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par

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Microlocal analysis from quantum fields to hyperbolic dynamics.

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Introduction.

1. Introduction en Français.

La théorie quantique des champs est une théorie qui a pour object de décrire les interactions entres particules élémentaires. De façon analogue, la mécanique quantique est apparue comme une théorie décrivant la physique atomique. La mécanique quantique a été un moteur pour le développement de plusieurs domaines des mathématiques comme les équations aux dérivées partielles, les algèbres d'opérateurs, l'analyse fonctionnelle et la géométrie. Mais il semble que la théorie quantique des champs nécessite plus de complexité mathématique. En théorie des champs, le calcul perturbatif de n'importe quel processus physique induit un processus de sommation sur une infinité d'états intermédiaires qui génère des quantités divergentes, donc produit de façon naturelle des infinis. Les divergences en théorie des champs perturbative sont directement liées aux singularités à courtes distances des fonctions de corrélations qui ont une structure hautement non triviale, elle proviennent de l'existence d'une infinité d'états à plusieurs particules.

Dans un sujet totalement différent tirant son origine de la "mécanique classique", les systèmes dynamiques, pour décrire les propriétés fines du comportement à grand temps des systèmes dynamiques hyperboliques dits Axiom A [**171, 21, 61**], il est nécessaire de quitter le monde des fonctions pour considérer des distributions de régularité de Sobolev négative. Ils apparaissent de façon inévitable car la dynamique contracte dans certaines directions et dilate dans d'autres directions.

De part la nature physique de ces deux problèmes, un point commun entre les deux sujets sont les **singularités** des objets étudiés : d'une part, les fonctions de corrélation en théorie des champs sont des fonctions (et même des distributions en théorie des champs sur les espace–temps lorentziens)

$$\langle \phi(x_1)\dots\phi(x_n)\rangle \text{ sur l'espaces de configuration } \{(\mathbb{R}^d)^n \setminus \text{toutes les diagonales}\}$$

qui deviennent singuliers quand $|x_i - x_j| \to 0$ (le long des cônes de lumière plus la diagonale en Lorentz puisque la métrique n'est plus positive définie). D'autre part, les corrélateurs dynamiques des systèmes Axiome A peuvent être exprimés en fonction des états résonants de Pollicott–Ruelle qui sont des distributions singulières. Par exemple, quand V est un champ de gradient Morse–Smale sur une variété compacte, nous montrons avec Rivière (voir le chapitre 1) que

$$\left\langle \Psi_2, e^{-tX}\Psi_1 \right\rangle = \sum_{a\in Crit(X)} \langle \Psi_2, U_a\rangle \langle S_a, \Psi_1\rangle + o(1)$$

où les U_a, (resp S_a) sont des courants au sens de de Rham qui ont des singularités anisotropes : lisses (resp singulières) dans les directions instables et singulières (resp lisses) dans les directions stables. De plus la notion de déterminant apparait dans les deux sujets sous des formes variées : sous forme de fonctions zeta dynamiques pour compter les orbites périodiques de systèmes dynamiques, de fonctions de partitions de fermions chiraux comme nous le verrons dans le chapitre 2 en théorie des champs. Dans le continuum, on utilise des déterminants zeta régularisés d'opérateurs elliptiques pour généraliser le déterminant usuel qui suffit pour définir les fonctions de partition de théories discrétisées sur réseaux.

L'analyse semiclassique est une branche des mathématiques qui a pour origine l'étude de la mécanique quantique alors que l'analyse algébrique a été inventée par Sato pour décrire les singularités de systèmes d'EDP dans le cotangent. Les méthodes de Sato ont été vite appliquées à l'étude des amplitudes de Feynman par l'école japonaise d'analyse algébrique. Ces méthodes ont été généralisées au cadre C^∞ par Hörmander dans les années 70 et ce sont vite révélées comme un outil incontournable dans l'étude des EDP linéaires comme nonlinéaires. Elles ont trouvé d'importants champs d'application pour décrire de façon quantitative les singularités de distributions et d'opérateurs dans l'espace des phases, qui contient des informations sur la position et aussi la codirection des singularités, en notant que les méthodes semiclassiques peuvent dans certains cas donner des résultats de localisation précis des singularités hautes fréquences. Ceci explique l'efficacité des méthodes microlocales, semiclassiques et spectrales pour ces deux types de problèmes.

Depuis la thèse de Radzikowski dans les années 90, les méthodes microlocales ont permis des avancées sur des problèmes conceptuels en théorie des champs : définition des champs en espace–temps courbes grace à la notion d'états d'Hadamard [**165**] et la preuve de la renormalisabilité perturbative des champs en espace–temps courbes Lorentziens [**26**]. En dynamique hyperbolique, parfois à l'interface avec la topologie ou la théorie analytique des nombres, plusieurs conjectures de Bowen, Fried [**62**], Smale [**171**] ont motivé des dizaines d'années de recherche dans le domaine. De façon relativement surprenante, des idées provenant du semiclassique et du microlocal ont produit des avancées récentes dans le sujet conduisant à la résolution de nombreuses conjectures [**75, 46, 45, 43, 82, 5, 54**]. De façon inattendue, ce sont des méthodes profondes provenant de la mécanique quantique qui permettent de résoudre des problèmes difficiles de mécanique classiques.

Le manuscript résume la série d'articles [**12, 13, 14, 15, 16, 17, 9**] qui illustrent des applications de méthodes microlocales, semiclassiques et spectrales à des problèmes de dynamique et théorie des champs [10]. Nous nous sommes efforcés de souligner les idées communes (bien sûr ces domaines sont différents et ont leur spécificité propre). Passons maintenant en revue le plan de notre mémoire :

— Le premier chapitre est introductif et couvre la série de travaux [**12, 13, 14, 15, 16**] en collaboration avec Rivière, qui sont tous reliés d'une façon ou d'une autre avec la théorie de Morse et la topologie. Nous débutons en introduisant les notions d'opérateur de transfert en dynamique et de spectre de Pollicott–Ruelle, en motivant par des analogies avec la mécanique quantique. Nous traitons un modèle jouet très simple sur un graphe où des raisonnements simples avec des matrices nous permettent de parler du lien entre spectre de Ruelle et fonction zeta. Dans un second temps, nous abordons le formalisme supersymétrique popularisé par Witten dans ses travaux sur la théorie de Morse. Nous passons en revue plusieurs exemples pour illustrer le formalisme comme les formes différentielles, les courants de de Rham, la théorie de Hodge, le laplacien de Witten et la formule de Lie Cartan. Dans un troisième temps, nous rappelons les définitions de flots Anosov et Morse–Smale. Puis nous expliquons un résultat obtenu avec Rivière [**12, 16**] qui donne le spectre des champs de gradient Morse–Smale. Ensuite, nous en donnons une autre approche qui fait le lien avec le laplacien de Witten [**16**]. Nous discutons brièvement des flots Morse–Smale généraux [**13, 14**] où le spectre de Pollicott–Ruelle a une structure en bandes verticales. Dans la dernière partie un peu plus "topologique", nous relions le spectre de Pollicott–Ruelle à des propriétés topologiques de la variété qui porte le flot. Nous en déduisons des inégalités de Morse pour les flots Morse–Smale et Anosov [**15**].

— Le second chapitre résume [**9**] qui porte sur la renormalisation de déterminants fonctionnels. Notre motivation dans ce travail est de donner un sens à la fonction de partition de fermions chiraux en interaction avec un potentiel de jauge externe, le potentiel est considéré comme un champ "lentement variable" donc il reste classique alors que le champ fermionique est quantifié c'est le champ "rapide". Ce chapitre et l'article résumé sont des tentatives pour comprendre une note conjecturale de Quillen [**148**]. Notre présentation diffère de l'article original [**9**] qui repose sur des

10. Ces travaux forment une partie de nos travaux après la thèse. Par manque de temps et de place, nous n'avons pas pu couvrir l'article [**18**] écrit avec Zhang.

méthodes de type noyau de la chaleur. Ici les méthodes sont plus microlocales bien que nous traitons de champs euclidiens. C'est tout à fait volontaire car nous souhaitons dans l'avenir étendre nos résultats aux champs quantiques sur un espace–temps Lorentzien où seule l'approche microlocale survit. L'exposé est inspiré d'un travail en cours avec Brouder–Zhang. Nous esquissons seulement certaines preuves et par contre nous insistons beaucoup sur les motivations physiques.

— Le troisième chapitre aborde des travaux communs avec Guillarmou-Rivière-Shen [**10**] et avec Chaubet [**6**]. L'objectif commun des deux articles est la conjecture de Fried reliant dynamique et topologie. Le but est de relier la torsion de Reidemeister, qui est un invariant topologique définit à la manière d'un déterminant d'un complexe de chaîne acyclique, et la valeur en zéro de la fonction zeta de Ruelle qui est une fonction qui compte les orbites périodiques d'un système dynamique donné. Comme les énoncés des résultats principaux sont lourds et mettent en jeu beaucoup d'objets, par souci de clarté et de brièveté nous avons presque exclusivement expliqué les énoncés des résultats.

— Le dernier chapitre porte sur [**17**] avec Rivière qui applique les méthodes abordées au chapitre 1 à un problème classique en géométrie et dynamique : le comptage orbital. Sur une surface à courbure négative, nous montrons que les séries de Poincaré comptant les arcs géodésiques orthogonaux à une paire de courbes géodésiques fermées admet un prolongement méromorphe au plan complexe. Quand les deux courbes sont homologiquement triviales, nous montrons que les séries de Poincaré ont une valeur rationnelle explicite en 0 en interprétant ce nombre comme un enlacement de noeuds legendriens. En particulier, pour n'importe quelle paire de points sur la surface, la longueur des arcs géodésiques reliant les deux points détermine le genre. De plus, pour n'importe quelle paire de géodésiques fermées homologiquement triviales, les longueurs des arcs géodésiques orthogonaux aux deux arcs fermés détermine l'enlacement des deux géodésiques.

2. Introduction in English.

Quantum field theory was developed as a theory describing interactions of elementary particles. In a similar way, quantum mechanics appeared as a theory describing atomic physics. Quantum mechanics stimulated the development of many areas of mathematics, such as the theories of partial differential equations, operator algebras, functional analysis, geometry. But the mathematical complexity of quantum field theory and the sophistication of related mathematical problems seem to be of a different magnitude. In quantum field theory, the perturbative calculation of any physical process involves a summation over an infinite number of virtual intermediate states which is generically divergent, hence produces infinities. The divergences of perturbation theory in quantum field theory are directly linked to its short distance structure which is highly non-trivial because its description involves the infinity of multi–particle states.

In a totally different subject namely dynamical systems, in order to describe analytically the fine properties of the long time behaviour of Axiom A dynamical systems [**171, 21, 61**], it is necessary to leave the world of functions to consider instead singular distributions of negative Sobolev regularity. They arise naturally because the dynamics contracts in certain directions and expands in others.

Because of the physical nature of the problems considered, a common feature in both subjects are the **singularities** of the objects of interest : on the one hand, correlation functions of quantum fields are functions

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \text{ on configuration space } \{(\mathbb{R}^d)^n \setminus \text{all diagonals}\}$$

which become singular when $|x_i - x_j| \to 0$. On the other hand, dynamical correlators for Axiom A dynamical systems can be expressed in terms of Ruelle resonant states which are singular. For instance when X is a Morse–Smale gradient field, we expect :

$$\left\langle \Psi_2, e^{-tX}\Psi_1 \right\rangle = \sum_{a\in Crit(X)} \left\langle \Psi_2, U_a \right\rangle \left\langle S_a, \Psi_1 \right\rangle + o(1)$$

where U_a, (resp S_a) are currents having anisotropic singularities : smooth (resp singular) in the unstable directions and singular (resp smooth) on the stable ones. Moreover, the ubiquitous notion of **determinant like object** appears in the two topics under various forms : determinant like objects counting periodic orbits in dynamical systems, determinants of chiral fermions, partition functions in quantum field theory which are either zeta regularized determinants of elliptic operators for field theories in the continuum or discrete determinants for lattice field theories.

Semiclassical analysis is a branch of mathematical analysis which is deeply rooted in the study of quantum mechanics whereas algebraic microlocal analysis was invented by Sato to describe the singularities of PDEs in the cotangent space and was immediately applied to the study of Feynman amplitudes by the Japanese school of algebraic analysis. They were adapted to the smooth setting by Hörmander [**93**] in the 70's and quickly became a powerful tool in the study of linear and nonlinear PDEs. These methods give important tools to describe quantitatively singularities of distributions and operators in **phase space**, this contains informations on the location but also the **direction** of the singularities at high frequency. This explains why microlocal, semiclassical methods and spectral theory are relevant for both problems. Starting from the 90's, microlocal analysis allowed to solve important conceptual problems in field theory : give a consistent formulation of quantum fields in curved space–times with the notion of Hadamard states [**165**] and prove the perturbative renormalizability of quantum field theory on curved Lorentzian space–times [**26**]. At the interface of dynamical systems and topology, many conjectures by Bowen, Fried [**62**], Smale [**171**] have motivated decades of research in the field. Recently surprisingly, microlocal and semiclassical ideas revolutionized the analytic study of hyperbolic flows in dynamical systems allowing to solve several longstanding conjectures [**75, 46, 45, 43, 82, 5, 54**]. Surprisingly, hard classical mechanics problems are solved using deep quantum mechanical techniques.

Our manuscript summarizes a series of articles [**12, 13, 14, 15, 16, 17, 9**] which illustrate some applications of microlocal, semiclassical and spectral techniques in dynamics and quantum field theory. We tried to highlight some ideas lying at the intersection of the two subjects. Let us give a detailed plan of our manuscript :

— The first chapter is introductory in nature and deals with the series of joint works [**12, 13, 14, 15, 16**] with Rivière which are all related to Morse theory and topology. We start with a soft introduction to the concept of transfer operators in dynamics and Pollicott–Ruelle spectrum, with some motivation and parallels with quantum mechanics. We completely treat a toy model involving graphs to illustrate the main ideas in the simplest possible case. We then move on to describe the formalism of supersymmetric quantum mechanics which was popularized by Witten. We give many examples to illustrate the formalism, such as de Rham forms, Hodge theory, the Witten Laplacian and the Lie–Cartan formula. We next recall some definitions of Anosov and Morse–Smale flows in dynamical systems. We state and briefly sketch a proof of our result with Rivière [**12, 16**] which gives the Pollicott–Ruelle spectrum for Morse–Smale gradient flows. We explain how this result relates to the Witten Laplacian [**16**]. We briefly discuss the results for general Morse–Smale flows [**13, 14**] whose Pollicott–Ruelle spectrum has vertical band structure. In the last more topological part, we relate spectras and topological properties of the underlying manifold carrying the dynamics by giving Morse inequalities for both Morse–Smale and Anosov flows[**15**].

— The second chapter is devoted to our paper [**9**] which deals with the renormalization of determinant like functions. Our goal is to make sense of the partition function of chiral fermions interacting with some external gauge potential. This adresses a conjectural picture of Quillen [**148**]. Our exposition of the result differs from the original paper and relies heavily on microlocal methods even though we are dealing with Euclidean quantum fields. This is on purpose since our longterm goal would be to extend these results to the Lorentz case where the microlocal viewpoint plays an essential role. It is inspired by some joint work in progress with Brouder–Zhang. We only sketch proofs and we provide lots of motivations from quantum field theory.

— The third chapter discusses some joint works with Guillarmou-Rivière-Shen [**10**] and with Chaubet [**6**]. Both articles are devoted to the Fried conjecture which relates dynamics and topology. It aims to connect the Reidemeister torsion, which is a topological invariant similar to the determinant of acyclic chain complexes, with the value at 0 of the Ruelle zeta function which is a complex function counting periodic orbits. Since the statements of the main results are complicated and involve many objects, this chapter contains almost no indication of proofs and only tries to explain the results.

— The last chapter discusses the work [**17**] with Rivière which gives application of microlocal analysis to some classical problem in geometry : orbital counting. On a negatively curved surface, we show that the Poincaré series counting geodesic arcs orthogonal to some pair of closed geodesic curves has a meromorphic continuation to the whole complex plane. When both curves are homologically trivial, we prove that the Poincaré series has an explicit rational value at 0 interpreting it in terms of linking number of Legendrian knots. In particular, for any pair of points on the surface, the lengths of all geodesic arcs connecting the two points determine its genus. Finally, for any pair of homologically trivial closed geodesics, the lengths of all geodesic arcs orthogonal to both geodesics determine the linking number of the two geodesics.

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Chapitre 1

Transfer operators, quantum theory and supersymmetry.

In this thesis, we take the opportunity to make some analogies and give more examples than what could be found in the published articles. We hope that they might serve the reader as they were very helpful to us. The goal of this first chapter is to introduce the main motivations and ideas behind all the works presented in the next chapters concerning hyperbolic dynamics.

1. The formalism of transfer operators.

1.1. Motivation. In dynamical systems and classical mechanics, we are given a phase space (or configuration space) that we may think as some smooth compact manifold M and some smooth vector field $V \in C^\infty(TM)$ on M which generates some flow $e^{-tV} : M \mapsto M$ acting on M. Throughout the manuscript, a dynamical system will consist in a triple (M, V, μ) where μ is a measure on M.

EXAMPLE 1.1 (The circle $\mathbb{S}^1$). *On $\mathbb{S}^1$, consider the dynamics generated by the vector field ∂_θ which generates the rotations whose invariant measure is $d\theta$.*

Roughly speaking, in dynamical systems, one deals with the study of statistical properties of trajectories of some discrete or continuous evolution in some space. To motivate the formalism of transfer operators, we follow Omri Sarig's lectures [**157**]. Let us consider the following :
Thought experiment. Drop a little bit of ink into a glass of water, and then stir it with a tea spoon.

— Can you predict where individual ink particles will end after one minute ? NO : the motion of ink particles is chaotic.
— Can you predict the density of the ink particles after one minute ? YES : it will be nearly constant, equal to |mass of ink| / |volume of water+ink|.

Gibbs's insight : For chaotic systems, it is often easier to predict the behavior of densities of large collections of initial conditions, then to predict the behavior of individual initial conditions.

The transfer operator : instead of studying the **individual trajectories** of $\varphi^t(x)$ for every $x \in M$ for large times t, we consider the action of the dynamical system on **extended objects** such as mass densities, functions, differential forms, currents...This idea will be particularly fruitful when we study Morse gradient flows whose "individual trajectories" are very simple. The next paragraph aims to compare the formalism of transfer operators with quantum mechanics.

1.2. From classical to quantum mechanics. In the case of the geodesic flow, we would like to compare the quantum and classical formalism. In the classical mechanics of the geodesic flow, the configuration space $M = S^*\mathcal{M}$ is a contact manifold, the cosphere bundle of some manifold $\mathcal{M}$. We are given some Hamiltonian function on $S^*\mathcal{M}$, the evolution of some particle in $S^*\mathcal{M}$ is dictated by some ordinary differential equation $\frac{d\varphi^t}{dt} = X_H \circ \varphi^t$ where X_H is the Hamiltonian vector field defined from the Hamiltonian $H \in C^\infty(S^*\mathcal{M})$.

EXAMPLE 1.2 (Free particle on $\mathbb{R}$). *The phase space reads $S^*\mathbb{R} = \mathbb{R} \times \{\pm 1\}$ with coordinates (x, p) where $p = \pm 1$. Then the geodesic flow takes the very simple form $t \mapsto (x + tp, p)$.*

Example 1.3 (Free particle on $\mathbb{T}^d$, geodesic flow.). *The phase space reads* $S^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{S}^{d-1}$ *with coordinates* (x, p) *where* $p \in \mathbb{S}^{d-1}$*. Then the geodesic flow takes the very simple form* $t \mapsto (x+tp, p)$*, the momentum component remains constant along the geodesic flow since the curvature vanishes.*

In quantum mechanics, the state of the system is modelled by some Hilbert space $\mathcal{H} = L^2(M)$ and observables are operators acting on the state space $\mathcal{H}$. The evolution of the quantum system is given by some operator evolution operator $e^{-itH} : \mathcal{H} \mapsto \mathcal{H}$ with infinitesimal generator the Hamiltonian $H : \mathcal{D} \subset \mathcal{H} \mapsto \mathcal{H}$ which is often selfadjoint so that the dynamics is unitary. In quantum mechanics, starting from some initial state of the system $\Psi_1 \in \mathcal{H}, \|\Psi_0\|_{\mathcal{H}} = 1$, the probability amplitude that the evolved system at time t to be in the state Ψ_2 reads $\langle \Psi_2, e^{itH}\Psi_1 \rangle$ and the actual probability is $| \langle \Psi_2, e^{itH}\Psi_1 \rangle |^2$.

Example 1.4 (Free particle on $\mathbb{T}^d$). *On the torus* $\mathbb{T}^d$*, the Hilbert space is* $L^2(\mathbb{T}^d)$ *and the quantum evolution is given by the Schrödinger propagator* $e^{it\Delta}$*, where* Δ *is the Laplacian on* $\mathbb{T}^d$ *which quantizes the geodesic flow since the symbol of* Δ *is the Hamiltonian of the geodesic flow.*

	classical	*quantum*
Hamiltonian	ξ^2	$\mathbb{H} = -\Delta$
Evolution	$t \mapsto (x + t\xi; \xi)$	$e^{it\Delta}\Psi$

1.3. From quantum mechanics to transfer operators in dynamics. The formalism of transfer operators is somewhat midway between classical dynamics and quantum mechanics. At first approximation, the state space $\mathcal{H}$ will be the space of smooth functions $C^\infty(M)$ on the configuration space M and more generally some anisotropic Sobolev space of distributions as we will see in section 5 [1]. Instead of viewing the flow e^{-tV} acting on points of M, we will rather view it as acting on functions and Sobolev distributions by **pull–back** :

$$\Psi \in \mathcal{H} \longmapsto e^{-tV}\Psi = \varphi^{-t*}\Psi. \tag{1}$$

The above linear evolution will be denoted naively as the transfer operator. The infinitesimal generator of the evolution is the Lie derivative $\mathcal{L}_V$ acting on functions and distributions. In the sequel, for simplicity of notations, we will simply denote this operator by V. In the language of PDE, the evolved state $\Psi(t) = \varphi^{-t*}\Psi$ is in fact the solution of the **transport equation**

$$\partial\Psi(t) = -V\Psi(t), \Psi(0) = \Psi \tag{2}$$

where we transported the initial Cauchy data Ψ by the flow e^{-tV}.

A fundamental question in dynamical systems, let $\Psi(t) = \varphi^{-t*}\Psi$ be the solution of the above transport equation, what is the long time behaviour

$$\lim_{t\to+\infty} \varphi^{-t*}\Psi?$$

1.4. Dynamical correlators and their Laplace transform. How to extract dynamical informations from the transfer operators ?

Example 1.5. *Assume we are given a subset* $\Omega \subset M$ *of our configuration space and the initial density of particles is* Ψ_1*. We would like to know how to write the number of particles in the region* Ω *when the system evolved at time t ? The answer is very simple*

$$\int_{\Omega\subset M} e^{-tV}\Psi_1 d\mu = \langle 1_\Omega, e^{-tV}\Psi_1 \rangle .$$

1. Traditionnaly in dynamics for instance in the book of Baladi, the transfer operator is viewed as acting on probability measures by push–forward

This example illustrates the meaning of the dynamical correlators. If there is a measure μ on the compact manifold M, then one can consider the dynamical correlators defined as :

$$C_{\Psi_1,\Psi_2}(t) = \int_M \Psi_2\left(e^{-tV}\Psi_1\right) d\mu. \tag{3}$$

To emphasize the analogies with quantum mechanics, we shall sometimes denote such correlator as $\left\langle \Psi_2, e^{-tV}\Psi_1\right\rangle$.

In the analytical approach to dynamical systems we shall follow, we will be interested in the long time behaviour of the dynamical correlators $C_{\Psi_1,\Psi_2}(t)$ when $t \to +\infty$, in particular the asymptotics for large $t \to +\infty$. The study of the asymptotic behaviour of correlators in hyperbolic dynamics has a long history with many contributions [**151, 123, 31, 112, 39**]. These asymptotics can be captured in some sense by the Laplace transform of $C_{\Psi_1,\Psi_2}(t)$ w.r.t. variable t . This leads us to a formal definition of the Pollicott-Ruelle resonances [**143, 152**] as the poles of the Laplace transformed correlators

DEFINITION 1.6 (Pollicott-Ruelle resonances). The Pollicott-Ruelle resonances of V are the poles, when they exist, of the Laplace transformed correlators

$$\widehat{C}_{\Psi_1,\Psi_2}(z) = \int_0^\infty C_{\Psi_1,\Psi_2}(t)e^{-tz}dt = \int_0^\infty \left(\int_M \Psi_2\left(e^{-tV}\Psi_1\right) d\mu\right) e^{-tz}dt. \tag{4}$$

So one first fundamental problem reads

Given some vector field V on some compact manifold M. Study the analytic properties of the Laplace transformed dynamical correlators. What is the structure of its poles ?

1.5. The concept of Atiyah–Bott flat trace. Let M be some compact manifold with smooth density μ. To some linear map $T : C^\infty(M) \mapsto \mathcal{D}'(M)$ corresponds a distributional Schwartz kernel $K(x, y) \in \mathcal{D}'(M \times M)$, we define the flat trace of T :

$$Tr^\flat(T) = \int_{x\in M} K(x, x)d\mu(x)$$

the pull–back of K on the diagonal followed by integration, when these operations are well–defined. Let us consider the following simple example to understand the most important case of flat trace [**1**].

EXAMPLE 1.7 (Flat trace and transfer operators). *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a C^∞ diffeomorphism which is transverse to the identity i.e. $f(x) = x \implies f'(x) \neq 1$ and $|f(x)| \to +\infty$ when $|x| \to +\infty$. The pull–back operator $T_f : \psi \in C^\infty_c(\mathbb{R}) \mapsto f^*\psi = \psi \circ f \in C^\infty_c(\mathbb{R})$ defines a transfer operator. Its Schwartz kernel reads $\delta(y - f(x))$ and is supported by the graph of f since $\int_{\mathbb{R}} \delta(y - f(x))\psi(y)dy = \psi(f(x))$.*

Then the flat trace of T_f is expressed in terms of fixed points of f as $Tr^\flat(T_f) = \sum_{x=f(x)} \frac{1}{|1-f'(x)|}$.

Observe that we already see in the simple example that the flat trace **does not count** the number of fixed points and we will see in subsection 1.1 that it is only by adding supersymmetry to the picture that we will be able to count the number of fixed points.

2. Resonances, zeta functions and graphs.

2.0.1. *Graphs and transfer operators.* We illustrate the main ideas and techniques used in the sequel in some simple example involving only matrices and graphs called subshifts of finite type in the dynamical systems litterature. Consider a finite alphabet $A = \{1, \ldots, n\}$, the letters in the alphabet are represented by the vertices of some directed graph Γ where for every pair of vertices $(i, j) \in A^2$, there is at most one directed edge $i \mapsto j$ in Γ. For every such pair, we assign a non negative number $m_{ij} \geqslant 0$ and $m_{ij} = 0$ if there are no edges connecting (i,j). In certain situations, m_{ij} can be some integer but we do not assume this a priori. If $m_{ij}\in\{0,1\}$, then M is the *adjacency matrix* of Γ. The matrix M is *irreducible* if any pair of vertices can be connected by some path in Γ. The notion of irreducibility is similar to ergodicity in the sense that, starting from any vertex, the dynamics can access any other vertex of the graph. The state space is $L^2(V(\Gamma))$ which are functions on the vertices. The matrix M defines a transfer operator $T:\Psi\in L^2(V(\Gamma))\mapsto M\Psi\in L^2(V(\Gamma))$. Dynamical correlators read $C_{\Psi_1,\Psi_2}(n)=\langle\Psi_1,T^n\Psi_2\rangle$ where $n\in\mathbb{Z}$ plays the role of discrete time. In this case, the Pollicott–Ruelle resonances are defined as the poles of the discrete Laplace transform

$$\widehat{C}_{\Psi_1,\Psi_2}(z)=\sum_{n=0}^{\infty}z^n\left\langle\Psi_1,T^n\Psi_2\right\rangle=\left\langle\Psi_1,(Id-zT)^{-1}\Psi_2\right\rangle.$$

Of course, these poles counted with multiplicity coincide with the inverse spectrum $\sigma(M)^{-1}$ of M.

2.0.2. *Relation with the periodic orbits.* A periodic orbit in Γ is a sequence $\{i_1,\dots,i_n\}$ of vertices s.t. $i_1\to i_2\to\cdots\to i_n\to i_1$, $n=\ell(\gamma)$ is the period of γ. A periodic orbit γ can be a multiple $\tilde{\gamma}^p$ of some periodic orbit $\tilde{\gamma}$ of smaller period. Also note that if $\gamma=\{i_1,\dots,i_n\}$ is a periodic orbit then together with $\{i_2,\dots,i_n,i_1\},\dots,\{i_n,i_1,\dots,i_{n-1}\}$ there are potentially several periodic orbits passing through the physical loop $i_1\to i_2\to\cdots\to i_n\to i_1$. To kill all these multiplicities in the definition of periodic orbits, we define the *equivalence classes of prime periodic orbits* that we denote by $[\gamma]$ which means that a representative γ of the class $[\gamma]$ cannot be a multiple $\tilde{\gamma}^p,|p|>1$ of some periodic orbit $\tilde{\gamma}$ of lesser period and we identity $[\{i_1,\dots,i_n\}]\sim[\{i_2,\dots,i_n,i_1\}]\sim\cdots\sim[\{i_n,i_1,\dots,i_{n-1}\}]$. This is a "geometric closed path on the graph" without repetition. Let us make the following trivial observation which is the simplest instance of *trace formulas.*

LEMMA 2.1 (The simplest trace formula). *Let M be a matrix corresponding to the finite directed graph Γ as above. To every periodic orbit $\gamma=(i_1\dots i_k)$, we associate the weight $w(\gamma)=m_{i_1i_2}\dots m_{i_ki_1}$. One could think of this weight as "exponential of the integral of some potential" on the discrete loop or the holonomy of some discrete connection on the graph. Observe that we have the identity :*

$$\boxed{Tr(M^n)=\underbrace{\sum_{i\in A}(M^n)_{ii}=\sum_{\gamma}w(\gamma)=\sum_{[\gamma]\in\mathcal{P}}\ell([\gamma])\sum_{k=1}^{\infty}w([\gamma])^k\delta_{n=k\ell([\gamma])}}_{\textit{geometric side}}=\underbrace{\sum_{\lambda\in\sigma(T)}\lambda^n}_{\textit{spectral side}}}\tag{5}$$

where the sum runs over all the periodic orbits γ of length n.

One should think of the geometric side as the discrete version of the Atiyah–Bott flat trace whereas the spectral side is like a spectral trace. In case M is the adjacency matrix, we always get $w(\gamma)=1$. In analogy with combinatorics and analytic number theory, one can associate a weighted zeta function which counts the periodic orbits in Γ :

$$\zeta(z)=\prod_{[\gamma]\in\mathcal{P}}(1-w(\gamma)z^{\ell(\gamma)})^{-1}\tag{6}$$

where the product runs over the **equivalence classes of prime periodic orbits** in Γ denoted by $\mathcal{P}$. This is called the *Artin–Mazur* zeta function in analogy with the Weil zeta function in number theory. The following theorem, which is easy to prove in this combinatorial setting, is the key to understand the proofs of analytic continuation of zeta functions. It gives a direct relation between periodic orbits and spectrum of M.

THEOREM 2.2 (Bowen–Landford [**61**]). *The identity $\zeta(z)=\det(I-zM)^{-1}$ holds true hence ζ is rational with poles at $\sigma(M)^{-1}$ which is the resonance spectrum of $T:L^2(V(\Gamma))\mapsto L^2(V(\Gamma))$. We also have*

$$\frac{d}{dz}\log\zeta(z)=Tr\left(M(Id-zM)^{-1}\right)=\sum_{\gamma}\ell(\gamma)w(\gamma)z^{\ell(\gamma)}.\tag{7}$$

The second formula relates the discrete resolvent $(Id - zM)^{-1}$ to the log derivative of the zeta function. A similar formula will play a crucial role in Chapter 2 where we also express the log-derivative of the partition function (derivative of the free energy) in terms of traces of resolvents and also in our work on the Fried conjecture where we also calculate the log–derivative of the Ruelle zeta function.

Démonstration. By the simple trace formula and using $(1-x)^{-1} = \exp(-\log(1-x))$:

$$\begin{aligned}
\zeta(z) &= \prod_{[\gamma]\in\mathcal{P}} (1 - w([\gamma]) z^{\ell([\gamma])})^{-1} = \exp\left(-\sum_{[\gamma]\in\mathcal{P}} \sum_{n=1}^{\infty} \frac{w([\gamma])^n z^{n\ell([\gamma])}}{n}\right) \\
&= \exp\left(-\sum_{k=1}^{\infty} \sum_{\ell(\gamma)=k} w(\gamma)\frac{z^k}{k}\right) = \exp\left(-\sum_{k=1}^{\infty} Tr\left(M^k\right)\frac{z^k}{k}\right) \\
&= \exp\left(-Tr\log(Id - zM)\right) = \det(Id - zM)^{-1}.
\end{aligned}$$

The second formula is proved similarly using Lemma 2.1 and equation (41). $\square$

This Theorem shows in the simple case of subshifts of finite type that the poles of the zeta function counting periodic orbits are contained in the poles of the dynamical correlators defined above and are therefore Pollicott–Ruelle resonances of the transfer operator T.

3. Supersymmetric formalism.

In this memoir, we have applications to topology in mind. We will ask the more general question, for a differential form $\omega \in C^\infty(\Lambda^\bullet T^*M)$, what is the long time behaviour of $\varphi^{-t*}\omega$ when times t goes to $+\infty$? Do we have weak convergence to de Rham currents ? So one may ask, what do we gain by considering the transport equation on differential forms instead of functions ? The short answer would be **supersymmetry** and it is the purpose of the present section to introduce differential forms which have an extra *supersymmetric* structure which makes them richer and contains topological information about the underlying manifold M.

3.1. Some motivating examples. We start by the important example of De Rham differential forms in differential geometry. Then we explain how some simple formalism appearing in the mathematical physics litterature relates Hodge theory, the Witten Laplacian and the Lie–Cartan formula. Before we describe differential forms, we need to recall the important notion of $\mathbb{Z}_2$ graded vector space which is used repeatedly in this memoir.

Definition 3.1. A vector space E is called $\mathbb{Z}_2$–graded if E is endowed with some involution $\Gamma : E \mapsto E$ s.t. $\Gamma^2 = Id$ and hence E reads as a direct sum of eigenspaces for $\Gamma : E = E_+ \oplus E_-$ where $\Gamma|_{E_\pm} = \pm Id|_{E_\pm}$.

3.1.1. *Differential forms and the de Rham differential* d. Given a vector space V with given basis $(e_1, \dots, e_n)$, the exterior algebra $\Lambda^\bullet V$ of V is the $\mathbb{Z}_2$–graded algebra generated by elements of the form $(e_{i_1} \wedge \dots \wedge e_{i_k})_{1\leqslant i_1 < \dots < i_k \leqslant n}$. The exterior algebra $\Lambda^\bullet V$ decomposes as $\Lambda^{\text{even}} V \oplus \Lambda^{\text{odd}} V$ where the operator Γ acts on homogeneous differential forms $u \in \Lambda^k V$ as $\Gamma u = \pm u$ depending on the parity of k. It is endowed with the exterior product

$$\wedge : (u, v) \in \Lambda^k V \times \Lambda^l V \mapsto u \wedge v \in \Lambda^{k+l} V$$

which satisfies the relation $u \wedge v = (-1)^{\deg(u)\deg(v)} v \wedge u$.

Remark 3.2. Geometrically, one can intuitively visualize the element $u_1 \wedge \dots \wedge u_k$ as the oriented k-volume element spanned by the k-uple $(u_1, \dots, u_k)$. This explains why this volume element vanishes if the family $(u_1, \dots, u_k)$ is not linearly independent.

On some smooth manifold M, starting from the cotangent bundle T^*M, taking the fiberwise union $\bigcup_{x\in M}\Lambda^\bullet T_x^*M$ of the exterior algebras of T^*M defines the vector bundle $\Lambda^\bullet T^*M$ of differential forms. The smooth differential forms on M are smooth sections of ΛT^*M which are denoted by $C^\infty(\Lambda^\bullet T^*M)$. The vector space $C^\infty(\Lambda^\bullet T^*M)$ is in fact a finitely generated module over the algebra $C^\infty(M)$. The vector space $C^\infty(\Lambda^\bullet T^*M)$ splits as a direct sum

$$C^\infty(\Lambda^\bullet T^*M) = C^\infty(\Lambda^{\mathrm{even}}T^*M) \oplus C^\infty(\Lambda^{\mathrm{odd}}T^*M)$$

of even and odd elements. In local coordinates $(x^1,\dots,x^n)$ near a point m on a manifold M, a differential form ω of degree k reads $\omega = \sum_{1\leqslant i_1<\dots<i_k\leqslant n}\omega_{i_1\dots i_k}dx^{i_1}\wedge\dots\wedge dx^{i_k}$ where $\omega_{i_1\dots i_k}\in C^\infty(M)$. There is a local operator d, the *de Rham differential*, which raises the degree of differential forms by 1 and acts in local coordinates as

$$d\omega = \sum_{1\leqslant i_1<\dots<i_k\leqslant n}\sum_{i=1}^{n}\frac{\partial\omega_{i_1\dots i_k}}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\dots\wedge dx^{i_k}.$$

One can verify that d defined as above is intrinsic and that d squares to zero i.e. $d^2=0$. This means that the differential forms $C^\infty(\Lambda^\bullet T^*M)$ endowed with the de Rham differential d forms a cochain complex

$$\dots \xmapsto{d} C^\infty(\Lambda^k T^*M) \xmapsto{d} C^\infty(\Lambda^{k+1}T^*M) \xmapsto{d} \dots$$

3.2. De Rham currents. The de Rham currents play a fundamental role in our works. They generalize differential forms in the same way as the distributions of Laurent Schwartz generalize smooth functions. There are two possible definitions of de Rham currents, as forms whose coefficients are distributions or as the topological dual of smooth differential forms for some appropriate notion of pairing. In practice, the de Rham currents form a vector space of differential forms with distributional coefficients. In local coordinates $(x^1,\dots,x^n)$ in some neighborhood Ω of a point m on a manifold M, a current $[T]$ of degree k reads

$$[T] = \sum_{1\leqslant i_1<\dots<i_k\leqslant n} T_{i_1\dots i_k}dx^{i_1}\wedge\dots\wedge dx^{i_k}$$

where $T_{i_1\dots i_k}\in\mathcal{D}'(\Omega)$. The currents of degree k on M will be denoted by $\mathcal{D}'^{,k}(M)$ and the de Rham differential also acts on the de Rham currents. The second definition is a bit more conceptual and uses duality.

Definition 3.3. We think of the de Rham currents of **dimension** k as the topological dual of differential forms of degree k, $[T]\in\mathcal{D}'_k(M)$ is a continuous linear map on $C^\infty(\Lambda^k T^*M)$:

$$[T] : \omega\in C^\infty(\Lambda^k T^*M)\mapsto\langle[T],\omega\rangle \tag{8}$$

where the duality pairing $\langle .,.\rangle$ generalizes the notion of integration.

The graded vector space $\mathcal{D}'_\bullet(M)$ is endowed with a **boundary operator** ∂ which is defined as the transpose of the de Rham differential d, by definition :

$$\langle\partial[T],\omega\rangle = \langle[T],d\omega\rangle\,. \tag{9}$$

Conceptually, formula 9 takes the Stokes formula as inherent definition of the boundary operator ∂, it is similar to the notion of distributional derivative in the theory of distribution which is defined by duality.

Example 3.4 (Integration currents on submanifolds). *Let S be a smooth, oriented, compact submanifold of M of dimension $d\leq n$. This defines an integration current $[S]$ in $\mathcal{D}'_d(X)$ which acts on test forms $\omega\in C^\infty(\Lambda^d T^*M)$ by $[S](\omega)=\int_S\omega$.*

For more on integration currents and the relation with delta distributions, we refer the reader to the appendix of **[11]** and **[161, 74, 38]**.

3.3. The general formalism of supersymmetric quantum mechanics, Hodge theory, Lie–Cartan formula and Witten Laplacians. The language described in this section was strongly motivated by a landmark paper of Witten [**183**] which related for the first time supersymmetry and Morse theory. We are given a $\mathbb{Z}_2$–graded vector space of states $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with parity operator Γ, $\Gamma u = u$ if $u \in \mathcal{H}_0$ and $\Gamma u = -u$ if $u \in \mathcal{H}_1$. As explained above, Γ is an involution $\Gamma^2 = 1$ and $\mathcal{H}_0 \oplus \mathcal{H}_1$ is a decomposition of $\mathcal{H}$ into a direct sum of eigenspaces for Γ. A linear operator $A : \mathcal{H} \mapsto \mathcal{H}$ is called even (resp odd) if it does not change the parity of elements (resp if it exchanges parities). Sometimes, we will also say that A has degree 0 if it is even and A has degree 1 if it is odd. Let us give the definition of the commutator of two linear maps in graded linear algebra [2] :

Definition 3.5. The supercommutator $[A, B]$ between two operators $\mathcal{H} \mapsto \mathcal{H}$ is defined to be

$$[A, B] = AB - (-1)^{\deg(A)\deg(B)} BA.$$

Definition 3.6. A supersymmetric quantum mechanical system consists of the following data :

- A $\mathbb{Z}_2$–graded Hilbert space of states $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with parity operator Γ. This represents some Hilbert space of sections of some $\mathbb{Z}_2$–graded bundle E and we have continuous dense inclusions $C^\infty(E) \subset \mathcal{H} \subset \mathcal{D}'(E)$ where $C^\infty(E)$ are smooth sections and $\mathcal{D}'(E)$ are E valued distributions.
- Two odd operators $Q, Q^\dagger$ which square to zero : $Q^2 = Q^{\dagger 2} = 0$.
- The supercommutator $[Q, Q^\dagger] = H$ defines the Hamiltonian of the quantum system which is the infinitesimal generator of the dynamics. H is an **even, unbounded** operator (**not necessarily self–adjoint**) from some dense domain $\mathcal{D} \subset \mathcal{H}$ into $\mathcal{H}$. We have the important commutations relations $[Q, H] = [Q^\dagger, H] = 0$.
- The operator H is such that the resolvent $(H - z)^{-1} : \mathcal{H} \mapsto \mathcal{H}$ is Fredholm analytic on some half–space $Re(z) > -a$ for some $a > 0$.

Let us remark that in concrete applications, the structure we find will be slightly more complicated. We have some family $(\mathcal{H}^m)$, where $m \in C^\infty(S^*M)$ is some order function, of anisotropic Sobolev spaces. These are Hilbert spaces of distributional sections of some vector bundle which depends on the application in mind. In all **applications from the present thesis**, Q equals the De Rham operator d or a twisted version of the De Rham operator (see 7.1 for the definition of twisting) which is a differential operator of order 1. Therefore, Q acts as a bounded operator $\mathcal{H}^{m+1} \mapsto \mathcal{H}^m$ where both spaces are anisotropic spaces.

Let us now give three fundamental examples covered by the supersymmetric formalism that we will meet several times in the manuscript.

3.3.1. *The Hodge Laplacian.* Obviously $C^\infty(\Lambda^\bullet T^*M)$ is $\mathbb{Z}_2$–graded by the parity of the differential form. Assume that there is a Riemannian metric g on M. This defines a Hodge star operator $* : C^\infty(\Lambda^\bullet T^*M) \mapsto C^\infty(\Lambda^{n-\bullet} T^*M)$ which is an involution on $C^\infty(\Lambda^\bullet T^*M)$ s.t. $\langle u, v\rangle = \int_M u \wedge *v$ where $\langle ., .\rangle$ is the natural scalar product on differential forms induced by the metric g on M. Using the scalar product $\langle ., .\rangle$, we may define the L^2 adjoint d^* of the de Rham differential d. So we have a pair of differentials (d, d^*) such that

$$[d, d^*] = dd^* - (-1)^{\deg(d)\deg(d^*)} d^*d = dd^* + d^*d = \Delta_g \tag{10}$$

where Δ_g is the Laplace–Beltrami operator.

3.3.2. *Lie–Cartan formula.* On a smooth compact manifold M with a smooth vector field V, we describe a natural supersymmetric structure. The states space is the vector space of differential forms which has a natural $\mathbb{Z}_2$ grading which comes from taking the degree of the differential form modulo 2. The de Rham operator d plays the role of Q since it is an odd operator squaring to 0. There is another operator denoted by ι_V : $C^\infty(\Lambda^\bullet T^*M) \mapsto C^\infty(\Lambda^{\bullet-1} T^*M)$ defined on one forms as $\iota_V \alpha = \alpha(V)$ and extended to all degrees by $\iota_V(u \wedge v) = (\iota_V u) \wedge v + (-1)^{\deg(u)} u \wedge \iota_V v$.

2. Sometimes called the Koszul rule of signs in the supergeometry litterature

Then $(C^\infty(\Lambda^\bullet T^*M), d, \iota_V)$ forms a supersymmetric quantum mechanical system whose dynamics is generated by

$$(11) \qquad [d, \iota_V] = d\iota_V - (-1)^{\deg(d)\deg(\iota_V)}\iota_V d = d\iota_V + \iota_V d = \mathcal{L}_V$$

by the Lie–Cartan formula.

3.3.3. *The Witten Laplacian.* This example interpolates in some sense between Hodge and the Lie–Cartan supersymmetric structures. Let f be some Morse function on M. Consider the twisting $e^{-\frac{f}{\hbar}}de^{\frac{f}{\hbar}} = d_f = d + \hbar^{-1}df\wedge$ and its adjoint d_f^* as in the Hodge theory. Then the rescaled supercommutator

$$(12) \qquad \frac{\hbar}{2}[d_f, d_f^*] = \Delta_f = \frac{\hbar}{2}\Delta + \text{order zero term} + \frac{\|\nabla f\|^2}{2\hbar}$$

is the celebrated Witten Laplacian in a rescaled form. Then $\left(C^\infty(\Lambda^\bullet T^*M), d_f, d_f^*\right)$ forms a supersymmetric quantum mechanical system whose dynamics is generated by the Witten Laplacian Δ_f.

Let us summarize the common features of the above three examples in the following array

	Lie-Cartan	Hodge	Witten
State space	$C^\infty(M)\subset\mathcal{H}\subset\mathcal{D}'(M)$	$L^2(M)$	$L^2(M)$
Hamiltonian	$i\mathcal{L}_V$	Δ	Δ_f
Evolution	$e^{-t\mathcal{L}_V}$	$e^{-t\Delta}$	$e^{-t\Delta_f}$
SUSY states	$\mathcal{H}\otimes_{C^\infty(M)}\Omega^\bullet(M)$	$L^2(M)\otimes_{C^\infty(M)}\Omega^\bullet(M)$	$L^2(M)\otimes_{C^\infty(M)}\Omega^\bullet(M)$
SUSY generators	$Q=d, Q^*=\iota_V$	$Q=d, Q^*=d^*$	$Q=d_f, Q^*=d_f^*$
Key identity	$\mathcal{L}_V=[d,\iota_V]$	$\Delta=[d,d^*]$	$\Delta_f=[d_f,d_f^*]$
Physical states	$\frac{\ker(Q)}{\mathrm{Im}(Q)}$	idem	idem
Hodge decomp.	$\mathcal{H}=\ker(\mathcal{L}_{V_f})\oplus\mathrm{Im}(d)\oplus\mathrm{Im}(\iota_V)$	$\mathcal{H}=\ker\Delta\oplus Im(d)\oplus Im(d^*)$	$\mathcal{H}=\ker\Delta_f\oplus Im(d_f)\oplus Im(d_f^*)$
Intersection pairing	$\langle u,v\rangle=\int_M u\wedge v$	idem	idem
Scalar product	?	$(u,v)=\langle u,\star v\rangle$	$(u,v)=\langle u,\star v\rangle$

Remark 3.7. There is a subtlety in the above comparison table which were brought to our attention by Maciej Zworski. Actually, the quantum analogue of the operator $i\mathcal{L}_V$ (which has real principal symbol) should be the operator Δ and therefore the analogue of the flow $e^{-t\mathcal{L}_V}$ should be the unitary Schrödinger propagator $e^{it\Delta}$. The correspondance is even more true in case the flow is volume preserving in which case $e^{-t\mathcal{L}_V}: L^2(M)\mapsto L^2(M)$ is also unitary. However, when acting on the anisotropic space, the operator $i\mathcal{L}_V$ is no longer "self adjoint" since the propagator $e^{-t\mathcal{L}_V}$ acting on the anisotropic spaces is no longer unitary and behaves more like a semigroup. Furthermore when V is a Morse–Smale gradient, the analogy with the heat flow in Hodge theory is striking since the flow $e^{-t\mathcal{L}_V}: \mathcal{H}\mapsto\mathcal{H}, t\geqslant 0$ will damp exponentially fast all Ruelle resonant states except the states for the resonance 0.

In sections 8 and 6 devoted to the Morse inequalities and the Witten Laplacian, we will come back to these correspondences in more detail.

3.4. Quasi–isomorphism and formal Morse inequalities. We recall a bit of terminology, two cochain complexes are called *quasi–isomorphic* if they have isomorphic cohomology groups. Let us indicate how one can prove Morse inequalities in the abstract setting of supersymmetric quantum mechanics. This sketch of proof will be applied repeatedly to Anosov and Morse–Smale flows as well as the Witten Laplacian. The mechanism works as soon as one can check the abstract structure of definition 3.6.

Observe that $\ker(H)$ is stable by Q since Q and H commute by definition of $H=[Q,Q^\dagger]$. The key idea is to show that the complex $(C^\infty(E),Q)$ is quasi–isomorphic to the complex $(\ker(H),Q)$ of zero modes for the Hamiltonian. Let Π_0 be the spectral projector on $\ker(H) \subset \mathcal{H}$. We have the following commutative diagram :

$$\begin{array}{r|ccc} \text{Infinite dim. complex} & C^\infty(E) & \xrightarrow{Q} & C^\infty(E) \\ & \downarrow & & \downarrow \\ \ker(H) & \Pi_0\left(C^\infty(E)\right) & \xrightarrow{Q} & \Pi_0\left(C^\infty(E)\right) \end{array}$$

where we need to prove that the projector Π_0 induces an isomorphism in cohomology. The key method is to establish the chain homotopy equation. We have a sequence of identities :

$$Id - \Pi_0 = H\underbrace{H^{-1}(Id-\Pi_0)}_{\text{key term}} = Q\underbrace{\Big(Q^\dagger H^{-1}(Id-\Pi_0)\Big)}_{R} + \underbrace{\Big(Q^\dagger H^{-1}(Id-\Pi_0)\Big)}_{R}Q.$$

This can be compactly written as :

$$\boxed{Id - \Pi_0 = QR + RQ} \tag{13}$$

which is called chain homotopy equation. The consequence of the above chain homotopy equation is that one has a Hodge type decomposition

$$\mathcal{H} = \ker(H) \oplus Im(Q) \oplus Im(Q^\dagger) \tag{14}$$

and also that a Q closed state $u \in \mathcal{H}$ satisfies $u = \Pi_0(u) + QRu$ hence $u - \Pi_0(u)$ is Q-exact and belong to the same Q-cohomology class.

Remark 3.8. Let us remark that the above Hodge type decomposition holds true in all the situations described in the present thesis, for instance when $Q = d$ is the de Rham differential, $Q^\dagger$ is the contraction ι_V. The decomposition does not require $Q^\dagger$ to be the adjoint of Q for some Hilbertian structure. The key idea used in the proof is that the resolvent $(V-z)^{-1} : \mathcal{H} \mapsto \mathcal{H}$ acting on the anisotropic spaces has discrete spectrum at 0 which has finite multiplicity. It follows that the spectral projector on the generalized eigenspace $\ker(V)$ reads $\Pi_0 = \frac{i}{2\pi}\int_\gamma (V-z)^{-1}dz$ where γ is some Jordan path which encloses $\{0\}$ and does not meet the rest of $\sigma(V)$. Then $V : (Id-\Pi_0)\mathcal{H} \mapsto (Id-\Pi_0)\mathcal{H}$ is invertible and this is what is needed to obtain the chain homotopy equation (13) and the Hodge type decomposition (14).

4. Morse–Smale and Anosov flows as hyperbolic dynamics.

In the present manuscript, we deal exclusively with Morse–Smale and Anosov flows. So in the present section, we shall give a simple introduction to both kind of dynamics emphasizing the similarities and differences.

4.1. Anosov flows.

We shall start with Anosov flows.

Definition 4.1. The flow $\varphi^t = e^{tV} : M \mapsto M$ generated by some vector field $V \in C^\infty(TM)$, is Anosov if the tangent bundle TM splits as a direct sum of invariant bundles :

$$TM = E_s \oplus E_u \oplus \mathbb{R}\langle V\rangle$$

where $\mathbb{R}\langle V\rangle$ is the line bundle generated by V,

$$\|d\varphi^t_x(v)\| \leqslant Ce^{-Kt}\|v\|, (x,v) \in E_s \tag{15}$$

$$\|d\varphi^{-t}_x(v)\| \leqslant Ce^{-Kt}\|v\|, (x,v) \in E_u. \tag{16}$$

Example 4.2 (Cotangent of negatively curved surface). *Let $\mathcal{M}$ be a surface endowed with a metric of negative curvature $K<0$ everywhere, visually one should imagine that all geodesic triangles have the sum of inner angles $<\pi$. Then it is a well known fact that the geodesic flow $(\varphi^t)_t$ on $S^*\mathcal{M}$ is Anosov* **[102]**.

4.2. Morse–Smale flows. Morse–Smale flows are slightly more complicated to describe since we need to introduce a bit of terminology from dynamical systems. We say that a point x in M is wandering if there exist some open neighborhood U of x and some $t_0 > 0$ such that

$$U \cap \left(\cup_{|t|>t_0}\varphi^t(U)\right) = \emptyset.$$

This means intuitively that after a certain time, some neighborhood U of x pushed by the flow will never intersect itself again. The *nonwandering set* of the flow is given by the points which are not wandering. The set of nonwandering points is denoted by $\mathrm{NW}(\varphi^t)$. For any invariant closed subset Λ of M, we define the *unstable and stable manifolds* of Λ :

$$W^u(\Lambda) := \{x \in M : \mathrm{dist}(\varphi^t(x), \Lambda) \to 0, t \to -\infty\},$$

and

$$W^s(\Lambda) := \{x \in M : \mathrm{dist}(\varphi^t(x), \Lambda) \to 0, t \to +\infty\}.$$

These notions are intuitively clear, the stable (resp unstable) manifold of Λ are the points which are attracted to Λ in the future (resp past).

The concept of hyperbolicity is central. It starts with linear maps, for E a real vector space, a linear map $L : E \mapsto E$ is hyperbolic if none of its eigenvalues lie in the unit circle. Geometrically, this means that E splits as the direct sum $E = E_s \oplus E_u$ of invariant subspaces E_s, E_u where E_s is the stable subspace where the dynamics contracts exponentially fast and E_u is the unstable subspace where the dynamics expands exponentially fast.

Example 4.3 (Fundamental example 1). *Let M be the matrix*

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

acting on the plane $\mathbb{R}^2$ by $x \mapsto M.x$. The unstable subspace is the Ox axis and the stable subspace is the Oy axis.

A fixed point m of some flow φ^t is **hyperbolic** if $d\varphi^t_m : T_mM \mapsto T_mM$ is hyperbolic for some $t > 0$.

Example 4.4 (Fundamental example 2). *Let A be an $n \times n$ matrix such that for every eigenvalue $\lambda \in \sigma(A)$, $Re(\lambda) \neq 0$. This implies that the exponential matrix e^{tA} is **hyperbolic** in the linear algebra sense as explained above.*

The vector field $\langle Ax, \partial_x\rangle = A^j_i x^i \partial_{x^j}$ which generates the flow

$$x \in \mathbb{R}^n \mapsto e^{tA}x \in \mathbb{R}^n$$

has $0 \in \mathbb{R}^n$ as unique hyperbolic fixed point.

A periodic orbit γ of a dynamical system is **hyperbolic** if the Poincaré return map $\mathcal{P}_\gamma$ mapping a transversal piece of hypersurface Σ to itself fixing $m \in \Sigma$ is such that $d\mathcal{P}_\gamma : T_m\Sigma \mapsto T_m\Sigma$ is hyperbolic.

Definition 4.5 (Morse–Smale flows). A flow $e^{tV} = \varphi^t : M \mapsto M$ is Morse–Smale if

- its non wandering set $NW(\varphi^t)$ is a **finite union** of closed orbits and fixed points, we will denote by $\Lambda \in NW(\varphi^t)$ the critical elements of V [3],
- the critical elements $\Lambda \in NW(\varphi^t)$ are **hyperbolic**,
- for every pair (Λ_1, Λ_2) of critical elements (they can be the same), the intersection $W^u(\Lambda_1) \cap W^s(\Lambda_2)$ is **transverse**. This condition will be called **Smale transversality**.

Let us comment the above definition. The fact that the stable and unstable manifolds are actually smooth submanifolds (but non properly immersed) follows from the hyperbolicity of the critical elements. The transversality of $N_1 \cap N_2$ where N_1, N_2 are submanifolds means that at every $x \in N_1 \cap N_2$, $T_xN_1 + T_xN_2$ spans T_xM. The most important example of Morse–Smale flows one should keep in mind are the Morse–Smale **gradient flows**

3. That is the connected components of $NW(\varphi^t)$

THEOREM 4.6 (Kupka–Smale). *For a generic pair of function and metric $(f,g) \in C^\infty(M)\times \boldsymbol{Met}(M)$, the gradient flow $\varphi^t : M \mapsto M$ generated by $V = \nabla f$ is a Morse–Smale flow whose nonwandering set coincides with the critical locus of f : $\{a \in M; df(a) = 0\}$.*

The manifold M is ***partitioned*** *as a union of unstable manifolds*

$$M = \bigcup_{a\in Crit(f)} W^u(a).$$

The observation that M is partitioned by unstable manifolds is due to Thom [**177**] and is an important precursor for applications of Morse theory to topology. In the case of Morse–Smale gradient flows, there are no periodic orbits and only critical points in $NW(\varphi^t)$ but this was historically an important example that motivated Smale to introduce the larger class of Axiom A flows [**171, 21, 61**].

EXAMPLE 4.7. *Consider the function $f(x,y) = ax^2 - by^2$ for $(a,b) > 0$ and the Euclidean metric $g = dx^2 + dy^2$. Then f has critical point $(0,0) \in \mathbb{R}^2$ and $V = \nabla f$ reads $ax\partial_x - by\partial_y$ with hyperbolic fixed point $(0,0)$. Integrating V, the flow reads $(e^{at}x, e^{-bt}y)$ thus $W^u((0,0)) = \{x = 0\}$ and $W^s((0,0)) = \{y = 0\}$.*

4.3. Comparing Anosov to Morse–Smale flows. Let us state some important classical results on Anosov and Morse–Smale flows. Then we will compare them and show their structural differences. If the vector field $V \in C^\infty(TM)$ is Anosov then there exists a measure of maximal entropy μ on M whose support is the whole manifold M such that the flow $\varphi^t : M \mapsto M$ preserves μ and is ergodic w.r.t. μ. This means that almost every orbit of the flow is everywhere dense by the Birkhoff ergodic Theorem : $\overline{\cup_{t\geqslant 0}\varphi^t(x)} = M$ for μ almost all $x \in M$ and the non wandering set $NW(\varphi^t)$ in the Anosov case is the whole manifold M itself.

EXAMPLE 4.8 (Geodesic flows and Liouville measure). *If $(\mathcal{M}, g)$ is smooth, compact negatively curved Riemannian manifold, then the geodesic flow $\varphi^t : S^*\mathcal{M} \mapsto S^*\mathcal{M}$ is Anosov and preserves the Liouville measure on $S^*\mathcal{M}$.*

Intuitively, Anosov flows are typical examples of chaotic dynamics where it is difficult to predict the history of individual orbits and it is therefore very fruitful to follow the transfer operator approach and study the evolution of extended objects.

The situation for Morse–Smale flows is drastically different. In fact, there is a deep Theorem of Smale whose complete detailed proof can be found in [**13**] which describes the geometric structure of Morse–Smale flows

THEOREM 4.9 (Smale). *Let $V \in C^\infty(TM)$ be Morse–Smale. Then the manifold M is partitioned as a union of unstable manifolds of critical elements*

$$M = \bigcup_{\Lambda\in NW(\varphi^t)} W^u(\Lambda), \tag{17}$$

and the ***relation*** *$\prec$ on the critical elements $NW(\varphi^t)$ defined as*

$$\Lambda_1 \prec \Lambda_2 \text{ if } W^u(\Lambda_1) \subset \overline{W^u(\Lambda_2)}$$

is a ***partial order relation****.*

Moreover, we have a precise description of the closure of any unstable manifold as

$$\overline{W^u(\Lambda)} = \cup_{\Lambda'\preceq\Lambda} W^u(\Lambda'). \tag{18}$$

The partial order relation gives a kind of ordering in the way in which M is stratified by the unstable manifolds. From the above Theorem, we see that for Morse–Smale flows, the only invariant measures are convex combinations of delta measures supported by the critical points and periodic orbits of the flow so their support has empty interior in contrary to the Anosov case.

4.4. Motivations and questions. In the case of Morse Smale gradient flows, observe that for every $x \in M$, x must belong to some stable manifold $W^s(a)$ and unstable manifold $W^u(b)$ and therefore $\varphi^t(x) \to a$ when $t \to +\infty$. So in contrast with the Anosov flows, the Morse–Smale trajectories exhibit very predictable and unsurprising behaviour. This is why it is more fruitful to ask different questions about the long times dynamics of Morse–Smale flows, more functional analytic in nature :

(1) In the paper [**86, 87**], Harvey–Lawson address the following problem : let u be a smooth differential form, what can we say about $\lim_{t\to+\infty} \varphi^{-t*}u$? Is there convergence in a weak sense to some current in $\mathcal{D}'(M)$? Can the topology of M influence the possible weak limits ? In fact with Rivière, we show there is an exponential rate of convergence to the limit, moreover we show that there is an asymptotic expansion in t when $t \to +\infty$.

(2) The first problem in Bowen's notebook [**16**] reads : *To what extent does the gradient flow near a critical point depend on the metric ?* With Rivière, we show that the asymptotics of correlators of global observables is entirely determined by the Lyapunov exponents of V at the critical points.

(3) Is there a relation between large times dynamics $\lim_{t\to+\infty} \varphi^{-t*}u$ and semiclassical limit of low energy quasimodes of the Witten Laplacian ? This question was asked by Harvey–Lawson and will be discussed in section 6.

Similar questions can be asked for Morse–Smale and Anosov flows, both types might contain periodic orbits contrary to the gradient flows. The goal of the remaining parts of this chapter is to provide partial answers to all above questions first for gradient flows then we shall discuss more general Morse–Smale and Anosov flows.

4.4.1. *The results of Laudenbach and Harvey–Lawson.* We state a Theorem in two parts for gradient flows which motivated our results. The first part is due to Laudenbach [**106**] and deals with the mass of unstable manifolds and the second due to Harvey-Lawson deals with the convergence of dynamical correlators, $f \in C^\infty(M)$ a given Morse–Smale function :

Theorem 4.10 (Laudenbach 1992, Harvey-Lawson 2000). *Let M be a smooth, compact, boundaryless manifold. Then for every pair (f, g) where $V = \nabla f$ satisfies the Smale transversality condition (see definition 4.5) and g is flat in Morse coordinates near $Crit(f)$:*

— *(Laudenbach) the unstable manifolds $W^u(a)$ define integration currents $U_a = [W^u(a)]$ of **finite mass** called* Laudenbach currents *:*

$$\langle U_a, \alpha\rangle = \int_{W^u(a)} \alpha.$$

— *(Harvey-Lawson)* the dynamical correlators converge and their limit reads *:*

$$\langle \beta, \varphi^{-t*}\alpha\rangle \underset{t\to+\infty}{\to} \sum_{a\in Crit(f)} \left(\int_{W^u(a)} \beta\right)\left(\int_{W^s(a)} \alpha\right). \tag{19}$$

For $a \in Crit(f)$, the current of integration on the stable manifold $W^s(a)$ will be denoted by S_a in the sequel.

Remark 4.11. One can reformulate the above result. Introduce a Schwartz kernel $[\Pi](x, y)$ in $\mathcal{D}'(M \times M)$ which is a current of degree n :

$$[\Pi](x, y) = \sum_{a\in Crit(f)} U_a(x, dx) \wedge S_a(y, dy) \tag{20}$$

then in terms of $[\Pi]$, the above result reads :

$$\lim_{t\to+\infty} \langle \beta, \varphi^{-t*}\alpha\rangle = \int_{M\times M} \beta(x) \wedge [\Pi](x, y) \wedge \alpha(y). \tag{21}$$

Note that for every critical point $a \in Crit(f)$, $\dim(W^u(a)) + \dim(W^s(a)) = n = \dim(M)$ which implies that the wedge product $U_a(x, dx) \wedge S_a(y, dy)$ is a current of degree n.

The Laudenbach currents $(U_a)_{a\in \mathrm{Crit}(f)}$ are global versions of the Ruelle–Sullivan [**154**] currents [4] in the specific situation of Morse–Smale flows. In the next section, we discuss our result with Rivière generalizing the above Theorem.

5. Ruelle spectrum of Morse–Smale flows.

In this section, we will discuss the particular case of the Ruelle spectrum of Morse Smale gradient flows which can be completely calculated. Let us observe that there are few examples of systems where the spectrum can be calculated [**52, 6**]. Given $V = \nabla f \in C^\infty(TM)$, we denote by $\lambda_j(a)_{j=1}^n, a \in Crit(f)$ the Lyapunov exponents of V at the critical points. These are defined to be eigenvalues of the matrix $(\partial_{x^i} a^j)(0)$ where the vector field reads $V = a^j(x)\partial_{x^j}$ in local chart where $a^j(0) = 0$. Given V, we define the subset $\sigma = \cup_{a\in Crit(f)} \sum_{j=1}^n \mathbb{N}|\lambda_j(a)|$ which is constructed from the Lyapunov exponents.

Theorem 5.1 (D-Rivière). *Let $(f, g) \in C^\infty(M) \times \mathbf{Met}(M)$ s.t. $V = \nabla f$ satisfies the Smale transversality condition. If the Lyapunov exponents $(\lambda_j(a))_{j=1,a\in Crit(f)}^n$ are $\mathbb{Q}$-independent, then there exists a sequence $([\Pi_\lambda])_{\lambda\in\sigma}$ of kernels in $\mathcal{D}'^{,n}(M \times M)$ s.t. :*

$$\langle \beta, \varphi^{-t*}\alpha\rangle \underset{t\to+\infty}{\sim} \sum_{\lambda\in\sigma} e^{-t\lambda} \int_{M\times M} \beta(x) \wedge [\Pi_\lambda](x, y) \wedge \alpha(y). \tag{22}$$

If V is only C^1 linearizable near $Crit(f)$, then σ is the Ruelle spectrum of V and we have an asymptotic expansion :

$$\langle \beta, \varphi^{-t*}\alpha\rangle \underset{t\to+\infty}{\sim} \sum_{\lambda\in\sigma} e^{-t\lambda} P_\lambda(t; \alpha, \beta) \tag{23}$$

where $P_\lambda(t; \alpha, \beta)$ is polynomial in t and bilinear continuous in $(\alpha, \beta) \in C^\infty(\Lambda^\bullet T^*M) \times C^\infty(\Lambda^\bullet T^*M)$. In both cases, $\ker(V)$ *contains no Jordan blocks.*

When the Lyapunov exponents $(\lambda_j(a))_{j=1,a\in Crit(f)}^n$ are $\mathbb{Q}$-independent the vector field V is C^∞ linearizable near $Crit(f)$ which means that the C^1 linearizability assumption is weaker. However, if V is only C^1 linearizable then the Ruelle eigenspaces have eventual Jordan blocks which explains the presence of terms which are polynomials in t. $[\Pi_\lambda]$ is the Schwartz kernel of the spectral projector on the eigenspace $\ker(V - \lambda)$. $[\Pi_0](x, y) = \sum_{a\in Crit(f)} U_a(x, dx) \wedge S_a(y, dy)$. In particular, $\ker(V)$ is spanned by the Laudenbach currents $\ker(V) = \left\langle (U_a)_{a\in Crit(f)}\right\rangle$.

5.1. Proof sketch. Our proof applies to currents of all degree since we can construct anisotropic Sobolev spaces of currents of all degree, but without loss of generality, we will sketch the proof in the case of degree 0. We first explain that we can construct a family of anisotropic Sobolev spaces $\mathcal{H}^m(M)$, $m \in C^\infty(S^*M)$ is an order function adapted to the dynamics, such that $V : \mathcal{D} \subset \mathcal{H}^m(M) \mapsto \mathcal{H}^m(M)$ has discrete spectrum on $Re(z) \geqslant -a$ for $a > 0$ arbitrarily large. The method follows from a long series of contributions by many people working in various contexts [**2, 15, 28, 101, 113, 79, 178, 179**], we refer to [**3**] for a detailed exposition and an extensive bibliography. We mostly follow the original method of Faure–Roy–Sjöstrand [**53**] with some ideas inspired by Lefeuvre [**108**, 2.4], Bonthonneau–Guillarmou–Hilgert–Weich [**25**]. It is similar in spirit to the "quantum scattering in phase space" as can be found in the works of Helffer–Sjöstrand [**89**] and Gérard–Sjöstrand [**71, 72**].

4. Ruelle–Sullivan constructed eigencurrents for Axiom A diffeomorphisms near each basic set

5.1.1. *Anisotropic spaces.* Given a flow $\varphi^t = e^{tV} : M \mapsto M$, we lift the flow to the unit cotangent space S^*M

$$\Phi^t : (x;\xi) \in S^*M \mapsto \Phi^t(x;\xi) = (\varphi^t(x); \frac{({}^T d\varphi^t_x)^{-1}\xi}{\|({}^T d\varphi^t_x)^{-1}\xi\|}) \in S^*M$$

and find the attractor Σ_s and repeller Σ_u for Φ^t. For every submanifold $S \subset M$, we will denote by $N^*S \subset T^*M$ its conormal bundle. In the Morse–Smale case, if there are only critical points in $NW(\varphi^t)$ (for instance in the gradient case), we need to consider the sets :

$$\Sigma_u = \cup_{\Lambda \in NW(\varphi^t)} N^*W^s(\Lambda), \;\; \Sigma_s = \cup_{\Lambda \in NW(\varphi^t)} N^*W^u(\Lambda).$$

The fundamental Theorem proved in [**12**] reads :

THEOREM 5.2. *The sets Σ_u, Σ_s are disjoint* ***closed, conical*** *subsets of T^*M and Σ_s (resp Σ_u) is the attractor (resp repeller) for the lifted flow $\Phi^t : S^*M \mapsto S^*M$ on the unit cosphere.*

In the recent microlocal terminology, Σ_u acts as a source whereas Σ_s acts as a sink for the lifted flow. Since the sets Σ_u, Σ_s are **disjoint**, we can build an order function $m \in C^\infty(S^*M)$, homogeneous of degree 0, which is very positive on Σ_u, very negative on Σ_s and m is **decreasing** along the flow Φ^t. Then using the order function m, we construct a symbol of variable order $A_C(x;\xi) = (1+\|\xi\|_{g(x)})^{m(x;\frac{\xi}{\|\xi\|})}$. Then by quantization, we define a pseudodifferential operator with variable order $Op(A_C) \in \Psi^m(M)$ and the corresponding anisotropic Sobolev space $\mathcal{H}^m(M) = Op(A_C)^{-1}L^2$. These Sobolev spaces of **variable order** contain distributions which are regular near Σ_u and irregular near Σ_s. The next step is to represent the resolvent $R(z) = (V+z)^{-1}$ in terms of the propagator of the flow e^{-tV}, in the spirit of the Hille–Yosida Theorem in semigroup theory [**51**]. Then conjugate the resolvent with $Op(A_C)$. Note that for $Re(z)$ large enough, we have $R(z) = (Id - e^{-T(V+z)})^{-1}\int_0^T e^{-t(V+z)}dt$ in the sense of bounded operators on L^2 for $T>0$. Given $a>0$, to prove the analytic continuation of the resolvent up to the half–plane $Re(z) > -a$, we formally conjugate $R(z)$ with $Op(A_C)$. This yields :

$$\begin{aligned} Op(A_C)R(z)Op(A_C)^{-1} &= (Id - Op(A_C)e^{-T(V+z)}Op(A_C)^{-1})^{-1} \\ &\circ \underbrace{Op(A_C)\left(\int_0^T e^{-t(V+z)}dt\right)Op(A_C)^{-1}}_{\text{holomorphic}} \end{aligned}$$

where the second term on the r.h.s is bounded and holomorphic in z on the whole complex plane. The key observation is a decomposition $Op(A_C)e^{-T(V+z)}Op(A_C)^{-1} = \mathcal{E}(z) + K(z)$ where $\mathcal{E}(z) = \mathcal{O}(e^{-(C+Re(z))T})$ for C arbitrarily large depending on the order function $m \in C^\infty(S^*M)$ hence $\mathcal{E}$ has small operator norm and $K(z)$ is a **compact operator**. If the order function $m \in C^\infty(S^*M)$ is chosen so that C is much larger than a, then the family of bounded operators $(Id - K(z) - \mathcal{E}(z))$ is Fredholm analytic on the half–plane $Re(z) > -a$ and therefore $Op(A_C)R(z)Op(A_C)^{-1}$ is meromorphic in $z \in \{Re(z) > -a\}$ with residues which are finite rank operators. This shows the meromorphic continuation of the resolvent acting on the anisotropic Sobolev space $\mathcal{H}^m(M)$.

5.1.2. *Resonances by zeta function.* But the above construction does not a priori give the location of the spectrum. To put constraints on the spectrum, in the original papers [**12, 14**], we mostly make a local study of the resonant states near the critical points relying on wave front set arguments and Taylor expansions. This shows the resonances of V coincide with σ. We also manage to eliminate Jordan blocks in $\ker(V)$ but we can eliminate Jordan blocks in $\sigma(V)$ only under the $\mathbb{Q}$–independence condition on the Lyapunov exponents of V. In the C^1 linearizable case, Viviane Baladi suggested to us another approach to put constraints on the spectrum that we now sketch. This method relies on the relation between resonances with poles and zeroes of the zeta function [**153**] as we sketched in the toy model of graphs 2. Recall the sets $\Sigma_{u/s} = \cup_{a \in Crit(f)} N^*W^{s/u}(a)$ are the repeller/attractor of the lifted flow in the cotangent. Fix $0 \leqslant k \leqslant d$ and we shall consider all operators as acting on

k-forms. Using the radial estimates of Melrose [**120**], Vasy [**181**] as in [**46**], we can show that for every $\varepsilon > 0$, the shifted resolvent $e^{-\varepsilon V}R(z)$ of the Morse–Smale flow has wave front set $\Gamma \subset T^*(M\times M)$ which does not meet the conormal $N^*(d_2)\subset T^*(M\times M)$ of the diagonal $d_2\subset M\times M$. Then it means that for every Pollicott-Ruelle resonance $\lambda\in\sigma$ of V acting on k-forms, we have the expansion for the resolvent :

$$R(z) = \sum_{1\leqslant \ell\leqslant p} \frac{\Pi_\ell}{(z+\lambda)^\ell} + R_+(z) \tag{24}$$

where $R_+(z)$ is a holomorphic germ near $-\lambda$ whose wave front set is also contained in Γ [5]. Π_1 is a projector of finite rank whose wave front set is contained in $\Sigma_u\times\Sigma_s$ since we know the wave front set of the resonant states. Since this has empty intersection with $N^*(d_2)$, its distributional flat trace $Tr^\flat_{\Lambda^kT^*M}(\Pi_1)$ is well–defined and equals $rk(\Pi_1)$.

By the bound on the wave front of the resolvent, together with the expansion 24, we obtain :

$$Tr^\flat_{\Lambda^kT^*M}\left(e^{-\varepsilon V}R(z)\right) = e^{-\varepsilon\lambda}\frac{rk\left(\Pi_1\right)}{(z+\lambda)} + \underbrace{Tr^\flat_{\Lambda^kT^*M}\left(e^{-\varepsilon V}R_+(z)\right)}_{\text{holomorphic near } -\lambda}.$$

This means that the Pollicott–Ruelle resonances can be identified with the **poles** of the flat trace $Tr^\flat_{\Lambda^kT^*M}\left(e^{-\varepsilon V}R(z)\right)$. So our next goal is to calculate explicitely these poles using the Atiyah–Bott flat trace. The Hille–Yosida Theorem tells us that we have the equality $e^{\varepsilon z}\int_\varepsilon^\infty e^{-tz}e^{-tV}dt = e^{-\varepsilon V}R(z) : \mathcal{H}^m(M)\mapsto\mathcal{H}^m(M)$ as operators acting on suitable anisotropic Sobolev spaces $\mathcal{H}^m(M)$ for $Re(z) > -a, a > 0$. Now an approximation argument similar to the one used in [**46**] yields [6] :

$$\begin{aligned} Tr^\flat_{\Lambda^kT^*M}\left(e^{-\varepsilon V}R(z)\right) &= e^{\varepsilon z}Tr^\flat_{\Lambda^kT^*M}\left(\int_\varepsilon^\infty e^{-tz}e^{-tV}dt\right) \\ = e^{\varepsilon z}\int_\varepsilon^\infty e^{-tz}Tr^\flat_{\Lambda^kT^*M}\left(e^{-tV}\right)dt &= \sum_{a\in Crit(V)} e^{\varepsilon z}\int_\varepsilon^\infty e^{-tz}\frac{Tr(\Lambda^k de^{-tV}|_{T_aM})}{|\det(Id-de^{-tV}|_{T_aM})|}dt, \end{aligned}$$

where we used the Atiyah–Bott flat trace identity [**1**] :

$$Tr^\flat_{\Lambda^kT^*M}\left(e^{-tV}\right) = \sum_{a\in Crit(V)} \frac{Tr(\Lambda^k de^{-tV}|_{T_aM})}{|\det(Id-de^{-tV}|_{T_aM})|}.$$

Now it is possible to calculate exactly the r.h.s using the Lyapunov exponents of the Morse–Smale flow along critical elements of the flow. Assume the critical point a has Morse index $p=\operatorname{ind}(a)$, then the Lyapunov exponents of V at a read $\{\lambda_1(a),\dots,\lambda_p(a),\lambda_{p+1}(a),\dots,\lambda_n(a)\}$ where $\lambda_i(a)<0, \forall i\in\{1,\dots,p\}$ and $\lambda_i(a)>0$ otherwise. Using these notations, we obtain the explicit formula for the flat trace :

$$\begin{aligned} \frac{Tr(\Lambda^k de^{-tV}|_{T_aM})}{|\det(Id-de^{-tV}|_{T_aM})|} &= \frac{\sum_{I\subset\{1,\dots,n\}}\prod_{i\in I}e^{-t\lambda_i(a)}}{\prod_{i=p+1}^n(1-e^{-t\lambda_i(a)})\prod_{i=1}^p(e^{-t\lambda_i(a)}-1)} \\ = \sum_{I\subset\{1,\dots,n\},|I|=k}\ \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n} &\exp\left(-t(\sum_{i\in I}\lambda_i(a)-\sum_{i=1}^p\lambda_i(a)+\sum_{i=1}^n k_i|\lambda_i(a)|)\right) \end{aligned}$$

where the sum runs over the k–elements subsets I of $\{1,\dots,n\}$ and we used the Neumann series decomposition. We recognize the integral $e^{\varepsilon z}\int_\varepsilon^\infty e^{-tz}\frac{Tr(\Lambda^k de^{-tV}|_{T_aM})}{|\det(Id-de^{-tV}|_{T_aM})|}dt$ as some Laplace transform which equals :

$$\sum_{I\subset\{1,\dots,n\},|I|=k}\ \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n} \frac{\exp\left(-\varepsilon(\sum_{i\in I}\lambda_i(a)-\sum_{i=1}^p\lambda_i(a)+\sum_{i=1}^n k_i|\lambda_i(a)|)\right)}{z+(\sum_{i\in I}\lambda_i(a)-\sum_{i=1}^p\lambda_i(a)+\sum_{i=1}^n k_i|\lambda_i(a)|)}.$$

5. This is a consequence of the Cauchy formula and of the completeness of the space $\mathcal{D}'_\Gamma$ of distributions whose wave font set is in Γ [**34**]

6. the approximation argument allows to invert flat traces and integral over t

This concludes that the pole λ must have the form $\sum_{i=1}^{n}\mathbb{N}|\lambda_i(a)|$ hence the set σ is the Ruelle spectrum of V.

Theorem 5.1 is related to Bowen's question since it expresses the Ruelle resonances of the gradient flow e^{-tV} in terms of the Lyapunov exponents of $V=\nabla f$ at the critical points of f. The next section relates Pollicott-Ruelle resonances to Witten Laplacians.

6. Witten Laplacian.

6.1. Problem. We are given a Morse function $f\in C^\infty(M)$ s.t. $V=\nabla f$ is Morse Smale. Recall from subsubsubsection 3.3.3 the definition of the rescaled Witten Laplacian $\Delta_{f,\hbar}=\frac{\hbar}{2}(d_fd_f^*+d_f^*d_f)$. Observe that when $\hbar>0$, this operator is elliptic, self–adjoint with compact resolvent therefore it has discrete spectrum contained on the nonnegative reals $\{0=\lambda_1(\hbar)\leqslant\dots\leqslant\lambda_k(\hbar)\dots\}$.

Moreover, a classical intuition from quantum mechanics tells us that the low energy states of $\Delta_{f,\hbar}$ are concentrated at wells of the potential $|df|^2$ which correspond to $\mathrm{Crit}(f)$.

What is the asymptotic behaviour of $\sigma(\Delta_{f,\hbar})$ when $\hbar\to 0^+$? How do we compare it to $\lim_{t\to+\infty}\varphi^{-t*}u$?

6.2. A fundamental identity. Motivated by question from theoretical physics, in [**58**], the authors study a system that interpolates between the transport by the gradient flow and some stochastic perturbation of it. A similar study was done by Dyatlov–Zworski [**48**] for perturbations of the transport by Anosov flows (see also [**186**] for an interpretation of scattering resonances as viscosity limits and [**42**] for similar investigations in the hypoelliptic case). The authors of [**58**] start from the rescaled Witten Laplacian

$$\Delta_{f,\hbar}=\frac{1}{2}\left(\hbar\Delta+\hbar^{-1}|df|^2+V+V^*\right)\tag{25}$$

then by conjugating with $e^{\frac{f}{\hbar}}$, they observe that one has a new Hamiltonian

$$H=e^{\frac{f}{\hbar}}\Delta_{f,\hbar}e^{-\frac{f}{\hbar}}=V+\hbar\frac{\Delta}{2}\tag{26}$$

so the Witten Laplacian is conjugated to some stochastic perturbation of the transport by the gradient flow and where at the limit $\hbar\to 0$, this yields $H=V$.

Remark 6.1. Let us remark that such observation is also made in the work of Bismut and the book by Helffer [**90**, p. 16–17] : one considers the following Dirichlet action functional [7]

$$S(\varphi)=\int_M\langle\nabla\varphi,\nabla\varphi\rangle_g\,e^{-\frac{f^2}{\hbar}}dv$$

where dv is the Riemannian volume and the action functional is perturbed by some exponential weight term $e^{-\frac{f^2}{\hbar}}$ which acts as a density w.r.t. the Riemannian volume. In the semiclassical limit, this density will localize the action functional near the critical points. This quadratic functional induces a new scalar product on $L^2(M,e^{-\frac{f^2}{\hbar}}dv)$. Start from the usual de Rham differential d and consider its adjoint d_f^* for the new scalar product, then the operator $P=V+\frac{\hbar}{2}\Delta$ is defined as the supercommutator $[d,d_f^*]$. This means that the operator P satisfies the structure of supersymmetric quantum mechanical system described in subsection 3.3.

Let us give a simple calculation to illustrate the phenomenon in the case of the rescaled and shifted Harmonic oscillator : set $A=\frac{x}{\sqrt{2\hbar}}+\sqrt{\frac{\hbar}{2}}\partial_x$ and $A^*=\frac{x}{\sqrt{2\hbar}}-\sqrt{\frac{\hbar}{2}}\partial_x$. We find $AA^*=-\frac{\hbar}{2}\partial_x^2+\frac{x^2}{2\hbar}+\frac{1}{2}$, $A^*A=-\frac{\hbar}{2}\partial_x^2+\frac{x^2}{2\hbar}-\frac{1}{2}$ where the Hamiltonian reads $H=-\frac{\hbar}{2}\partial_x^2+\frac{x^2}{2\hbar}-\frac{1}{2}$. We recall the crucial commutator relations responsible for spacing in the spectrum

7. Called Dirichlet form in Helffer's book

$[H,A^*]=A^*$ and $[H,A]=-A$. If $Hu=\lambda u$ then $HA^*u-A^*Hu=HA^*u-\lambda A^*u=A^*u \implies HA^*u=A^*u+\lambda A^*u$ hence A^*u is an eigenfunction for $\lambda+1$. Similarly, Au is an eigenfunction for $\lambda-1$. Hence eigenvalues differ by 1 after rescaling. To see the path going from the harmonic oscillator to the gradient field :

$$e^{\frac{x^2}{2\hbar}}\left(\frac{x^2}{2\hbar}-\frac{\hbar}{2}\partial_x^2-\frac{1}{2}\right)e^{-\frac{x^2}{2\hbar}}=\left(x\partial_x-\frac{\hbar}{2}\partial_x^2\right)$$

which tends to $x\partial_x$ when $\hbar\to 0$. To summarize, we have the following three operators on $\mathbb{R}$

Witten Laplacian	Stochastic perturbation	Gradient field
$-\frac{\hbar}{2}\partial_x^2+\frac{x^2}{2\hbar}-\frac{1}{2}$	$\left(x\partial_x-\frac{\hbar}{2}\partial_x^2\right)$	$x\partial_x$

The motivation of **[58]** was to exhibit a phenomenon of localization on instanton moduli space (the space of gradient flowlines), Morse theory is the typical example of such theory. Let us try to explain the heuristics. The propagator of the Witten Laplacian is represented like a path integral where one integrates over all paths.

$$\underbrace{K_t(x,y)=\int_{\gamma(0)=x,\gamma(t)=y}[\mathcal{D}\gamma]\exp\left(\int_0^t\frac{\lambda}{2}\left(|\dot\gamma(s)|^2_{g(\gamma(s))}+|V_f(\gamma(s))|^2_{g(\gamma(s))}\right)ds+\text{ fermions}\right)}_{\text{Schwartz kernel of } e^{-t\Delta_{f,\hbar}}}$$

At the semiclassical limit, it is believed that the integrand under the path integral is some Gaussian shaped differential form on path space localized around instantons and converges to some delta form in the infinite dimensional space of path which is supported on the moduli space of instantons. At an intuitive level, the path integral localizes as an integral over the instanton moduli space which represents the propagator of V.

$$K_t(x,y)=\int_{\gamma(0)=x,\gamma(t)=y}[\prod_{0\leqslant s\leqslant t}\mathcal{D}\gamma(s)]\delta(\dot\gamma(s)-V_f(\gamma(s)))\times\det\dots$$

Let us give a concrete and rigorous simple example :

EXAMPLE 6.2 (Localization). *Consider the vector field $V=\partial_{x^1}$ on $\mathbb{R}^n$ with coordinates $(x^1,\dots,x^n)$. The propagator e^{-tV} has Schwartz kernel $e^{-tV}(x,y)=\delta(x^1-t,\dots,x^n,y^1,\dots,y^n)$ whereas the propagator $e^{-t(V+\hbar\frac{\Delta}{2})}$ has Schwartz kernel $e^{-t(V+\hbar\frac{\Delta}{2})}(x,y)=\frac{e^{-\frac{(x_1-t-y_1)^2+\dots+(x_n-y_n)^2}{2t\hbar}}}{(2\pi t\hbar)^{\frac{n}{2}}}$ which has Gaussian shape around the classical trajectory and converges to the δ distribution when $\hbar\to 0^+$.*

One of the examples of **[58]** is the calculation of the Pollicott–Ruelle spectrum of $V=\nabla f$ for the height function f on the Riemann sphere $\mathbb{CP}^1$ in which case they find that the Ruelle spectrum $\sigma(\nabla f)$ is the integers $\mathbb{N}$. Our result concerns the explicit determination of the spectrum both from the asymptotics of the correlators and from the limit spectrum of the Witten Laplacians when $\hbar\to 0^+$.

6.2.1. *Strategy of the proof.* The key idea is the identity

$$e^{\frac{f}{\hbar}}\Delta_{f,\hbar}e^{-\frac{f}{\hbar}}=V+\frac{\hbar}{2}\Delta \tag{27}$$

relating the Witten Laplacian $\Delta_{f,\hbar}$ and the stochastic perturbation $V+\hbar\frac{\Delta}{2}$. So in practice we study the transport equation perturbed by some **viscosity term**. For $\hbar\geqslant 0$, we start from some initial data $u(0)=\Psi\in C^\infty(\Lambda^\bullet T^*M)$ and we look for a solution on $\mathbb{R}_{\geqslant 0}\times M$ of the equation :

$$\boxed{\partial_t u=-\left(\frac{\hbar\Delta}{2}+V\right)u.}$$

The solution reads $u(t)=e^{-t\left(\frac{\hbar\Delta}{2}+V\right)}\Psi$ where $e^{-t\left(\frac{\hbar\Delta}{2}+V\right)}$ is the semigroup generated by the operator $\frac{\hbar\Delta}{2}+V$. The natural idea is to study $V+\hbar\frac{\Delta}{2}$ by the microlocal methods of Faure–Roy–Sjöstrand used to study V. For every $a\in\mathbb{R}$, we construct the anisotropic

Sobolev spaces $\mathcal{H}^m(M)$ adapted to V using the results of section 5. The operator $(V+z) : \mathcal{D}\subset\mathcal{H}^m \mapsto \mathcal{H}^m$ is Fredholm analytic with resolvent $(V+z)^{-1}$ which is meromorphic on the half–plane $Re(z) > -a$. This anisotropic space $\mathcal{H}^m(M)$ is defined independently of $\hbar$. Then we prove that the spectrum of the elliptic operator $V+\frac{\hbar\Delta}{2}$ on the anisotropic space $\mathcal{H}^m(M)$ coincides with its L^2 spectrum [8] which follows from the ellipticity of $V+\frac{\hbar\Delta}{2}$. Then the resonance spectrum and states of V are obtained as viscosity limits of $\sigma(\frac{\hbar\Delta}{2}+V)$ and eigenstates of $\frac{\hbar\Delta}{2}+V$ acting on $\mathcal{H}^m(M)$. This shows stochastic stability of resonances as in [**48**]. Technically, the convergence follows from the interpretation of the spectrum of both V and $\frac{\hbar\Delta}{2}+V$ as zeroes of some Fredholm determinant which depends continuously on $\hbar$.

Let us state the main Theorem from [**16**]. First, we recall that $\Gamma=\cup_{a\in Crit(f)}N^*W^u(a)$ is a closed conical subset of T^*M and $\sigma=\cup_{a\in Crit(f)}\sum_{j=1}^n\mathbb{N}|\lambda_j(a)|$ where $(\lambda_j(a))_{j=1}^n$ are the Lyapunov exponents of V at the critical point $a\in Crit(f)$.

THEOREM 6.3 (D-Rivière). *For $(f,g)\in C^\infty(M)\times \boldsymbol{Met}(M)$ s.t. $V=\nabla f$ is C^1-linearizable near critical points and V satisfies the Smale transversality condition. Then :*

— *σ is the set of Pollicott–Ruelle resonances of V and is obtained as the limit of the usual discrete spectrum of the Witten Laplacian $\Delta_{f,\hbar}$*

$$\boxed{\lim_{\hbar\to 0}\sigma(\Delta_{f,\hbar})=\sigma.} \tag{28}$$

— $\ker(V)$ *is spanned by **invariant currents** $U_a\in\mathcal{D}'_\Gamma, a\in Crit(f)$ [9] supported by the closure of unstable manifolds $(W^u(a))_{a\in Crit(f)}$,*

— *The family $U_a(\hbar), a\in Crit(f)$ which spans the **low energy** eigenspaces of $\Delta_{f,\hbar}$ is such that $e^{\frac{f-f(a)}{\hbar}}U_a(\hbar)$ is a quasimode of $V+\frac{\hbar}{2}\Delta$ and $e^{\frac{f-f(a)}{\hbar}}U_a(\hbar)\underset{\hbar\to 0}{\to}U_a$ in $\mathcal{D}'_\Gamma$* [10]

6.3. Construction of the quasimodes. Our goal is to outline the construction of quasimodes of $\Delta_{f,\hbar}$ which satisfy exactly the Witten–Helffer–Sjöstrand instanton formula without exponential correction. The construction of low energy states follows a procedure that we call **cut–project–correct–conjugate**. We will make extensive use of the three operators in our problem : the Lie derivative V along the gradient field, the stochastic perturbation $H=V+\frac{\hbar}{2}\Delta$ which is conjugated to the Witten Laplacian $\Delta_{f,\hbar}$.

6.3.1. *Cut.* Near $a\in \mathrm{Crit}(f)$, consider a cut–off function $\chi=1$ near a and χ vanishes near all other critical points. Then consider the "germ of integration current" $[W^u(a)]\chi$ on $W^u(a)$ near a. This is well–defined since $W^u(a)$ is a smooth submanifold near a.

6.3.2. *Project.* Recall that the resonant states of V have their wave front set contained in $\Gamma=\cup_{a\in Crit(f)}N^*(W^u(a))$ and also that the eigenfunctions of $V+\frac{\hbar}{2}\Delta$ converge to resonant states in the space $\mathcal{D}'_\Gamma$. The wave front set of $[W^u(a)]\chi$ is obviously contained in the conormal bundle $N^*W^u(a)$ hence in Γ. Therefore, the current $[W^u(a)]\chi$ belongs to some anisotropic space $\mathcal{H}^m(M)$ for some well-chosen order function $m\in C^\infty(S^*M)$, then we apply the spectral projector Π_0 on $\ker(V)$ and define an element $U_a=\Pi_0([W^u(a)]\chi)$. We repeat the operation for all $a\in\mathrm{Crit}(f)$, the currents $(U_a)_{a\in Crit(f)}$ span $\ker(V)$ in $\mathcal{H}^m(M)$ and are the Laudenbach currents from Theorem 4.10.

6.3.3. *Correct.* We fix $0<\varepsilon<\inf_{\lambda\in\sigma\setminus\{0\}}\lambda$ strictly smaller than the smallest non zero resonance. Then for $\hbar << 1$, the projected quasimode converges in the semiclassical limit :

$$1_{[0,\varepsilon]}(H)(U_a)\underset{\hbar\to 0}{\to}U_a$$

which shows the family $(1_{[0,\varepsilon]}(H)(U_a))_{a\in Crit(f)}$ spans the low energy states of $H=V+\frac{\hbar}{2}\Delta$.

8. Recall that the elliptic operator $V+\frac{\hbar\Delta}{2}$ has compact resolvent
9. The existence of these currents was established by Laudenbach and Harvey–Lawson
10. for the **normal topology** of $\mathcal{D}'_\Gamma$ introduced in [**34**]

6.3.4. *Conjugate.* The Witten Laplacian is related to H by conjugation $\Delta_{f,\hbar} = e^{-\frac{(f-f(a))}{\hbar}} H e^{\frac{(f-f(a))}{\hbar}}$ hence the family

$$U_a(\hbar) = e^{-\frac{(f-f(a))}{\hbar}} 1_{[0,\varepsilon]}(H)(U_a) = 1_{[0,\varepsilon]}(\Delta_{f,\hbar})(e^{-\frac{(f-f(a))}{\hbar}} U_a), a \in Crit(f)$$

spans the low energy states of the Witten Laplacian $\Delta_{f,\hbar}$. We therefore obtained some purely spectral construction of the low energy states of $\Delta_{f,\hbar}$. To summarize all four steps of the construction, we obtain the low energy quasimode of $\Delta_{f,\hbar}$ concentrated at $a \in Crit(f)$ from the "germ of integration current" $[W^u(a)]\chi$ using the formula
(29)

$$\boxed{U_a(\hbar) = 1_{[0,\varepsilon]}(\Delta_{f,\hbar}) \left(e^{-\frac{(f-f(a))}{\hbar}} \Pi_0 \left([W^u(a)]\chi \right) \right) = e^{-\frac{(f-f(a))}{\hbar}} 1_{[0,\varepsilon]}(H) \left(\Pi_0 \left([W^u(a)]\chi \right) \right).}$$

Our construction is also related to the approach of Bismut-Zhang [**14, 13**]. The idea of constructing approximate objects and then correct them by application of the spectral projector goes back to the seminal paper of Witten [**183**]. In the next section, we leave the gradient flows to consider general Morse–Smale flows with might have periodic orbits.

7. Vertical bands for Morse–Smale flows.

General Morse–Smale flows with periodic orbits are not of gradient type therefore Bowen's question makes no sense for these flows. Instead of studying the convergence of $\lim_{t\to+\infty} \varphi^{-t*}u$, we will ask what is the behaviour of the Laplace transformed dynamical correlators $\widehat{C}_{\Psi_1,\Psi_2}(z)$ when X is Morse Smale or Anosov ? By [**56**] in the Anosov case and [**13**] for Morse–Smale flows, it has meromorphic continuation to the whole complex plane then it means these works proved the existence of Pollicott–Ruelle resonances for these flows. But in the same spirit as Bowen's question one may ask : can we express the resonances in terms of the jets of X at critical elements in $NW(\varphi^t)$ in the Morse–Smale case ? We generalize previous works [**12**] by coupling the flow with a flat connection in which case the dynamics acts on sections of some flat bundle E in the more general Morse–Smale case with periodic orbits. We find in the work [**14**] that the Ruelle spectrum has vertical band structure which comes from the Lyapunov exponents of the periodic orbits and the monodromies of some flat connection at the periodic orbits. We next explain how to lift the flow on sections of some bundles.

7.1. Lifting the flows on some flat vector bundle. Let (E, ∇) be some flat vector bundle over M, this is the data of the vector bundle E and of some flat connection ∇. A connection ∇ always reads locally as $d + \omega$ where $\omega \in C^\infty(T^*M \otimes End(E))$ and the flatness of the connection reads

$$d\omega + [\omega \wedge \omega] = 0 \tag{30}$$

which means that ω solves the Maurer–Cartan equation. More geometrically, it is equivalent to the fact that the parallel transport along any path γ with endpoints $(a, b) \in M^2$ depends only on the homotopy class of the path γ among paths with endpoints $(a, b) \in M^2$.

Using any connection ∇, one can lift the dynamics of some flow $\varphi^t = e^{tV}$ acting on functions on M to some dynamics acting on smooth sections of E as follows :

Definition 7.1. For every section $s \in C^\infty(\Lambda^\bullet T^*M \otimes E)$, we define $e^{-tV}s$ as the unique solution of the transport equation

$$\partial_t u + \mathcal{L}_V^\nabla u = 0, u(0, .) = s \tag{31}$$

where $\mathcal{L}_V^\nabla = [d^\nabla, \iota_V]$.

Only in this section, we use the notation $\mathcal{L}_V^\nabla$ to insist on the fact that the Lie derivative is twisted by the connection ∇, the Lie derivative acting on **functions** is still denoted by V. Intuitively, twisting the dynamics with a connection has the effect of *adding a potential term* in the transport equation. In fact this observation is totally rigorous in case the bundle

E is trivial since the connection ∇ always has the form $\nabla = d+\omega$, $\omega \in C^\infty(T^*M\otimes End(E))$ and therefore :

$$\mathcal{L}_V^\nabla u = Vu + \omega(V)u$$

and $\omega(V) \in C^\infty(End(E))$ acts as a potential term in the transport equation.

EXAMPLE 7.2 (Abelian flat connection). *For instance, if $E = M\times\mathbb{C}$, the flat connection just reads $d+\omega$ where ω is a 1-form and the transport equation is easily solved. Then $e^{-tV}s$ would simply read*

$$e^{-tV}s(x) = e^{\int_{-t}^{0}\omega(V)(e^{uV}(x))du}s(e^{-tV}(x)) \tag{32}$$

*The Maurer–Cartan equation in this case just amounts to solve $d\omega + \omega\wedge\omega = d\omega = 0$ which means the 1-form ω is **closed**.*

EXAMPLE 7.3 (On the torus). *We give an example on the torus $\mathbb{T}^d$. Consider a flat connection $d+\omega$ on the trivial bundle $E = \mathbb{C}\times\mathbb{T}^d$ where ω is closed. In fact, it is easy to show there is a gauge transformation which transforms the flat connection in the form $d+\omega = d+\sum_{i=1}^d \alpha_i d\theta^i$ where $\alpha_i \in \mathbb{C}$ and $(d\theta^i)_{i=1}^d$ are the generators of $H^1(M,\mathbb{Z})$. For any vector field $V(\theta)^i\partial_{\theta^i}$, $\mathcal{L}_V^\nabla$ reads $V(\theta)^i\partial_{\theta^i} + V(\theta)^i\alpha_i$ where we see the twisting with the flat connection produced a potential term.*

More geometrically, the value $(e^{-tV}s)(m) \in E_m$ at m is obtained by taking the value of the section $s(e^{-tV}(m)) \in E_{e^{-tV}(m)}$ at $e^{-tV}(m)$ and parallel transport this vector w.r.t. the connection ∇ along the path $s \in [-t,0] \mapsto e^{sX}(m)$ to end up with some vector $(e^{-tV}s)(m) \in E_m$.

The above action extends naturally to E-valued differential forms that we will denote as $C^\infty(\Lambda^\bullet T^*M\otimes E)$.

REMARK 7.4. The notion of twisting also plays a crucial role in QFT when we study fermionic particles interacting with some gauge potential. The twisted connection allows to couple the fermionic field with the gauge potential inside the action functional of the theory. See subsection 1.1 in Chapter 2.

7.2. Pollicott–Ruelle resonances, the bundle case. To define Pollicott–Ruelle resonances in the bundle case, we only need to define a suitable notion of dynamical correlators in some more general setting and we briefly explain that almost nothing should be changed in our existence proof of Pollicott–Ruelle resonances for scalar functions and distributions.

Assume E is Hermitian, then dynamical correlators of two sections (Ψ_1,Ψ_2) of E are defined as

$$C_{\Psi_2,\Psi_1}(t) = \int_M \left\langle \Psi_2, e^{-tV}\Psi_1\right\rangle \tag{33}$$

where we denote by $\left\langle \Psi_2, e^{-tV}\Psi_1\right\rangle$ the fact that we consider the exterior product of the differential form part but we must use the scalar product on the fibers of E to get numbers. As usual the Pollicott–Ruelle resonances are defined as the poles of the Laplace transformed correlators. The existence proof of Ruelle resonances just follows 5.1.1 except that the pseudodifferential and Fourier integral operators act on the bundle E but the argument is similar because the symbol of $\mathcal{L}_V^\nabla$ has scalar principal part which coincides with the scalar case,

$$\sigma\left(\mathcal{L}_V^\nabla\right)(x,\xi) = \xi(V)(x)\otimes Id_{E_x} + \text{lower order terms.} \tag{34}$$

7.3. Vertical bands for Morse–Smale flows. In the more general Morse–Smale case, we found in [**14**] that the Ruelle spectrum has vertical band structure. Instead of giving the proof, we would like to explain the main ideas with the following example.

Example 7.5 (The circle). *Consider the trivial line bundle $\mathbb{C}\times\mathbb{S}^1$ over the circle $\mathbb{S}^1$ of period 2π. Let $V=g(\theta)\partial_\theta$ be a vector field on $\mathbb{S}^1$ without zeroes and $d+\alpha(\theta)d\theta$ a given flat connection. The differential operator $\mathcal{L}_V^\nabla$ reads $g(\theta)(\partial_\theta+\alpha(\theta))$. We try to find conditions on z s.t. the equation $(\mathcal{L}_V^\nabla+z)u=0$ has a non trivial solution $u\in C^\infty(\mathbb{S}^1)$.*

The solution reads $u(\theta)=e^{-\int_0^\theta \alpha(s)+zg^{-1}(s)ds}u(0)$ and the periodicity condition implies that $\int_0^{2\pi}\alpha(s)+zg^{-1}(s)ds=2i\pi k$ for $k\in\mathbb{Z}$ which implies that $z=\frac{2i\pi k-\int_{\mathbb{S}^1}\alpha}{\mathcal{P}}$, $k\in\mathbb{Z}$ where $\mathcal{P}$ is the period of the periodic orbit $\mathbb{S}^1$ and $\int_{\mathbb{S}^1}\alpha$ is the integral of the connection 1-form on the orbit. So the possible solutions of the eigenvalue equation forms a vertical band in $\mathbb{C}$ and the vertical spacing between the resonances is inverse proportional to the period of the orbit. We also see that the monodromy of the flat connection has an effect on the spectrum, it can translate the vertical band.

7.4. Pollicott–Ruelle resonant states meets Epstein–Glaser renormalization. In our work, we follow an observation of Frenkel–Losev–Nekrasov [**58**] that there is a deep analogy between constructing the resonant states of Morse–Smale flows and renormalization in quantum field theory formulated as the problem of extension of currents by Epstein–Glaser (later revisited by Stora, Brunetti–Fredenhagen [**26**], Hollands–Wald [**91, 92**], Nikolov–Todorov–Stora [**128**], see [**7**]).

Let us be precise on the analogy. In perturbative QFT, one is interested in making sense of correlation functions as distributions on configuration space M^n where M is space–time. Consider a 2-point function in ϕ^4 theory for instance,

$$\langle\Omega|T\left(\phi^4(x)\phi^4(y)\right)|\Omega\rangle=C\Delta_F^4(x,y)$$

where $\Delta_F\in\mathcal{D}'(M\times M)$ is the Feynman propagator. Δ_F is a distribution on $M\times M$ which is a fundamental bisolution of the wave equation.

Example 7.6 (On $\mathbb{R}^{1+3}$). *The Feynman propagator $\Delta_F(t-s,x-y)$ on Minkowski space reads $e^{i|t-s|\sqrt{\Delta}}\Delta^{-\frac{1}{2}}(x-y)=C((t-s)^2-\sum_{i=1}^3(x_i-y_i)^2+i0)^{-1}$ where C is some constant.*

By classical wave front set arguments, it is well–known that the product Δ_F^4 is well–defined on $M\times M\setminus d_2$, so the issue is to extend this distribution on the whole manifold $M\times M$. The idea is to use some Euler vector field ρ[11] which is defined near the diagonal $d_2\subset M\times M$. Recall Morse–Bott vector fields are gradients of functions f s.t. the critical set $\{df=0\}$ is a closed submanifold and whose Hessian is nondegenerate in the normal direction to $\{df=0\}$. The Euler vector field ρ is therefore of Morse–Bott type since its critical set is the submanifold $d_2\subset M\times M$, it is nondegenerate in the normal direction of d_2 and the unstable manifold $W^u(d_2)$ of d_2 is some neighborhood $U\subset M\times M$ of d_2. It means that all elements in U are attracted to d_2 by $e^{-t\rho},t\geqslant 0$. Set $\Gamma\subset T^*(M\times M)$ to be the Feynman wave front set : for $Q(\xi)=\xi_0^2-\sum_{i=1}^3\xi_i^2$ the quadratic form of signature $(1,3)$,

$$\Gamma=\{(x,y;\xi,\eta);Q(x-y)=0,\xi=\tau dQ(x),\eta=-\tau dQ(x),\tau\in\mathbb{R}_{>0}\}\cup N^*d_2\subset T^*\left(\mathbb{R}^4\times\mathbb{R}^4\right).$$

We have $\Gamma\cap T^*_{d_2}(M\times M)=N^*(d_2)$ which means the restriction of Γ on the diagonal d_2 is the conormal. We assume the vector field ρ is chosen in such a way that $e^{-t\rho*}\Gamma\subset\Gamma$, for all $t\geqslant 0$. When the Lorentzian spacetime M, $\dim(M)=4$ is globally–hyperbolic, one has an asymptotic expansion in $\mathcal{D}'_\Gamma(U\setminus d_2)$ of the form :

$$e^{-t\rho*}\Delta_F^4\underset{t\to+\infty}{\sim}\sum_{k=-8}^{\infty}e^{-kt}(v_{k,0}+v_{k,1}t+\cdots+v_{k,4}t^4)$$

where $(v_{k,p})\in\mathcal{D}'_\Gamma(U\setminus d_2)$. This is very similar to the resonance expansion for the Morse–Smale gradient flows but it is only localized near the critical element d_2. The existence of the asymptotic expansion is a consequence of the Hadamard parametrix approximation of the Feynman propagator Δ_F. The fact there are polynomials in t on the r.h.s. of the

11. These are non unique

expansion shows there are Jordan blocks in the generalized eigenspaces of ρ, generated by $(v_{k,0})_k$. These blocks come from the log term in the Hadamard parametrix of Δ_F. Then the asymptotic expansion suggests we decompose Δ_F^4 as $\Delta_F^4 = \sum_{k=-8}^{-4} v_{k,0} + R \in \mathcal{D}'_\Gamma(U \setminus d_2)$ where $\sum_{k=-8}^{-4} v_{k,0}$ is the singular part and R is some distribution which behaves nicely with respect to scaling. R is the regular part and has a unique extension $\overline{R}$ in $\mathcal{D}'(U)$ s.t. $e^{-t\rho*}\overline{R} \underset{t\to+\infty}{\rightarrow} 0$. Now each $(v_{k,0})_{k=-8}^{-4}$ solves a transport equation of the type $(\rho - k)^{p+1} v_{k,0} = 0$ in $\mathcal{D}'_\Gamma(U \setminus d_2)$ where p is the dimension of the Jordan block generated by $v_{k,0}$. So one way to phrase the renormalization problem would be : find solutions $\overline{v_{k,0}} \in \mathcal{D}'_\Gamma(U), k \in \{-8, \ldots, -4\}$ of the linear PDE : $(\rho - k)^{p+1}\overline{v_{k,0}} = 0$. Then the extension of Δ_F^4 reads $\sum_{k=-8}^{-4} \overline{v_{k,0}} + \overline{R}$. The existence of a distributional extension is possible by the results in [**7**]. However, one may loose the property of being a solution of $(\rho - k)^{p+1}\overline{v_{k,0}} = 0$, renormalization may increase the rank of the Jordan block and one can only solve $(\rho - k)^N \overline{v_{k,0}} = 0$ for some N large enough.

EXAMPLE 7.7 (On $\mathbb{R}^d$). *The function $(\frac{1}{\|x\|^d})$ obviously solves the linear PDE : $(\rho + d)(\frac{1}{\|x\|^d}) = 0$ on $\mathbb{R}^d \setminus \{0\}$ where $\rho = x^i\partial_{x^i}$ is the radial vector field. The distribution $FP(\frac{1}{\|x\|^d})$ on $\mathbb{R}^d$ solves the equation*

$$(\rho + d)FP(\frac{1}{\|x\|^d}) = C\delta_{\{0\}}(x) \implies (\rho + d)^2 FP(\frac{1}{\|x\|^d}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

where $C > 0$ and there is no extension of $(\frac{1}{\|x\|^d})$ which solves the initial PDE. So the renormalization process has created a Jordan block which is referred in [**58**] *as logarithmic mixing.*

In our work [**12, 14**] and in [**58**], we solve equations of the form $Vu = \lambda u$ near some critical element $\Lambda \in NW(\varphi^t)$, where u must be supported by $\overline{W^u(\Lambda)}$. We can construct the current u first as a germ at Λ using some linearized chart near Λ and then propagate the germ by the flow using $e^{-t(V-\lambda)}u = u$ to get an element u in $\mathcal{D}'(M \setminus \partial W^u(\Lambda))$. So to define a global object, we need to extend the current u to $\mathcal{D}'(M)$ which still solves $Vu = \lambda u$. In concrete examples, u has moderate growth near the boundary $\partial W^u(\Lambda)$ of the unstable manifold $W^u(\Lambda)$ and can be extended as a current $\overline{u}$ by the renormalization à la Epstein–Glaser. However in our case, we use the spectral projector to extend the current. We pick some cut–off function $\chi = 1$ near Λ and χ vanishes near all other critical elements. Then we set $\overline{u} = \Pi_\lambda(\chi u)$, all the difficulties are hidden in the construction of the anisotropic spaces and of the spectral theory for V. Still, the renormalization viewpoint might help to understand the Jordan block structure in the resonant states, this phenomenon was called logarithmic mixing by [**58**].

Up to now, our discussion only dealt with spectras. We next give topological applications of these spectral techniques.

8. Morse inequalities for Anosov and Morse–Smale flows.

We recall the notion of twisted cohomology since the Morse inequalities are expressed in these terms.

8.0.1. *Twisted de Rham complex.* Using the flat connection ∇, we may defined a differential $d^\nabla : C^\infty(\Lambda^\bullet M \otimes E) \mapsto C^\infty(\Lambda^{\bullet+1} M \otimes E)$ where the condition $d^\nabla \circ d^\nabla = 0$ comes from the flatness of ∇. This defined a complex of E–valued differential forms $C^\infty(\Lambda^\bullet M \otimes E), d^\nabla$ whose cohomology $H^\bullet(M, E)$ is called the twisted de Rham cohomology.

8.0.2. *Morse inequalities for twisted cohomology.* We then state the Morse inequalities for a Morse–Smale or Anosov flow e^{-tV} acting on sections of some flat vector bundle $(E, \nabla) \to M$ in terms of Pollicott–Ruelle resonances.

THEOREM 8.1. *Let M be a closed compact manifold, $(E, \nabla) \mapsto M$ some flat vector bundle over M and $V \in C^\infty(TM)$ a vector field on M which is Anosov or Morse–Smale. Set $C^k(M)$ to be the vector space generated by the degree k resonant states of $\mathcal{L}_V^\nabla$ for the* *resonance* 0 *and* $c_k = \dim\left(C^k(M)\right)$, $b_k = \dim(H^k(M,E))$ *are the Betti numbers of the twisted cohomology. Then :*

— *the complex* $(C^\infty(\Lambda^\bullet T^*M\otimes E), d^\nabla)$ *and* $(C^k(M,E), d^\nabla)$ *are quasi–isomorphic,*

— *we have the Morse inequalities*

$$\forall 0\leqslant p\leqslant n, \sum_{k=0}^{p}(-1)^{n-k}c_k \geqslant \sum_{k=0}^{p}(-1)^{n-k}b_k \tag{35}$$

with equality when $p=n$.

Recall that d^∇ commutes with $\mathcal{L}_V^\nabla=[d^\nabla,\iota_V]$ acting on $C^\infty(\Lambda^\bullet T^*M\otimes E)$ hence with the spectral projector Π_0 on $\ker(\mathcal{L}_V^\nabla)$. This implies that $\ker(\mathcal{L}_V^\nabla)$ is stable by d^∇ hence $\left(\ker(\mathcal{L}_V^\nabla), d^\nabla\right)$ forms a cochain complex. Once we show that the operator V has good spectral properties, the proof of the Morse inequalities follows immediately from the quasi–isomorphism whose proof is sketched in subsection 3.4. In the Anosov case, the result of Theorem 8.1 is new and in the Morse–Smale case, it is related to the Morse inequalities established by Smale [**172**]. What is surprising about the above Theorem is that the Betti number $b_k(M)$ gives some abstract topological constraints that force the existence of currents U of degree k, which are killed by the Lie derivative $\mathcal{L}_V^\nabla$ and whose wave front set is in E_u^*. The author has no idea how these currents look like, we just know they exist, we know their wave front set and their support which is the whole manifold M.

8.1. The instanton complex. In the particular case of a Morse–Smale gradient V, we saw that $Im\,(\Pi_0)=\ker(V)$ is a complex quasi–isomorphic to the de Rham complex $(C^\infty(\Lambda^\bullet T^*M), d^\nabla)$. But our proof is abstract and used only spectral arguments. It gives no geometric interpretation on the complex $(\ker(V), d)$. A natural question one could ask is what is the complex $\left(\ker(V), d^\nabla\right)$ counting ?

We show in our microlocal context that we can recover a result of Laudenbach who was able to determine precisely the boundary of the unstable currents $(U_a)_{a\in\mathrm{Crit(f)}}$.

Theorem 8.2 (Laudenbach). *The chain complex* $(\ker(V), d)$ *generated over* $\mathbb{Z}$ *by* $(U_a)_{a\in Crit(f)}$ *has differential which satisfies the equation :*

$$\boxed{dU_a=\sum_{b\in Crit(f);ind(b)=ind(a)-1} n_{ab}U_b} \tag{36}$$

where $n_{ab}\in\mathbb{Z}$ *counts the number of instantons connecting* (a,b).

Let us sketch the proof of the exact instanton formula. We would like to compute dU_a. Because we know that the finite dimensional space $\ker(V)=Span(U_a)_{a\in Crit(f)}$ forms a cochain complex, we know already dU_a expresses as a linear combination $\sum_{b\in\mathrm{Crit}(f);\mathrm{ind}(b)=\mathrm{ind}(a)-1} c_{ab}U_b$ where a priori $c_{ab}\in\mathbb{R}$ and we know the sum runs over critical points b s.t. $\mathrm{ind}(b)=\mathrm{ind}(a)-1$ since U_a is a current of degree $k=n-\mathrm{ind}(a)$, dU_a has degree $k+1$ which implies that the currents U_b must have degree $k+1$ hence the Morse index of b is fixed.

Consider the dual basis $(S_b)_b$ of coresonant states, then observe that $c_{ab}=\langle dU_a, S_b\rangle$. To prove that c_{ab} is an integer, we just consider a family of currents $e^{-tV}(U_a\chi)$ approximating U_a in the anisotropic space $\mathcal{H}^m(M)$ for $t>0$ large, $\chi=1$ near a and vanishes near other critical points . Then we prove that $\left\langle de^{-tV}(U_a\chi), S_b\right\rangle=(-1)^{\deg(U_a)}\left\langle e^{-tV}(U_a d\chi), S_b\right\rangle$ converges to c_{ab} for large t and also that c_{ab} is an integer counting the number of instantons between the critical points a and b.

8.2. The exact instanton formula for Witten quasimodes. Let us now show how to obtain the exact instanton formula from the quasimodes of the Witten Laplacian constructed in subsubsection 6.3. Recall the quasimodes in $1_{[0,\varepsilon]}(\Delta_{f,\hbar})$ are the smooth forms $(U_a(\hbar))_{a\in Crit(f)}$. They span the image of $1_{[0,\varepsilon]}(\Delta_{f,\hbar})$ and are not necessarily eigenstates for the eigenvalue 0 since for small $\hbar$, there are small eigenvalues that approach 0. The twisted de Rham differential reads $d_{f,\hbar}=e^{-\frac{f}{\hbar}}de^{\frac{f}{\hbar}}$. Since it commutes with the Witten Laplacian $\Delta_{f,\hbar}$, it commutes with all functions of $\Delta_{f,\hbar}$ obtained by the spectral functional calculus.

Therefore $d_{f,\hbar}$ commutes with the spectral projector $1_{[0,\varepsilon]}(\Delta_{f,\hbar})$ for all $\epsilon > 0$. For all $\varepsilon > 0$, up to choosing $\hbar_0$ small enough, $\varepsilon \notin \sigma(\Delta_{f,\hbar})$ for all $\hbar \in [0, \hbar_0]$. So this means that the quasimodes $(U_a(\hbar))_{a\in Crit(f)}$ in $1_{[0,\varepsilon]}(\Delta_{f,\hbar})$ generate a cochain complex for the twisted differential $d_{f,\hbar}$.

THEOREM 8.3 (Witten, Helffer–Sjöstrand, D–Rivière). *Under the notations as above, consider the low energy cochain complex $(U_a(\hbar), d_{f,\hbar})$. Then the quasimodes satisfy the* ***exact tunneling*** *formula*

$$\boxed{d_{f,\hbar}U_a(\hbar) = \sum_{b\in Crit(f)} \underbrace{e^{\frac{f(a)-f(b)}{\hbar}}}_{Quantum\ correction} n_{ab}U_b(\hbar).} \tag{37}$$

The integer n_{ab} appearing on the r.h.s. is the same as the integer n_{ab} of Theorem 8.2. Let us sketch the proof whose principle is extremely simple. The relation $d_{f,\hbar} = e^{-\frac{f}{\hbar}}de^{\frac{f}{\hbar}}$ yields :

$$d_{f,\hbar}U_a(\hbar) = d_{f,\hbar}\underbrace{1_{[0,\varepsilon]}(\Delta_{f,\hbar})(e^{-\frac{(f-f(a))}{\hbar}}U_a)}_{\text{by definition of } U_a(\hbar)} = \underbrace{1_{[0,\varepsilon]}(\Delta_{f,\hbar})(e^{-\frac{(f-f(a))}{\hbar}}dU_a)}_{\text{commmute } d_{f,\hbar}}$$

$$= \underbrace{\sum_b n_{ab}1_{[0,\varepsilon]}(\Delta_{f,\hbar})(e^{-\frac{(f-f(a))}{\hbar}}U_b)}_{\text{by Thm 8.2}} = \sum_b n_{ab}e^{\frac{f(a)-f(b)}{\hbar}}\underbrace{1_{[0,\varepsilon]}(\Delta_{f,\hbar})(e^{-\frac{(f-f(b))}{\hbar}}U_b)}_{=U_b(\hbar)}$$

where $f(a) - f(b) < 0$ in above sum. We would like to insist on the fact that the tunneling formula is really due to Witten, Helffer–Sjöstrand but it holds true only up to some exponential factor. The only new feature of our work [**16**] is the exact formula without exponential error.

There is a number of results from [**16**] we did not cover in the present chapter. The most important is the proof of Fukaya's conjecture using our methods and we refer the reader to the original paper for details.

9. Perspectives.

The notion of transfer operator comes from statistical physics where it appears in the analysis of spin chains. It was also motivated by these deep analogies that mathematical physicists, such as Sinai, Ruelle and Bowen [**20**] among many others, developped the thermodynamic formalism for hyperbolic dynamics [**132, 20**]. There is a dictionary which relates the analysis of 1d spin chains and hyperbolic dynamics, the relation is via Markov partitions.

In statistical physics, the deep variational principle states that Gibbs measures of some spin system maximize the free energy. The (potential free) dynamical analogue of Gibbs measures are the Bowen–Margulis measures of maximal entropy. Is there a microlocal interpretation of these measures and of the variational principle ? Another naive question would be how to estimate precisely the Hölder regularity of the resonant states for Anosov and Axiom A flows and diffeomorphisms using microlocal methods. Can it help in understanding the nature of the measures of maximal entropy ?

Conversely, can the semiclassical methods be used to revisit the study of higher dimensional spin chains, in particular lattice gauge theories that admit a transfer operator formulation ? This would be in the spirit of the beautiful book of Helffer [**90**]. In fact, Nelson axiomatized the properties of quantum systems that admit a transfer operator description. This is related to reflection positivity and covers many examples of interest [**127**], among them, lattice gauge theories. Still inspired by quantum field theories, it could be interesting in the future to investigate the simplest examples of dynamical systems from [**59**] which are infinite dimensional in nature. For instance, study the space of maps from the cylinder [12] $\mathbb{R} \times \mathbb{S}^1$ to the torus $\mathbb{T}^{2n}$, viewed as a Kähler manifold, as some gradient flow

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on loop space for the Floer functional [**59**, p. 14–15]. Can we rigorously define a notion of Pollicott–Ruelle spectrum ?

Finally, something quite striking in the works of Faure–Sjöstrand, Dyatlov–Zworski in hyperbolic dynamics and works of Vasy, Wrochna, Gérard on wave equations is that hyperbolic problems are treated using Fredholm techniques. This is possible in both cases because there is some sort of scattering. A natural question one could ask, at the interface of analysis and geometry, is can one find some new index formulas using these machineries ? For us, an index formula should be understood in the broad sense as giving a geometric interpretation of some formula of spectral nature : Fredholm index, winding numbers of some eigenvalues, counting some special resonances. These index formulas would be hyperbolic counterparts of the Atiyah–Singer index formula in the elliptic case.

Chapitre 2

A conjectural picture of Quillen on determinant lines.

The goal of the present chapter is to present our work [**9**] which aims at understanding a short three page note of Quillen [**148**] where he asks questions about the meaning of renormalization for *chiral fermions interacting with external gauge potential.* We will explain the meaning of these words in the next paragraph. Our exposition is strongly influenced by surveys by Perrot [**139, 140, 141**] and Singer [**170**]. Since the results in the original paper [**9**] are rather technical, we would like to present a more leisurely exposition with some emphasis on the physical motivations from Quantum Field Theory (QFT). Then, we shall present a new point of view on the main results which is part of some joint work in progress with C. Brouder and B. Zhang.

1. The motivations and geometrical set–up.

Consider a quantum system which contains an infinite number of fermionic particles, for instance a gas of electrons, interacting with external currents or charges which produce some electromagnetic field which is supposed to have *slow variation.* One can imagine that the gauge potential is generated by some nuclei or magnets...Slow–fast regime is very useful in QFT applied to material science, condensed matter theory, solid state physics, quantum chemistry but also in the study of chiral anomalies as discussed in [**148, 67, 66, 65**]. We would like to acknowledge Jan Derezinski for insisting that the renormalization of such systems should be investigated [**37**].

The mathematical formalism which describes the interaction is linear response theory [**65**, p. 67]. Under some variation of the external potential, the quantum system is supposed to respond by producing some quantum current which can be measured [**65**, p. 45, 64]. In the quantum formalism, one has an effective action S_{eff} defined as the log of some renormalized determinant $\det_{ren}(D_{+A})$ of a chiral Dirac operator D_{+A} coupled to some external gauge potential s.t. $S_{eff}(A) = \log\det_{ren}(D_{+A})$. S_{eff} is a **functional** of the external potential A. Since we deal with a quantum system, the presence of currents can be accessed from the **functional derivatives** of $S_{eff}(A) = \log\det_{ren}(D_{+A})$ in A which give **currents correlators**. Our goal in the present chapter is to sketch a construction of the renormalized determinants based on elementary microlocal analysis and following some insights of Quillen [**148**], Perrot [**139**].

Let us describe the ingredients which appear in such study. The fermionic particles we would like to describe have some internal degree of freedom, namely they have spin. At the classical level, the fermion fields are sections of some spin bundle, more concretely the reader may think about vector valued functions. The Dirac operator describes the classical equation of motion and action functional satisfied by the fermionic field. Then in paragraph 1.2, we shall explain the quantization of the Dirac operator which is the functional determinant $\det_{ren}$ and why it requires **renormalization** to remove infinities.

1.0.1. *The Dirac operator.* Looking for a relativistic quantum theory of the electron, Dirac tried to find some first order operator D that squares to the wave operator $D^2 = \Box$. In the present manuscript, since we are motivated by Euclidean QFT, we rather consider Dirac operators that square to the nonnegative Laplacian $D^2 = \Delta$. Dirac soon realized that such operator should be matrix valued. First, let us describe the modern point of view on Dirac operators essentially following [**107**]. Given the vector space $\mathbb{R}^{2n}$ endowed with its usual Euclidean metric, we look for an algebra $Cl(\mathbb{R}^{2n})$ which is generated by abstract elements $\gamma^\mu, \mu = (1, \dots, 2n)$ satisfying the commutation relation $\gamma^i\gamma^j + \gamma^j\gamma^i = -\delta_{ij}$. In practice, we deal with **representations** of the Clifford algebra hence we look for matrices $\gamma^\mu, \mu=(1,\dots,2n)$ that satisfy the above commutation relations. Therefore, the first order operator $D=\sum_{\mu=1}^{2n}\gamma^\mu\partial_{x^\mu}$ squares to the non negative Laplace operator acting on vector valued functions. We give the fundamental examples following [**107**, p. 119-120].

EXAMPLE 1.1 (Dimension 2 and 4). *Let us give the fundamental examples in dimension 2 and 4. First we may identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}\simeq\mathbb{R}^2$. The Dirac operator is a 2×2 matrix, $D=\begin{pmatrix}0 & -\frac{\partial}{\partial z}\\ \frac{\partial}{\partial\bar z} & 0\end{pmatrix}$ and is expressed in terms of the Cauchy–Riemann operator $\bar\partial$. Identifying $\mathbb{R}^4$ with the quaternion plane $\mathbb{H}^4$, the Dirac operator can also be defined in terms of Cauchy–Riemann operators associated to the quaternions. It reads $D=\begin{pmatrix}0 & -\frac{\partial}{\partial q}\\ \frac{\partial}{\partial\bar q} & 0\end{pmatrix}$, for the Cauchy Riemann operator : $\frac{\partial}{\partial\bar q}=\frac{\partial}{\partial x_0}+\sigma_1\frac{\partial}{\partial x_1}+\sigma_2\frac{\partial}{\partial x_2}+\sigma_3\frac{\partial}{\partial x_3}$ where $(\sigma_i)_i$ are the 2×2 Pauli matrices which represent the action of quaternions on $\mathbb{C}^2$.*

REMARK 1.2. There is yet another representation of the Clifford algebra hence of the Dirac operator on $\mathbb{R}^4$ [**67**, p. 20,50]. We set the following four antihermitian matrices $\gamma^\mu, \mu=1,\dots,4$ which generate a representation of the Clifford algebra $Cl(\mathbb{R}^4)$:

$$\gamma^1 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\ \gamma^2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},\ \gamma^4=\begin{pmatrix}-i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&i\end{pmatrix}$$

In this representation, the chirality operator γ [1] reads

$$\gamma=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \tag{38}$$

1.1. Spinors, Yang–Mills theories and chiral fermion coupled to gauge potentials. In this paragraph, we mostly follow [**139, 170**]. Let M be a compact even dimensional spin manifold. Concretely, the reader can take $M=\mathbb{T}^4,\mathbb{S}^4$. The spinor bundle $S(M)$ splits as a direct sum $S=S_+\oplus S_-$ [2] and there is a chirality operator denoted by $\gamma\in End(S)$ such that $\gamma S_\pm=\pm S_\pm$. Each bundle is made of eigensections for the chirality operator γ. The full Dirac operator $D: C^\infty(S)\mapsto C^\infty(S)$ reads $D=\begin{pmatrix}0 & D_+\\ D_- & 0\end{pmatrix}=\sum_{\mu=1}^4\gamma^\mu\nabla_\mu$ in the decomposition $S=S_+\oplus S_-$ where $D_\pm: C^\infty(S_\pm)\mapsto C^\infty(S_\mp)$ where ∇_μ is the Levi–Civita connection naturally lifted to the spinor bundle $S\mapsto M$. Let $E=\mathbb{C}^N\times M$ be some trivial auxiliary bundle over M and a connection ∇ on E can be identified with some matrix valued 1-form $A\in C^\infty(T^*M\otimes End(E))=C^\infty(T^*M)\otimes_{C^\infty(M)}M_N(\mathbb{C})$ called **gauge potential** in the physics litterature [3]. The coupling of the Dirac operator with the gauge potential A defines a twisted Dirac operator [**67**, p. 50, 74]

$$D_A=\gamma^\mu\left(\nabla_\mu\otimes Id+Id\otimes A_\mu\right)=\begin{pmatrix}0 & D_{+A}\\ D_{-A} & 0\end{pmatrix} \tag{39}$$

where $D_{\pm A}: C^\infty(S_\pm\otimes E)\mapsto C^\infty(S_\mp\otimes E)$ and D_{+A} is called **chiral Dirac operator**.

1. the celebrated $\gamma_5=\gamma_1\dots\gamma_4$ from physics textbooks
2. right and left handed spinors
3. In electromagnetism, E is a complex line bundle

Remark 1.3. Let us make the following remark about an abuse of notation. If we vary the gauge potential A, then the two twisted Dirac operators D_A and D_{A_0} differ by a term of order 0 which reads

$$\left(\gamma^\mu \left(Id \otimes A_\mu - Id \otimes A_{0\mu}\right)\right).$$

Using the decomposition $S \otimes E = S_+ \otimes E \oplus S_- \otimes E$ given by the chirality operator γ, so $\left(\gamma^\mu \left(Id \otimes A_\mu - Id \otimes A_{0\mu}\right)\right) \in End(S \otimes E)$ can be decomposed as a 2×2 block part and in reality

$$D_{+A} - D_{+A_0} = \left(\gamma^\mu \left(Id \otimes A_\mu - Id \otimes A_{0\mu}\right)\right)_{-+} \in Hom(S_- \otimes E, S_+ \otimes E).$$

But in the sequel, for simplicity, we will make the notational abuse $D_{+A} - D_{+A_0} = A - A_0$ even though the reader has to think it is the complicated block part we just describe above.

1.2. Chiral gauge theories and chiral fermions. In the convention of [**67**, p. 20, p. 57], the Dirac conjugate variable $\tilde{\Psi}$ reads $\tilde{\Psi} = -i\overline{\Psi}^t\gamma^4$. The action now reads [**67**, p. 65]

$$S[\Psi, \tilde{\Psi}, A] = i \int_M \tilde{\Psi}(x) D_A \Psi(x) d^4x \tag{40}$$

where $\tilde{\Psi} = -i\overline{\Psi}^t\gamma^4$. This means $i\int_M \tilde{\Psi}(x) D_A \Psi(x) d^4x = \int_M \overline{\Psi}^t\gamma^4\gamma^\mu \left(\nabla_\mu + A_\mu\right)\Psi$.

Now the physicists want to compute the functional integral

$$\int D\Psi D\tilde{\Psi} e^{iS[\Psi,\tilde{\Psi},A]}$$

where the fields $\Psi \in C^\infty(M, \mathbb{C}^4), \tilde{\Psi} \in C^\infty(M, \mathbb{C}^4)$ are considered as independent variables in the formal functional integration. This is similar as considering $(z, \overline{z})$ as independent coordinate functions on $\mathbb{R}^2$ which is identified with $\mathbb{C}$. $D\Psi D\tilde{\Psi}$ is a **formal infinite dimensional analogue** of the Berezin measure, since the $\Psi, \tilde{\Psi}$ fields are fermions. Since we deal with massless fermions, we can split the fermionic action in two parts $S[\Psi, \tilde{\Psi}, A] = S_+[\Psi_+, \tilde{\Psi}_-, A] + S_-[\Psi_-, \tilde{\Psi}_+, A]$ where

$$S_\pm[\Psi, \tilde{\Psi}, A] = \int_M \tilde{\Psi}_\mp D_{\pm A} \Psi_\pm.$$

The functional integral now reads

$$Z\left(A\right) = \int D\Psi_+ D\tilde{\Psi}_- D\Psi_- D\tilde{\Psi}_+ e^{iS_+[\Psi_+,\tilde{\Psi}_-,A]} e^{iS_-[\Psi_-,\tilde{\Psi}_+,A]}.$$

By formal application of the Fubini Theorem under the functional integral, the partition function **factorizes** as a product of partition functions for chiral fermions :

$$Z\left(A\right) = Z_+\left(A\right) Z_-\left(A\right), \text{ where } Z_\pm(A) = \int D\Psi_\pm D\tilde{\Psi}_\mp e^{iS_\pm[\Psi_\pm,\tilde{\Psi}_\mp,A]}.$$

By analogy with probability theory, the fields $(\Psi_+, \tilde{\Psi}_-)$ and $(\Psi_-, \tilde{\Psi}_+)$ should be viewed as **independent random variables** distributed w.r.t. the "measure" $e^{iS_+[\Psi_+,\tilde{\Psi}_-,A]}$ and $e^{iS_-[\Psi_-,\tilde{\Psi}_+,A]}$ respectively. Indeed, for any pair of observables $F_1(\Psi_+, \tilde{\Psi}_-)$ and $F_2(\Psi_-, \tilde{\Psi}_+)$ [4], we have

$$\begin{aligned}
& \left\langle F_1(\Psi_+, \tilde{\Psi}_-) F_2(\Psi_-, \tilde{\Psi}_+) \right\rangle \\
= \ & \frac{1}{Z_+} \int D\Psi_+ D\tilde{\Psi}_- e^{iS_+[\Psi_+,\tilde{\Psi}_-,A]} F_1(\Psi_+, \tilde{\Psi}_-) \times \frac{1}{Z_-} \int D\Psi_- D\tilde{\Psi}_+ e^{iS_-[\Psi_-,\tilde{\Psi}_+,A]} F_2(\Psi_-, \tilde{\Psi}_+) \\
= \ & \left\langle F_1(\Psi_+, \tilde{\Psi}_-) \right\rangle_{(\Psi_+,\tilde{\Psi}_-)} \left\langle F_2(\Psi_-, \tilde{\Psi}_+) \right\rangle_{(\Psi_-,\tilde{\Psi}_+)}
\end{aligned}$$

So this decoupling of the full quantum theory in the two *chiral sectors* gives the chiral gauge theory which is concerned only with the fields $\left(\Psi_+, \tilde{\Psi}_-\right)$ called *chiral fermions.*

4. Mathematically just some polynomial functionals of $(\Psi_+, \tilde{\Psi}_-)$ and $(\Psi_-, \tilde{\Psi}_+)$

1.3. The conjectural picture of Quillen. The motivation from the note of Quillen is to make sense of the *half functional integral* [**148**, p. 282]

$$A \mapsto Z_+(A) = \int D\Psi_+ D\tilde{\Psi}_- e^{iS_+[\Psi_+,\tilde{\Psi}_-,A]}$$

quantizing the chiral fermions emphasizing the **dependence on the gauge potential** A. Note that the action S_+ is **quadratic** in the variables $\Psi_+, \tilde{\Psi}_-$ and therefore the above looks formally like a Gaussian integral. In finite dimension, a Gaussian integral wrt to the Berezin measure would equate a determinant and we expect that in the infinite dimensional case [**129**, p. 37-38] :

$$Z_+(A) = \int D\Psi_+ D\tilde{\Psi}_- e^{iS_+[\Psi_+,\tilde{\Psi}_-,A]} = \det_{ren}(D_{+A})$$

for a suitable notion $\det_{ren}$ of renormalized determinant, where $S''_+ = D_{+A}$ is the Hessian of the quadratic function S_+ [5]. So the finite partition function is expected to be some analytic function of A.

	Chiral fermions coupled to gauge potential
Bundles	$S_\pm \otimes F$
Fast	$\left(\Psi_+, \tilde{\Psi}_-\right)$ spinor
Slow	(gauge field A, metric g)
Operator	$D_{+A} \underbrace{\gamma^\mu \left(\nabla_\mu \otimes Id + Id \otimes A_\mu\right)}_{\text{twisted Dirac}}$
Action	$S_+(\Psi_+, \tilde{\Psi}_-, A, g) = i\int_M \left\langle \tilde{\Psi}_-, D_{+A}\Psi_+ \right\rangle$
Partition f. integrate fast field	$Z_+\left(A\right) = \int [D\Psi_+ D\tilde{\Psi}_-] e^{S(\Psi_+,\tilde{\Psi}_-,A,g)}$

The main goal of the present chapter is to make sense of such renormalized determinants and study their analytical properties. In general, we encounter two problems :

1.3.1. *Divergences.* Assume we fix an invertible element D_{+A_0} where A_0 is often called background connection, then we would like to define our determinant relative to D_{+A_0}. This is very natural, since as a function of A, we just fix a multiplicative normalization $Z_+(A_0) = 1$. Set the affine space $\mathcal{A} = D_{+A}$ of twisted Dirac operators D_{+A} for all gauge potentials $A \in C^\infty(T^*M \otimes End(E))$. Following Quillen [**148**, p. 282] and Perrot [**139**], a formal computation yields

$$\begin{aligned}
\frac{\det(D_{+A})}{\det(D_{+A_0})} &= \frac{\det(D_{+A_0} + (A - A_0))}{\det(D_{+A_0})} = \det(Id + (A - A_0)D^{-1}_{+A_0}) \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} Tr\left(\left((A - A_0)D^{-1}_{+A_0}\right)^k\right)\right)
\end{aligned}$$

where we formally expanded the ratio of determinants as a power series of the perturbation $A - A_0$ being careless about convergence issues. The problem is that there are a finite number of traces $Tr\left(\left((A - A_0)D^{-1}_{+A_0}\right)^k\right)$ to be renormalized since for small k, the operators $\left((A - A_0)D^{-1}_{+A_0}\right)^k$ are not trace class. So we need to deal with divergences produced by variations of A ?

5. We find the determinant and not the Pfaffian since $\Psi_+, \tilde{\Psi}_-$ are viewed as independent variables as in [**129**, p. 35]

1.3.2. *The chiral Dirac operator acts on different vector spaces.* The chiral Dirac operator D_{+A} sends sections from $C^\infty(S_+\otimes E)$ to sections of $C^\infty(S_-\otimes E)$ which are sections of bundles of different chirality. So even in the finite dimensional analogy, it is similar to making sense of the determinant of an invertible linear map T acting between different vector spaces. Let (E,F) be two finite dimensional vector spaces and $T: E\mapsto F$ an invertible linear map. What meaning can we possibly give to $\det(T)$? How to get numbers ?

For every pair of translation invariant volume forms $vol(E), vol(F)$ in $\Lambda^{top}E^*, \Lambda^{top}F^*$, the ratio $\frac{T_*vol(E)}{vol(F)}$ defines a number, hence the natural object to consider is rather the **determinant line** $\simeq \frac{\Lambda^{top}(E)}{\Lambda^{top}(F)}$ over the element T. In the present situation, we consider familie of such T's parametrized by some affine space $\mathcal{A}$ hence varying the determinant line for each $T\in\mathcal{A}$ yields the determinant line bundle over $\mathcal{A}$. Quillen [**149**] constructed the **holomorphic line bundle** $\mathcal{L}\mapsto\mathcal{A}$ in the Cauchy–Riemann case which was later generalized by Bismut–Freed [**12**] to families of Dirac operators.

Our work aims to deal with both issues. Now let us directly quote Quillen's note [**148**, p. 284] to give a reader a sense of what one should expect from renormalization of determinants applied to QFT.

These considerations lead to the following conjectural picture. Over the space $\mathcal{A}$ of gauge fields there should be a principal bundle for the additive group of polynomial functions of degree $\leqslant d$ where d bounds the trace which have to be regularized. The idea is that near each $A\in\mathcal{A}$ we should have a well–defined trivialization of $\mathcal{L}$ up to $\exp$ of such a polynomial. Moreover, we should have a flat connection on this bundle.

1.4. Recollection on the usual determinant.

To motivate our definitions of renormalized determinants, let us recall some important properties of the traditional determinant which will be relevant for our discussion.

PROPOSITION 1.4. *The determinant* $\det: M_n(\mathbb{C})\mapsto\mathbb{C}$ *is the unique entire*[6] *function on the complex vector space* $M_n(\mathbb{C})$ *s.t.*

(1) $\det(Id)=1$,

(2) $\det$ *vanishes over non invertible elements, the order of vanishing at a non invertible* $A\in M_n(\mathbb{C})$ *equals* $\dim(\ker(A))$,

(3) $\det$ *has polynomial growth along complex rays* $z\in\mathbb{C}\mapsto\det(A+zB)$.

It satisfies the following equation, for any branch $\log$ *of the logarithm, we have*

$$d\log\det(A)=Tr(A^{-1}dA) \tag{41}$$

where $A\in GL_n(\mathbb{C})$ *and the equality holds in the sense of* 1*–forms on* $GL_n(\mathbb{C})$.

The above results are well known facts about determinant. The only unusual result is the uniqueness claim. Assume someone else constructed some function f which satisfies the same three properties as det so in particular it has the same zeroes with the same multiplicities. On every complex ray $z\mapsto Id+zH$ emanating from the identity matrix Id, the entire function $z\in\mathbb{C}\mapsto\frac{\det}{f}(Id+zH)$ never vanishes and has polynomial growth, it must be constant by Hadamard factorization Theorem. But $f(Id)=\det(Id)=1$ therefore $\det=f$ along all such complex rays hence $\det=f$ everywhere on $GL_n(\mathbb{C})$.

In particular the determinant can be *reconstructed by path integration* as follows, for every $A\in GL_n(\mathbb{C})$, we have the **Abel–Liouville–Jacobi–Ostrogradskii** formula [**35**, p. 15865] :

$$\det(A)=\exp\left(\int_\gamma Tr(A^{-1}dA)\right) \tag{42}$$

6. This means holomorphic on $M_n(\mathbb{C})$

where we integrate along some path γ connecting Id and A in $GL_n(\mathbb{C})$ which is always possible since $GL_n(\mathbb{C})$ is connected.

Replacing the ring $M_n(\mathbb{C})$ by $Id +$ Trace class and Tr with the functional trace Tr_{L^2} defines the **Fredholm determinant** $\det_F$ [**121**] [7].

2. Renormalized determinants.

Following a suggestion from Singer [**170**] one may use the famous zeta regularization pioneered by Ray–Singer [**166**] and Hawking [**88**]. Assume we work on the sphere $M = \mathbb{S}^d$ for even d with metric of positive curvature. Fix a base connection A_0 s.t. D_{+A_0} is invertible which means $\ker(D_{+A_0}) = \{0\}$ and $\text{Index}(D_{+A_0}) = 0$ [8]. Observe that $D^*_{+A_0}D_{+A} : S_+ \otimes E \mapsto S_+ \otimes E$ is elliptic with Laplace type principal symbol but $D^*_{+A_0}D_{+A}$ is **non selfadjoint**. Then we may define the zeta determinant as

$$\boxed{A \mapsto \det_\zeta\left(D^*_{+A_0}D_{+A}\right) = \exp(-\frac{d}{ds}|_{s=0}\zeta(s)),}$$

where $\zeta(s) = \underbrace{\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(e^{-tD^*_{+A_0}D_{+A}}\right)t^{s-1}dt}_{\textbf{spectral zeta}}$. So for the moment, let us summarize the two approaches to defining renormalized determinants. Both cases involve fixing some base connection A_0 :

(1) Following Singer consider $A \in C^\infty(T^*M \otimes End(E)) \mapsto \det_\zeta(D_{+A}D^*_{+A_0})$, the question is about the regularity in A.
(2) Following Quillen and Perrot, these authors suggest to start formally from the perturbative series

$$A \mapsto \det(I + (A - A_0)D^{-1}_{+A_0}) = \sum \frac{(-1)^{k+1}}{k} Tr\left(((A - A_0)D^{-1}_{+A_0})^k\right)$$

and then try to renormalize the divergent traces on the r.h.s. of the above identity.

If we manage to make sense of the second approach, a natural question is to compare the renormalized determinant obtained with the zeta regularized version. Choose a background gauge potential A_0. We want to define analytic functions of A $A \mapsto \det_{Ren}(D_A)$ in a similar way as formula 42.

2.0.1. *Pseudodifferential calculus and* ***power counting****.* Here we recollect some properties of pseudodifferential operators of order $\leqslant 0$ with polyhomogeneous symbols that will be used in the sequel. An exposition which is very close to the spirit of this chapter can be found in [**49**, Appendix E].

— On M, we have a **filtered algebra** $\cdots \subset \Psi^{-k-1}(M) \subset \Psi^{-k}(M) \subset \cdots \subset \Psi^0(M)$ of operators, $\Psi^{-\infty}$ is the ideal of smoothing operators they map distributions to smooth functions, $\Psi^{<-\dim(M)}$ is the ideal of trace class operators,
— We have the composition Theorem $\Psi^a \circ \Psi^b \subset \Psi^{a+b}$ which is why we speak about filtered algebra,
— Polyhomogeneous symbols of order 0 can be defined as smooth functions in $C^\infty(\overline{T^*M})$ where $\overline{T^*M}$ is the **radial compactification** of T^*M whose boundary is the cosphere at infinity $\partial\overline{T^*M} = S^*M$. There is a linear quantization map which maps symbols to operators $a \in C^\infty(\overline{T^*M}) \mapsto Op(a) \in \Psi^0(M)$, the construction is detailed in any textbook of microlocal analysis [**175**, p. 3] and also in [**138**, Chapter 6]. Then in some sense, one should think of $\Psi^0(T^*M)$ as a quantization of $C^\infty(\overline{T^*M})$ and $\Psi^{-k}(M)$ as a quantization of the **ideal** Ψ^{-k} of functions on $\overline{T^*M}$ vanishing at order k at boundary infinity $\partial\overline{T^*M}$.

Example 2.1. *$\Delta^{-1} \in \Psi^{-2}, D^{-1}_{+A} \in \Psi^{-1}$, multiplication operator by $C^\infty \in \Psi^0$.*

7.

8. Since both conditions imply that $\text{coker}(D_{+A_0})$ and therefore the Fredholm operator D_{+A_0} is surjective from its domain to L^2 sections

2.0.2. *Divergences and renormalized traces.* Inspired by [**148, 139, 121**] assume we have solved the problem of defining some renormalized determinant $\det_{ren}$. Then along some path $(A_t)_{t\in[0,1]} \in \Psi^0(M)$ where all elements along the path are invertible, we formally differentiate $\log\det_{ren}(D_{+A_t})$:

$$\frac{d\log\det_{ren}\left(D_{+A_t}\right)}{dt} = \underbrace{Tr\left(\frac{dA_t}{dt}D_{+A_t}^{-1}\right)}_{\in\Psi^{-1}}$$

But we immediately see that the r.h.s. is ill–defined since $\frac{dA_t}{dt}D_{+A_t}^{-1} \in \Psi^{-1}(M)$ is **not trace class**. Now the idea is to perturb D_{+A}^{-1} around the fixed potential A_0, set $H = D_{+A_0}^{-1}(A-A_0)$ then we have the **decomposition** :

$$D_{+A_t}^{-1} = \underbrace{(Id+H)^{-1}}_{\text{expand in } H}D_{+A_0}^{-1} = \underbrace{(\sum_{k=0}^{p-1}(-1)^kH^k)D_{+A_0}^{-1}}_{=P(A)} + \underbrace{(-1)^pH^p(Id+H)^{-1}D_{+A_0}^{-1}}_{=R(A)}$$

where the **singular part** $P(A) \in \Psi^{-1}(M)$ is a **polynomial** in A and $R(A) \in \Psi^{-p-1}(M)$ is the **regular part** and is trace class when $p = \dim(M)$. Choose $p = \dim(M)$ and **extend continuously** the L^2–trace [**139**, eq (197) p. 32]

$$Tr_{L^2} : \Psi^{<-\dim(M)}(M) \mapsto \mathbb{C}$$

as a **linear map** to handle the singular part $P(A)$. In doing so, we often loose the cyclicity of trace. The extension is denoted by

$$Tr_{ren} : \text{Divergent terms} + \Psi^{<\dim(M)} \subset \Psi^{-1} \longmapsto \mathbb{C}.$$

The extension is always possible in the polyhomogeneous setting as we will see in the next example. These extension procedures are operator theoretic analogues of the Epstein–Glaser technique used to renormalize Feynman amplitudes as we discuss in [**18**] and is related to the works of Brunetti–Fredenhagen [**26**] and Hollands–Wald [**91, 92**].

Example 2.2 (Trace extensions in terms of regularized integrals of the symbol). *In terms of symbols $a \in C^\infty(\overline{T^*M})^{<-\dim(M)}$, we can identify the L^2-trace with integrals of the symbol as follows : $Tr_{L^2}(Op(a)) = \int_{\overline{T^*M}} a(x;\xi)dxd\xi$, where $dxd\xi$ is the Liouville form. Divergences occur because of some finite jets of a at* ***boundary infinity*** *$\partial\overline{T^*M}$. The idea is to* ***extend*** *$\int_{\overline{T^*M}}$ from the closed ideal $C^\infty(\overline{T^*M})^{<-\dim(M)}$ to $\int^{\mathcal{R}}_{\overline{T^*M}}$ as* ***continuous linear map*** *on $C^\infty(\overline{T^*M})$:*

$$Tr_{ren}(Op(a)) = \int^{\mathcal{R}}_{\overline{T^*M}} a(x;\xi)dxd\xi.$$

This is always possible using the Hadamard finite part or Riesz regularization as beautifully explained in Paycha's book [**138**, Chapter 3].

Remark 2.3. In fact, choosing a continuous extension of the L^2 trace is not quite enough. We need to add an extra condition of microlocal nature. We consider the compactified cotangent $\overline{T^*M}$ endowed with the Liouville measure μ which **diverges** near the boundary $\partial\overline{T^*M}$. If ρ is a boundary defining function for $\partial\overline{T^*M}$ and $(y^i)_{i=1,\dots,2n-1}$ are local coordinates on $\partial\overline{T^*M}$, then in local coordinates near the boundary, the Liouville measure has a local expression as $\mu = \rho^{-(n+1)}d\rho dy^1\dots dy^{2n-1}$ + lower order terms where it is polyhomogeneous in ρ. Then the renormalized trace should be interpreted as a distributional extension $\overline{\mu}$ of μ :

$$Tr_{ren}\left(Op(a)\right) = \langle\overline{\mu}, a\rangle_{\overline{T^*M}}$$

where the extension $\overline{\mu} \in C^\infty(\overline{T^*M})'$ has minimal distributional order and the wave front set $WF(\overline{\mu})$ is contained in the conormal $N^*\left(\partial\overline{T^*M}\right)$ which is the minimal possible wave front set. We will later briefly discuss why this wave front condition guarantees that the second derivative of the $\log\det_{ren}$ has wave front set in the conormal of the diagonal.

It is also possible to preserve some form of **covariance** which means the regularized trace depends only on the metric and finite jets of A_0 :

EXAMPLE 2.4. *Recall for f admitting an asymptotic expansion $f(\varepsilon) \sim_{\varepsilon\to 0^+} \sum_{n=0}^{\infty} \varepsilon^{n-k} a_{n-k} + \log(\varepsilon) b$ the finite part is defined as $FP(f)(0) = a_0$ where we only keep the constant term in the asymptotic expansion. So we may define two regularizations of the trace. The heat regularization :*

$$Tr_{ren}(P(A)) = FP|_{\varepsilon=0} Tr_{L^2}\left(e^{-\varepsilon D^*_{+A_0} D_{+A_0}} P(A)\right).$$

The zeta regularization :

$$Tr_{ren}(P(A)) = FP|_{s=0} Tr_{L^2}\left((D^*_{+A_0} D_{+A_0})^{-s} P(A)\right)$$

which is called weighted trace in [**138**, 7.3, def 7.12 p. 136].

2.0.3. *Renormalized determinant.* Once we possess a renormalized trace Tr_{ren}, we can very simply define $\det_{ren}$ using the Abel–Liouville–Jacobi–Ostrogradskii formula as [9] :

DEFINITION 2.5. Choose Tr_{ren}. For every D_{+A}, choose some smooth path $\gamma = (A_t)_{t\in[0,1]}$ connecting 0 and A in the space of potentials s.t. $(D_{+A_t})_{t\in[0,1]}$ is always non invertible, then :

$$\det{}_{ren}(D_{+A}) = \exp\left(\int_0^1 Tr_{ren}\left(\frac{dA_t}{dt} D^{-1}_{+A_t}\right) dt\right). \tag{43}$$

Recall $D^{-1}_{+A_t} = P(A_t) + R(A_t)$ then $\det_{ren}$ factors as a product of some $e^{\text{Polynomial}(A)}$ times the Gohberg–Krein's determinant $\det_{p+1}$:

$$\det{}_{ren}(D_{+A}) = \underbrace{\exp\left(\int_\gamma Tr_{ren}\left(\frac{dA_t}{dt} P(A_t)\right) dt\right)}_{1-\text{loop renorm}} \underbrace{\exp\left(\int_\gamma Tr\left(\frac{dA_t}{dt} R(A_t)\right) dt\right)}_{\det_{p+1}(Id+H)},$$

where we used that $Tr_{ren} = Tr_{L^2}$ on the trace class operators.

We next discuss properties of $\det_{ren}$ related to locality which is a central concept in field theory.

2.0.4. *Locality.* In field theory, we always manipulate functionals which are functions of functions or functions of sections of some bundles over spacetime manifold M. Since functions on M have a notion of support, this notion will have consequences for the functionals we consider. We usually work with polynomial functionals, which means continuous maps $P : A \in C^\infty(T^*M \otimes End(E)) \mapsto P(A) \in \mathbb{C}$ such that $t \in \mathbb{R} \mapsto P(tA) \in \mathbb{C}$ is polynomial of fixed degree $\deg(P) \in \mathbb{N}$ for all $A \in C^\infty(T^*M \otimes End(E))$. We first explain the notion of local functional.

DEFINITION 2.6 (Local polynomial functionals). $P : A \in C^\infty(T^*M \otimes End(E)) \mapsto P(A) \in \mathbb{C}$ is a C^∞, local polynomial functional iff it can be represented as $P(A) = \int_M \Lambda(j^k A(x)) dv$, where dv is a C^∞ density, $\Lambda(j^k A(x))$ is a polynomial in k-jets of A at x for some k. We denote this vector space by $\mathcal{O}_{loc}$ [10].

EXAMPLE 2.7. $V \in C^\infty(\mathbb{R}) \mapsto P(V) = \int_{-1}^1 V^4(x) dx$ *is a local polynomial functional of degree* 4 *in* V.

There is a functional characterization of polynomial local functionals which we shall call the Hammerstein condition. We learned this property working with Brouder, Laurent–Gengoux, Rejzner on [**3**]. Before we discuss this condition, we should explain the notion of differential of order 2 of a smooth functional. To simplify the discussion, just assume that

9. Beware that in the above definition the renormalized determinant is defined up to multiplicative constant since we need to choose an invertible basepoint D_+ and decide that $\det_{ren}(D_+) = 1$.

10. It is not even an algebra, for instance $V \mapsto \int V$ is local but not $V \mapsto \int V \int V$

A is a function on M instead of some complicated section. If F is a smooth functional of $A \in C^\infty(M)$, then it has a Taylor expansion near any element $A_0 \in C^\infty(M)$ which reads

$$F(A) = F(A_0) + DF(A_0; A - A_0) + \frac{1}{2} D^2F(A_0; A - A_0, A - A_0) + \mathcal{O}(\|A - A_0\|^3)$$

where $H \mapsto DF(A; H)$ and $(A_1, A_2) \mapsto D^2F(A_0; A_1, A_2)$ is linear and bilinear continuous for the Fréchet topology of $C^\infty(M)$. Bilinear continuous maps can be represented by a pairing of $A_1 \otimes A_2$ against a distribution in $\mathcal{D}'(M \times M)$ that we will call the Schwartz kernel of the second derivative.

Proposition 2.8 (Hammerstein condition). *For P a polynomial functional, $P \in \mathcal{O}_{loc} \Leftrightarrow D^2P(A; A_1, A_2) = 0$ for all $(A, A_1, A_2) \in C^\infty(T^*M \otimes End(E))$ s.t. A_1, A_2 have **disjoint supports**.*

Equivalently, it means that the Schwartz kernel of the bilinear map $D^2F(A_0; ., .)$ is supported on the diagonal $d_2 \subset M \times M$ and $WF\left(D^2F(A_0; ., .)\right)$ is contained in the conormal bundle $N^*(d_2)$ [11].

The intuition behind the previous result is that in the second differential $D^2P(A; A_1, A_2)$ of the local functional, we have to multiply derivatives of A_1 with derivatives of A_2 which vanishes by the condition on the support. For the converse sense, we refer to **[3]** for a pedagogical exposition.

Example 2.9. *For $V \in C^\infty(\mathbb{R}) \mapsto P(V) = \int_{-1}^{1} V^4(x)dx$, the Schwartz kernel of $D^2P(V; ., .)$ reads $12V^2(x)1_{[-1,1]}(x)\delta(x-y) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$ which is supported by the diagonal $\{x = y\}$ in $\mathbb{R} \times \mathbb{R}$.*

2.1. Subtracting local counterterms in the Lagrangian. In **[9]**, we were trying to find **analytic maps** $A \mapsto \det_{ren}(D_{+A})$ vanishing over $Z = \{A \text{ s.t. } \ker(D_{+A}) \neq 0\}$, of minimal growth when some norm of A goes to $+\infty$, which is obtained by **local renormalization**. Let us briefly explain this concept first with a simple toy example.

Example 2.10. *Assume we work on the torus $\mathbb{T}^2$ with flat metric and we would like to define $\det_F(Id + \Delta^{-1}V)$ for some C^∞ function V. This is not possible since $\Delta^{-1}V$ is not trace class. So we can mollify it by applying the heat operator, $e^{-\varepsilon\Delta}\Delta^{-1}V$ then the regularized determinant $\det_F(Id + e^{-\varepsilon\Delta}\Delta^{-1}V)$ is well–defined provided $\varepsilon > 0$ but we still have a logarithmic divergence :*

$$\log det_F(Id + e^{-\varepsilon\Delta}\Delta^{-1}V) = \pm \frac{\int_{\mathbb{T}^2} V}{4\pi} \log(\varepsilon) + \mathcal{O}(1).$$

To remove the ultraviolet cut–off ε, we must find some way to subtract the divergences by multiplicative renormalization : we must find exponential of some local functional of V, local is understood in the sense of paragraph 2.0.4, such that

$$det_F(Id + e^{-\varepsilon\Delta}\Delta^{-1}V) \times \exp \underbrace{\left(\mp \frac{\int_{\mathbb{T}^2} V}{4\pi} \log(\varepsilon)\right)}_{\text{local functional of } V \text{ depends on } \varepsilon}$$

has a limit when $\varepsilon \to 0^+$.

In our case, regularize the propagator $D_{+A_0}^{-1}$ by the heat operator $e^{-\varepsilon(D_{+A_0}^* D_{+A_0})} \in \Psi^{-\infty}$. The operator $Id + (A - A_0)e^{-\varepsilon(D_{+A_0}^* D_{+A_0})} D_{+A_0}^{-1}$ has the form $Id +$ smoothing and the Fredholm determinant $\det_F \left(Id + (A - A_0)e^{-\varepsilon(D_{+A_0}^* D_{+A_0})} D_{+A_0}^{-1}\right)$ is well–defined. Of

11. In fact, the condition is slightly more technical since we have to formulate everything in terms of the topology of $\mathcal{D}'_{N^*(d_2)}$ and the C^∞ topology for A but we prefer to give the unprecise statement for simplicity

course, the divergences occur when $\varepsilon \to 0^+$ since the operator $(A-A_0)e^{-\varepsilon(D^*_{+A_0}D_{+A_0})}D^{-1}_{+A_0}$ converges to some non trace class operator. The regularized determinant

$$\det_F\left(Id+(A-A_0)e^{-\varepsilon(D^*_{+A_0}D_{+A_0})}D^{-1}_{+A_0}\right)$$

can be renormalized by subtraction of local counterterms if there exists a sequence of **local polynomial functionals** $P_\varepsilon \in \mathcal{O}_{loc}\otimes\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ such that the following limit

$$\det_{ren}(D_{+A}) = \lim_{\varepsilon\to 0^+}\det_F\left(Id+(A-A_0)e^{-\varepsilon(D^*_{+A_0}D_{+A_0})}D^{-1}_{+A_0}\right)\exp\underbrace{(P_\varepsilon(A))}_{\textbf{local counterterm}}$$

exists and defines an analytic functional of A. Our goal is to classify all possible renormalized determinants $\det_{ren}$ that can be obtained by local renormalization as above. So in [**9**], we introduced some set of axioms of functional analytic nature, in the spirit of the three properties of Proposition 1.4, they aim to characterize in a functional analytic way all possible renormalized determinants $\det_{ren}$ that can be obtained by local renormalization. Before we give the axioms, we would like to draw an analogy between our problem of finding "good" renormalized determinants and the problem of finding some entire function with given zeroes whose answer is well–known and given by Hadamard's factorization Theorem.

2.1.1. *Digression on Hadamard's factorization Theorem.*

THEOREM 2.11 (Hadamard's factorization Theorem). *Let $(a_n)_n$ be some sequence s.t. $\sum_n |a_n|^{-(p+1)} < +\infty$ but $\sum_n |a_n|^{-p} = \infty$. Then any entire function with $f(a_n)=0$ and $|f(z)| \leqslant Ce^{K|z|^p}$ has **unique representation** :*

$$\boxed{f(z) = z^m e^{P(z)}\prod_{n=1}^{\infty}E_p\left(\frac{z}{a_n}\right)} \tag{44}$$

where P polynomial of deg p, $E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^p}{p}}$ Weierstrass factor of order p and m vanishing order at 0.

So the problem of finding f with a prescribed divisor is **non unique** when the exponent p is positive, there is a **polynomial ambiguity** P which is very similar to the polynomial ambiguity we will find which was conjectured by Quillen.

2.1.2. *Functional analytic axioms and the main Theorem.* We can now state our axioms for $\det_{ren}$.

DEFINITION 2.12 (Axioms). $\det_{ren}$ is a renormalized determinant on $\mathcal{A}$ if

— **Zeroes of determinants**, $A \in C^\infty(T^*M\otimes End(E)) \mapsto \det_{ren}(D_{+A})$ is an analytic functional of A which vanishes exactly on noninvertible elements D_{+A}.

— **Growth** order $d+1$: $|\det_{ren}(D_{+A})| \leqslant Ce^{K\|A\|^{d+1}_{C^m}}$ for some norm $\|.\|_{C^m}$.

— **Locality** $\det_{ren}$ satisfies the Kontsevich–Vishik [**104**, p. 4] [12] equation :

$$\boxed{D^2\log\det_{ren}(D_{+A};A_1,A_2) = Tr_{L^2}\left(D^{-1}_{+A}A_1D^{-1}_{+A}A_2\right)} \tag{45}$$

if $\mathrm{supp}(A_1)\cap\mathrm{supp}(A_2)$ are **disjoint**.

— **Smoothness of the counterterms**, the Schwartz kernel of the second derivative $D^2\log\det_{ren}(D_{+A};.,.)$ has its wave front set contained in the conormal bundle of the diagonal $d_2\subset M\times M$.

Once we give the axioms for $\det_{ren}$, we can now state our main Theorem that the reader can think of as some infinite dimensional version of the Hadamard factorization Theorem.

12. In the original paper, they attribute this equation to Witten

THEOREM 2.13 (Brouder–D–Zhang). *For every renormalized trace Tr_{ren},*

$$det_{ren}(D_{+A}) = \exp\left(\int_0^1 Tr_{ren}\left(\frac{dA_t}{dt}A_t^{-1}\right)dt\right)$$

satisfies the first three axioms of definition 2.12 and is renormalized by subtraction of local counterterms.

To prove the fourth axiom which states that $WF\left(D^2\log\det_{ren}(D_{+A};.,.)\right)\subset N^*d_2$, we need the supplementary condition from remark 2.3. This will be discussed in the joint work with Brouder–Zhang.

In particular, in the original paper [**9**] we proved :

THEOREM 2.14. *The zeta determinant $det_\zeta\left(D^*_{+A_0}D_{+A}\right)$ satisfies all the axioms of definition 2.12.*

The solutions to the axioms of definition 2.12 are not unique. We also obtain a result which classifies the solutions, this is very closed to what was conjectured by Quillen [**148**].

THEOREM 2.15. *Let $\mathcal{O}_{loc,\leqslant d}$ denotes the local functionals of degree $\leqslant d$. The group $(\mathcal{O}_{loc,\leqslant d},+)$ acts **freely and transitively** on renormalized determinants :*

$$P\in\mathcal{O}_{loc,\leqslant d}\mapsto\exp\left(P(A)\right)det_{ren}\left(D_{+A}\right). \tag{46}$$

There is a Hadamard factorization formula which reads :

$$det_{ren}\left(D_{+A}\right) = e^{P(A)}det_p\left(Id+(A-A_0)D^{-1}_{+A_0}\right) \tag{47}$$

where P is a nonlocal polynomial functional in A.

We shall not give the proof of the full Theorem but we will just discuss an important point concerning the renormalization ambiguities since the solutions to the axioms of definition 2.12 are not unique.

2.2. Locality and renormalization ambiguities.

2.2.1. *Renormalization ambiguities are local.* We want to explain the power of the equation (45) found in Kontsevich–Vishik and why it forces the renormalization ambiguities to be local. Assume $\det_{ren,1}$ and $\det_{ren,2}$ are two renormalized determinants solutions of the axioms from definition 2.12. We compute the second derivative of $\log\left(\frac{\det_{ren,1}(D_{+A})}{\det_{ren,2}(D_{+A})}\right)$.

$$D^2\log\frac{\det_{ren,1}(D_{+A})}{\det_{ren,2}(D_{+A})}(A_1,A_2) = Tr_{L^2}\left(D^{-1}_{+A}A_1D^{-1}_{+A}A_2\right) - Tr_{L^2}\left(D^{-1}_{+A}A_1D^{-1}_{+A}A_2\right) = 0.$$

for all (A_1,A_2) with disjoint supports. This allows to conclude by the Hammerstein condition that $A\mapsto\log\left(\frac{\det_{ren,1}(D_{+A})}{\det_{ren,2}(D_{+A})}\right)\in\mathcal{O}_{loc}$. So this proves that the renormalization ambiguities are of the form exponential of some element in $\mathcal{O}_{loc}$.

2.2.2. *Wodzicki residue and renormalization ambiguities.* In this short paragraph, we will try to answer some questions which were asked to the author by Sylvie Paycha, Jan Derezinski and Michal Wrochna about the relation between renormalization ambiguities and the notion of Wodzicki residue. Following Paycha's book [**138**, Prop 7.24 p. 139], given two Laplacians Δ_1,Δ_2, one can define the renormalized traces $Tr_{ren,1}$ and $Tr_{ren,2}$ on $\Psi^0(M)$ as follows :

$$Tr_{ren,1}(B) = FP|_{s=0}Tr_{L^2}\left(\Delta_1^{-s}B\right),\quad Tr_{ren,2}(B) = FP|_{s=0}Tr_{L^2}\left(\Delta_2^{-s}B\right). \tag{48}$$

These are called weighted traces in Paycha's book [**138**, def (7.12) p. 136]. We would like to compare the corresponding renormalized determinants $\det_{ren,1}$ and $\det_{ren,2}$. They are defined from the renormalized trace using formula 42. By construction of the renormalized determinants,

$$\log \frac{\det_{ren,1}(D_{+A})}{\det_{ren,2}(D_{+A})} = \int_0^1 \left(Tr_{ren,1}(\frac{dA_t}{dt}P(A_t)) - Tr_{ren,2}(\frac{dA_t}{dt}P(A_t)) \right) dt$$
$$= \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k} Tr_{ren,1}\left(((A-A_0)D^{-1}_{+A_0})^k \right) - \frac{(-1)^{k+1}}{k} Tr_{ren,2}\left(((A-A_0)D^{-1}_{+A_0})^k \right)$$

where $P(A_t)$ was the singular part appearing in the decomposition of $D^{-1}_{+,A_t} = P(A_t) + R(A_t)$ and is polynomial in A_t and $(A_t)_{t\in[0,1]}$ is a smooth path in $C^\infty(T^*M \otimes End(E))$.

Now we use a result proved by Melrose–Nistor [**122**], Cardona–Ducourtioux–Magnot–Paycha comparing different weighted traces, for any pseudofifferential operator B with polyhomogeneous symbol [**138**, Prop 7.24 p. 139] :

$$FP|_{s=0} Tr\left(\Delta_1^{-s}B\right) - FP|_{s=0} Tr\left(\Delta_2^{-s}B\right) = \mathbf{Res}\left(B\left(\frac{\log(\Delta_2)}{2} - \frac{\log(\Delta_1)}{2} \right) \right)$$

where **Res** on the r.h.s is the celebrated Wodzicki residue. Hence

$$\boxed{\frac{\det_{ren,1}(D_{+A})}{\det_{ren,2}(D_{+A})} = \exp\left(\sum_{k=1}^{p} \frac{(-1)^{k+1}}{k} \mathbf{Res}\left(((A-A_0)D^{-1}_{+A_0})^k \left(\frac{\log(\Delta_2)}{2} - \frac{\log(\Delta_1)}{2} \right) \right) \right)}$$

and the locality is recovered from the fact that Wodzicki residues vanish on trace class pseudodifferential operators.

3. Perspectives

There is a question by Jan Derezinski that we plan to address in future works. It is related to the fact that our space of renormalized determinants depends on all possible extension of the trace which is infinite dimensional. Physically, one would like to put more constraints on the renormalized traces and determinants so that they depend only on the metric and a finite number of choices. The general philosophy is to reduce the renormalization ambiguities to the strict minimum. In our approach, one has to renormalize a finite number of traces contained in the singular term $P(A)$ which depend polynomially on A. But the best approach to covariance would be to impose extra conditions in definition 2.12 on our determinants so that divergences in $P(A)$ are linear combinations of universal polynomials of finite degree in covariant derivatives of the metric up to some finite order and polynomials of finite degree in finite jets of $A-A_0$ where A_0 is the background connection. This will be treated in the joint work with Brouder–Zhang.

As noticed by Christian Brouder, one should add another axiom to 2.12 which is the fact that functional derivatives of odd orders of $\log\det$ should vanish. Physically, this should be a consequence of symmetry by charge conjugation and is called Furry's Theorem in the classical litterature in QED. We refer such investigation to our future work.

A natural line of investigation would be to prove some index Theorem for our renormalized determinants : it gives a topological formula for the winding number of the partition function when the gauge potential describes some non trivial loop in the space of gauge potentials. Then relate the index to quantum anomalies as in the work [**139, 170**] which was the original purpose of Quillen. Motivated by Quantum Field Theory on Lorentzian space times, we could try to generalize all this to the Lorentz Dirac operator. This means :

— define the correct functional framework to define some notion of determinant to the hyperbolic Dirac operator, this would probably use similar microlocal tools as discussed in Chapter 1,

— prove analyticity in the gauge potential, this has the same flavour as the analytic continuation of zeta function,

— give a geometric versus topological formula for the winding number of the partition function when the gauge potential describes some non trivial loop in the space of gauge potentials.

Another interesting direction would be to show that the limit of quantum partition functions of a discretized quantum field converges to our renormalized determinants when the mesh of the discretization goes to zero and after suitable renormalization. The next step would be to explore discrete versions of index Theorems that were found by physicists [**67**].

4. Appendix

4.1. Meaning of currents in gauge theories. First one should not get confused between de Rham currents and currents in field theory which are different objects [13]. In this appendix, $M = \mathbb{T}^{2d}$ is a torus of even dimension.

4.1.1. *Group action.* Consider the action of $C^\infty(M, U(1))$ on the configuration space $(\Psi, \tilde{\Psi})$: $\Psi \mapsto e^{i\alpha\gamma}\Psi = \left(e^{i\alpha}\Psi_+, e^{-i\alpha}\Psi_-\right)$, $\tilde{\Psi} \mapsto \tilde{\Psi}e^{+i\alpha\gamma} = \left(\tilde{\Psi}_+ e^{i\alpha}, \tilde{\Psi}_- e^{-i\alpha}\right)$. The fact that there is a $+$ sign on the action on the component $\tilde{\Psi}$ comes from the fact that γ_5 anticommutes with γ_4 $\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0$ and the Dirac conjugate $\tilde{\Psi} = -i\overline{\Psi}^t\gamma^4$ is defined in terms of γ^4. When α is constant this is a symmetry of the action $S[\Psi_+, \tilde{\Psi}_-, A]$ but for general $\alpha \in C^\infty(M, \mathbb{R})$ this acts as a symmetry only when the functional S is restricted to solution of the Dirac equation $D_{+A}\Psi_+ = 0, D_{-A}\tilde{\Psi}_- = 0$ [14]. Currents measure the response of the action functional S under an infinitesimal action $\varepsilon\alpha$:

$$\frac{d}{d\varepsilon}S[e^{i\varepsilon\alpha\gamma}\Psi, \tilde{\Psi}e^{i\varepsilon\alpha\gamma}, A] = i\int_M \tilde{\Psi}\gamma^\mu \frac{d\alpha}{dx^\mu}\gamma\Psi = -i\int_M \alpha\partial_{x^\mu}\left(\tilde{\Psi}\gamma^\mu\gamma\Psi\right) = -i\int_M \alpha\partial_{x^\mu}J^\mu_A$$

where $J^\mu_A = \left(\tilde{\Psi}\gamma^\mu\gamma\Psi\right)$ is called axial current. So this yields an equation $-i\int_M \alpha\partial_{x^\mu}J^\mu_A + \int_M \frac{\delta S}{\delta A_\mu(x)}\partial_\mu\alpha(x) = 0$.

4.1.2. *Conserved charges and Noether's Theorem.* Since

$$\frac{d}{d\varepsilon}S[e^{i\varepsilon\alpha\gamma}\Psi, \tilde{\Psi}e^{i\varepsilon\alpha\gamma}, A] = -i\int_M \alpha\partial_{x^\mu}\tilde{\mathcal{J}}^\mu = 0$$

for all $\alpha \in C^\infty_c(M)$ where Ψ is a **solution of the Dirac equation**, we find out that $\partial_\mu\tilde{\mathcal{J}}^\mu(\Psi, \tilde{\Psi}) = 0$. This implies that the $(n-1)$-form $Q_A(\Psi, \tilde{\Psi}) = \tilde{\mathcal{J}}^\mu(\Psi, \tilde{\Psi})\iota_{\partial_\mu}dx^1 \wedge \dots dx^n$ which depends quadratically on the solutions $(\Psi, \tilde{\Psi})$ of the Dirac equation are closed forms. By Stokes Theorem, their integral over $(n-1)$–cycles in $\mathbb{T}^n$ is called *charge* $\int_\Sigma Q_A(\Psi, \tilde{\Psi})$ which does not depend on the homology class of the cycles. One speaks of conservation of the charges.

4.1.3. *Inserting currents in the action functional and current correlators.* In Quantum theories describing the interaction of *charged particles*, in our case the fermion fields $(\Psi_+, \tilde{\Psi}_-)$, the *currents* measure the response of the action functional under some group action which come from the degrees of freedom of our fermion fields (here it is just a phase represented by $U(1)$ action) and are quadratic functionals of the fields valued in tensors : the vectorial current $J^\mu_V = \tilde{\Psi}_-\gamma^\mu\Psi_+$ and the axial current $J^\mu_A = \tilde{\Psi}_-\gamma\gamma^\mu\Psi_+$. Note the difference between both currents comes from the appearance of the chirality operator γ. In physical applications, one is interested in correlation functions of quantum currents of the form

$$\left\langle J^{\mu_1}_A(x_1)\dots J^{\mu_k}_A(x_k)J^{\nu_1}_V(y_1)\dots J^{\nu_l}_V(y_l)\right\rangle_{A,V}.$$

Therefore, following the exposition of Fröhlich [**66, 65**] and Leutwyler [**110, 111**], one defines a general action functional which contains both vector $(V_\mu)_\mu$ and axial $(A_\mu)_\mu$ potentials which are coupled to the corresponding currents.

$$S[\Psi, \tilde{\Psi}, A, V] = \int_M \tilde{\Psi}\gamma^\mu\left(\partial_{x^\mu}\otimes Id + Id\otimes(V_\mu + A_\mu\gamma)\right)\Psi = \int_M \tilde{\Psi}\gamma^\mu\partial_{x^\mu}\Psi + (J^\mu_A A_\mu + J^\mu_V V_\mu).$$

13. Even though sometimes they can be related
14. In physics terminology, this is called on–shell symmetry

Take any branch of the log, or just view the log as a multivalued function, define the correlation functions as Schwartz kernels of the differentials of $\log Z(A,V)$

$$\frac{\delta^n \log Z(A,V)}{\delta A_{\mu_1}(x_1)\dots\delta A_{\mu_k}(x_k)} = \left\langle J_A^{\mu_1}(x_1)\dots J_A^{\mu_k}(x_k)\right\rangle_{A,V}.$$

So quantum correlators come from the response of the log of the partition function, called free energy or effective action, to external excitations of the gauge potential. They give correlators of quantum currents because the currents are coupled to the gauge potential in the Lagrangian density.

An important observation is that the correlator $\left\langle J_A^{\mu_1}(x_1)\dots J_A^{\mu_k}(x_k)\right\rangle_{A,V}$ reads as a product of Dirac propagators which is smooth on $M^n\setminus$ all diagonals. But the renormalization of the partition function implies that the quantum correlators $\left\langle J_A^{\mu_1}(x_1)\dots J_A^{\mu_k}(x_k)\right\rangle_{A,V}$ **extend as distributions** on the configuration space M^n which relates the renormalization of the partition function with the Epstein–Glaser viewpoint on renormalization as explained to us by Denis Perrot.

4.1.4. *Relation with the phase of the S-matrix.* For quantum fields interacting with external fields, the partition function, given in terms of the functional integral, actually represents the vacuum expectation value which is a coefficient of the S–matrix of the system. For example the term

$$\int [D\Psi_+ D\overline{\Psi_-}] e^{\int_M \overline{\Psi_-}D_+\Psi_+ + f_\mu :L_+^\mu:} = \left\langle 0|e^{\int_M f_\mu :L_+^\mu:}|0\right\rangle \tag{49}$$

represents a vacuum to vacuum **transition amplitude**. The modulus square $|\left\langle 0|e^{\int_M f_\mu :L_+^\mu:}|0\right\rangle|^2$ gives the probability amplitude not to create a Fermion pair after interaction with the external field L_μ controlled by sources $f_\mu\in C_c^\infty(M)$. In this language, there is a quantum anomaly whenever the phase $Im(W)$ of $\left\langle 0|e^{\int_M f_\mu :L_+^\mu:}|0\right\rangle$ is no longer gauge invariant.

Chapitre 3

The Fried conjecture.

1. Motivations.

The Fried conjecture relates the Reidemeister torsion and the value at 0 of some twisted Ruelle zeta function. The torsion is the homological counterpart of the determinant in linear algebra. In a nutshell, one can argue that the Fried conjecture states that under certain conditions which have to be found, torsion counts periodic orbits of some specific classes of flows in the same way as the Lefschetz trace counts fixed points of diffeomorphisms. Let us start by presenting a dictionary which motivates the present chapter.

Algebra	Topology	Dynamics
$\dim(V)$	Euler $\chi(V,d)$	zeroes of vector fields $\sum_{c\in Crit(V)}(-1)^{\mathrm{ind}_V(c)}$
trace(T)	Lefschetz $\mathcal{L}(T)$ $\sum_{i=0}^{\dim(M)}(-1)^i Tr(T\|_{H^i(M)})$	fixed points of maps $\sum_{x=T(x)}\mathrm{ind}_T(x)$
determinants	Torsion τ	periodic orbits flows $\prod_{\gamma\in \text{ prime}}\det\left(Id-\rho(\gamma)\Delta(\gamma)\right)^{(-1)^{\mathrm{ind}(\gamma)}}$

We next illustrate the Fried conjecture with the simpler example of the Lefschetz formula for the Euler characteristic χ essentially following Atiyah–Bott [**1**]. We realize the topological invariant χ as the superdimension of the space of harmonic forms of the Hodge Laplacian, which gives a *quantum interpretation* and also as a weighted count of critical points of a Morse function f which yields a *dynamical interpretation.*

1.1. Lefschetz principle. Let M be a compact manifold and $[d_2]\in\mathcal{D}^{\prime,d}(M\times M)$ be the current of integration on the diagonal $d_2\subset M\times M$. We would like to make sense of the self–intersection of the diagonal defined formally as

$$\int_{M\times M}[d_2]\wedge[d_2] = \ ? \tag{50}$$

Of course the wedge product of currents is ill–defined since we multiply a current with itself hence the wave front sets are not transverse and the product is forbidden. This classical example is meant to illustrate the usefulness of the notion of supersymmetry by calculating in three different ways this renormalized self–intersection. We calculate it using three “renormalization schemes” :

— the heat regularization and extract the finite part,
— the zeta regularization and use analytic continuation,
— by Morse theory by making transverse perturbations.

Supersymmetry will ensure that the three results are the same hence the computation is *scheme independent* [1].

We would like to interpret the wedge product with the diagonal current $[d_2]$ as a supersymmetric generalization of the flat trace.

1. There is a fourth way using excess intersection and Chern classes that we do not sketch here but it is also related to the above three methods

LEMMA 1.1 (Superflat trace). *Let* $T : C^\infty(\Lambda^\bullet T^*M) \mapsto \mathcal{D}'(\Lambda^\bullet T^*M)$ *be a linear map of degree* 0*. Then for* $[K] \in \mathcal{D}'^{,d}(M \times M)$ *the Schwartz kernel of* T*, we have the equality*

$$\int_{M\times M} [K] \wedge [d_2] = \sum_{k=0}^{d} (-1)^k Tr^\flat \left(T|_{\Lambda^k T^*M}\right)$$

whenever both sides are defined.

1.1.1. *Method 1, heat regularization.* The current $[d_2]$ is the Schwartz kernel of the identity map $Id : C^\infty(\Lambda^\bullet T^*M) \mapsto C^\infty(\Lambda^\bullet T^*M)$. So at least **formally**, we are calculating

$$\int_{M\times M} [d_2] \wedge [d_2] = \sum_{k=0}^{d} (-1)^k Tr^\flat \left(Id|_{\Lambda^k T^*M}\right)$$

where both sides are ill–defined by the same lack of transversality reason. But one may approximate the identity map by the heat operator $e^{-\varepsilon\Delta}$ where Δ is the Hodge Laplacian since $e^{-\varepsilon\Delta} \underset{\varepsilon\rightarrow 0^+}{\rightarrow} Id$. Then one can introduce a regularized superflat trace

$$\sum_{k=0}^{d} (-1)^k Tr^\flat \left(e^{-\varepsilon\Delta}|_{\Lambda^k T^*M}\right) = \sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(e^{-\varepsilon\Delta}|_{\Lambda^k T^*M}\right)$$

which makes sense when $\varepsilon > 0$ since the heat kernel is smoothing and a priori has divergent asymptotic expansion when $\varepsilon \rightarrow 0^+$. We next define the renormalized superflat trace as the finite part in the sense of Hadamard :

$$FP|_{\varepsilon=0} \sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(e^{-\varepsilon\Delta}|_{\Lambda^k T^*M}\right).$$

Let us calculate the finite part by the following spectral argument. Assume that $\lambda \neq 0$ lies in the discrete spectrum of Δ, then if $u \in \ker(\Delta - \lambda) \cap C^\infty(\Lambda^k T^*M)$ then $du \in \ker(\Delta - \lambda) \cap C^\infty(\Lambda^{k+1} T^*M)$ since d and Δ commute. It means that in the alternate sum of traces : $\sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(e^{-\varepsilon\Delta}|_{\Lambda^k T^*M}\right)$, the contributions of all non zero eigenvalues of Δ cancel out and only the zero modes survive. This yields $FP|_{\varepsilon=0} \sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(e^{-\varepsilon\Delta}|_{\Lambda^k T^*M}\right) = \dim(\ker(\Delta))_{\text{even}} - \dim(\ker(\Delta))_{\text{odd}} = \chi(M)$ since the harmonic forms are quasi–isomorphic to the de Rham complex by the argument in paragraph 3.4 so the superdimension of the space of harmonic forms equals the Euler characteristic.

1.1.2. *Method 2, zeta regularization.* In the same way as above, one could regularize by complex powers of the Laplace operator Δ^{-s} instead of using the heat regularization. It is well–known that $\sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(\Delta^{-s}|_{\Lambda^k T^*M}\right)$ admits an analytic continuation to the complex plane and we woud like to compute

$$FP|_{s=0} \sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(\Delta^{-s}|_{\Lambda^k T^*M}\right).$$

Then by a similar spectral argument as above only the zero modes survive and we get

$$FP|_{s=0} \sum_{k=0}^{d} (-1)^k Tr_{L^2} \left(\Delta^{-s}|_{\Lambda^k T^*M}\right) = \dim(\ker(\Delta))_{\text{even}} - \dim(\ker(\Delta))_{\text{odd}} = \chi(M).$$

1.1.3. *Method 3, superflat traces and transversal perturbations.* Instead of perturbing the identity with the heat operator or with complex powers of Δ, we use $e^{-\varepsilon V}$ where $V = \nabla f$ is the gradient of a Morse function. V has nondegenerate zeroes therefore the Schwartz kernel of $e^{-\varepsilon V}$ is supported on the graph of $e^{-\varepsilon V}$ and is transverse to the identity. So we would like to calculate the limit $\lim_{\varepsilon\rightarrow 0^+} \sum_{k=0}^{d} (-1)^k Tr^\flat \left(e^{-\varepsilon V}|_{\Lambda^k T^*M}\right)$ provided it exists. We give an example which shows that the quantity $\sum_{k=0}^{d} (-1)^k Tr^\flat \left(e^{-\varepsilon V}|_{\Lambda^k T^*M}\right)$ gives a weighted count of the critical points of f.

EXAMPLE 1.2 (Counting fixed points and superflat traces). *Let us return to example 1.7 in Chapter 1 but now we consider the action of the pull–back operator T_f on the full differential graded algebra $C_c^\infty(\Lambda^\bullet T^*\mathbb{R}) = C_c^\infty(\mathbb{R}) \oplus C_c^\infty(T^*\mathbb{R})$. Observe that for a 1-form ψdx, $T_f(\psi dx) = f'(x)\psi(f(x))dx$ which implies that the superflat trace $\sum_{k=0}^1 Tr^\flat(T_f|_{\Lambda^k T^*\mathbb{R}})$ reads*

$$\boxed{\sum_{k=0}^{1} Tr^\flat(T_f|_{\Lambda^k T^*\mathbb{R}}) = \sum_{x=f(x)} \frac{1}{|1-f'(x)|} - \frac{f'(x)}{|1-f'(x)|} = \sum_{x=f(x)} (-1)^{sgn(1-f'(x))}}$$

which is a weighted sum over the fixed points of f and is of topological nature[2].

The previous example should convince the reader that for every $\varepsilon > 0$, we find that

$$\sum_{k=0}^{d}(-1)^k Tr^\flat\left(e^{-\varepsilon V}|_{\Lambda^k T^*M}\right) = \sum_{x\in Crit(V)} (-1)^{\mathrm{ind}(V)} = \chi(M)$$

as a consequence of the Morse inequalities, this quantity does not depend on ε hence it has a well–defined limit. Moreover, it is a consequence of our result with Rivière on the Ruelle spectrum of $V = \nabla f$ acting on differential forms, that $\chi(M) = \dim(\ker(V))_{\text{even}} - \dim(\ker(V))_{\text{odd}}$ where $\ker(V)$ is generated by the Laudenbach currents which act as dynamical replacement of the harmonic forms.

1.2. Conclusion. What we just proved is that the fundamental additive topological invariant $\chi(M)$ has both a quantum representation and a dynamical realization

$$\chi(M) = \underbrace{\dim(\ker(\Delta))_{\text{even}} - \dim(\ker(\Delta))_{\text{odd}}}_{\text{quantum side}} = \underbrace{\sum_{x\in Crit(V)} (-1)^{\mathrm{ind}(V)} = \dim(\ker(V))_{\text{even}} - \dim(\ker(V))_{\text{odd}}}_{\text{dynamical side}}.$$

The Fried conjecture is the multiplicative analogue of the above Lefschetz formula where torsion takes the place of the Euler characteristic and we are counting the periodic orbits instead of fixed points.

2. Geometric context.

The geometric context of the present chapter can be quickly described as follows. We work on a closed, compact, contact manifold (M,θ), $\dim(M) = 2d+1$, $\theta \in C^\infty(T^*M)$ is the contact 1-form which means that $\theta \wedge d\theta^{\wedge d}$ is a **volume form**. The contact form θ defines a vector field $V \in C^\infty(TM)$ called the Reeb vector field $\theta(V) = 1, \iota_V d\theta = 0$. We assume that the Reeb flow $e^{-tV} : M \mapsto M$ is Anosov, we refer the reader to subsection 4.1 of Chapter 1 for precise definitions. We are given some representation ρ of the fundamental group in $GL_n(\mathbb{C})$. In practice, this is implemented using a flat bundle (E,∇) and $\rho : \pi_1(M) \mapsto GL_n(\mathbb{C})$ is realized by the monodromy of the flat connection ∇ as described in subsection 7.1.

EXAMPLE 2.1 (Abelian representations). *If α is a closed 1-form, then $\rho(\gamma) = \exp\left(\int_\gamma \alpha\right)$ is a* ***character*** *on $\pi_1(M)$: $\rho(\gamma_1+\gamma_2) = \exp\left(\int_{\gamma_1\circ\gamma_2}\alpha\right) = \exp(\int_{\gamma_1}\alpha)\exp(\int_{\gamma_2}\alpha) = \rho(\gamma_1)\rho(\gamma_2)$ hence $\rho : \pi_1(M) \mapsto \mathbb{C}^*$.*

We may denote the representation a bit abstractly as $\rho = e^{\langle\alpha,.\rangle} : \pi_1(M) \mapsto \mathbb{C}^*$, $[\alpha] \in H^1(M,\mathbb{R})$.

2. In case of the real line it must vanish

2.1. The twisted Ruelle zeta function. The main object of our study is the twisted Ruelle zeta. Let us motivate its structure by considering some classical complex functions from analytic number theory. The motivation is to count primes. A natural idea is to associate some complex function to the counting problem. The Riemann zeta function reads $\zeta(s) = \sum_{n\geqslant 1} n^{-s} = \underbrace{\prod_{p\in\text{Primes}} (1-p^{-s})^{-1}}_{\text{factorized}}$. Its analytical properties are related to the distribution of primes. A generalization of the Riemann zeta function which is used if one wants to count primes subject to congruence conditions are the Dirichlet L-functions. They depend on some character $\chi : \mathbb{N} \mapsto \mathbb{S}^1$ and are functions of (s, χ) :

$$\boxed{L(s,\chi) = \prod_{p\in\text{Primes}} (1-\chi(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi(n)n^{-s}.}$$

The analytical properties of L–functions can be used to prove a generalization of the prime number Theorem due to Dirichlet. Note the similarity of structures between the ζ and L, the main difference being the fact we use a character χ to twist zeta.

Definition 2.2. Given a contact Anosov flow $e^{-tV} : M \mapsto M$ and some character $\chi \in Hom(\pi_1(M), \mathbb{C}^*)$, we can form the twisted Ruelle zeta function (dynamical L–functions) :

$$\zeta_{V,\chi}(s) = \prod_{\gamma\in\mathcal{P}} \left(1 - \chi(\gamma)e^{-s\ell(\gamma)}\right),$$

where the product runs over the set $\mathcal{P}$ of prime periodic orbits of e^{tV}, $\ell(\gamma)$ period of γ. More generally, given a representation $\rho : \pi_1(M) \mapsto GL_n(\mathbb{C})$, we form the twisted Ruelle zeta function

$$\zeta_{V,\rho}(s) = \prod_{\gamma\in\mathcal{P}} \det\left(1 - \rho(\gamma)e^{-s\ell(\gamma)}\right). \tag{51}$$

The twisted Ruelle zeta function $\zeta_{V,\rho}(s)$ is defined by some infinite product which converges when $s > h_{top}$ where h_{top} is the topological entropy of the Anosov flow. The reader unfamiliar with h_{top} just has to think of it as some exponent measuring the exponential growth rate when $T \to +\infty$ of the number of prime geodesics of length less than T. We refer to subsection 2.2 of Chapter 4 for a more detailed discussion of h_{top}.

Let us give the simplest example of Ruelle zeta function in the case of the circle $\mathbb{S}^1$.

Example 2.3. *On $\mathbb{S}^1$ of length ℓ, flow ∂_θ, u generator of $\pi_1(M)$,* ***monodromy*** *$\rho(u) \in \mathbb{C}^*$, $\zeta_{V,\rho}(s) = (1-\rho(u)e^{-s\ell})$.*

2.2. Natural properties of $\zeta_{V,\rho}$. It is well known $\zeta_{V,\rho}$ is holomorphic when $Re(s) > h_{top}$. There are two natural questions one could ask : Is there an analytic continuation result for $\zeta_{V,\rho}$? This was conjectured by Smale motivated by the analogy with number theory since both the Riemann zeta function and the Dirichlet L-functions have meromorphic continuations. Smale's conjecture was adressed using Markov partition techniques by Rugh [**155**] for 3d **analytic** Axiom A flows building on the works of Ruelle. Then Fried [**64**] generalized the result of Rugh to all analytic Anosov flows. Using functional analytic techniques, a similar problem was solved in the discrete case by Liverani for Anosov diffeomorphisms, then Kitaev [**101**] and Baladi–Tsujii [**4**] for Axiom A diffeomorphisms. Going back to flows, it is only recently (2013) that Giuletti–Liverani–Pollicott [**75**] proved the meromorphic continuation for C^∞ Anosov flows. This result was recovered by Dyatlov–Zworski [**46**] using a microlocal proof, in the spirit of the work of Faure–Sjöstrand [**56**] and relying on the radial estimates of Melrose [**120**], Vasy [**181**]. Finally, Dyatlov–Guillarmou [**45, 43**] settled Smale's conjecture for general C^∞ Axiom A flows.

Theorem 2.4. *The function $\zeta_{V,\rho}$ has* ***meromorphic continuation*** *to the complex plane for all $V \in C^\infty(TM)$, nonsingular Axiom A hence for V Anosov.*

A second question one could ask is what is the topological content of $\zeta_{V,\rho}$ which are questions posed by Bowen and Fried [**62**]. Both problems, the analytical continuation and the topological content are deeply related. A result which is simple to state reads :

THEOREM 2.5 (Dyatlov–Zworski [**47**]). *For a surface $\mathcal{M}$ of variable negative curvature, V generates the geodesic flow on $S^*\mathcal{M}$ then we have near $s=0$:*

$$\zeta_{V,Id}(s)=\prod_{\gamma}(1-e^{-s\ell(\gamma)})=s^{2g-2}\left(c+\mathcal{O}(s)\right) \tag{52}$$

where g is the ***genus*** *of $\mathcal{M}$. In particular, the* ***length spectrum determines the genus****.*

The above result was generalized by Hadfield in the boundary case [**84**] and Cekić–Paternain [**32**] for volume preserving Anosov flows in terms of **asymptotic cycles and helicity** of the flow.

In some sense, our situation is the opposite. To define torsion, we need to kill the cohomology groups (hence the Euler characteristic would vanish) to make our complexes acyclic. This is done by twisting the de Rham complex by a flat connection.

2.2.1. *Abstract torsion of cochain complexes.* The goal of this short paragraph is to introduce the notion of torsion for abstract cochain complexes.

Let us start by discussing a motivating example.

EXAMPLE 2.6. *$T:E\mapsto F$ isomorphism, corresponding complex $0\mapsto E\mapsto F\mapsto 0$. How from T do we get* ***numbers*** *? If we choose volume elements $\mu_1\in\Lambda^{top}E,\mu_2\in\Lambda^{top}F$ then $T_*\mu_1=\lambda\mu_2$ where λ is a number. Torsion generalizes the notion of determinant for* ***based*** *cochain complexes*[3]*.*

Given some cochain complex $(C^\bullet,\partial)$

$$0\mapsto C^0\overset{\partial}{\mapsto}C^1\mapsto\dots\overset{\partial}{\mapsto}C^N\mapsto 0$$

$\partial^2=0$, acyclicity means that $Im(\partial)=\ker(\partial)$, we choose some volume element $[c]$ in $C^\bullet$, this means a volume element c_i in each C^i, then the torsion τ of the based cochain complex is defined as :

$$\tau(C^\bullet,\partial)=|\prod_{i=0}^{N}[\partial b_{i+1}b_i/c_i]^{(-1)^{i+1}}| \tag{53}$$

where b_i is basis of $\operatorname{coker}(\partial)_i$ and $[(\partial b_{i+1})b_i/c_i]\in\mathbb{R}$ just compares the volume elements $(\partial b_{i+1})b_i$ and c_i. Here $(\partial b_{i+1})b_i$ is a basis of C^i by acyclicity of ∂ and $[(\partial b_{i+1})b_i/c_i]$ can be thought of as the determinant of the matrix going from the basis c_i to the basis $(\partial b_{i+1})b_i$. The abstract torsion $\tau(C^\bullet,\partial)$ does not depend on the choice of basis $(b_i)_i$.

2.2.2. *Geometric implementation on manifolds.* Once we defined torsion for abstract cochain complexes, we need to discuss how to define torsion of some manifold M endowed with some acyclic representation ρ of the fundamental group. Our recipe uses Morse theory but this could equally be well–defined using cell decompositions of M. It goes as follows, for simplicity we give it for Abelian representations $\rho\in Hom(\pi_1(M),\mathbb{C}^*)$: on the manifold M and given some closed form α with complex coefficients, we choose some Morse function f s.t. $V=\nabla f$ satisfies the usual transversality conditions of definition 4.5 discussed in Chapter 1. The Morse complex generated by $\mathbf{Crit}(f)$, twisted by the representation $\rho=e^{\langle\alpha,.\rangle}$ has a differential defined as :

$$\partial a=\sum_{\gamma:a\mapsto b}\pm\underbrace{e^{\int_\gamma\alpha}}_{\text{twisting}}b \tag{54}$$

where the sum runs over the instantons connecting (a,b) s.t. $ind(b)=ind(a)+1$ and the $\pm$ depend on the choices of orientations and we choose to be a bit unprecise here for the sake of simplicity. So this is exactly the instanton formula for the Witten complex discussed in Chapter 1 except in the present case, there is a correction term $e^{\int_\gamma\alpha}$ which

3. More precisely for cochain complexes equipped with some volume element

represents the parallel transport w.r.t. the flat connection $d+\alpha$ along the path γ. Note the similarity between this formula and the tunneling formula of Helffer–Sjöstrand which is not a coincidence but comes from the fact that we twist the de Rham differential d. In the Witten Laplacian case, the differential was twisted by an exact form $\hbar^{-1}df$ since $d_{f,\hbar}=d+\hbar^{-1}df$ which explains that the correction reads $e^{-\frac{\int_\gamma df}{\hbar}}=e^{-\frac{f(b)-f(a)}{\hbar}}$ where (a,b) is a pair of critical points connected by some instanton.

THEOREM 2.7 (Whitehead, Milnor). *Let M be a smooth compact manifold without boundary, ρ is a unitary representation s.t. the twisted Morse complex $(C_f^\bullet, d_\rho)$ is **acyclic**. Then $\tau_R(\rho):=\tau(C_f^\bullet, d_\rho)$ does not depend on the choice of Morse function f satisfying the Smale transversality condition. It is a topological invariant of the pair (M,ρ).*

Let us give a complete example of the above recipe in the case of the circle where we choose the height function as Morse function.

EXAMPLE 2.8. ***Acyclicity***. *First, let us briefly explain how twisting can kill cohomology groups in the simplest case of $\mathbb{S}^1$. On $\mathbb{S}^1$ of perimeter 2π, let $\alpha\in i\mathbb{R}$ s.t. $e^{-2\pi\alpha}\neq 1$. We set the twisted de Rham differential to be $d+\alpha d\theta$. The parallel transport w.r.t. the connection $d+\alpha d\theta$ generates a unitary representation of $\pi_1(\mathbb{S}^1)\simeq\mathbb{Z}$ s.t. $\rho(u)=e^{-2\pi\alpha}\in\mathbb{S}^1\subset\mathbb{C}^*$ where u is the generator of $\pi_1(\mathbb{S}^1)$.*

Let us look for $u\in C^\infty(\mathbb{S}^1,\mathbb{C})$ solutions of $(d+\alpha d\theta)u=0$. We solve $\partial_\theta u+\alpha u=0$ with $u(0)=u(2\pi)$ by periodicity. The solution reads $u(\theta)=u(0)e^{-\alpha\theta}$. But periodicity and smoothness impose $u(2\pi)=u(0)e^{-2\pi\alpha}$ hence $u(0)=u(2\pi)=0$ since $e^{-2\pi\alpha}\neq 1$. Therefore $\ker(d+\alpha d\theta)\cap C^\infty(\mathbb{S}^1)=0$ *and the same can be said about the 1-forms. So we see that a **monodromy** condition $e^{-2\pi\alpha}\neq 1$ implies the **acyclicity** of the twisted de Rham complex $C^\infty(\Lambda^\bullet T^*\mathbb{S}^1), d+\alpha d\theta$.*

Torsion. *Consider $\mathbb{S}^1$ with the height function as Morse function, we denote by (a,b) the two critical points. The basis of the Morse complex is (a,b). The differential reads*

$$\partial a=e^{-\pi\alpha}b-e^{\pi\alpha}b=(e^{-\pi\alpha}-e^{\pi\alpha})b$$

but this implies that

$$|\det(\partial)|=|1-e^{2\pi\alpha}|$$

and therefore the definition of Reidemeister torsion gives :

$$\boxed{\tau_R(\rho)=|\det(Id-\rho(\gamma))|^{-1}=|\zeta_{V,\rho}(0)|^{-1}.} \tag{55}$$

3. Some results on the Fried conjecture.

In the example of the circle, we saw that the twisted Ruelle zeta function equals $|\det(Id-\rho(\gamma))|^{-1}$ which is related to the value at 0 of $\zeta_{V,\rho}$. Such observations motivated Fried to conjecture a relation between $\zeta_{V,\rho}(0)$ and τ_R. Let us state a Theorem [**63**] in such direction :

THEOREM 3.1 (Fried). *Let $\mathcal{M}$ be a hyperbolic manifold, $M=S^*\mathcal{M}$,* $\dim(\mathcal{M})=d$, *$V\in C^\infty(TM)$ generates the geodesic flow, $\rho:\pi_1(\mathcal{M})\mapsto U_n$ an **acyclic unitary** representation. Then :*

$$\tau_R(\rho)=|\zeta_{V,\rho}(0)|^{(-1)^{d-1}}. \tag{56}$$

The above result was extended to locally symmetric spaces by Moscovici–Stanton [**125**], Shen [**164**], then to complex torsions and arbitrary representations by Müller [**126**], Spilioti [**168**] and Shen [**167**]. In another direction, Sanchez–Morgado [**156**] proved the Fried conjecture for V **analytic Anosov** when $\dim(M)=3$ under some conditions on the Anosov flow e^{-tV} on M and the representation ρ.

Then building on both above results, together with Guillarmou–Rivière–Shen, we could show :

Theorem 3.2 (D–Guillarmou–Rivière–Shen). *Let M be smooth compact manifold. If for some flat bundle $(E,\nabla)\mapsto M$ and Anosov $V_0\in C^\infty(TM)$, we have* $\ker(V_0)=\{0\}$ *then*

$$\zeta_{V,\rho}(0)=\zeta_{V_0,\rho}(0) \tag{57}$$

for all V near V_0. In particular, the Fried conjecture :

$$\tau_R(\rho)=|\zeta_{V,\rho}(0)|^{(-1)^{d-1}}. \tag{58}$$

holds true for V Anosov in $3d$ if $b_1(M)>0$ and in $5d$ near geodesic flows of hyperbolic manifolds.

Remark 3.3. The absolute values are already present in Theorem 3.1 due to Fried himself, $\tau_R(\rho)$ is positive and equals $|\zeta_\rho|$ on hyperbolic manifolds, what Theorem 3.2 says is that the equality still holds when V is close to the geodesic flow of some hyperbolic manifold.

The assumption that $\ker(V)=\{0\}$ implies the acyclicity of the twisted de Rham complex since we showed that $(C^\infty(\Lambda^\bullet T^*M\otimes E),d^\nabla)$ is quasi–isomorphic to $(\ker(V),d^\nabla)$.

4. The Fried conjecture for Turaev's refined torsions.

We would like to point out two weaknesses in the above Theorem 3.2. It does not cover the case where ρ is acyclic but $\ker(V)\neq\{0\}$ since in such case, $\zeta_{V,\rho}$ might vanish or have a pole at 0, hence the value at 0 is a priori ill–defined. The next issue is that we would like to remove the absolute value $|.|$ to capture the phase of $\zeta_{V,\rho}(0)$. Then a natural question is to compare

$$\zeta_{V,\rho}(0)$$

with complex torsions that refine τ_R. Recall that we proved that $(\ker(V),d^\nabla)$ is a chain complex which is quasi–isomorphic to the twisted de Rham complex $(C^\infty(\Lambda^\bullet T^*M\otimes E),d^\nabla)$. The problem is that when we compute the torsion of a chain complex, we need some distinguished basis. But there is no special basis in $\ker(V)$ for V Anosov. For Morse–Smale flows, there is always a distinguished basis of currents of integration on unstable manifolds. Similarly, when we consider a cell decomposition of a manifold M, there is also a natural basis of the chain complex $(C_\bullet,\partial)$ generated by the cells of the decomposition. The key idea to resolve this issue is the observation that the contact structure on M induces some involution $\Gamma:\ker(V)\mapsto\ker(V)$. This begins with the definition of Lefschetz maps due to Lepage :

Proposition 4.1 (Lepage 1946). *Let (M,θ) be some contact manifold. Then there exists **bundle isomorphisms***

$$\mathrm{L}^{d-k}:\varphi\in C^\infty\left(\Lambda^kT^*M\right)\cap\ker(\iota_V)\mapsto\varphi\wedge d\theta^k\in C^\infty\left(\Lambda^{2d-k}T^*M\right)\cap\ker(\iota_V),\forall k\leqslant d.$$

These isomorphisms are defined only on differential forms which are killed by contraction with the Reeb field V. We next extend the construction to all differential forms.

Definition 4.2. Every k-form $\varphi\in C^\infty(\Lambda^kT^*M)$ decomposes as a sum $\varphi=f\wedge\theta+g,(f,g)\in\ker(\iota_V)$ and we define a *chirality* operator Γ as the unique involution satisfying :

$$\Gamma\varphi=\mathrm{L}^{d-k}g\wedge\theta+\mathrm{L}^{d-k+1}f,k\leqslant d. \tag{59}$$

By construction, the involution Γ commutes with the action of V on anisotropic Sobolev spaces and therefore induces some involution Γ on $\ker(V)$. Now we use an important result of Braverman–Kappeler [**17, 18, 19**], who noticed the fact that an involution Γ on any finite dimensional complex $C^\bullet$ defines a normalized torsion $\tau(\Gamma,C^\bullet)$ without choosing some volume element $[c]$ in $C^\bullet$. In fact, the involution Γ selects some class of Γ–invariant volume elements which fixes the value of torsion.

Proposition 4.3 (Braverman–Kappeler). *Let $(C^\bullet,d)$ be a finite dimensional cochain complex endowed with some involution Γ. Then Γ defines an **intrinsic torsion** $\tau_\Gamma(C^\bullet)$. In particular, there is an intrinsic torsion $\tau_\Gamma(\ker(V))$ which depends only on the contact form θ.*

Once we have a torsion $\tau_\Gamma(\ker(V))$ of $\ker(V)$, we may define a new object, named dynamical torsion, which corrects the value at $s = 0$ of the Ruelle zeta function in the case $\ker(V) \neq \{0\}$.

Definition 4.4 (Dynamical torsion). Inspired by the work of Braverman–Kappeler [**17, 18, 19**] and Hutchings–Lee [**97**], we define the dynamical torsion as the product

$$\boxed{\tau(V,\rho) = \underbrace{\tau_\Gamma(\ker(V))}_{\text{torsion of ker}} \times \underbrace{\lim_{s\to 0^+} s^{-m}\zeta_{V,\rho}(s)}_{\text{renormalized zeta}}.} \tag{60}$$

When $\ker(V) = \{0\}$, $\tau(V,\rho) = \zeta_{V,\rho}(0)$ and therefore $\tau(V,\rho)$ **generalizes** the value $\zeta_{V,\rho}(0)$.

Now our next goal is to briefly describe Turaev's refined torsion which is some complex torsion which depends holomorphically on the representation. This will allow us to compare $\tau(V,\rho)$ with the Turaev torsion as holomorphic functions on the acyclic part of the representation variety.

4.0.1. *Turaev torsion.* It is well–known that there are ambiguities in the definition of the Reidemeister torsion which seem to disappear by miracle since we take the modulus $|.|$ and also since the representation ρ is unitary. However, if we keep the same definition of τ_R as in equation (53) except we remove the modulus $|.|$ and consider arbitrary representations in $GL_n(\mathbb{C})$, we see that there are ambiguities in the definition of τ_R and it was proved by Turaev that these ambiguities are fixed once we choose some Euler structure $\mathfrak{e} \in Eul(M)$ and some homology orientation. To fix ambiguities of τ_R, Turaev defined a torsion τ as a **holomorphic function** of the representation $\rho \in Hom\,(\pi_1(M), GL_n(\mathbb{C})$ and as a function of the Euler structure $\mathfrak{e} \in Eul(M) : (\rho,\mathfrak{e}) \in Hom\,(\pi_1(M), GL_n(\mathbb{C})\times Eul(M) \mapsto \tau_{\mathfrak{e}}(\rho) \in \mathbb{C}^*$. This extends the Reidemeister torsion in the sense that $|\tau_{\mathfrak{e}}(\rho)| = \tau_R(\rho)$ for all unitary representation ρ and all choices of Euler structure $\mathfrak{e} \in Eul(M)$.

The main Theorem we proved with Chaubet can be stated as follows. Let us denote by $\mathcal{A}$ the space of Anosov vector fields on M, this is an open subset of $C^\infty(TM)$ by structural stability. We also denote by Rep_0 the subset of acyclic representations in $Hom(\pi_1(M), GL_n(\mathbb{C}))$ which is open in $Hom(\pi_1(M), GL_n(\mathbb{C}))$.

Theorem 4.5 (Chaubet–D). *The map $V \in \mathcal{A} \mapsto \tau(V,\rho)$ is **locally constant**. For all **connected open** sets $\mathcal{U} \subset Rep_0$ and $\mathcal{V} \subset \mathcal{A}$, $\exists \mathfrak{e} \in Eul(M)$, $C \in \mathbb{C}^*$ such that*

$$\boxed{\underbrace{\tau(V,\rho)}_{\textit{dynamical torsion}} = C\ \underbrace{\tau_{\mathfrak{e}}(\rho)^{-1}}_{\textit{Turaev torsion}}\ ,\forall V \in \mathcal{V}, \forall \rho \in \mathcal{U}}$$

*where the constant C does not depend on $(X,\rho) \in \mathcal{U}\times\mathcal{V}$ and both sides are **holomorphic functions** of $\rho \in Rep_0$.*

There is a strong analogy between our work on Quillen's conjectural picture and torsion as a function on the character variety. Let us explain the similarities. The torsion plays the role of the partition function of chiral fermions. They are both **determinants** viewed as holomorphic functions of the connection ∇. In both cases, the key idea is to differentiate the log of the determinant under variation of ∇ and to find a heuristic formula of the type :

$$\delta \log\det(\nabla) = Tr\,(\text{Variation of connection } \delta\nabla \circ \text{ Resolvent of some operator})$$

where the resolvent on the r.h.s is the Dirac inverse in the case of QFT or some operator K which satisfies $[d^\nabla, K] = Id$ [4] in case of torsion.

5. Perspectives.

In the paper [**85**], the authors have given evidence that the Ruelle zeta function can be defined as the partition function of some topological field theory of BF type where the gauge fixing uses the vector field V. It would be interesting to push this analogy further and see if one can define correlations of observables in BF theories in terms of counting of

4. This is called chain contraction in algebraic topology

some dynamical objects such as closed geodesics, geodesic arcs or even geodesic nets. This would give new formulas in the spirit of the Fried conjecture of the form :

$$\begin{aligned} \text{Topological invariant} \quad &= \quad \text{value at 0 of generating function counting} \\ &\quad\ \ \text{dynamical objects associated to } V. \end{aligned}$$

Chapitre 4

Orbital counting.

1. Introduction

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold with negative curvature, X_1, X_2 are two oriented submanifolds in $\mathcal{M}$, for simplicity the reader can just take a pair of points, consider the function

$$\eta_{X_1,X_2}(z) = \sum_{\gamma} e^{-z\ell(\gamma)} \tag{61}$$

where the sum runs over geodesic arcs orthogonal to X_1, X_2 and going from X_1 to X_2, $\ell(\gamma)$ is the length of the geodesic arc γ.

A similar problem can be considered when $\mathcal{M}$ is a surface with negative curvature, and given a pair of closed geodesic arcs X_1, X_2 which may intersect or have some selfintersection.

2. Motivations for the problem under study.

Let us give three motivations to study the Poincaré series η which are all related to dynamical systems : analytic number theory, topological entropy and orbital counting. For more informations, we refer the reader to the surveys [**130, 131**].

2.1. Counting with complex functions. The first is the analogy of counting objects in dynamics with analytic number theory. The following array summarizes the different correspondences between counting primes in analytic number theory, counting periodic orbits in dynamics and counting arcs which is the subject of this chapter.

object	Prime numbers	Periodic orbits	Geodesic arcs
Counting function	$N_T = \lvert\{p \leqslant T\}\rvert$	$N_T = \lvert\{\gamma; \ell(\gamma) \leqslant T\}\rvert$	$N_T = \lvert\{\gamma; \ell(\gamma) \leqslant T\}\rvert$
Asymptotics	$N_T \sim \frac{T}{\log(T)}$	$N_T \sim C\frac{e^{h_{top}T}}{T}$	$N_T \sim Ce^{h_{top}T}$
Complex function	$\zeta(s) = \sum_1^\infty n^{-s}$ $= \prod_p (1-p^{-s})^{-1}$ Riemann	$\zeta(z) = \prod_\gamma (1 - e^{-z\ell(\gamma)})$ Smale	$\eta(z) = \sum_\gamma e^{-z\ell(\gamma)}$ Poincaré
Holomorphy domain	$Re(s) > 1$	$Re(z) > h_{top}$	$Re(z) > h_{top}$
Continuation	yes	yes for Axiom A flows	curvature -1, variable negative curvature ?
Zeroes, poles	?	Selberg	?
Value at $s = 0$	$\zeta(0) = 1 + \cdots + 1 + \cdots = -\frac{1}{2}$	$\zeta(z) = z^{2g-2}(C + \mathcal{O}(z))$	$\eta(0) = 1 + \cdots + 1 + \cdots = ?$

2.2. Riemannian geometry and topological entropy. The topological entropy is an invariant of a flow which measures the complexity of a flow $\varphi^t : M \mapsto M$ in long times. It is usually defined from the asymptotics number of Bowen balls [1] needed to cover the manifold M carrying the dynamics :

$$\lim_{\varepsilon\to 0^+} \limsup \frac{1}{T} \log\left(\text{Min number of Bowen balls } B(., T, \varepsilon)\text{needed to cover } M\right).$$

This is a definition of dynamical nature. Yet, for geodesic flows, there is another definition of topological entropy which is simpler. In a sense, it is purely geometric. Given two points (x, y) on any Riemannian manifold $(\mathcal{M}, g)$, let $N_T(x, y)$ be the number of geodesic arcs of

1. We refer to [**100**] for the definition of Bowen balls

length $\leqslant T$ from x to y, Mañé [**114**] and Paternain–Paternain [**136, 134**] showed that one could recover h_{top} by the formula :

$$\boxed{h_{top} = \lim_{T\to+\infty} \frac{1}{T} \log \int_{\mathcal{M}\times\mathcal{M}} N_T(x,y)dv(x)dv(y)} \tag{62}$$

where dv is the Riemannian volume. This was later generalized by G. Paternain [**135**] in the case with a potential to recover the topological pressure.

Then what could we prove if the pair (x, y) is fixed ? Mañé–Freire [**115**] showed that if the metric g has no conjugate points, h_{top} can be given by :

$$\boxed{h_{top} = \lim_{T\to+\infty} \frac{1}{T} \log N_T(x,y)} \tag{63}$$

for all pairs (x, y) ! Still in the case without conjugate points, Mañé has proved an identification with the growth of volumes of balls. He shows with Freire [**115**] that if we take a ball $B_\rho(r)$ centered at ρ on the universal cover $\tilde{\mathcal{M}} \mapsto \mathcal{M}$, then the volume growth of the ball when the radius r goes to infinity is related to h_{top} by the formula

$$h_{top} = \lim_{r\to+\infty} \frac{1}{r} \log \mathrm{Vol}\left(B_\rho(r)\right).$$

Hence for metrics without conjugate points, which contains the negative curvature case, the counting of arcs $N_T(x, y)$ connecting x to y gives h_{top} by the following formulas :

$$\boxed{h_{top} = \lim_{r\to+\infty} \frac{1}{r} \log \mathrm{Vol}\left(B(r)\right) = \lim_{T\to+\infty} \frac{1}{T} \log \int_{\mathcal{M}\times\mathcal{M}} N_T(x,y)dv(x)dv(y) = \lim_{T\to+\infty} \frac{1}{T} \log N_T(x,y).}$$

REMARK 2.1. $N_T(x, y)$ is deeply related to orbital counting. On the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$, fix x and consider the orbit of $\Gamma = \pi_1(\mathcal{M})$ through $y \in \tilde{\mathcal{M}}$, then

$$N_T(x,y) = |\{g \in \Gamma; g.y \in B_x(T)\}| \tag{64}$$

hence the number of points in the Γ orbit of y contained in balls of large radius.

2.3. Relation with Anosov flows and the work of Selberg, Margulis. Similarly to the study of periodic orbits, very precise informations on the arc counting are obtained either in strict negative curvature, either in algebraic situations using the powerful tools from homogeneous dynamics. In fact, in constant negative curvature, using the relation with the analysis of the Laplacian, precise results about arc counting were obtained by Delsarte [**36**], Huber [**95, 96**] and Selberg [**162, 163**] :

THEOREM 2.2 (Delsarte, Huber, Selberg). *If $\mathcal{M}$ has constant negative curvature then $\eta_{x,x}(s)$ has a meromorphic continuation to the complex plane with a simple pole at $s = 1$ which is the only pole on the vertical axis $Re(s) = 1$.*

The counting of orthogeodesics in constant negative curvature seems to be first studied by Good [**76**] and recently revisited by [**30, 29**]. In variable negative curvature, Margulis [**117**] showed in his thesis :

THEOREM 2.3 (Margulis). *The function η is holomorphic in the half–plane $Re(z) > h_{top}$ and $N_T(x,y) \sim C_{x,y}e^{h_{top}T}$.*

This was later generalized by Pollicott and Pollicott–Sharp in several other cases [**142, 144, 146, 145**].

2.4. Examples of Poincaré series and the quantum classical correspondence. Poincaré series deal with counting of geodesic arcs and the first results on their analytic continuation relied on the quantum–classical correspondence by expressing the Poincaré series in terms of quantities related to the Laplacian. In the spirit of the Selberg trace formula which relates some formula in terms of periodic geodesics to the trace of the heat kernel. This is possible in homogeneous situations because of the strong ties between the Laplacian and the geodesic flow. Working in variable curvature requires a different approach that we later discuss. Let (x,y) be some pair of points on the torus and we want to study the Poincaré series $\eta_{x,y}(z)=\sum_\gamma e^{-z\ell(\gamma)}$ where the sum runs over the geodesics connecting x and y. By the subordination identity $e^{-za}=\frac{z}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{z^2}{4u}}e^{-ua^2}u^{-\frac{3}{2}}du$ [**174**, p. 247 Chapter 5],[**184**, 3.6],[**173**, (β)p. 61] :

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} e^{-z\|x-y+k\|} &= \frac{z}{2\sqrt{\pi}}\int_0^\infty \sum_{k\in\mathbb{Z}^d} e^{-\frac{z^2}{4u}}e^{-u\|x-y+k\|^2}u^{-\frac{3}{2}}du\\ &= \frac{z}{\sqrt{\pi}}\int_0^\infty e^{-tz^2}\sum_{k\in\mathbb{Z}^d} e^{-\frac{\|x-y+k\|^2}{4t}}\frac{t^{\frac{3}{2}}}{t^2}dt\end{aligned}$$

with variable change $u=\frac{1}{4t}$. This makes appear the heat kernel on the torus since by Poisson summation one recognizes $\sum_{k\in\mathbb{Z}^d} e^{-\frac{\|x-y+k\|^2}{4t}}=(4\pi t)^{\frac{d}{2}}e^{-t\Delta}(x,y)$, Δ is the Laplacian on the torus, which yields a further simplification

$$\sum_{k\in\mathbb{Z}^d} e^{-z\|x-y+k\|} = \frac{z(4\pi)^{\frac{d}{2}}}{\sqrt{\pi}}\int_0^\infty e^{-t(\Delta+z^2)}(x,y)t^{\frac{d+1}{2}}\frac{dt}{t}$$

Now we use the Mellin transform identity : $(\Delta+z^2)^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t(\Delta+z^2)}t^s\frac{dt}{t}$ which yields :

$$\sum_{k\in\mathbb{Z}^d} e^{-z\|x-y+k\|} = \frac{z(4\pi)^{\frac{d}{2}}\Gamma(\frac{d+1}{2})}{\sqrt{\pi}}(\Delta+z^2)^{-\frac{d+1}{2}}(x,y).$$

Finally, we get an equation relating geodesic arcs and the Laplacian :

$$\boxed{\underbrace{\eta_{x,y}(z)=\sum_{k\in\mathbb{Z}^d} e^{-z\|x-y+k\|}}_{\text{classical}} = \underbrace{z2^d\pi^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})(\Delta+z^2)^{-\frac{d+1}{2}}(x,y)}_{\text{quantum}}} \tag{65}$$

which proves that $\eta(z)$ has meromorphic continuation to the whole complex plane with poles at $\pm i\lambda$ where λ is in the spectrum of $\sqrt{\Delta}$.

2.4.1. *Poincaré series on hyperbolic* 3*-manifolds.* Let $\mathcal{M}$ be a compact hyperbolic 3-manifold of the form $\mathcal{M}=\mathbb{H}^3/\Gamma$. We start from $\mathbb{H}^3$ before going to the quotient. Denote by L the shifted Laplacian $L=\Delta-1$ on $\mathbb{H}^3$ [2], the important idea is that L **satisfies the strong Huygens principle**. Then in Taylor–Metcalfe [**176**, equation 2.3 p. 3492], we find the explicit formula for the wave propagator :

$$\frac{\sin(t\sqrt{L})}{\sqrt{L}}\delta_y(x)=\frac{\delta(t-d(x,y))}{4\pi\sinh(t)} \tag{66}$$

where $d(x,y)$ is the hyperbolic distance between (x,y).

This implies when we go to the quotient $\mathcal{M}=\mathbb{H}^3/\Gamma$ that the counting measure reads

$$\mu(t)=\sum_\gamma \delta(t-\ell(\gamma))=4\pi\sinh(t)\frac{\sin(t\sqrt{L})}{\sqrt{L}}\delta_y(x)$$

where the operator $L=\Delta-1$ defined on the r.h.s is defined on $\mathcal{M}$. By Laplace transform, we get

$$\begin{aligned}\eta_{x,y}(z) &= 4\pi\int_0^\infty e^{-tz}\sinh(t)\frac{\sin(t\sqrt{L})}{\sqrt{L}}\delta_y(x)dt\\ \underbrace{=\sum_\gamma e^{-z\ell(\gamma)}}_{\text{classical}} &= \underbrace{2\pi\left((z-1)^2+\Delta-1\right)^{-1}(x,y)-2\pi\left((z+1)^2+\Delta-1\right)^{-1}(x,y)}_{\text{quantum}}\end{aligned}$$

2. Recall that Δ has spectrum in $[\frac{(3-1)^2}{4}=1,+\infty)$ in $\mathbb{H}^3$ so adding -1 shifts the spectrum to $[0,+\infty)$

[3] So the r.h.s has analytic continuation to the complex plane with poles still related to the spectrum of $\sqrt{\Delta-1}$ which seems to be a common feature of these algebraic situations.

3. Main Theorem.

Let us state the main Theorem of this chapter.

THEOREM 3.1 (D–Rivière). *Let $(\mathcal{M},g)$ be a negatively curved surface and $V\in C^\infty(TS^*\mathcal{M})$ is the generator of the geodesic flow. The function $\eta_{X_1,X_2}(z)$ extends as a meromorphic function on the complex plane. The poles are contained in the set of Pollicott–Ruelle resonances of the vector field $\mathcal{L}_V$ acting on 1-forms. If X_1,X_2 are two points*

$$\eta_{X_1,X_2}(0)=\frac{1}{\chi(\mathcal{M})}\ \text{if } X_1\neq X_2 \tag{67}$$

$$\eta_{X_1,X_1}(0)=\left(\frac{1}{\chi(\mathcal{M})}-1\right). \tag{68}$$

If X_1,X_2 are homologically trivial, there exists a couple of surfaces $\mathbf{S}_1,\mathbf{S}_2$ which bound the projections of X_1,X_2 s.t.

$$\boxed{\eta_{X_1,X_2}(0)=\varepsilon(X_2)\left(\frac{\chi(\mathbf{S}_1)\chi(\mathbf{S}_2)}{\chi(\mathcal{M})}-\chi(\mathbf{S}_1\cap\mathbf{S}_2)+\frac{1}{2}\chi(X_1\cap X_2)\right).} \tag{69}$$

In a certain sense, we generalize the results of Delsarte, Huber [**95, 96**], Selberg [**162, 163**] (1956) to the variable curvature case and this answers some question of Pollicott, Sharp (appendix to the thesis of Margulis [**117**]). Our main identity is somewhat reminiscent of the hyperbolic identities found by Basmajian, McShane which express some function of the orthogeodesic spectrum in terms of some Euler characteristic or volume [**8, 22, 23, 118, 119**]. There are relation of our results with the works of Bergeron–Charollois–Garcia–Venkatesh [**10, 11**] on the rationality of the value at zero of certain zeta functions arising in number theory (Klingen, Siegel, Shintani see Bergeron [**10, 11**] and also [**40, 73**]).

4. Sketch of proof.

Let us fix some conventions for the present chapter. The base surface reads $\mathcal{M}$ whereas the unit cotangent $S^*\mathcal{M}$ is denoted by M. The conormal of a curve $X\subset\mathcal{M}$ is the set $N^*X=\{(x;\xi);x\in X,\xi\in T_xX^\perp\}\subset S^*\mathcal{M}$. The proof of Theorem 3.1 has four parts.

(1) We use the theory of currents to give a new integral formula to represent the counting functions $N_T(x,y)$ and η. The important idea is to lift the counting problem to the cotangent $S^*\mathcal{M}$ and interpret arc countings as a counting of Reeb chords between Legendrian curves N^*X_1 and N^*X_2 in $S^*\mathcal{M}$.

(2) We use the results sketched in chapter 1 to relate to the resolvent $(V+z)^{-1}$ corresponding to the Lie derivative $\mathcal{L}_V$ along the flow, the resolvent acts on Sobolev anisotropic currents described in chapter 1.

(3) We show there are no poles at 0 based on the works of Dyatlov–Zworski [**47**]. We identify the value at 0 with minus the linking number of Legendrian knots.

(4) We calculate the linking number by using some constructible function which *quantizes* the Legendrian knot. We will later explain this notion of *quantization.*

3. There is a notation abuse, $\frac{\sin(t\sqrt{L})}{\sqrt{L}}\delta_y(x)$ is the Schwartz kernel of $\frac{\sin(t\sqrt{L})}{\sqrt{L}}$ taken at (x,y) which are fixed since we count arcs from x to y. But this is viewed as distribution of t, in the same way as $\delta(t-\|x-y\|)$ in the representation of the retarded propagator for wave equations in $\mathbb{R}^{1+3}$.

4.1. The integral formula on the cotangent space. We consider the conormals N^*X_1 and N^*X_2 to the submanifolds (X_1, X_2) in $\mathcal{M}$, these are Legendrian curves in $S^*\mathcal{M}$: the Liouville form pdq vanishes on the curves N^*X_1 and N^*X_2. In the present paragraph, for any oriented submanifold $S \subset M$, we denote by $[S]$ the corresponding integration current. The function $N_T(X_1, X_2)$ counts the number of Reeb chords from $\Sigma_1 = N^*X_1$ to $\Sigma_2 = N^*X_2$ in $S^*\mathcal{M}$. Consider the Legendrian curve $\Sigma_1 = N^*X_1$, we push it by the flow until time T, this describes a surface of dimension 2. The image of Σ_1 by

$$\Phi : (t, x) \in [0, T] \times S^*\mathcal{M} \mapsto \varphi^t(x) \in S^*\mathcal{M}$$

describes a current $\Phi_* ([[0, T] \times \Sigma_1])$ [4]. **Physically, this is the surface which is bounded by $\varphi^T(\Sigma_1) - \Sigma_1$.** With a natural choice of orientation, we get the homotopy formula due to Elie Cartan, used by De Rham **[38]**, Federer **[57]**, Harvey–Lawson **[86, 87]** in the context of currents :

$$\partial\Phi_* ([[0, T] \times \Sigma_1]) = \Phi_* ([(\partial[0, T]) \times \Sigma_1]) = \varphi^T_*[\Sigma_1] - [\Sigma_1]. \tag{70}$$

The integral formula for the surface reads :

$$\Phi_* ([[0, T] \times \Sigma_1]) = -\int_0^T \left(\iota_V e^{-tV*}[\Sigma_1]\right) dt. \tag{71}$$

Instead of giving the proof, we give a simple example.

Example 4.1 (From point to interval). *The integration current on the point $(0,0)$ is defined by the equation $[(0,0)] = \delta(x,y)dx \wedge dy$. $[0,0]$ plays the role of $[\Sigma_1]$. From the relation $\int_0^T \delta(x-t)dt = 1_{[0,T]}(x)$, we find*

$$\int -\iota_{\partial_x} e^{-t\partial_x *}\delta(x,y)dx \wedge dy = -\int_0^T (\delta(x-t,y)dy)\, dt = -1_{[0,T]}(x)\delta(y)dy$$

which is the current of integration on $[0,T]$ on the x axis which plays the role of the surface traced out by flowing $[\Sigma_1]$.

Now we just want to verify that the boundary of the current $-\int_0^T \left(\iota_V e^{-tV*}[\Sigma_1]\right) dt$ really matches the boundary of $\Phi_* ([[0, T] \times \Sigma_1])$. $\partial[\Sigma] = (-1)^{\deg([\Sigma])-1}d$ compares the boundary operator with the de Rham d differential and the Cartan magic formula $\mathcal{L}_V = d\iota_V + \iota_V d$.
$[\Sigma_1]$ is a closed current of integration on the knot, this is a 2-form.

$$\begin{aligned}\partial \int_0^T \left(-\iota_V e^{-tV*}[\Sigma_1]\right) dt &= d\int_0^T \left(-\iota_V e^{-tV*}[\Sigma_1]\right) dt \\ = \int_0^T -\mathcal{L}_V e^{-tV*}[\Sigma_1]dt &= e^{TV}_*[\Sigma_1] - [\Sigma_1].\end{aligned}$$

If we intersect this surface with Σ_2 which has dimension 1, under suitable transversality hypothesis, the intersection $\Phi_* ([0, T] \times \Sigma_1) \cap \Sigma_2$ is a finite number of points. Assuming the positivity of all oriented intersection numbers, we get :

$$\langle \Phi_* ([[0, T] \times \Sigma_1]), [\Sigma_2]\rangle = \int_M \left(\int_0^T \left(-\iota_V e^{-tV*}[\Sigma_1]\right) dt\right) \wedge [\Sigma_2] = N_T(X_1, X_2) \tag{72}$$

where we view the pairing as some intersection product of currents on M. Let us state the transversality assumption that guarantees that such intersection product is possible.

Definition 4.2. The transversality assumption : Σ_1 (resp Σ_2) is transverse to $E_s \oplus E_0$ (resp $E_u \oplus E_0$).

In that case the tangent spaces $de^{tV}(T\Sigma_1), de^{-tV}(T\Sigma_2)$ will approach E_u, E_s without reintersecting the bundles (E_s, E_u). The transversality assumption is satisfied by the vertical fibers of S^*M and we also proved in the original paper that it is satisfied by the conormal $N^*X \subset S^*\mathcal{M} = M$ where X is a geodesic arc in $\mathcal{M}$.

4. Recall Φ_* means pushforward by Φ

4.1.1. *From the counting measure to the Poincaré series.* Before we discuss the Poincaré series η, we will first describe the counting measure $\mu(t)$ defined as :

$$\mu(t) = \sum_{\gamma} \varepsilon(\gamma)\delta(t-\ell(\gamma)) \in \mathcal{D}'(\mathbb{R}_{>0}),$$

$\varepsilon(\gamma) = \pm 1$ depending on the orientations but the sign remains constant for all $t \geqslant T$ for some time $T > 0$ if Σ_1, Σ_2 satisfy the transversality condition 4.2. We use an integral formula involving de Rham currents :

$$\mu(t) = -\int_M \left(\left(\iota_V e^{-tV*}[\Sigma_1]\right)\right) \wedge [\Sigma_2]. \tag{73}$$

Remark 4.3. In fact, some supersymmetric version of the flat trace of Guillemin may be recast in the above language by doubling the variables as follows. A flow $\varphi^t : M \mapsto M$ with hyperbolic periodic orbits induces a flow on $M \times M$ as $\Phi^t : (x, y) \in M \times M \mapsto (x, \varphi^t(y)) \in M \times M$. Let $[\Delta] \in \mathcal{D}'^{,n}(M \times M)$ denotes the current of integration on the diagonal of $M \times M$ which is oriented once an orientation of M is chosen. Then a weighted counting of the periodic orbits can be recovered by considering :

$$\mu(t) = \int_{M\times M} \Phi^t_*[d_2] \wedge [d_2] = \sum_{\gamma} (-1)^{ind(\gamma)}\delta(t-\ell(\gamma)) = \sum_{k=0}^{\dim(M)} (-1)^k Tr^{\flat}_{\Lambda^k T^*M}(e^{-tV}) \tag{74}$$

where $\text{ind}(\gamma)$ is the index of the Poincaré return map induced by the periodic orbit γ. $\mu(t)$ is a distribution in the variable t exactly as in the above formula and the Guillemin trace formulas.

The Poincaré series is just obtained by Mellin transform of the counting measure μ :

$$\boxed{\eta(z) = \sum_{\gamma} e^{-\ell(\gamma)z}\varepsilon(\gamma) = \int_0^\infty e^{-tz}\mu(t)dt = \int_M \left(\int_0^\infty \left(-\iota_V e^{-tV*}[\Sigma_1]\right) e^{-tz}dt\right) \wedge [\Sigma_2]}$$

where the series converges when $Re(z) > h_{top}$ thanks to the results from the thesis of Margulis.

4.2. Relating to the resolvent. On the anisotropic Sobolev spaces of chapter 2, the propagator $e^{-tV} : \mathcal{H} \mapsto \mathcal{H}$ generates a strongly continuous semigroup. By the Hille–Yosida Theorem, one can then relate the propagator e^{-tV} with the resolvent via

$$\int_0^\infty e^{-tV}e^{-tz}dt = \underbrace{(V+z)^{-1}}_{\text{resolvent } R(z)} : \mathcal{H}^m(M) \mapsto \mathcal{H}^m(M). \tag{75}$$

If we transport our current $[\Sigma_1]$ by the forward flow for sufficiently long time, we will push the wave front set of $\varphi^{-T/2*}[\Sigma_1], \varphi^{T/2*}[\Sigma_2]$ in some conical neighborhoods of E_u^*, E_s^* respectively where $E_u^* \subset T^*M$ (resp $E_s^* \subset T^*M$) is the dual unstable (resp stable) bundle. For T large enough, these currents will belong to anisotropic spaces of currents (here we are cheating a bit since we did not discuss the points in the neutral direction but they cause no problems since they are in the elliptic region of V).

So up to transporting the currents by the flow, we may reformulate η in terms of the resolvent acting on anisotropic currents :

$$\boxed{\eta_{\Sigma_1,\Sigma_2}(z) = \int_M (-\iota_V R(z)[\Sigma_1]) \wedge [\Sigma_2].} \tag{76}$$

Then the upper bound on the wave front set of $R(z)$ **[46]** :

$$\begin{aligned} WF(R(z)) &= N^*d_2 \cup \Omega_+ \cup E_u^* \times E_s^* \\ \Omega_+ &= \{\left(x, \varphi^s(x); -\xi, ((d\varphi^s)^{-1})^t(\xi)\right); \xi(X) = 0, s \geqslant 0\}, \end{aligned} \tag{77}$$

where d_2 is the diagonal in $M \times M$ and N^*d_2 its conormal, yields the meromorphic continuation of η.

4.3. Elimination of the pole and linking of Legendrian knots. Now we shall prove that there is no pole at $z = 0$ and identify the value $\eta(0)$ with a linking number. Linking numbers can easily be visualized. Start from two oriented knots T_1, T_2 in $\mathbb{R}^3$, since $\mathbb{R}^3$ is contractible, we can consider a Seifert surface $\mathbf{S}_1$ bounded by T_1. Then we compute the oriented intersection number $\langle [\mathbf{S}_1], [T_2] \rangle$ which counts with orientation how many times the knot T_2 will intersect the surface $\mathbf{S}_1$. The important idea to keep in mind is that of taking a primitive of $[T_1]$ in the current theoretic sense. A formal definition reads :

DEFINITION 4.4 (Linking of knots). If T_1, T_2 are two oriented knots in a 3–manifold M s.t. the currents $[T_1], [T_2]$ are trivial in $H^2(\mathcal{D}'(M), \mathbb{R})$. Then for any primitive $[\mathbf{S}_i]$ of $[T_i]$, $\partial[\mathbf{S}_i] = [T_i]$, we define

$$\mathbf{Lk}(T_1, T_2) = \langle [\mathbf{S}_1], [T_2] \rangle = \langle [T_1], [\mathbf{S}_2] \rangle$$

where $\mathbf{Lk}(T_1, T_2)$ does not depend on the choice of primitive.

There is another closely related approach to linking numbers whose root goes back to Gauss. We learned about this from topological quantum field theory. In particular in papers by Harvey–Lawson [**86, 87**], Fukaya [**68, 69, 70**], Lescop [**109**]. In some perturbative treatment of Chern–Simons theory, many authors express 3-manifolds and knot invariants by **counting configurations of graphs**. To perform such counting, one writes integral formulas on configuration space which involve products of some current on $M \times M$ called the *propagator* of the theory [5]. This propagator is some sort of de Rham primitive of the current $[d_2]$ of integration on the diagonal $d_2 \subset M \times M$.

EXAMPLE 4.5 (Linking propagator). *Let T_1, T_2 be two knots in $\mathbb{R}^3$. Their linking admits an integral formula going back to Gauss*

$$\int_{T_1 \times T_2} \omega(x - y, dx - dy) \text{ where } \omega(x, dx) = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{4\pi \|x\|^3}$$

is a 2-form called linking form.

4.3.1. *Resolvent and the linking form.* To extract the de Rham primitive of the current $[\Sigma_1]$ carried by the Legendrian knot Σ_1, we shall use the resolvent $R(z)$. In our work, the operator $\iota_V R_+(0)$, where $R_+(0) = \lim_{z \to 0} R(z) - \frac{\Pi_0}{z}$, will take the primitive and its Schwartz kernel is the analogue of the Gauss linking propagator. This operator plays an essential role in our work with Chaubet, see Chapter 3, since it appears in the derivative of the torsion w.r.t. the flat connection.

The key result that we shall use is due to Dyatlov–Zworski [**47**, Lemma 3.5]. We give a reformulation of it as follows :

THEOREM 4.6. *Let $\mathcal{M}$ be a negatively curved surface. Let $\mathcal{D}'^{,1}_{E_u^*}(S^*\mathcal{M})$ denotes currents of degree 1 whose wave front is contained in the dual unstable bundle E_u^*. There are **no Jordan blocks** in the resonant states in $\ker(V) \cap \mathcal{D}'^{,1}_{E_u^*}(M)$ which is generated by $(\theta, U_1, \dots, U_{b_1(\mathcal{M})})$ where θ is the contact form and $dU_i = \iota_V U_i = 0$ and $(U_1, \dots, U_{b_1(\mathcal{M})})$ generate $H^1(S^*M, \mathbb{R})$.*

*Dually, there are **no Jordan blocks** in the coresonant states in $\ker(V) \cap \mathcal{D}'^{,2}_{E_s^*}(M)$ which is generated by $(d\theta, S_1, \dots, S_{b_1(\mathcal{M})})$ where $\theta \wedge S_i = 0$ and $\langle U_i, S_j \rangle = \delta_{ij}$.*

Near $z = 0$, the resolvent acting on $C^\infty(T^*M)$ admits a decomposition

$$R(z) = \frac{\Pi_0}{z} + R_+(z) : C^\infty(T^*M) \mapsto \mathcal{D}'(T^*M) \tag{78}$$

where $R_+(z)$ is holomorphic near $z = 0$. Using the Lie–Cartan formula $V = [d, \iota_V]$, $d[\Sigma] = 0$ and Poincaré duality, this implies that :

$$d(V + z)^{-1} \iota_V [\Sigma] = V(V + z)^{-1}[\Sigma] = -z \underbrace{V \frac{\Pi_0}{z}([\Sigma])}_{=0} - z R_+(z) \iota_V [\Sigma] + [\Sigma].$$

5. This propagator has nothing to do with the propagator of the flow

This implies that $\iota_V R_+(0)[\Sigma_2]$ is a primitive of $[\Sigma_2]$ using the holomorphy of $R_+(z)$ at $z=0$. We conclude that

$$\eta_{X_1,X_2}(0) = -\left\langle \iota_V R_+(0)[\Sigma_2], [\Sigma_1] \right\rangle \tag{79}$$

which is minus the linking between the Legendrian knots Σ_1, Σ_2.

5. Linking between cotangent fibers.

Recall in case X_1, X_2 are two **points**, the value $\eta_{X_1,X_2}(0)$ equals the linking between the Legendrian knots $\Sigma_1 = N^*X_1$ and $\Sigma_2 = N^*X_2$. We will present two proofs of linking between cotangent fibers. The first one uses Morse theory and the second one is a hyperbolic geometry proof.

5.1. The Morse theoretic proof. The result $\mathbf{Lk}(S_x^*\mathcal{M}, S_y^*\mathcal{M}) = -\frac{1}{\chi(\mathcal{M})}$ was communicated to us at an early stage by Baptiste Chantraine and the present proof was suggested and strongly inspired by discussions with Jean Yves Welschinger. We would like to thank both of them warmly. We want to compute $\eta_{x,y}(0) = -\mathbf{Lk}(S_x^*\mathcal{M}, S_y^*\mathcal{M})$ which does not depend on $x \neq y$, it is purely topological. Given $X \in C^\infty(T\mathcal{M})$ with hyperbolic zeroes, assume w.l.o.g. that $X = \nabla f$ for a Morse function f and a metric of the form $\sum dx_i^2$ in Morse coordinates. This implies that the Lyapunov exponents of ∇f at critical points belong to $\{\pm 1\}$. The graph Γ of $\frac{X}{\|X\|}$ defines an integration current in $S^*\left(\mathcal{M} \setminus Crit(X)\right)$. The important fact is that it **extends uniquely as a de Rham current of finite mass** denoted by $[\Gamma]$ on $S^*\mathcal{M}$. But the current $[\Gamma]$ is not closed and satisfies the equation :

$$\partial[\Gamma] = -\sum_{a \in Crit(X)} (-1)^{ind(a)}[S_a^*\mathcal{M}]. \tag{80}$$

So the current theoretic boundary of $[\Gamma]$ is the union of the currents of integration on the cotangent fibers weighted by the Morse indices of f. Γ is the Seifert surface of the link $\sum_{a\in Crit(X)}(-1)^{ind(a)+1}[S_a^*\mathcal{M}]$.

We conclude by observing that for $y \in \mathcal{M} \setminus Crit(X)$:

$$1 = \left\langle [\Gamma], S_y^*\mathcal{M} \right\rangle = -\sum_{a \in Crit(X)} (-1)^{ind(a)}\mathbf{Lk}(S_a^*\mathcal{M}, S_y^*\mathcal{M}) = -\chi(\mathcal{M})\mathbf{Lk}(S_x^*\mathcal{M}, S_y^*\mathcal{M})$$

and we are done.

5.2. The hyperbolic geometry proof. The proof was strongly inspired by discussion with Gabriel Paternain and we wish to warmly thank him here. The starting point is to reformulate the integral $\int_{y\in\mathcal{M}} \eta_{x,y}(s)dv(y)$ as some integral in the universal cover $\tilde{\mathcal{M}} \mapsto \mathcal{M}$. Let $\tilde{y} \in \tilde{\mathcal{M}}$ be some representative of y, then we reformulate the counting as a sum over the fundamental group :

$$\eta_{x,y}(s) = \sum_{g \in \pi_1(\mathcal{M})} e^{-sd(x,g.\tilde{y})}.$$

Let D be a fundamental domain of $\tilde{\mathcal{M}}$, then by Fubini we get

$$\int_{y\in\mathcal{M}} \eta_{x,y}(s)dv(y) = \int_{\tilde{y}\in D} \sum_{g\in\pi_1(\mathcal{M})} e^{-sd(x,g.\tilde{y})}dv(\tilde{y})$$

$$= \sum_{g\in\pi_1(\mathcal{M})} \int_{\tilde{y}\in D} e^{-sd(x,g.\tilde{y})}dv(\tilde{y}) = \int_{y\in\tilde{\mathcal{M}}} e^{-sd(x,y)}dv(y).$$

We pull–back the last integral over $T_x\mathcal{M}$ by the exponential map $\exp_x : T_x\mathcal{M} \mapsto \tilde{\mathcal{M}}$ which is a global diffeomorphism. Using the tautological fact that for $y = \exp_x(tv)$ where $v \in S_x\mathcal{M}$, we find $d(x,y) = t$, this yields :

$$\boxed{\int_{y\in\mathcal{M}} \eta_{x,y}(s)dv(y) = \int_0^\infty e^{-ts}\left(\int_{v\in S_x\mathcal{M}} |\det(d\exp_x(tv))|d\theta(v)\right)dt.} \tag{81}$$

But in hyperbolic geometry, the exponential map can be explicitely calculated based on the relation with Jacobi fields. By the properties of the Jacobi fields in [**9**, Prop 67 p. 249], for every $h \in T_x\mathcal{M}, v \in T_x\mathcal{M}$, we find that

$$d\exp_x(tv)(h) = Y(t)$$

where $Y(t)$ solves the Jacobi field equation $Y'' = R(\gamma'(t), Y(t))\gamma'(t)$ with $Y(0) = 0, Y'(0) = h$ where $\gamma(t)$ is the geodesic $\exp_x(tv)$. In constant curvature $K = -1$, $Y(t) = -\sinh(t)h$ for every vector $h \perp v$ [6]. It follows that $|\det(d\exp_x(tv))| = |\sinh(t)|$ and

$$\int_{y\in\mathcal{M}} \eta_{x,y}(s)dv(y) = \int_0^\infty e^{-st}\left(\int_{v\in S_x\mathcal{M}} \sinh(t)d\theta(v)\right)dt = 2\pi\int_0^\infty e^{-st}\sinh(t)dt = \frac{2\pi}{s^2-1}.$$

In particular $\int_{y\in\mathcal{M}} \eta_{x,y}(0)dv(y) = -2\pi$. But we know already that $\eta_{x,y}(0) = -\mathbf{Lk}(S_x^*\mathcal{M}, S_y^*\mathcal{M})$ which is a topological **constant** c of y which is independent of $y \neq x$. Therefore

$$\int_{y\in\mathcal{M}} \eta_{x,y}(0)dv(y) = cVol(\mathcal{M}) = -2\pi.$$

By the Gauss–Bonnet formula, $\int_{\mathcal{M}} KdA = 2\pi\chi(\mathcal{M})$, since $K = -1$ we find that $Vol(\mathcal{M}) = -2\pi\chi(\mathcal{M})$ therefore $c = \frac{1}{\chi(\mathcal{M})}$ as desired.

5.3. How to think about orientations. Orientations are a central issue in our work. There is a part of arbitrariness but certain choices must be made consistently with what we are counting. The reader should keep in mind that the Poincaré series does not know about orientations, it only counts geodesic arcs, but we represent this counting formula in terms of intersection of currents and this requires to make some choices consistent with the fact we count positive intersections.

EXAMPLE 5.1. *There is only* 1 *point at the intersection of x and y axis on $\mathbb{R}^2$ and the number* 1 *carries no orientation information! However one can realize* 1 *as the oriented intersection $Ox \cap Oy$ where $\mathbb{R}^2$ is oriented by $\partial_x \wedge \partial_y$ or as $Oy \cap Ox$ where $\mathbb{R}^2$ is oriented by $\partial_y \wedge \partial_x$.*

For an oriented submanifold $N \subset \mathcal{M}$, denote by $[N]$ the integration current on N then $-[N]$ is the integration current on N with the **opposite orientation**. We choose some orientation on the base surface $\mathcal{M}$, this is our **first choice**. This means any small disc D in $\mathcal{M}$ inherits a canonical orientation from $\mathcal{M}$ which implies that the boundary ∂D also has an induced orientation.

EXAMPLE 5.2. *In the plane $\mathbb{R}^2$ with canonical orientation (∂_x, ∂_y), the disc $x^2+y^2 \leqslant 1$ has boundary the unit circle oriented* ***counterclockwise****.*

We need to orient $M = S^*\mathcal{M}$ before we can intersect currents in $S^*\mathcal{M}$. We choose an orientation on the fibers $\mathbb{S}^1$ of $S^*\mathcal{M}$ in such a way that the image $e^{\varepsilon V}(S_x^*\mathcal{M})$ of a fiber by the geodesic flow at time $\varepsilon > 0$, once projected on the base $\mathcal{M}$, bounds a disc of radius $\varepsilon > 0$ around x has canonical orientation. The orientation of the base $\mathcal{M}$ plus the fiber induces an orientation $Or(S^*\mathcal{M})$ of $S^*\mathcal{M}$ with the following property : in this orientation, if we fix an arbitrary fiber $S_x^*\mathcal{M}$, the orientation induces a current $[S_x^*\mathcal{M}]$. For any germ of section $\sigma : U_x \hookrightarrow S^*\mathcal{M}$ defined in some neighborhood U_x of x, $[\sigma]$ is a current induced by the orientation of U_x, then $\int_{S^*\mathcal{M}}[\sigma] \wedge [S_x^*\mathcal{M}] = 1$.

EXAMPLE 5.3. *If $S^*\mathbb{R}^2$ with coordinates (x, y, θ) is oriented by $\partial_x \wedge \partial_y \wedge \partial_\theta$ then the fibers should be oriented by ∂_θ.*

6. In fact, the metric g in polar normal coordinates (r, θ) reads

$$g = dr^2 - r^2\sinh^2(r)d\theta$$

where $d\theta$ is the canonical volume on the unit circle $\mathbb{S}^1$.

Now, we start from two fibers $S_x^*\mathcal{M}$ and $S_y^*\mathcal{M}$ with their orientations. Then the question is how to orient the surface $\cup_{t\in[0,T]}\varphi^t(S_x^*\mathcal{M})$ obtained by flowing out $S_x^*\mathcal{M}$ in $S^*\mathcal{M}$ by the geodesic flow so that its oriented intersection with $S_y^*\mathcal{M}$ yields only **positive integers**. It suffices that the projection of this surface on the base $\mathcal{M}$ has the same orientation as $\mathcal{M}$. For small $0<\varepsilon<T<<1$, we therefore expect that the projection of $\cup_{t\in[\varepsilon,T]}\varphi^t(S_x^*\mathcal{M})$ on $\mathcal{M}$ looks like a ring and the projection of the oriented boundary of $\cup_{t\in[\varepsilon,T]}\varphi^t(S_x^*\mathcal{M})$ should look like a circle of radius T minus a circle of radius ε both centered around x and oriented counterclockwise. The current of integration on $\cup_{t\in[\varepsilon,T]}\varphi^t(S_x^*\mathcal{M})$ with correct orientation is therefore exactly given by $\int_\varepsilon^T -\iota_V e^{-tV*}[S_x^*\mathcal{M}]dt$ since

$$\begin{aligned} & \partial\pi_*\int_\varepsilon^T -\iota_V e^{-tV*}[S_x^*\mathcal{M}]dt = \pi_*\partial\int_\varepsilon^T -\iota_V e^{-tV*}[S_x^*\mathcal{M}]dt \\ = \; & \pi_*\left(e^{-TV*}[S_x^*\mathcal{M}] - e^{-\varepsilon V*}[S_x^*\mathcal{M}]\right) = \pi_* e_*^{TV}[S_x^*\mathcal{M}] - \pi_* e_*^{\varepsilon V}[S_x^*\mathcal{M}]. \end{aligned}$$

6. The case X_1 is a point and X_2 is a curve.

Assume that Ω is some domain bounded by some simple curve X_1 without selfintersections and $X_2=y$ which does not belong to X_1. We would like to calculate the linking $\mathbf{Lk}(N^*\Omega, S^*y\mathcal{M})$ between the conormal of the domain Ω and the fiber $S_y^*\mathcal{M}$. Before, we need to state a Poincaré-Hopf Theorem we will use.

6.0.1. *Poincaré-Hopf for manifold with boundaries by Morse 1929.* Our workhorse Theorem is a generalisation of the Poincaré–Hopf Theorem by Morse. We give some interpretation of the Poincaré–Hopf formula for manifolds with boundary, due to Morse (Theorem A_0 p. 170-171), in terms of intersection of some graph of vector field with the normal cycle

$$N\Omega_+ = \{(x;v) | x\in\partial\Omega, v\in T_x\partial\Omega^\perp\}$$

of a domain with boundary :

THEOREM 6.1. *Let Ω be a planar domain in some closed oriented surface $\mathcal{M}$ with smooth boundary $\partial\Omega$, $W\in C^\infty(T\mathcal{M})$ a smooth vector field such that the graph of $\frac{W}{\|W\|}$ in $S\mathcal{M}$ intersects the outward normal $N\Omega_+$ **transversally**, in particular W does not vanish on $\partial\Omega$.*

Let $W^\perp$ be the orthogonal projection of W on $\partial\Omega$ and for every $a\in Crit(W^\perp)$, $ind_{W^\perp}(a)$ is the Poincaré index of a for $W^\perp$. Then :

$$\chi(\Omega) = \underbrace{\sum_{a\in Crit(W)\cap\Omega} (-1)^{ind_W(a)}}_{\textbf{bulk term}} - \underbrace{\sum_{a \text{ s.t. } W\perp\partial\Omega_+} (-1)^{ind_{W^\perp}(a)}}_{\textbf{boundary term}}. \tag{82}$$

Denote by $[\underline{0}|_\Omega]$ the current of integration on the zero section restricted to Ω. Then denoting by S the graph of W :

$$\chi(\Omega) = \underbrace{\langle [S], [\underline{0}|_\Omega]\rangle}_{\textbf{bulk term}} + \underbrace{\langle [S], [N\Omega_+]\rangle}_{\textbf{boundary term}}$$

where we can count in terms of both outgoing or ingoing normals to get the Euler characteristic.

The reformulation in terms of intersection with the normal cycle plays a central role in our approach and is a classical idea from microlocal geometry. This Theorem has some strong analogies with the Gauss–Bonnet Theorem for surfaces with boundaries that we would like to stress with the next :

EXAMPLE 6.2 (Gauss–Bonnet–Chern). *Under the assumptions of Theorem 6.1, $Kd\sigma$ is the scalar curvature times the volume form for the metric, α is the angular form :*

$$2\pi\chi(\Omega) = \underbrace{\int_\Omega Kd\sigma}_{bulk} + \underbrace{\int_{\partial\Omega}\alpha}_{boundary}. \tag{83}$$

REMARK 6.3. In fact, both Theorems can be unified if one represents the Euler class with currents in $T\mathcal{M}$ and formalizes the notion of transgression, originally due to Chern, in terms of currents.

We also have the extremely naive impression that this is the simplest instance of some sort of bulk–boundary phenomenon in elementary differential topology.

6.0.2. *Idea of the proof.* Choose some vector field X with hyperbolic critical points outside $\partial\Omega$ s.t. the graph of $\frac{X}{\|X\|}$ over $\partial\Omega$ coincides with the inward normal of Ω. X has isolated critical points in Ω. The key idea is that the graph Γ of $\frac{X}{\|X\|}$ over Ω is a cobordism between the normal $N\Omega$ and the link $\cup_{a\in\Omega\cap\mathrm{Crit}(X)}S^*_a\mathcal{M}$:

$$\partial[\Gamma] = [N\Omega] - \sum_{a\in\Omega\cap\mathrm{Crit}(X)} (-1)^{ind(a)}[S^*_a\mathcal{M}].$$

By Stokes Theorem, observe that

$$\left\langle[\Gamma],[S^*_y\mathcal{M}]\right\rangle = \langle\partial[\Gamma],[R]\rangle$$

where $d[R]=[S^*_y\mathcal{M}]$, hence R is a de Rham primitive of $[S^*_y\mathcal{M}]$. Therefore

$$\begin{aligned}\left\langle[\Gamma],[S^*_y\mathcal{M}]\right\rangle &= \langle[N\Omega],[R]\rangle - \sum_{a\in\Omega\cap\mathrm{Crit}(X)} (-1)^{ind(a)}\left\langle[S^*_a\mathcal{M}],[R]\right\rangle \\ &= \mathbf{Lk}(N^*\Omega,S^*y) - \underbrace{\sum_{a\in\Omega\cap\mathrm{Crit}(X)} (-1)^{ind(a)}}_{=\chi(\Omega)}\underbrace{\mathbf{Lk}(S^*_a\mathcal{M},S^*_y\mathcal{M})}_{=-\frac{1}{\chi(\mathcal{M})}}.\end{aligned}$$

Applying the result of Morse yields $\mathbf{Lk}(N^*\Omega,S^*y) = -\frac{\chi(\Omega)}{\chi(\mathcal{M})}$ si $y\notin\Omega = 1-\frac{\chi(\Omega)}{\chi(\mathcal{M})}$ si $y\in\Omega$. Finally :

$$\mathbf{Lk}(N^*\Omega,S^*y) = \chi(\Omega\cap\{y\}) - \frac{\chi(\Omega)\chi(\{y\})}{\chi(\mathcal{M})}$$

7. The general case.

In this part, we will be very sketchy and we refer to the original paper for further details. We start from a Legendrian knot whose projection on the surface has only transverse double self–intersections. In the original paper, we first killed all multiple self–intersections by pushing the Legendrian knot by the geodesic flow.

The key idea is to decompose some arc with selfintersections as a finite union of simple closed curves to reduce to the simple case of computing the linking of $N^*\gamma_1$ with $N^*\gamma_2$ for a pair γ_1,γ_2 of smooth simple closed curves. For each of these closed curves γ_i appearing in the decomposition, we try to generate the Seifert surface S_i which is bounded by $N^*\gamma_i$.

Our derivation of the topological content of $\eta_{X_1,X_2}(0)$ relies crucially on the *Poincaré-Hopf index formula* as it was derived by Morse in [**124**], and we use this formula from a point of view which is inspired by microlocal geometry. In fact, the microlocal index theorems of Brylinski–Dubson–Kashiwara [**24**] and Kashiwara [**98**], later revisited by Kashiwara–Schapira [**99**, p. 384] and Grinberg–McPherson [**80**], can be understood as generalizations of the Poincaré–Hopf index formula.

7.0.1. *Constructible functions.* Σ is an oriented curve hence it defines a current $[\Sigma]$, the push–forward by the projection π : $\pi_*[\Sigma]=[\gamma]$ where $[\gamma]$ is some current which lives downstairs on the base space. The curve γ downstairs is a geodesic arc, its complement $M\setminus\gamma=\cup_{i\in I}\Omega_i$ is a finite union of connected components. We fix $f|_{\Omega_1}=0$ and therefore

(84) $\qquad f|_{\Omega_i} =$ oriented intersection numbers of a path from Ω_1 to Ω_i with γ.

So f is some piecewise constant function which satisfies the equation :

$$\boxed{\partial f = [\gamma]} \tag{85}$$

in the sense of currents where ∂ is the de Rham boundary operator.

For the moment f is defined up to some integer constant but we fixed the constant by imposing the minimum of f to be 0 therefore f does not take negative values and 0 is attained. The constructible function f is then viewed as a *quantization* of the Legendrian Σ. In general, given a real algebraic manifold X and a stratification $\mathcal{S}$ of X, one says that a function $f : X \mapsto \mathbb{Z}$ is constructible if it is constant on each stratum. Our next goal is to use the constructible function to decompose the curve γ as a union of simple closed curves and extract the Seifert surfaces bounded by the conormals of these simple closed curves.

7.0.2. *Euler integral.* The notion of Euler characteristic generalizes to constructible functions [**182, 158, 159, 160**] : $f \mapsto \chi(f) := \int_X f d\chi$, and it is referred as Euler (characteristic) integrals – see [**7, 33**] for an introduction to this notion and various applications to motion sensors. We just say here that, for the characteristic function $\mathbf{1}_\Omega$ of a domain Ω, $\chi(\mathbf{1}_\Omega) = \int_\Omega d\chi$ coincides with the usual Euler characteristic $\chi(\Omega)$ and that the extension to constructible functions follows from $\mathbb{Z}$–linearity [**7**, Def 2.6 p. 831]. An important property is that χ satisfies the inclusion exclusion relation therefore for reasonable subsets (A, B) [7], we have $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. One can view χ as some sort of topological measure, which explains the basic identity $\int 1_A d\chi = \chi(A)$. Euler integrals are defined by the fundamental relation

$$\int f d\chi = \sum_{s \in \mathbb{Z}} s\chi\{f^{-1}(s)\}. \tag{86}$$

In our case this definition is not practical. The function f is defined as in paragraph 7.0.1 and we will decompose f by level sets. We set $\Omega_i = \{f \geqslant i\}$ the region where $f \geqslant i$ and observe that we have the elementary identities :

$$\boxed{f = \sum_{i=0}^{\infty} 1_{\Omega_i}, \quad \int f d\chi = \sum_{i=0}^{\infty} \chi(\Omega_i).} \tag{87}$$

In microlocal geometry, given a constructible function f one associates a conical Lagrangian cycle Λ_f which generalizes the simple example 1_Ω indicator of $\Omega \mapsto \Lambda_{1_\Omega} = N^*\Omega$ which is the conormal of Ω. Then, for every pair f_1, f_2 of constructible functions on X which satisfy some appropriate transversality conditions, the microlocal index formula reads [**80**, p. 269] :

$$\boxed{\underbrace{\chi(f_1 f_2)}_{\text{Euler integral}} = \underbrace{[\mathrm{Ch}(f_1)] \cap [\mathrm{Ch}(f_2)]}_{\text{Lagrangian intersection}}} \tag{88}$$

where $[\mathrm{Ch}(f_1)] \cap [\mathrm{Ch}(f_2)]$ is the intersection of the two corresponding Lagrangian cycles. Hence, the microlocal index formula gives an interpretation of Lagrangian intersections as the Euler characteristic of some product of constructible functions. Our formula is in the spirit of the above microlocal index formula. But instead of computing the intersection of Lagrangian cycles, we rather consider the linking of Legendrian cycles and we also express it in terms of constructible functions. More precisely, for every pair of Legendrian cycles Σ_1, Σ_2 which are small deformations by Hamiltonian isotopies of the unit conormal bundle of our homologically trivial geodesic representatives c_1 and c_2, we associate a pair (f_1, f_2) of constructible functions quantizing the two knots Σ_1, Σ_2. Then we prove the microlocal index formula :

$$\boxed{\underbrace{\frac{\chi(f_1)\chi(f_2)}{\chi(X)} - \chi(f_1 f_2) + \frac{1}{2}\chi(\mathbf{1}_{c_1 \cap c_2})}_{\text{Euler integral}} = \underbrace{\pm\mathbf{Lk}(\Sigma_1, \Sigma_2)}_{\text{Legendrian linking}} = \underbrace{\lim_{s \to 0} \sum_{\gamma \in \mathcal{P}_{c_1, c_2} : \ell(\gamma) > 0} e^{-\ell(\gamma)s}}_{\text{Poincaré series at zero}}.} \tag{89}$$

In the framework of symplectic topology, the Poincaré series is understood as a sum over the Reeb chords of the geodesic flow joining the two Legendrian curves Σ_1 and Σ_2. Hence,

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this index formula [8], gives an interpretation of some linking of two Legendrian curves in terms of Euler integrals but also as a zeta regularized sum over the Reeb chords from Σ_1 to Σ_2. While the first equality is obtained by purely topological means, the second one is a consequence of our spectral approach to the problem.

7.0.3. *The decomposition Theorem for our Legendrian knots.* Now everything relies on a decomposition of our initial Legendrian knot Σ as a union of conormals of simple closed curves :

THEOREM 7.1. *Let Σ be a Legendrian knot whose projection on M has only double transverse intersections. Then there exists $(\Omega_i)_i$ oriented domains with piecewise C^∞ boundaries s.t.*

$$\boxed{[\mathcal{L}] = \sum_i [N^*\Omega_i]} \tag{90}$$

and f is the corresponding constructible function s.t. $f = \sum 1_{\Omega_i}$.

Then using smoothing arguments we can round the corners of our piecewise smooth simple curves to reduce to smooth simple closed curves. Then using the bilinearity of the linking, we reduce the calculation to the simpler case of the linking between conormals of smooth simple closed curves where we managed to apply Theorem 6.1, yielding the final result.

8. Perspectives.

There are many directions we would like to explore together with Rivière. First, we would like to gain a better *microlocal* understanding of the quantum versus classical correspondance for the problem of arc countings in both negative variable curvature and more *homogeneous* cases, this includes exotic spaces such as metric graphs where it would be already interesting to define the geodesic flow. In the light of the recent result of Guillarmou–Lefeuvre [**82**] related to rigidity questions, one could also ask if the marked length spectrum of all geodesic arcs between two given points on some Riemannian manifold (M, g) of negative curvature determines the isometry class of g for small perturbations of the metric ?

8. However see [**180**, Th.4] and [**147**, Eq. (10)] for related results of Turaev regarding the first equality on $S^*\mathbb{S}^2$.

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