8.11. *Let M be a smooth, compact, oriented manifold without boundary. Then, for every $0 \leq k \leq n$, $\mathcal{P}_0^{(k)}$ induces an isomorphism between $\mathbb{H}^k(C^*(f), d)$ and $\mathbb{H}^{n-k}(C^*(-f), d)'$. In particular, $b_k(M) = b_{n-k}(M)$ for every $0 \leq k \leq n$.*

Remark 8.12. This discussion could be generalized to the complex $(C^k(f, A), d)$ defined in Remark 8.10. For that purpose, we should observe that, for every A inside $\bigcup_{0\leq k\leq n} \mathcal{R}_k(f, g)$, $C^*(f, A)$ is a subspace of $\mathcal{D}'^{,*}_{\Gamma_-}(M)$ while $C_*(f, A)$ is a subspace of $\mathcal{D}'^{,*}_{\Gamma_+}(M)$. The duality pairing $\langle u, v\rangle$ is then given by the duality pairing $\int_M u \wedge v$ between $\mathcal{D}'^{,*}_{\Gamma_-}(M)$ and $\mathcal{D}'^{,n-*}_{\Gamma_+}(M)$ which is well defined as $\Gamma_- \cap \Gamma_+ = M \times \{0\}$. Note that it coincides with the duality pairing between the anisotropic Sobolev spaces when we consider elements inside the complexes $C(\pm f, A)$. This would yield a generalized Poincaré isomorphism:

$$\mathcal{P}_A^{(k)} : U_{a,k}^{\alpha,j} \in C^k(f,A) \mapsto \langle U_{a,k}^{\alpha,j}, . \rangle \in C^{n-k}(-f,A)'.$$

Yet, from the point of view of the cohomology, it would not give more informations – see Remark 8.10.

8.6. **Koszul complex associated with i_{V_f}.** As was already alluded in the introduction, the Cartan formula

$$\mathcal{L}_{V_f} = d \circ i_{V_f} + i_{V_f} \circ d,$$

replaces in our context the formula $\Delta = d \circ d^* + d^* \circ d$ in Hodge theory. Hence, what plays the role in the Morse context of the complex $(\Omega^*(M), d^*)$ from Hodge theory is the Koszul complex induced by the contraction operator i_{V_f}.

Remark 8.13. We underline that the Cartan formula combined with our spectral decomposition yields an analogue of the Hodge decomposition in our framework:

$$u = \mathbb{P}^{(k)}(u) + d\left(i_{V_f} \circ (\mathcal{L}_{V_f}^{(k)})^{-1} \circ (\mathrm{Id} - \mathbb{P}^{(k)})(u)\right) + i_{V_f}\left(d \circ (\mathcal{L}_{V_f}^{(k)})^{-1} \circ (\mathrm{Id} - \mathbb{P}^{(k)})(u)\right).$$

In other words, any u in $\Omega^k(M)$ can be decomposed as the sum of an invariant current, of a coboundary (for d) and of a boundary (for i_{V_f}).

We now consider the Morse-Koszul homological complex $(C^*(f), i_{V_f})$

$$0 \rightarrow C^n(f) \rightarrow C^{n-1}(f) \rightarrow \ldots \rightarrow C^0(f) \rightarrow 0.$$

Again, this is a well defined complex as i_{V_f} commutes with the Lie derivative $\mathcal{L}_{V_f}$. It is naturally associated with the homological complex $(\Omega^*(M), i_{V_f})$:

$$0 \rightarrow \Omega^n(M) \rightarrow \Omega^{n-1}(M) \rightarrow \ldots \rightarrow \Omega^0(M) \rightarrow 0.$$

Recall that the Euler characteristic of a homological complex (C^*, i) is given by

$$\chi(C^*, i) = \sum_{j=0}^{n} (-1)^j \dim\left(Z_j(C^*, i)/B_j(C^*, i)\right),$$

where

$$Z_j(C^*, i) := \mathrm{Ker}(i) \cap C^j, \text{ and } B_j(C^*, i) := \mathrm{Im}(i) \cap C^j.$$

We start our discussion on the Morse-Koszul complex with the following property

Proposition 8.14. *Let (f, g) be a smooth Morse pair inducing a Morse-Smale gradient flow. Then, one has*

$$\chi(C^*(f), i_{V_f}) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the manifold.

Proof. Recall that $C^k(f)$ is equal to the vector space generated by the Laudenbach currents U_a associated with critical points a of index k. According to Proposition 6.7, near a critical point a of index k, U_a can be written in the adapted coordinates of paragraph 3.5 as

$$U_a(x, y, dx, dy) = \delta_0^{\mathbb{R}^k}(x) dx^1 \wedge dx^2 \wedge \dots dx^k.$$

On the other hand, the vector field V_f can be written in this system of coordinates:

$$V_f(x, y, \partial_x, \partial_y) = \sum_{j=1}^{r} \chi_j(a) x_j \partial_{x_j} + \sum_{j=r+1}^{n} \chi_j(a) y_j \partial_{y_j}.$$

Hence, locally near a, one has

$$i_{V_f}(U_a)(x, y, dx, dy) = \sum_{j=1}^{r} \chi_j(a) x_j \delta_0^{\mathbb{R}^k}(x) dx_1 \wedge \dots \widehat{dx_j} \dots \wedge dx_r = 0.$$

As U_a is supported in $\overline{W^u(a)}$, we can deduce that $i_{V_f}(U_a)$ is also carried by $\overline{W^u(a)}$. As we have just shown that it is equal to 0 near a and as $\mathcal{L}_{V_f}(i_{V_f}(U_a)) = 0$, we can deduce that the support of $i_{V_f}(U_a)$ is contained in $\overline{W^u(a)} - W^u(a)$. According to Remark 3.7, we can then deduce that the support of $i_{V_f}(U_a)$ is contained in the union of unstable manifold $W^u(b)$ with $\operatorname{ind}(b) > k$. We now use Proposition 6.8 to write

$$i_{V_f}(U_a) = \sum_{b' : \operatorname{ind}(b') = k-1} \alpha_{b'} U_{b'}.$$

Using Proposition 6.7 and the fact that $i_{V_f}(U_a)$ is carried on a union of unstable manifold of index $> k$, we can deduce that $\alpha_{b'} = 0$ for every critical point b' of index $k-1$. In other words, $Z_k(C^*(f), i_{V_f}) = C^k(f)$ and $B_k(C^*(f), i_{V_f}) = \{0\}$. In particular, one has

$$\chi(C^*(f), i_{V_f}) = \sum_{j=0}^{n} (-1)^j c_{n-j}(f),$$

from which the result follows thanks to the case of equality in the Morse inequalities. □

Remark 8.15. We note that we have implicitely shown that the k-th homology $\mathbb{H}_k(C^*(f), i_{V_f})$ is equal to $C^k(f)$.

As in the case of the Thom-Smale-Witten complex, it would be natural to expect that, as i_{V_f} commutes with $\mathbb{P}^{(k)}$, $\mathbb{P}^{(*)}$ induces an isomorphism between the homological complexes $(C^*(f), i_{V_f})$ and $(\Omega^*(M), i_{V_f})$. In fact, one has also, for every time $t > 0$, a chain homotopy equation between the complexes $(\Omega^*(M), i_{V_f})$ and $(\varphi_f^{-t*}\Omega^*(M), i_{V_f})$. Precisely, one can write

$$u - \varphi_f^{-t*} u = i_{V_f} R_t u + R_t i_{V_f} u \tag{42}$$

$$R_t := \int_0^t d \circ \varphi_f^{-s*} ds. \tag{43}$$

Then, we could let $t \to +\infty$ and obtain a chain homotopy equation between $(C^*(f), i_{V_f})$ and $(\Omega^*(M), i_{V_f})$. Precisely, as before, we would write

$$u = \mathbb{P}(u) + i_{V_f} \circ R_\infty(u) + R_\infty \circ i_{V_f}(u),$$

where $R_\infty := d \circ \mathcal{L}_{V_f}^{-1} \circ (\mathrm{Id} - \mathbb{P})$. However, it is not the case anymore that the induced morphism is a bijection. In fact, it is neither injective nor surjective as the example on $\mathbb{S}^1$ below will show. The reason for that phenomenon is that the Morse-Koszul complex is sensitive to the regularity of the coefficients. In fact, compared with the case of the coboundary operator, this complex is not elliptic since the contraction with the vector field V_f is hyperbolic with degenerate points at $\mathrm{Crit}(f)$.

8.6.1. *A surprising example with the circle.* Let us now discuss the case of a Morse function on $\mathbb{S}^1$ to illustrate the previous observation. Fix f to be a smooth Morse function on $\mathbb{S}^1$ and g to be a smooth metric on $\mathbb{S}^1$. Note that such a metric necessarily satisfies the Morse-Smale condition. The fact that we have a smooth adapted coordinate chart near any critical point follows from the fact that we can apply the Sternberg-Chen Theorem in such a neighborhood as there is only one Lyapunov exponent. Consider now the Koszul complex $(\Omega^*(\mathbb{S}^1), i_{V_f})$ induced by i_{V_f}:

$$0 \to \Omega^1(\mathbb{S}^1) \to \Omega^0(\mathbb{S}^1) \to 0.$$

Let ω be a cycle in $\Omega^1(\mathbb{S}^1)$, i.e. an element satisfying $i_{V_f}(\omega) = 0$. We find that $\omega(x, dx) = 0$ except at the critical points of a but, as ω is smooth, it implies that $\omega = 0$. In particular, the homology $\mathbb{H}_1(\Omega^*(M), i_{V_f})$ is equal to 0 which is not isomorphic to $\mathbb{H}_1(C^*(f), i_{V_f})$ whose dimension is equal to $c_1(f) \neq 0$ from the proof of Proposition 8.14.

We could also consider the complex $(\mathcal{D}'^{,*}(\mathbb{S}^1), i_{V_f})$:

$$0 \to \mathcal{D}'^{,1}(\mathbb{S}^1) \to \mathcal{D}'^{,0}(\mathbb{S}^1) \to 0.$$

In that case, we find that any element u in $\mathcal{D}'^{,1}(\mathbb{S}^1)$ satisfying $i_{V_f}(u) = 0$ must be supported on the critical points of a. In particular, near every critical point a of f, it is of the form

$$u = \sum_{k \leq N(a)} \alpha_{a,k} \delta_a^{(k)}(x) dx,$$

where $\alpha_{a,k}$ are constants and $N(a)$ is a finite integer. Using the fact that the vector field is generated by a Morse function, we can verify that $N(a) = 0$ for every critical point a in order to satisfy $i_{V_f}(u) = 0$. Thus, the first homology $\mathbb{H}_1(\mathcal{D}'^{,*}(f), i_{V_f})$ is of dimension $c_0(f) + c_1(f) \neq c_1(f)$ as there is at least one critical point of index 0. This first calculation shows that the Koszul complex is sensitive to the regularity of the coefficients which was not the case for the de Rham complex.

We have shown the lack of surjectivity between the first homology of $(\Omega^*(M), i_{V_f})$ and $(C^*(f), i_{V_f})$. Let us now prove that the morphism between the homology of order 0 cannot be injective. For that purpose, we need to compute the image of $i_{V_f} : \Omega^1(\mathbb{S}^1) \to \Omega^0(\mathbb{S}^1)$. Let ψ be an element in $\Omega^0(\mathbb{S}^1)$ which is of the form $i_{V_f}(\omega)$ for some ω in $\Omega^1(\mathbb{S}^1)$. Fix now a critical point a of f. In a system of adapted coordinates near a, we find that ψ must be of the form $x\psi_a(x)$. Hence, if we fix χ_a a smooth cutoff function near a, then we find that

$$\mathrm{Im}(i_{V_f}) \simeq \left(1 - \sum_a \chi_a\right)\Omega^0(\mathbb{S}^1) \oplus \bigoplus_{a\in\mathrm{Crit}(f)} x\chi_a(x)\Omega^0(\mathbb{S}^1).$$

Thus, the dimension of $\mathbb{H}_0(C^*(f), i_{V_f})$ is equal to $c_0(f) + c_1(f) \neq c_0(f)$.

8.6.2. *The classical Koszul complex.* As a final remark on the comparison between Koszul and Morse–Koszul, we generalize the above discussion on $\mathbb{S}^1$ to the case of a general manifold M. This is done by recalling classical results on the Koszul complex associated to a smooth vector bundle E over a manifold M. In our work, the relevant bundle is $E = TM$. Recall that a Koszul complex associated with a smooth vector bundle is defined as follows:

Definition 8.16. Let $E \to M$ be a smooth vector bundle of rank k over M, E^* is the dual bundle and choose a section $s \in \Gamma(M; E)$ and define the contraction operator $i_s : \Gamma(M, \Lambda^{\bullet+1}E^*) \to \Gamma(M, \Lambda^\bullet E^*)$. For every open subset $U \subset M$ (U can be taken equal to M), the contraction operator $i_s : \Gamma(U, \Lambda^{\bullet+1}E^*) \to \Gamma(U, \Lambda^\bullet E^*)$ is $C^\infty(U)$ linear. The Koszul complex $(\Gamma(\Lambda^\bullet E^*); i_s)$ is a complex of sheaves of C^∞ modules :

$$0 \to \Gamma(\Lambda^k E^*) \to \cdots \to \Gamma(\Lambda^0 E^*) = C^\infty \to 0.$$

In each degree, $\mathrm{Ker}(i_s)$ and $\mathrm{Im}(i_s)$ are sheaves of C^∞ modules. It follows that the homology $\mathbb{H}_\bullet(\Gamma(\Lambda^\bullet E^*); i_s)$ defined as the quotient $\mathrm{Ker}(i_s|_{\Gamma(\Lambda^\bullet E^*)})/\,\mathrm{Im}(i_s|_{\Gamma(\Lambda^{\bullet+1}E^*)})$ are sheaves of C^∞ modules.

We next give well–known properties of the Koszul complex in the algebraic setting [39, see Proposition 1.4 p. 52], but formulated in the smooth case:

Proposition 8.17. *If $s \in \Gamma(U, E)$ does not vanish on U then the complex $(\Gamma(U, \Lambda^\bullet E^*); i_s)$ is acyclic i.e. $\mathbb{H}_i(\Gamma(U, \Lambda^\bullet E^*); i_s) = 0$ for all $i \in \{0, \dots, k\}$.*

If the zeros of $s \in \Gamma(M, E)$ are non degenerate then the homology vanishes in all degrees except in degree 0 where $\mathbb{H}_0(\Gamma(M, \Lambda^\bullet E^*); i_s) = \mathbb{R}^p$ where p is the ***number of zeros*** *of s.*

Beware that **the number of zeros of s is counted set theoretically.** Therefore, the Koszul complex represents **homologically** the zeros of our section s.

Proof. Our first step is to show that for all U, $\mathbb{H}_i(\Gamma(U, \Lambda^\bullet E^*), i_s) = 0$ for all $i > 0$. If s has isolated zeros on U, then we choose a moving coframe $(e_1, \dots, e_k)$ of $E^*|_U$ such that $e_i(s) = \delta_{1i}$ and we find that for every $\alpha = \sum_{i_1<\cdots<i_p} \alpha_{i_1\dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Gamma(U, \Lambda^p E^*)$, $i_s\alpha = 0$ is equivalent to the fact that $\alpha_{1i_2\dots i_p}|_{U\setminus\{s=0\}} = 0$ hence $\alpha_{1i_2\dots i_p}|_U = 0$ for all $1 < i_2 < \cdots < i_p$ since the zeros of s are isolated. The above means that e_1 does not appear in the decomposition of α hence $\alpha = i_s(e_1 \wedge \alpha)$ and $\mathrm{Ker}(i_s|_{\Gamma(U,\Lambda^i E^*)}) = \mathrm{Im}(i_s|_{\Gamma(U,\Lambda^{i+1}E^*)})$ for all $i > 0$.

For $i = 0$, if s does not vanish on U, then it is simple to see that $i_s\Gamma(U, \Lambda^1 E^*)$ is onto and therefore the cohomology vanishes in all degree. Otherwise, thanks to the above property, we can localize the proof over small open subsets U_a centered around each critical point a of s and $\mathbb{H}_0(\Gamma(M,\Lambda^\bullet E^*);i_s)=\oplus_{a\in\text{zeros}(s)}\mathbb{H}_0(\Gamma(U_a,\Lambda^\bullet E^*);i_s)$. On U_a, using the fact that the zeros of s are non degenerate, in some coordinates $(x_i)_i$ where $x_i(a)=0$, we find that $s=\sum_{i=1}^n x_is_i, s_i\in\Gamma(U_a,E)$ where $s_i(0)\neq 0,\forall i$. Therefore, an immediate calculation reveals that $\mathbb{H}_0(\Gamma(U_a,\Lambda^\bullet E^*);i_s)=\frac{C^\infty(U_a)}{\mathcal{I}_a}$ where $\mathcal{I}_a$ is the ideal of functions vanishing on a hence $\mathbb{H}_0(\Gamma(U_a,\Lambda^\bullet E^*);i_s)\simeq\mathbb{R}$ and $\mathbb{H}_0(\Gamma(M,\Lambda^\bullet E^*);i_s)=\oplus_{a\in\text{zeros}(s)}\mathbb{R}$. □

Appendix A. Proof of Lemma 4.1

In this appendix, we give the proof of Lemma 4.1, i.e. construct of the escape function $G_m(x,\xi)$. Let $N_0,N_1>4\|f\|_{\mathcal{C}^0}$ be some large parameters. As was already explained, up to some minor differences due to the special form of the dynamics, our construction is the one given in section 2 of [32]. Using the conventions of paragraph 3.4, we recall the following result [32, Lemma 2.1]:

Lemma A.1. *Let V^u and V^s be small open neighborhoods of Σ_u and Σ_s respectively, and let $\epsilon>0$. Then, there exist $\mathcal{W}^u\subset V^u$ and $\mathcal{W}^s\subset V^s$, $\tilde{m}$ in $\mathcal{C}^\infty(S^*M,[0,1])$, $\eta>0$ such that $\tilde{X}_{H_f}.\tilde{m}\geq 0$ on S^*M, $\tilde{X}_{H_f}.\tilde{m}\geq\eta>0$ on $S^*M-(\mathcal{W}^u\cup\mathcal{W}^s)$, $\tilde{m}(x,\xi)>1-\epsilon$ for $(x,\xi)\in\mathcal{W}^s$ and $\tilde{m}(x,\xi)<\epsilon$ for $(x,\xi)\in\mathcal{W}^u$.*

Proof. Let us recall the main lines of the proof of this Lemma which relies only on the compactness and on the attracting properties of Σ_u and Σ_s. First, we have to verify that, up to shrinking V_u and V_s a little bit, $V^u\cap V^s=\emptyset$,

$$\forall t\geq 0,\ \tilde{\Phi}_f^t(V^s)\subset V^s,\ \text{and}\ \tilde{\Phi}_f^{-t}(V^u)\subset V^u. \tag{44}$$

This follows from Lemmas 3.6 and 3.8. Once we have this property, we can follow the proof of [32]. More precisely, we know that

$$\mathcal{I}(x,\xi):=\{t\in\mathbb{R}:\ \tilde{\Phi}_f^t(x,\xi)\in S^*M-(V^u\cup V^s)\},$$

is a closed, connected interval whose length is uniformly bounded by some constant $\tau>0$. We then set $T>0$ such that $\tau/(2T)<\epsilon$ satisfying

$$\mathcal{W}^u:=\tilde{\Phi}_f^{-T}(S^*M-V^s)\subset V^u,\ \mathcal{W}^s:=\tilde{\Phi}_f^{-T}(S^*M-V^u)\subset V^s.$$

Once these parameters are fixed, one just has to verify that, if $m_0\in\mathcal{C}^\infty(S^*M,[0,1])$ is equal to 1 on V^s and to 0 on V^u, then the function

$$m_T(x,\xi):=\frac{1}{2T}\int_{-T}^T m_0\circ\tilde{\Phi}_f^t(x,\xi)dt$$

satisfies the assumption of the Lemma – see [32] for details. □

We now use this Lemma with V^u, V^s and $\epsilon>0$ small enough (to be precised). Thus, we have a function $\tilde{m}(x,\xi)$ defined on S^*M. We introduce a smooth function m_1 defined on T^*M which satisfies

$$m_1(x,\xi)=N_1\tilde{m}\left(x,\frac{\xi}{\|\xi\|_x}\right)-N_0\left(1-\tilde{m}\left(x,\frac{\xi}{\|\xi\|_x}\right)\right),\ \text{for}\ \|\xi\|_x\geq 1,$$

and

$$m_1(x,\xi)=0,\ \text{for } \|\xi\|_x\leq\frac{1}{2}.$$

We set the order function of our escape function to be

$$m(x,\xi)=-f(x)+m_1(x,\xi).$$

Set now

$$\tilde{\Gamma}_{\mp}:=\left\{(x,\xi)\in T^*M:\ \xi\neq 0 \text{ and } \frac{\xi}{\|\xi\|_x}\in\mathcal{W}^{s/u}\right\}.$$

From the definition of m, $\tilde{\Gamma}_-$ (resp. $\tilde{\Gamma}_+$) is a small conical neighborhood of Γ_- (resp. of Γ_+). Moreover, for every (x,ξ) in $\tilde{\Gamma}_-$ (resp. $\tilde{\Gamma}_+$) satisfying $\|\xi\|_x\geq 1$, one has

$$m(x,\xi)\leq -N_0(1-\epsilon)+N_1\epsilon+\|f\|_{\mathcal{C}^0}\quad (\text{resp.} \geq N_1(1-\epsilon)-N_0\epsilon-\|f\|_{\mathcal{C}^0}).$$

If we choose ϵ small enough, then the first items of Lemma 4.1 are proved. We now set the following escape function:

$$G_m(x,\xi)=m(x,\xi)\log(1+\|\xi\|_x^2),$$

and we have to compute the derivative $X_{H_f}.G_m$ of G_m along the Hamiltonian vector field X_{H_f} associated with H_f. Note that

$$X_{H_f}.G_m(x,\xi)=\log(1+\|\xi\|_x^2)X_{H_f}.m(x,\xi)+m(x,\xi)\frac{X_{H_f}.\|\xi\|_x^2}{1+\|\xi\|_x^2}. \tag{45}$$

Let $r>0$ be a small parameter. We shall estimate the derivative of G_m along the Hamiltonian function in $T^*B(a,r)$ for every critical points and in the complementary of this set.

Let us start with the case where (x,ξ) belongs to T^*M_{reg} where M_{reg} is the complementary set of $\cup_{a\in\text{Crit}f}B(a,r/2)$. In that case, we fix $R_0>1$. Then, there exists C_g depending only on the Riemannian metric and on f such that, for every (x,ξ) in T^*M satisfying $\|\xi\|_x\geq R_0$,

$$X_{H_f}.G_m(x,\xi)\leq -X_{H_f}.f(x)\log(1+R_0^2)+C_g\left(N_0+N_1+\|f\|_{\mathcal{C}^0}\right),$$

as $X_{H_f}.m_1\leq 0$ for $\|\xi\|>1$ according to Lemma A.1. As x is far from the critical points of f, one knows from (4) that there exists a constant $c(r)>0$ (depending only on $r>0$) such that $X_{H_f}.f(x)\geq c(r)$. In particular, one has

$$X_{H_f}.G_m(x,\xi)\leq -c(r)\log(1+R_0^2)+C_g\left(N_0+N_1+\|f\|_{\mathcal{C}^0}\right)\leq -\min\{N_0,N_1\},$$

where the last equality holds if we choose $R_0>1$ large enough (in a way that depends on r, N_0 and N_1).

It now remains to analyse the behaviour in a "neighborhood" of a critical point a in $\text{Crit}f$. In that case, we can as in [32] make use of the (local) hyperbolic structure of the flow. We fix (x,ξ) in T^*M such that $\|\xi\|_x\geq 1$ and x in $B(a,r)$, and we use (4) to write

$$X_{H_f}.G_m(x,\xi)\leq X_{H_f}.m_1(x,\xi)\log(1+\|\xi\|_x^2)+m(x,\xi)\frac{X_{H_f}.\|\xi\|_x^2}{1+\|\xi\|_x^2}.$$

We now distinguish three cases:

- Suppose that we could show that, if (x, ξ) belongs to $\tilde{\Gamma}_-$ and $\|\xi\| > 1$, then, one can find a constant[11] $c_- > 0$ *depending only on* f and g such that

$$X_{H_f}.(\|\xi\|_x^2) > c_- \|\xi\|_x^2. \tag{46}$$

  In particular, one could infer

$$X_{H_f}.G_m(x, \xi) \leq -\frac{N_0}{2} c_-,$$

  where we used Lemma A.1 to bound $X_{H_f}.m_1(x, \xi)$.

- Suppose that we could show that, if (x, ξ) belongs to $\tilde{\Gamma}_+$ and $\|\xi\| > 1$, then, one can find a constant $c_+ > 0$ *depending only on* f and g such that

$$X_{H_f}.(\|\xi\|_x^2) < -c_+ \|\xi\|_x^2. \tag{47}$$

  In particular, one could infer

$$X_{H_f}.G_m(x, \xi) \leq -\frac{N_1}{2} c_+,$$

  where we used again Lemma A.1 to bound $X_{H_f}.m_1(x, \xi)$.

- If (x, ξ) does not belong to $\tilde{\Gamma}_- \cup \tilde{\Gamma}_+$, then $\frac{\xi}{\|\xi\|}$ belongs to $S^*M - (\mathcal{W}^u \cup \mathcal{W}^s)$, and, by Lemma A.1, one finds

$$X_{H_f}.G_m(x, \xi) \leq -\eta(N_0 + N_1) \log(1 + \|\xi\|^2) + C_g \left(N_0 + N_1 + \|f\|_{\mathcal{C}^0}\right).$$

  Thus, if we choose $R_0 > 1$ large enough (in a way that depends on N_0, N_1), then one can ensure that $X_{H_f}.G_m(x, \xi) \leq -\min\{N_0, N_1\}$ whenever $\|\xi\| \geq R_0$ on this set.

This concludes the second part of the Lemma except for (46) and (47) that are still to be proved. The proof is similar in both cases and we will only treat the first case. We note that the compactness of Σ_u and Σ_s will one more time play a crucial role in the proof.

We start with the case where $(x, \xi) = (a, \xi)$ belongs to $E_s^*(a)$. In that case, we can make use of the fact that we have a smooth linearizing chart – see paragraph 3.5. Recall also that the linearized vector field $L_g(a)$ is diagonalizable in a basis of eigenvectors which is orthogonal for the metric $g^*(a)$ – see paragraph 3.3. This implies that, in the adapted coordinates (z, ζ) around a,

$$X_{H_f}.(\|\zeta\|_z^2) = 2 \sum_{j=1}^{r} |\chi_j(a)| \zeta_j^2 + \mathcal{O}_{f,g}(\|\zeta\|_a^2),$$

where the constant in the remainder depends only on f and g and $g^*(a)$ is the induced Riemannian metric on T_a^*M. In particular, provided that we take some small enough constant $c_- = c_-(f, g)$ depending on g and f, inequality (46) holds in the case where $(x, \xi) = (a, \xi)$ belongs $E_s^*(a)$. We then define

$$U_r := \left\{(x, \xi) \in \cup_{y \in B(a,r)} T_y^*M : \|\xi\|_x > 1, \text{ and (46) holds with } c_- = 2c_-(f, g)\right\},$$

[11] We note that we may have to take V^s small enough.

which is an open set in $T^*M-(M\times\{0\})$. Then, we have to prove that we can choose V_s small enough to ensure that the neighborhood

$$\tilde{V}_s^{(r)}:=\{(x,\xi)\in\cup_{y\in B(a,r)}T_y^*M:\|\xi\|>1\text{ and }(x,\xi/\|\xi\|_x)\in V_s\}$$

is contained in U_r. We proceed by contradiction, and we suppose that, for every $r>0$ small enough, there exists $m_0\geq 1$ such that, for any $m\geq m_0$ and for any neighborhood V_m of Σ_s of size $1/m$, one can find $(x_m^{(r)},\xi_m^{(r)})\notin U_r$ belonging to $\tilde{V}_m^{(r)}$. Without loss of generality, we can suppose that $\|\xi_m^{(r)}\|\leq 2$. By compactness, we can then extract a subsequence such that $\lim_{m\to+\infty}(x_m^{(r)},\xi_m^{(r)})=(x^{(r)},\xi^{(r)})$. Moreover, as Σ_s is compact, $(x^{(r)},\xi^{(r)})$ belongs to $\cup_{y\in\overline{B(a,r)}}(\Gamma_-\cap T_y^*M)$, and, by construction of the sequence, (46) does not hold at this point with the constant $c_-=2c_-(f,g)$. This holds for any $r>0$ small enough. We now extract a converging subsequence as $r\to 0^+$, and we find a point (x,ξ) in $E_s^*(a)$ where we know that (46) holds with the constant $c_-=c_-(f,g)$. This gives the expected contradiction as $\xi\neq 0$.

Appendix B. Proof of Proposition 4.2

The proof of this Proposition was given in great details in [32, Th. 1.4] for the case $k=0$. The adaptation to the case $0\leq k\leq n$ is almost identical except that we have to deal with pseudodifferential operators with values in $\Lambda^k(T^*M)$. The main point is that the (pseudodifferential) operators under consideration have a diagonal symbol. In fact, given any local basis $(e_j)_{j=1,\dots J_k}$ of $\Lambda^k(T^*M)$ and any family $(u_j)_{j=1,\dots J_k}$ of smooth functions $\mathcal{C}^\infty(M)$, one has

$$\mathcal{L}_{V_f}^{(k)}\left(\sum_{j=1}^{J_k}u_je_j\right)=\sum_{j=1}^{J_k}\mathcal{L}_{V_f}(u_j)e_j+\sum_{j=1}^{J_k}\mathcal{L}_{V_f}^{(k)}(e_j)u_j,$$

where the second part of the sum in the right-hand side is a lower order term (of order 0). This diagonal form allows to adapt the proofs of [32] to this vector bundle framework. For completeness, we briefly recall the main lines of the proof and just point a (minor) simplification due to the particular form of our flow. To make the comparison with that reference simpler, we shall consider the operator $-i\mathcal{L}_{V_f}$ instead of $-\mathcal{L}_{V_f}$.

Remark B.1. As was already mentionned, the case of currents was treated by Dyatlov and Zworski in [28] via a slightly different approach. Their method could also probably be adapted to deal with the case of Morse-Smale gradient flows.

The strategy is to consider the equivalent operator

$$\widehat{\mathcal{L}}_f^{(k)}:=\mathrm{Op}(\mathbf{A}_m^{(k)})\circ(-i\mathcal{L}_{V_f}^{(k)})\circ\mathrm{Op}(\mathbf{A}_m^{(k)})^{-1},\tag{48}$$

and to begin with, we recall the following result [32, Lemma 3.2]:

Lemma B.2. *The operator*

$$\mathrm{Op}(\mathbf{A}_m^{(k)})\circ(-i\mathcal{L}_{V_f}^{(k)})\circ\mathrm{Op}(\mathbf{A}_m^{(k)})^{-1}+i\mathcal{L}_{V_f}^{(k)}$$

is a pseudodifferential operator in $\Psi^{+0}(M)$ *whose symbol in any given system of coordinates is of the form*

$$P(x,\xi)=i(X_{H_f}.G_m)(x,\xi)\mathbf{Id}+\mathcal{O}(S^0)+\mathcal{O}_m(S^{-1+0}).$$

In this Lemma, the notation $\mathcal{O}(.)$ means that the remainder is independent of the order function m, while the notation $\mathcal{O}_m(.)$ means that it depends on m. In particular, this Lemma says that $\widehat{\mathcal{L}}_f^{(k)}$ is an element in $\Psi^1(M,\Lambda^k(T^*M))$. Then, combining this Remark to [32, Lemma A.1] which can be adapted directly to the case of operators with values in a vector bundle, one finds that $\widehat{\mathcal{L}}_f^{(k)}$ has a unique closed extension as an unbounded operator on $L^2(M,\Lambda^k(T^*M))$. This shows the first part of the Proposition in the case $k=0$.

Remark B.3. The proof of this Lemma was given for $k=0$ in [32] and the adaptation to the case $1\leq k\leq n$ follows from the diagonal structure of the operators involved. Let us recall that the key idea is to observe, by linearizing the exponential,

$$\begin{aligned}\mathrm{Op}(\mathbf{A}_m^{(k)})\circ(-i\mathcal{L}_{V_f}^{(k)})\circ\mathrm{Op}(\mathbf{A}_m^{(k)})^{-1}\simeq&\quad(1+\mathrm{Op}(G_m)+\ldots)\circ(-i\mathcal{L}_{V_f}^{(k)})\circ(1-\mathrm{Op}(G_m)+\ldots)\\ =&\quad -i\mathcal{L}_{V_f}^{(k)}+[\mathrm{Op}(G_m\mathbf{Id}),-i\mathcal{L}_{V_f}^{(k)}]+\ldots,\end{aligned}$$

which implies via symbolic calculus

$$\mathrm{Op}(\mathbf{A}_m^{(k)})\circ(-i\mathcal{L}_{V_f}^{(k)})\circ\mathrm{Op}(\mathbf{A}_m^{(k)})^{-1}\simeq -i\mathcal{L}_{V_f}^{(k)}+i\,\mathrm{Op}(X_{H_f}.G_m\mathbf{Id})+\ldots.$$

Then, up to the fact that we have to deal with $L^2(M,\Lambda^k(T^*M))$, the second part of Proposition 4.2 is exactly the content of Lemma 3.3 of [32] which only makes use of the properties of the escape function given in Lemma 4.1. We also note that they implicitely shows that, for every z in $\mathbb{C}$ satisfying $\mathrm{Im}z>C_0$, one has

$$\left\|\left(\widehat{\mathcal{L}}_f^{(k)}-z\right)^{-1}\right\|_{L^2(M,\Lambda^k(T^*M))\rightarrow L^2(M,\Lambda^k(T^*M))}\leq\frac{1}{\mathrm{Im}(z)-C_0}. \tag{49}$$

Remark B.4. Combining Proposition 4.2 to the Hille-Yosida Theorem [30, Cor. 3.6, p. 76], one knows that (by conjugation)

$$(\varphi_f^{-t})^*:\mathcal{H}_k^m(M)\rightarrow\mathcal{H}_k^m(M), \tag{50}$$

generates a strongly continuous semigroup which is defined for every $t\geq 0$ and whose norm is bounded by e^{tC_0}.

Finally, the last part of the Proposition is based on results from analytic Fredholm theory. It is in fact the only place where things differ with [32]. The situation is in fact slightly simpler here as we shall now briefly explain it. We write

$$\widehat{\mathbf{V}}_f^{(k)}:=\frac{i}{2}\left(\left(\widehat{\mathcal{L}}_f^{(k)}\right)^*-\widehat{\mathcal{L}}_f^{(k)}\right).$$

We denote by $\mathbf{V}_f^{(k)}(x,\xi)$ the symbol of this operator. Note that, from [32, Lemma A.1], $(\widehat{\mathcal{L}}_{V_f}^{(k)})^*$ also has a unique closed extension to $L^2(M,\Lambda^k(T^*M))$. Combining Lemma B.2 to

Lemma 4.1, one knows that, for every (x, ξ) in T^*M,

$$\mathbf{V}_f^{(k)}(x,\xi) \leq (-C_N + C)\mathbf{Id} + \mathcal{O}_m(S^{-1+0}),$$

for some constant $C > 0$ which is independent of m and for the constant C_N defined in Lemma 4.1. From the sharp Gårding inequality, one can deduce that, for every $0 < \mu < 1$, there exists a constant $C_{\mu,m} > 0$ such that, for every u in $\mathcal{C}^\infty(M)$

$$\langle (\widehat{\mathbf{V}}_f^{(k)} + C_N - C)u, u\rangle_{L^2(M,\Lambda^k(T^*M))} \leq C_{\mu,m}\|u\|^2_{H^{\frac{\mu-1}{2}}(M,\Lambda^k(T^*M))},$$

where the remainder $\mathcal{O}_m(S^{-1+0})$ has been absorbed in the RHS thanks to the Calderón-Vaillancourt Theorem. From this inequality, one can deduce that

$$\langle (\widehat{\mathbf{V}}_f^{k)} + C_N - C)u, u\rangle_{L^2(M,\Lambda^k(T^*M))} \leq \left\langle \tilde{C}_{\mu,m}\left(1 - \Delta_g^{(k)}\right)^{\frac{\mu-1}{2}} u, u\right\rangle_{L^2(M,\Lambda^k(T^*M))},$$

where $\Delta_g^{(k)}$ is the Laplace-Beltrami operator acting on k differential forms. We define then

$$\widehat{\chi}_k := \tilde{C}_{\mu,m}\left(1 - \Delta_g^{(k)}\right)^{\frac{\mu-1}{2}} \in \Psi^{\mu-1}(M, \Lambda^k(T^*M)),$$

which is a compact operator as $\mu - 1 < 0$. Hence, we can rewrite the last inequality as

$$\langle (\widehat{\mathbf{V}}_f^{(k)} - \widehat{\chi}_k + C_N - C)u, u\rangle_{L^2(M,\Lambda^k(T^*M))} \leq 0,$$

from which one can deduce[12] that the resolvent

$$\left(\widehat{\mathcal{L}}_f^{(k)} - i\widehat{\chi}_k - z\right)^{-1}$$

defines a bounded operator from $L^2(M, \Lambda^k(T^*M))$ to itself as soon as $\mathrm{Im}(z) > -(C_N - C)$. From the compactness of $\widehat{\chi}_k$ we can deduce that $\widehat{\chi}_k\left(\widehat{\mathcal{L}}_f^{(k)} - i\widehat{\chi}_k - z\right)^{-1}$ is also a compact operator which is exactly the content of Lemma 3.4 in [32]. The conclusion then follows by a classical argument from analytic Fredholm theory given in [32, Lemma 3.5].

Appendix C. Asymptotic expansions

In this appendix, we review some classical facts on asymptotic expansions (see [51, Chapter 1] for a nice review).

Definition C.1. Let $\mathcal{I}$ be a discrete[13] countable subset of $\mathbb{R}$ bounded from below. We call $\mathcal{I}$ the **index set**. Then $h \in C^\infty((0,1],\mathbb{R})$ has polyhomogeneous asymptotic expansion indexed by $\mathcal{I}$ if

$$\exists (a_\lambda)_{\lambda\in\mathcal{I}} \text{ such that } \forall \Lambda \in \mathbb{R}\setminus\mathcal{I}, \exists C \geqslant 0, \forall s \in (0,1]$$
$$\left| h(s) - \sum_{\lambda\in\mathcal{I},\lambda\leqslant\Lambda} a_\lambda s^\lambda \right| \leqslant C s^{\lambda_0}$$

where $\lambda_0 = \inf\{\lambda \in \mathcal{I}\cap[\Lambda, +\infty)\}$.

[12] The proof of this fact is similar to the proof of Lemma 3.3 in [32].

[13] We mean that it has no accumulation point.

The key property we use in this article is:

Proposition C.2. *If such an asymptotic expansion exists then it is* ***unique****.*

Proof. Assume that h has two asymptotic expansions $\sum_{\lambda\in\mathcal{I}} a_\lambda s^\lambda$ and $\sum_{\lambda\in\mathcal{J}} b_\lambda s^\lambda$. Note that, up to taking the union of two index sets, we can make the assumption that both expansions are with respect to the same set, i.e. $\mathcal{I}=\mathcal{J}$. Suppose by contradiction that there exists $\lambda\in\mathcal{I}$ such that $a_\lambda\neq b_\lambda$, and pick the smallest element enjoying this property. Let us call this element λ_0. Then the approximation property gives for every $\Lambda\in\mathbb{R}\setminus\mathcal{I}$ sufficiently large and as $s\to 0^+$,

$$\left|\sum_{\lambda\in\mathcal{I},\lambda\leqslant\Lambda} a_\lambda s^\lambda-\sum_{\lambda\in\mathcal{I},\lambda\leqslant\Lambda} b_\lambda s^\lambda\right|=o(s^\Lambda).$$

However $|\sum_{\lambda\in\mathcal{I},\lambda\leqslant\Lambda} a_\lambda s^\lambda-\sum_{\lambda\in\mathcal{I},\lambda\leqslant\Lambda} b_\lambda s^\lambda|=|a_{\lambda_0}-b_{\lambda_0}|s^{\lambda_0}(1+o(1))$, which contradicts the above upper bound. □

Now we have the following

Lemma C.3. *Let $(\lambda_i)_{i=1}^n$ be a collection of n positive real numbers. Let $\mathcal{I}$ be the index set defined as*

$$\mathcal{I}:=\left\{\sum_{j=1}^n k_j\lambda_j:\ \forall 1\leq j\leq n,\ k_j\in\mathbb{N}\right\}\subset\mathbb{R}.$$

Then, for all $\psi\in C^\infty(\mathbb{R}^n)$, the function

$$s\in(0,1]\longmapsto\psi(s^{\lambda_1},\dots,s^{\lambda_n})$$

has polyhomogeneous asymptotic expansion indexed by $\mathcal{I}$.

Proof. Use the fact that ψ has a Taylor expansion in $\mathbf{s}=(s_1,\dots,s_n)$ with remainder, in multi–index notations :

$$\psi(\mathbf{s})=\sum_{|\alpha|\leqslant k}\frac{\mathbf{s}^\alpha}{\alpha!}\partial_{\mathbf{s}}^\alpha\psi(0)+\mathcal{O}_f((|s_1|+\ldots+|s_n|)^{k+1}).$$

Then, making the substitution $s_i=s^{\lambda_i}$ yields an asymptotic expansion of the form

$$\psi(s^{\lambda_1},\dots,s^{\lambda_n})=\sum_{|\alpha|\leqslant k}\frac{s^{\lambda_1\alpha_1+\ldots+\lambda_n\alpha_n}}{\alpha!}\partial_{\mathbf{s}}^\alpha\psi(0)+\mathcal{O}_\psi((s^{\lambda_1}+\ldots+s^{\lambda_n})^{k+1}).$$

If we fix $\Lambda>0$ and k such that $(k+1)\min_i(\lambda_i)>\Lambda$, one can verify from the above expression that $\psi(s^{\lambda_1},\dots,s^{\lambda_n})$ has a polyhomogeneous expansion in $\mathcal{I}$. □

A function $h\in C^\infty((0,1],\mathbb{R})$ is said to be **weakly homogeneous** if

$$\exists C>0,\exists d\in\mathbb{R},\forall s\in(0,1],|h(s)|\leqslant Cs^d.$$

Recall that the Mellin transform of $h\mathbf{1}_{[0,1]}$ for f weakly homogeneous is then defined as

$$\mathcal{M}\left(h\mathbf{1}_{[0,1]}\right)(z)=\int_0^1 h(s)s^z\frac{ds}{s}, \tag{51}$$

and that it is holomorphic on the half–plane $\mathrm{Re}(z) > -d$. Finally, we note that the following holds:

Lemma C.4. *Under the above conventions, one has:*

(1) *For w in $\mathbb{C}$, the Mellin transform $\mathcal{M}(s^w \mathbf{1}_{[0,1]}(s))(z)$ equals $\frac{1}{w+z}$ and thus, it extends meromorphically with a simple pole at $z = -w$.*

(2) *For every polyhomogeneous h where $h \sim \sum_{\lambda\in\mathcal{I}} a_\lambda s^\lambda$, the Mellin transform $\mathcal{M}\left(h\mathbf{1}_{[0,1]}\right)(z)$ extends meromorphically to the complex plane with simple poles at $z \in -\mathcal{I}$.*

Proof. This is a particular case of [51, Thm 3.1 p. 11] □

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Institut Camille Jordan (U.M.R. CNRS 5208), Université Claude Bernard Lyon 1, Bâtiment Braconnier, 43, boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex
E-mail address: dang@math.univ-lyon1.fr

Laboratoire Paul Painlevé (U.M.R. CNRS 8524), U.F.R. de Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: gabriel.riviere@math.univ-lille1.fr