GAUSSIAN FREE FIELDS AND RIEMANNIAN RIGIDITY.

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ABSTRACT. On a compact Riemannian manifold (M, g) of dimension $d \leqslant 4$, we present a rigorous construction of the renormalized partition function $Z_g(\lambda)$ of a massive Gaussian Free Field where we explicitely determine the local counterterms using microlocal methods. Then we show that $Z_g(\lambda)$ determines the Laplace spectrum of (M, g) hence imposes some strong geometric constraints on the Riemannian structure of (M, g). From this observation, using classical results in Riemannian geometry, we illustrate how the partition function allows to probe the Riemannian structure of the underlying manifold (M, g).

1. INTRODUCTION.

In the present paper, we only consider smooth, compact, Riemannian manifolds (M, g) without boundary. For simplicity we also assume M to be connected and orientable. On such manifold, the Laplace–Beltrami operator Δ admits a discrete spectral resolution [11, Lemma 1.6.3 p. 51] which means there is an increasing sequence of eigenvalues :

$$\sigma(\Delta) = \{0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \to +\infty\}$$

and corresponding L^2-basis of eigenfunctions $(e_\lambda)_{\lambda\in\sigma(\Delta)}$ so that $\Delta e_\lambda = \lambda e_\lambda$.

1.0.1. *Gaussian Free Fields and Feynman amplitudes.* We next briefly recall the definition of the Gaussian free field (GFF) associated to Δ. Our definition is probabilistic and represents the Gaussian Free Field ϕ as a random distribution on M [14, Corollary 3.8 p. 21] [18, eq (1.7) p. 3] [20] (see also [21, section 4.2] for a related definition in a planar domain D). In the classical physics litterature, this object is called Euclidean bosonic quantum field and can be defined differently in terms of Gaussian measures on the space of distributions although the two definitions are equivalent.

Definition 1.1 (Gaussian Free Field)**.** *The Gaussian free field ϕ associated to (M, g) is defined as follows : denote by $(e_\lambda)_{\lambda\in\sigma(\Delta)}$ the spectral resolution of Δ. Consider a sequence $(c_\lambda)_{\lambda\in\sigma(\Delta)}, c_\lambda \in \mathcal{N}(0, 1)$ of independent, identically distributed, centered Gaussian random variables. Then we define the Gaussian Free Field ϕ as the random series*

$$\phi = \sum_{\lambda\in\sigma(\Delta)\setminus\{0\}} \frac{c_\lambda}{\sqrt{\lambda}} e_\lambda \tag{1.1}$$

where the sum runs over the positive eigenvalues of Δ and the series converges almost surely as distribution in $\mathcal{D}'(M)$.

The covariance of the Gaussian free field defined above is the Green function :

$$\boxed{\mathbf{G}(x,y)=\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\frac{1}{\lambda}e_\lambda(x)e_\lambda(y)}$$

where the above series converges in $\mathcal{D}'(M\times M)$.

Note that in our definition of the Gaussian Free Field, we choose the random field ϕ to be orthogonal to constant functions so that the covariance of ϕ is exactly the Green's function as defined above. The above means that the Gaussian measure $d\mu$ is constructed on the subspace $H^s(M)_0$, $\forall s<1-\frac{d}{2}$ of Sobolev distributions orthogonal to constants. This has the consequence that most of our arguments deal with the restriction of Δ to the orthogonal of constant functions. Remark that to really construct a measure on $H^s(M)=H^s(M)_0\oplus\mathbb{R}$ would require tensoring $d\mu$ with the Lebesgue measure dc on $\mathbb{R}$ which takes care of the zero modes : $d\mu\otimes dc$ [18, p. 30]. But we will not use this extended measure in the present paper.

We next recall the definition of polygon Feynman amplitudes.

Definition 1.2 (Feynman amplitudes)**.** *Let (M,g) be a closed compact Riemannian manifold and $\mathbf{G}$ the Green function of the Laplace–Beltrami operator Δ.*

For $n\geqslant 2$, define the formal product of Green function :

$$t_n(x_1,\dots,x_n)=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1) \tag{1.2}$$

as an element in $C^\infty(M^n\setminus \text{diagonals})$.

The amplitude t_n is well–defined outside diagonals because the Green function $\mathbf{G}$ is smooth outside the diagonal and develops singularities at coinciding points.

The study of Euclidean quantum fields has a long history in the constructive quantum field theory community with seminal contributions of Albeverio, Fröhlich, Gallavotti, Glimm, Guerra, Jaffe, Nelson, Seiler, Spencer, Simon, Symanzik and Wightman just to name a few, see [46, 54, 55, 56] and the references inside. We were inspired in part by the work of Seiler [25] who stressed the relation between quantum fields and functional determinants. Our main result is a rigorous construction of the renormalized partition function of some quadratic perturbation of the GFF by some smooth potential on Riemannian manifolds of dimension $\leqslant 4$. This includes the massive GFF but not only. We show how some simple microlocal methods allow to renormalize and give a rigorous mathematical meaning to the partition function :

$$Z_g(V)=\frac{\int e^{-\frac{1}{2}\int_M\langle\nabla\phi,\nabla\phi\rangle-V(x)\phi^2dv(x)}[d\phi]}{\int e^{-\frac{1}{2}\int_M\langle\nabla\phi,\nabla\phi\rangle dv(x)}[d\phi]}$$

where v is the Riemannian volume on M, $V\in C^\infty(M)$ is some smooth potential and $e^{-\frac{1}{2}\int_M\langle\nabla\phi,\nabla\phi\rangle dv(x)}[d\phi]$ is the functional GFF measure. We rely on the relation between the Feynman amplitudes from definition 1.2 and the probability distribution of the Wick square of the GFF. The general idea is to consider quantum fields interacting with external fields or depending on external parameters such as the potential $V\in C^\infty(M)$, then consider the response of the free energy $\log Z$ to variations of the external parameters as in linear response theory. A useful observation for our purpose is the relation to the Gohberg–Krein determinants and the fact that $\int_M V : \phi^2 : dv(x)$ is a random variable in dimension $d < 4$. This gives a direct relation to the spectral theory of the Laplacian which seemed to be unnoticed in the litterature. We hope that the rather simple microlocal approach presented in our work might be useful for people working on the GFF in low dimension, in particular in dimensions $d = 2, 3$. Also the result in dimension $d = 4$ might also be relevant since most interesting QFT appearing in nature are 4–dimensional.

1.0.2. *Probing Riemannian geometries with quantum fields.* A second purpose or our work is to relate the properties of the quantum field on the manifold with the geometric properties of the underlying manifold itself. This is in the spirit of the work of Osgood–Phillips–Sarnak who used the zeta determinant (which is the partition function of the massless GFF) as a height function on the space of $2d$ metrics. Our motivation was to show how the usual free Boson field can probe the geometrical features of the manifold. In physical terminology, we couple a massive free Boson scalar field with some potential which stays classical and we study $\delta \log(Z(V))$ and how it interacts with geometry. In the classical QFT litterature [40, 6], one considers instead the log partition function as a function of the metric g and the functional derivative $\frac{\delta \log Z(g)}{\delta g^{\mu\nu}(x)} = \langle T_{\mu\nu}(x) \rangle$ is interpreted as the vacuum expectation value of the Quantum stress–energy tensor. For free Bosonic fields, in local chart, $\langle T_{\mu\nu}(x) \rangle = \langle -\partial_{x^\mu} \phi \partial_{x^\nu} \phi + \dots \rangle$ which is ill–defined and requires renormalization to make sense [1]. Intuitively, one intends to *measure the response of the quantum system when it is coupled to gravity.* On the other hand, in the present work, we are interested in the fluctuations of the integrated Wick square $\int_M : \phi^2 :$ which is a simpler object than the quantum stress energy tensor [2] and is morally related to the loop space of the manifold as we discuss in paragraph 2.1.1 when we discuss the toy model of graphs. At this point, we should stress that our results on rigidity of Riemannian structures come in two flavors :

(1) the diffeomorphism type of M (topology and C^∞ structure) is fixed and the question is about the metric g up to isometry,
(2) the diffeomorphism type of M is not fixed and the question is about the **pair** (M, g) up to isometry.

In what follows, we denote by $\mathbb{C}[[\lambda]]$ the ring of formal power series in λ. For geometrical applications, we will be particularly interested in the renormalized partition function:

$$Z_g(\lambda) = \mathbb{E}\left(\exp\left(-\frac{\lambda}{2} \int_M : \phi^2(x) : dv \right) \right) \in \mathbb{C}[[\lambda]] \tag{1.3}$$

[1] In fact, there are several subtleties: for free massive scalar bosons, the field square $\phi^2(x)$ is proportional to the trace of the classical stress–energy tensor $Tr(T_{cl}(x))$ as showed in [28, Appendix B p. 67] but the zeta regularized average $\langle Tr(T_{cl}(x)) \rangle$ does not coincide with the quantum stress energy tensor by the anomaly formula [28, Lemma B2 p. 69]. However, it seems that the full renormalized stress energy tensor is not a random variable valued in distributions, one can only make sense of it inserted inside correlation functions.

[2] It is a random variable in dimension $d = 2, 3$

of a free bosonic theory when the dimension of (M, g) equals $2 \leqslant d \leqslant 4$ where we need some extra renormalization when $d = 4$, see Theorem 1. In fact, for $|\lambda|$ small enough, the partition function $Z_g(\lambda) \in \mathbb{C}[[\lambda]]$ actually converges thanks to the identification with Gohberg–Krein's determinants. It depends only on the isometry class of (M, g) and a natural question would be what informations on (M, g) can be extracted from $Z_g(\lambda)$ as formal power series.

2. Main results.

In what follows, we introduce some preliminary definitions on convergence of sequences of Riemannian manifolds and moduli spaces of metrics that we need to state our two main results on Riemannian rigidity from quantum fields.

2.0.1. *Convergent sequences of Riemannian manifolds in Lipschitz topology.* In the present paragraph, the diffeomorphism type of the manifolds is **not fixed**. Let us recall that the set of C^∞ Riemannian metrics on M with the usual Fréchet topology on smooth 2-tensors is denoted by $\mathbf{Met}(M)$. It is an open convex cone of the space of symmetric 2–tensors in the C^∞ topology. We have the natural action of $\mathbf{Diff}(M)$, the set of diffeomorphisms of M acting by pull–back on $\mathbf{Met}(M)$. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are equivalent if there exists a C^∞ diffeomorphism $\varphi : M_1 \mapsto M_2$ s.t. $\varphi^* g_2 = g_1$. Then $\mathfrak{Riemm}$ is the set of equivalence classes of Riemannian manifolds. We insist that elements in $\mathfrak{Riemm}$ can have **different diffeomorphism types**. In our work, we will only need to define Lipschitz convergence for sequences of smooth Riemannian manifolds. Concretely, a sequence of isometry classes of Riemannian manifolds $(M_n, g_n)_{n\in\mathbb{N}}$ converges to (M, g) in $\mathfrak{Riemm}$ for the Lipschitz topology if there exists a sequence $\varphi_n : M_n \mapsto M$ of bilipschitz homeomorphisms s.t. both $\sup_{x\in M_n} \|d\varphi_n\|$ and $\sup_{x\in M_1} \|d\varphi_n^{-1}\|$ tend to 1 when $n \to +\infty$. We will need the Lipschitz topology in the formulation of our main Theorem 1 and also when we discuss compactness properties of isospectral metrics in subsubsection 5.0.3.

2.0.2. *The moduli space of metrics.* In this paragraph, we fix the smooth manifold M and only the metrics on M will vary. We define the **moduli space of Riemannian metrics** as a quotient space [2, p. 381] :

$$\boxed{\mathcal{R}(M) = \mathbf{Met}(M)/\mathbf{Diff}(M)} \tag{2.1}$$

endowed with the quotient topology. In practice, a sequence of isometry classes $[g_n] \underset{n\to+\infty}{\to} [g]$ if there is a sequence of representatives g_n of $[g_n]$ which converges to g in the C^∞–topology [30, p. 602] [31, p. 233] (see also [32, p. 175]). For every $0 < \varepsilon < 1$, we use the notations $\mathcal{R}(M)_{\leqslant -\varepsilon}$ and $\mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]}$ for the moduli space of Riemannian metrics with negative sectional curvatures bounded from above by $-\varepsilon$ and whose sectional curvature is contained in $[-\varepsilon^{-1}, -\varepsilon]$ respectively.

It is a result of Ebin [7, 8] that $\mathcal{R}(M)$ endowed with the quotient topology is a Hausdorff **metric space** [10, p. 317–319]. In the sequel, we shall summarize the main properties of the metric structure on $\mathcal{R}(M)$.

2.0.3. *The moduli space $\mathcal{R}(M)$ as a metric space.* For every $s > \frac{\dim(M)}{2}$, Ebin considered the Hilbert manifold $\mathbf{Met}^s(M)$ of Sobolev metrics of regularity s and the topological group $\mathbf{Diff}^{s+1}(M)$ of bijective maps f s.t. both f and f^{-1} are Sobolev maps in $H^{s+1}(M,M)$ [9, 2.3 p. 158] acting on $\mathbf{Met}^s(M)$. He constructed a Riemannian metric $\mathbf{g}_s$ on $\mathbf{Met}^s(M)$, called Sobolev metric of degree s, which is invariant by the action of $\mathbf{Diff}^{s+1}(M)$ and is defined as follows. The tangent space $T_g\mathbf{Met}^s(M)$ at $g \in \mathbf{Met}^s(M)$ is naturally identified with the Sobolev space $H^s(S^2T^*M)$ of Sobolev sections of S^2T^*M of regularity s. So for every $h \in H^s(S^2T^*M) \simeq T_g\mathbf{Met}^s(M)$,

$$\langle h,h\rangle_{\mathbf{g}_s} = \sum_{k=0}^{s} \int_M \left\langle \nabla_g^k h, \nabla_g^k h\right\rangle_{S^{k+2}T^*M} dv_g \tag{2.2}$$

where dv_g is the volume form induced by g, ∇_g is the covariant derivative defined by g acting on $H^s(S^2T^*M)$ and $\langle .,.\rangle_{S^{k+2}T^*M}$ denotes the fiberwise scalar product on the bundle $S^{k+2}T^*M$ induced by g. The corresponding distance function on $\mathbf{Met}^s(M)$ is denoted by $\mathbf{d}^s$. Following Fischer [10, p. 319], we may define a distance $\mathbf{d}$ on $\mathbf{Met}(M)$ as follows :

$$\mathbf{d}(g_1,g_2) = \sum_{k>\frac{\dim(M)}{2}} \frac{1}{2^k}\frac{\mathbf{d}^k(g_1,g_2)}{1+\mathbf{d}^k(g_1,g_2)} \tag{2.3}$$

where the distance $\mathbf{d}$ is $\mathbf{Diff}(M)$ invariant by construction. Hence $\mathbf{d}$ induces a distance on the quotient space $\mathcal{R}(M)$ which generates the quotient topology. Now that we recalled the metric space structure of $\mathcal{R}(M)$, a natural question is if $\mathcal{R}(M)$ admits a smooth manifold structure.

2.0.4. *The regular part $\mathcal{G}$ of $\mathcal{R}(M)$.* Unfortunately, the answer is negative and $\mathcal{R}(M)$ should be understood as some kind of orbifold. The proper setting for the analysis in infinite dimensional space of metrics is that of inductive limit of Hilbert spaces (ILH) structures which are specializations of Fréchet manifolds defined by Omori [3, Def II.5 p. 4]. So all the words *submanifolds* or *diffeomorphisms* must be understood in the sense of ILH submanifolds and diffeomorphisms. The set $\mathcal{R}(M)$ does not have a manifold structure but it is a fundamental result of Ebin [7, 8] and Palais independently that the action of $\mathbf{Diff}(M)$ on $\mathbf{Met}(M)$ admits slices. Moreover Ebin [7, 8] proved that for a metric g whose isometry group I_g is trivial, $I_g = \{\varphi \in \mathbf{Diff}(M), \varphi^*g = g\} = Id$, the quotient $\mathcal{R}(M)$ has a manifold structure in some neighborhood of $[g]$. Such $[g]$ are called **regular points** of the moduli space $\mathcal{R}(M)$ and the set of regular points is denoted by $\mathcal{G}$. It is a result of Ebin that $\mathcal{G} \subset \mathcal{R}(M)$ is **open dense and has a smooth manifold structure**. In the sequel, for every $\varepsilon > 0$, we shall denote by $\mathcal{G}_{\geqslant\varepsilon}$ the set of $[g] \in \mathcal{G}$ s.t. $\mathbf{d}([g],\partial\mathcal{G}) \geqslant \varepsilon$ where $\partial\mathcal{G} = \mathcal{R}(M) \setminus \mathcal{G}$.

2.1. **Fluctuations of the integrated Wick square.** In quantum field theory on curved space times, one is interested in the behaviour of the stress–energy tensor and its fluctuations under quantization of the fields assuming that the metric stays classical. For instance, many works of Moretti [34, 35, 36, 37, 38] deal with the renormalization of various quantum field theoretic quantities, for instance the stress–energy tensor, using zeta regularization and local point splitting methods. A natural question one could be interested in is what would be a probabilistic interpretation of the renormalized stress energy tensor $:T^{\mu\nu}:$ and can it be related to the geometry of the underlying manifold ? However, in the present paper we study fluctuations of the integral of the Wick square $\int_M :\phi^2(x): dv$ on the manifold M which is a simpler observable and is the integral of the *field fluctuations* in Moretti's work. In aQFT, it also appears in the work of Sanders [41] and is interpreted as a local temperature. It was observed by the anonymous referee of the present paper that the lower order terms in the small mass perturbation theory for $\log(Z_g)$ have relevant geometric meaning once they are correctly defined using the correct regularization, see [33, section 4.1].

2.1.1. *DGFF and loop measures on graphs.* We argue that Wick squares allow us to probe the geometry of the manifold M since it gives us access to the loop space of M. In probability, starting from the work of Symanzik, the Wick square can be related to loop measures associated to some random walks on graphs [42, 43] and it is assumed that the continuous Wick square should be related to some continuum version of loop measures. They express the local times of the self–intersections of random Brownian paths.

To illustrate our main results in the most direct way, we focus on graphs as a toy model. Consider a finite oriented graph Γ whose vertices and edges are denoted by $E(\Gamma)$ and $V(\Gamma)$ respectively. To simplify our formulas, we shall make the assumption that for every pair of vertices $(i,j)\in\Gamma^2$ there is at most one edge connecting them. Start from a Laplacian $(\Lambda-T)$ where $\Lambda=\mathrm{Diag}(\Lambda(j))_{j\in V(\Gamma)}=\lambda(j)\delta_{ji}$ is diagonal and T is the incidence matrix of the graph Γ, $T_{ij}=1$ if $i\neq j$ are connected by an edge otherwise $T_{ij}=0$.

Path space $x\mapsto y$. We consider some measure on the space of paths $\gamma: x\mapsto y$ from x to y, the mass of a path γ reads $\prod_{j\in\gamma}\lambda(j)^{-1}$ where the product runs over the vertices visited by γ. Assume the diagonal part Λ is large enough [3], it means the mass of loops $\mu(\gamma)$ is small enough. The Green function associated to our Laplacian reads

$$(\Lambda-T)^{-1}_{xy}=\sum_{\gamma:x\mapsto y}\prod_{j\in V(\Gamma)}\lambda(j)^{-n_\gamma(j)}$$

where $n_\gamma(j)$ is the time the random path γ spends in j. To calculate the average time a random path spends in some given subset $\Omega\subset V(\Gamma)$, we first modify the diagonal part with a formal parameter, the previous identity yields immediately: $(\Lambda(z^{-1}1_\Omega+1_{\Omega^c})-T)^{-1}_{xy}=\sum_{\gamma:x\mapsto y}\prod_{j\in\Omega}(z\lambda(j)^{-1})^{n_\gamma(j)}\prod_{j\in\Omega^c}(\lambda(j)^{-1})^{n_\gamma(j)}$. The r.h.s. rewrites

$$\sum_{\gamma:x\mapsto y}\prod_{j\in\Omega}(z)^{n_\gamma(j)}\prod_{j\in V(\Gamma)}(\lambda(j)^{-1})^{n_\gamma(j)}=\sum_{\gamma:x\mapsto y}z^{\sum_{j\in\Omega}n_\gamma(j)}\prod_{j\in V(\Gamma)}(\lambda(j)^{-1})^{n_\gamma(j)},$$

therefore we get $\frac{1}{p!}(z\partial_z)^p(\Lambda(z^{-1}1_\Omega+1_{\Omega^c})-T)^{-1}_{xy}|_{z=1}=\sum_{\gamma:x\mapsto y}\left(\sum_{j\in\Omega}n_\gamma(j)\right)^p\mu(\gamma)$ which gives the moments of the function $\gamma\mapsto\ell(\gamma\cap\Omega)$ defined on path space. Now, we consider the DGFF (discrete Gaussian Free Field) measure whose covariance reads $(\Lambda-T)^{-1}$. Finally, the resolvent identity relates correlators of the DGFF, involving the Wick square and the

[3] $\lambda(j)>2\times$ the sup of incidence number of every vertex

local time a random path spends in Ω:

$$\boxed{\mathbb{E}\left(\phi(x)\left(\sum_{j\in\Omega}\frac{\lambda(j)}{2}:\phi^2(j):\right)\phi(y)\right)=\sum_{\gamma:x\mapsto y}\left(\sum_{j\in\Omega}n_\gamma(j)\right)\mu(\gamma).}$$

Loop measure. In what follows, our loops can backtrack on Γ. There are two space of loops: the marked loops $\mathcal{ML}$ [4] and the unmarked loops $\mathcal{L}$ where the space $\mathcal{ML}$ fibers over $\mathcal{L}$. Given a loop $\gamma\in\mathcal{L}$ in the graph Γ, its mass reads $\mu(\gamma)=\prod_{j\in\gamma}\lambda(j)^{-1}$ where the product runs over the vertices visited by γ. The measure defined in this way does not depend on the marking of the loop γ so induces a measure on the space of unmarked loops.
Recall $n_\gamma(j)$ is the number of visits of a loop γ at the vertex j, we have $\ell(\gamma)=\sum_{j\in V(\Gamma)}n_\gamma(j)$. Assume the diagonal part Λ is large enough so that we get:

$$\begin{aligned}-\log\det\left(Id-\Lambda^{-1}zT\right) &= \sum_{k=1}^\infty\frac{1}{k}Tr\left(\left(\Lambda^{-1}zT\right)^k\right)=\sum_{k=1}^\infty\frac{1}{k}\sum_{\ell(\gamma)=k,\gamma\in\mathcal{ML}}\frac{z^{\sum_{j\in V(\Gamma)}n_\gamma(j)}}{\ell(\gamma)}\left(\prod_{j\in V(\Gamma)}\lambda(j)^{-1}\right)^{n_\gamma(j)}\\ &= \sum_{\gamma\in\mathcal{ML}}\frac{z^{\ell(\gamma)}n_\gamma(j)}{\ell(\gamma)}\mu(\gamma)=\sum_{\gamma\in\mathcal{L}}z^{\ell(\gamma)}\mu(\gamma),\end{aligned}$$

where the last sum is over unmarked loops. In the last step, we need to divide the mass of each marked loop by $\ell(\gamma)$ in order to recover the correct measure on $\mathcal{L}$ after push–forward against the counting measure along the fibration $\mathcal{ML}\mapsto\mathcal{L}$.
Length spectrum from the DGFF. In the context of graphs, what we call length spectrum is for all $k\in\mathbb{N}$, the mass of all unmarked loops γ whose length is $\ell(\gamma)=k$, this is simply a sequence of real numbers defined as: $\sigma(\Gamma)=\{\sum_{\ell(\gamma)=k}\mu(\gamma),k\in\mathbb{N}\}$. Let us show how the partition function of some quadratic perturbation of the DGFF recovers the spectrum for $\sigma(\Gamma)$. The identity $-\log\det\left(Id-\Lambda^{-1}zT\right)=\sum_{\gamma\in\mathcal{L}}z^{\ell(\gamma)}\mu(\gamma)$ immediately implies:

$$\boxed{-2\log\mathbb{E}\left(\sum_{j\in V(\Gamma)}(z^{-1}-1)\lambda(j)\phi^2(j)\right)=C+\sum_{\gamma\in\mathcal{L}}z^{\ell(\gamma)}\mu(\gamma)}$$

where the expectation is taken w.r.t. the DGFF measure whose covariance is the propagator $(\Lambda-T)^{-1}$, the constant does not depend on z. We recover the length spectrum from the moments of the function $\gamma\in\mathcal{L}\mapsto\ell(\gamma)$ which are encoded in the free energy, for $p\geqslant 0$:

$$\boxed{\frac{i}{\pi}\int_{\tilde{\gamma}}\log\mathbb{E}\left(\sum_{j\in V(\Gamma)}(z^{-1}-1)\lambda(j)\phi^2(j)\right)\frac{dz}{z^{k+1}}=\sum_{\ell(\gamma)=k}\mu(\gamma)}$$

where $\tilde{\gamma}$ is some small contour surrounding 0.

We would like to summarize the correspondence between objects from the DGFF and path space:

[4] Loops together with a choice of basepoint

Discrete	Quantum correlator	Path space
Potential Wick squares DGFF	$-2\log\mathbb{E}\left(\sum_{j\in V(\Gamma)}(z^{-1}-1)\lambda(j)\phi^2(j)\right)$	$C+\sum_{\gamma\in\mathcal{L}} z^{\ell(\gamma)}\mu(\gamma)$
Two point function DGFF	$\mathbb{E}\left(\phi(x)\left(\sum_{j\in\Omega}\frac{\lambda(j)}{2}:\phi^2(j):\right)\phi(y)\right)$	$\sum_{\gamma:x\mapsto y}\left(\sum_{j\in\Omega}n_\gamma(j)\right)\mu(\gamma)$

On discrete graphs there seems to be no intuitive difference between geodesic loops or random loops whereas on Riemannian manifolds these are very different objects. However, on Riemannian manifolds of negative curvature, there is a strong relation between Brownian motion on the base manifold M, the continuous version of random walks, and the geodesic flow on the unitary cosphere bundle S^*M over M which is Anosov. This topic was studied by many authors like Ancona, Arnaudon, Guivarc'h, Kaimanovich, Kendall, Kifer, Ledrappier, Le Jan, Pinsky and Thalmaier among many others (see [1] and references therein). Our main results, Thereoms 1 and 2, give an explicit relation between fluctuations of the Wick squares, the partition function $Z_g(\lambda)$, periodic geodesics and rigidity on manifolds with negative curvature. This generalizes in the continuum setting the above formulas on graphs, the DGFF becomes the continuum GFF and discrete loops are replaced by periodic geodesics. An important challenge is already to show that the partition function makes sense using renormalization.

2.1.2. *Periods of the geodesic flow.* We recall the definition of the periods of the geodesic flow [32, section 10.5].

Definition 2.1 (Periods). *Let us consider the moduli space of Riemannian metrics $\mathcal{R}(M)$ on M. For every element of $\mathcal{R}(M)$, choose a representative g. We denote by $(\Phi^t)_g : S^*M \mapsto S^*M$ the geodesic flow acting on the unitary cosphere bundle S^*M. Then for every class $[g]\in\mathcal{R}(M)$, we define the* ***periods*** *$\mathcal{P}([g])$ as the set :*

$$\boxed{\mathcal{P}([g])=\{T>0 \text{ s.t. } \Phi_g^T(x;\xi)=(x;\xi) \text{ for some } (x;\xi)\in S^*M\}\subset\mathbb{R}_{>0}.} \tag{2.4}$$

The set $\mathcal{P}(g)$ is called the **length spectrum** of (M,g).

2.2. **Main results.** Recall we defined the formal product t_n of Green functions in definition 1.2. For a compact operator A, we will denote by $\sigma(A)$ the set of singular values of A. On any oriented smooth manifold X, we shall denote by $|\Lambda^{top}|X$ the bundle of densities on X and by $C^\infty\left(|\Lambda^{top}|X\right)$ its smooth sections. Our main result reads :

Theorem 1. *Given a closed compact Riemannian manifold (M,g) of dimension $2\leqslant d\leqslant 4$, a function $V\in C^\infty(M)$, define the sequence of numbers*

$$c_n(g,V)=\int_{M^n} t_n(x_1,\dots,x_n)V(x_1)\dots V(x_n)dv_n,$$

$n\in\mathbb{N}$ where dv_n is the Riemannian density in $C^\infty\left(|\Lambda^{top}|M^n\right)$. For $\varepsilon\in(0,1]$, let $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ be the heat regularized GFF, $:\phi_\varepsilon^2(x):=\phi_\varepsilon^2(x)-\mathbb{E}\left(\phi_\varepsilon^2(x)\right)$ and define the renormalized partition

functions :

$$Z_g(\lambda,V) = \lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv\right)\right),\ \textit{when } d=(2,3),$$
$$Z_g(\lambda,V) = \lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv - \frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\right),\ \textit{when } d=4.$$

Then the sequence $c_n(g,V)$ is well–defined for $n>\frac{d}{2}$ and the partition functions Z_g satisfies the following identity for small $|\lambda|$:

$$Z_g(\lambda,V) = \exp\left(P(\lambda)+\sum_{n>\frac{d}{2}}\frac{(-1)^n c_n(g,V)\lambda^n}{2n}\right) \tag{2.5}$$

where $P=c\lambda^2$ when $d=4$, $P=0$ when $d<4$ and Z_g^{-2} extends as an ***entire function*** *on the complex plane $\mathbb{C}$ whose zeroes lie in $-\sigma(V\Delta^{-1})$.*

Note that $V\Delta^{-1}$ is a pseudodifferential operator of negative degree hence a compact operator and $\sigma(\Delta^{-1}V)$ is well–defined. We observe that for $d=4$, the formal integral $c_2(g,V)$ is ill–defined. If it were well–defined, it would be understood as the limit when $\varepsilon\to 0^+$: $\lim_{\varepsilon\to 0^+}\frac{1}{4}\mathbb{E}\left(:\phi^2(V)::\phi^2(V):\right)$ which diverges logarithmically. This divergence is subtracted by the counterterm $\frac{\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|$. The resulting finite part **is hidden in the constant** c in the polynomial term $P(\lambda)$ which **depends on the metric g and the function V**. But in the special case where $V=1$, we will see that c depends on the metric g **only through the spectrum** of Δ.

At this point, it was pointed out to the author by Claudio Dappiaggi that there should be some explicit relation between the renormalization done here and the methods from the papers [22, 44, 23] on Euclidean algebraic Quantum Field Theory which use an Euclidean version of Epstein–Glaser renormalization. We hope that the rather simple microlocal approach presented in our work might be useful for people working on the GFF in low dimension, in particular in dimensions $d=2,3$.

From the above, we deduce the following corollaries when $V=1\in C^\infty(M)$:

Corollary 2.2. *Let $(M_1,g_1),(M_2,g_2)$ be a pair of compact Riemannian manifolds without boundary of dimension $2\leqslant d\leqslant 4$, then the following claims are equivalent :*

1. *$c_n(g_1)=c_n(g_2)$ for all $n>\frac{d}{2}$,*
2. *the partition functions coincide $Z_{g_1}=Z_{g_2}$,*
3. *$(M_1,g_1),(M_2,g_2)$ are* ***isospectral.***

In particular the Einstein–Hilbert action S_{EH}, hence the Euler characteristic $\chi(M)$ when M is a surface, can be recovered from Z_g by the formula :

$$\boxed{S_{EH}(g) = Res|_{s=\frac{d}{2}-1}\sum_{\lambda,Z_g(\lambda)^{-2}=0}\lambda^{-s}.}$$

and if (g_1, g_2) are metrics with negative sectional curvatures s.t. $Z_{g_1} = Z_{g_2}$, then $\mathcal{P}(g_1) = \mathcal{P}(g_2)$ where the length spectrum from definition 2.1 coincides with the singular support of the distribution :

$$\boxed{t \mapsto Re\left(\sum_{\lambda, Z_g(\lambda)^{-2}=0} e^{it\sqrt{\lambda}}\right) \in \mathcal{D}'(\mathbb{R}_{>0}).} \tag{2.6}$$

Real valued random variables X are entirely characterised by their probability distribution or equivalently by their generating function which is the formal Fourier–Laplace transform of the probability distribution. In dimension $d = (2, 3)$, the second main Theorem of our note deals with the rigidity of the Riemannian structure in negative curvature where the fluctuations of the Wick square are encoded by the probability distribution of the random variable $\int_M : \phi^2(x) : dv$ or the partition function Z_g which should be understood as some kind of Fourier–Laplace transform of the probability distribution of $\int_M : \phi^2(x) : dv$. One can realize the GFF measure as a Gaussian measure μ on the topological vector space $\mathcal{D}'(M)$. In fact, we prove that when $d = 2, 3$, the regularized real valued functionals $\phi \in \mathcal{D}'(M) \mapsto \int_M (e^{-\varepsilon\Delta}\phi)^2(x)dv - \mathbb{E}\left(\int_M (e^{-\varepsilon\Delta}\phi)^2(x)dv\right)$ converge in $L^p, p \in [2, +\infty)$ over the vector space $\mathcal{D}'(M)$ for the GFF measure μ.

For $d = 4$, $\int_M : \phi^2(x) : dv$ is no longer a random variable. Only the renormalized partition function Z_g is well–defined and we obtain a similar rigidity result fixing the renormalized partition function. The result illustrates the intricate relation between Wick squares and loop spaces that we described for graphs in paragraph 2.1.1 in the setting of manifolds, closed geodesics play the role of the loop space of a finite graph. Our second Theorem is stated in terms of finite dimensional submanifolds N of the regular part $\mathcal{G}$ of the moduli space of metrics. We will explain right after the statement of the Theorem some subtle aspects of our assumptions.

Theorem 2. *For every compact Riemannian manifold (M, g) of dimension $2 \leqslant d \leqslant 4$, ϕ is the Gaussian free field with covariance $\mathbf{G}$ with corresponding measure μ. Denote by $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ to be the heat regularized GFF.*

If $d = 2, 3$ then the limit $\int_M : \phi^2(x) : dv = \lim_{\varepsilon \to 0^+} \int_M \phi_\varepsilon^2(x)dv - \mathbb{E}\left(\int_M \phi_\varepsilon^2(x)dv\right)$ converges as a real valued random variable in $L^p(\mathcal{D}'(M), \mu), 2 \leqslant p < +\infty$ with the following properties :

1. *Let N be a* ***finite dimensional*** *submanifold of $\mathcal{G} \subset \mathcal{R}(M)$ s.t. its boundary ∂N is contained in the boundary $\partial\mathcal{G}$ of the regular part $\mathcal{G}$. For all $\varepsilon > 0$, the set of classes of metrics $[g] \in N \cap \mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ such that the random variable $\int_M : \phi^2(x) : dv$* ***has given probability distribution is finite****.*
2. *When $d = 3$, for a sequence $(M_i, g_i)_{i\in\mathbb{N}}$ of Riemannian 3–manifolds of negative curvature such that the random variable $\int_{M_i} : \phi^2(x) : dv_i$* ***has a fixed given probability distribution****, the number of diffeomorphisms types represented in the sequence $(M_i)_i$ is* ***finite*** *and one can extract a subsequence such that M_i has* ***fixed diffeomorphism type*** *and $g_i \to g$ for some metric g in the C^∞ topology.*

If $d = 4$ then

(1) *Let N be a **finite dimensional** submanifold of $\mathcal{G} \subset \mathcal{R}(M)$ s.t. its boundary ∂N is contained in the boundary $\partial\mathcal{G}$ of the regular part $\mathcal{G}$. For all $\varepsilon \in (0,1)$, the set of classes of metrics $[g] \in N \cap \mathcal{R}(M)_{[-\varepsilon^{-1},-\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$ **with given partition function** Z_g **is finite**.*

(2) *For a sequence $(M_i, g_i)_{i\in\mathbb{N}}$ of Riemannian 4–manifolds of negative sectional curvatures bounded in some compact interval such that the **partition function** Z_g **is given**, the number of diffeomorphisms types represented in the sequence $(M_i)_i$ is **finite** and one can extract a subsequence such that M_i has **fixed diffeomorphism type** and $(M_i, g_i) \to (M, g)$ in the **Lipschitz topology**.*

Our result gives an example of metric dependent (non topological) Quantum Field Theory where the knowledge of the partition function gives both some **topological and metrical** constraints on the Riemannian manifold (M, g). This result is also in the spirit of the work of Osgood–Phillips–Sarnak [63], who used the partition function of the massless GFF (the zeta determinant) as a height function on the classes of $2d$ metrics, which is a proper function in certain cases for surfaces. Our proof given in section 6 is very detailed because we use well–known but hard to find results in the litterature and we believe they might be relevant for people working on the GFF.

Let us comment on the definition of N. The set N is a submanifold of the regular part $\mathcal{G}$ since only $\mathcal{G}$ has a manifold structure. The boundary ∂N of N is defined by taking the closure of N in $\mathcal{R}(M)$ for the topology of $\mathcal{R}(M)$, then $\partial N = \overline{N} \setminus \mathrm{Int}\,(N)$ is considered as a subset of $\mathcal{R}(M)$. A subtle but important observation is that N is not **necessarily compact**. Let us explain why and then give some example. Both $\mathcal{R}(M)$ and $\mathcal{G}$, endowed with the Ebin metric $\mathbf{d}$, have finite diameter. But the point is that bounded subsets for the metric $\mathbf{d}$ are not necessarily bounded for the induced topology on $\mathcal{R}(M)$ as illustrated in the following :

Example 1. *Choose any metric $g \in \boldsymbol{Met}(M)$ on M whose isometry group is reduced to the identity element. Hence the corresponding class $[g]$ belongs to the regular part $\mathcal{G}$. Observe that the subset $\{tg \text{ s.t. } t > 0\} \subset \boldsymbol{Met}(M)$ is not bounded in the $C^\infty(M)$ topology [5]. The metrics $t_1 g$ and $t_2 g$ are not isometric if $t_1 \neq t_2$ since they give different volumes for M, therefore each tg gives a different class $[tg] \in \mathcal{G}$ and by quotient this defines a non trivial subset $N = \{[tg] \text{ s.t. } t > 0\} \subset \mathcal{G}$. By definition of the quotient topology, the subset N is not bounded in $\mathcal{G}$ for the topology of $\mathcal{R}(M)$.*

The next example provides a simple analogy with the above phenomena.

Example 2. *Consider the Fréchet space $C^\infty(\mathbb{S}^1)$ of smooth function on the circle $\mathbb{S}^1$. Then the smooth topology of $C^\infty\left(\mathbb{S}^1\right)$ is metrizable. Set $\|.\|_{H^s}$ to be the Sobolev norm of degree $s \in \mathbb{N}$ then consider the distance $\mathbf{d}(f,g) = \sum_{s=0}^{\infty} \frac{1}{2^s} \frac{\|f-g\|_{H^s}}{1+\|f-g\|_{H^s}}$ for $(f,g) \in C^\infty(\mathbb{S}^1)^2$. Then by construction the diameter of $C^\infty(\mathbb{S}^1)$ for the distance $\mathbf{d}$ equals 2 but $C^\infty(\mathbb{S}^1)$ itself is not bounded.*

If we denote by $\iota : N \hookrightarrow \mathcal{G}$ the abstract embedding, then the preimage of a bounded subset for $\mathbf{d}$ is not necessarily bounded.

[5] since it is not even bounded for the C^0 norm.

Example 3. *Consider again the* 1*–dimensional manifold* N *from example 1. Then this defines an embedding* $\iota : \mathbb{R} \simeq N \longmapsto \mathcal{G}$ *and the preimage of* $\mathcal{G}$ *itself which is a bounded subset for* $\boldsymbol{d}$ *is* $\mathbb{R}$ *which is not bounded.*

To overcome these difficulties we will make use of compactness Theorems for isospectral metrics and also the finite dimensionality of N will play an important role in our proof.

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3. Proof of Theorem 1.

The results of Theorem 1 are particular cases of the main results from [45]. However, since we are in low dimension $d \leqslant 4$ in the present case, we can give a simple, self–contained proof which relies on simple commutator arguments in pseudodifferential calculus and using the asymptotic expansion of the heat kernel.

3.0.1. *Quadratic perturbations of Gaussian measures.* We recall the content of [46, Proposition 9.3.1 p. 211], slightly adapted to our situation, which yields a relation between partition functions of small quadratic perturbations of some Gaussian field and some convergent power series. We shall denote by $L^2(M)_0$ and $\mathcal{D}'(M)_0$ the respective closed subspaces of $L^2(M)$ and $\mathcal{D}'(M)$ which are orthogonal to constants and $\|A\|_{HS} := \sqrt{Tr_{L^2}\left(A^*A\right)}$ denotes the Hilbert–Schmidt norm. For any Hilbert space H, we denote by $\mathcal{B}\left(H,H\right)$ the algebra of bounded operators on H.

Proposition 3.1. *Let* C *be a bounded positive self–adjoint operator on* $L^2(M)_0$ *and* b *real, symmetric s.t.* $0 < C^{-1} + b$ *as quadratic forms. Denote by* $d\mu_C$ *the Gaussian measure on* $\mathcal{D}'(M)_0$ *whose covariance is* C*. Set* $:\mathcal{V}:_C = \frac{1}{2}\int_{M\times M} :\phi(x)b(x,y)\phi(y):_C$ *where* $b(x,y)$ *denotes the Schwartz kernel of* b *and* $:\phi(x)b(x,y)\phi(y):_C$ *the Wick ordered operator w.r.t. the Gaussian measure* $d\mu_C$*. If* $\widehat{b} = C^{\frac{1}{2}}bC^{\frac{1}{2}}$ *is Hilbert–Schmidt then both* $:\mathcal{V}:_C$ *and* $e^{-:\mathcal{V}:_C}$ *are in* $L^p(d\mu_C)$ *for all* $p < +\infty$ *and*

$$\mathbb{E}\left(e^{-:\mathcal{V}:_C}\right) = \exp\left(-\frac{1}{2}Tr_{L^2}\left(\log(I+\widehat{b}) - \widehat{b}\right)\right)$$

where the expansion in powers of $:\mathcal{V}:_C$ *converges absolutely for* $\|\widehat{b}\|_{HS} < 1$*.*

In the sequel, we denote by Δ^{-1} the continuous linear map $\mathcal{D}'(M) \mapsto \mathcal{D}'\left(M\right)_0$ whose Schwartz kernel is the Green function $\mathbf{G} \in \mathcal{D}'(M\times M)$ defined in definition 1.1. For all distribution $u \in \mathcal{D}'(M)$, $\Delta\left(\Delta^{-1}u\right) = \Delta^{-1}\left(\Delta u\right) = u - \frac{\int_M u}{Vol(M)}$ which means Δ^{-1} acts as the inverse of Δ restricted to $\mathcal{D}'(M)_0$. In general in our paper, all powers $\Delta^{-s}, s \in \mathbb{R}$ of the Laplace operator Δ are defined using the spectral resolution as : $\forall u\in\mathcal{D}'(M), \Delta^{-s}u=\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\lambda^{-s}\left\langle u,e_\lambda\right\rangle e_\lambda$ where the r.h.s converges in $\mathcal{D}'(M)$.

Set $\widehat{V}_\varepsilon=e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}$, then by definition of $\Delta^{-1}:L^2(M)\mapsto L^2(M)_0$, $\ker(\Delta^{-1})$ is reduced to the constant functions. Therefore we find that for all $k\geqslant 1$,

$$Tr_{L^2_0}\left(\widehat{V}_\varepsilon^{\,k}\right)=Tr_{L^2}\left(\widehat{V}_\varepsilon^{\,k}\right). \tag{3.1}$$

We will use the above identity to switch between the two traces Tr_{L^2} and $Tr_{L^2_0}$ when dealing with analytic functionals of $\widehat{V}_\varepsilon$. The above proposition 3.1 applied to the covariance $C_\varepsilon=e^{-2\varepsilon\Delta}\Delta^{-1}$ and the quadratic perturbation $\frac{1}{2}\lambda\int_M V(x)\phi(x)^2dv$ together with identity 3.1 yields for $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|e^{-2\varepsilon\Delta}\Delta^{-1}\|_{HS}}$:

$$\begin{aligned}\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv\right)\right) &= \exp\left(-\frac{1}{2}Tr_{L^2_0}\left(\log\left(I+\lambda\widehat{V}_\varepsilon\right)-\lambda\widehat{V}_\varepsilon\right)\right)\\ &= \exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I+\lambda\widehat{V}_\varepsilon\right)-\lambda\widehat{V}_\varepsilon\right)\right)\end{aligned}$$

where $\widehat{V}_\varepsilon$ is smoothing hence Hilbert–Schmidt and both series are absolutely convergent in λ since

$$\begin{aligned}\|e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}\|_{HS}^2 &= \underbrace{Tr_{L^2}\left(e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-1}e^{-2\varepsilon\Delta}Ve^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}\right)}_{\text{by Lidskii since the operator is smoothing}}\\ =\underbrace{Tr_{L^2}\left(V\Delta^{-1}e^{-2\varepsilon\Delta}V\Delta^{-1}e^{-2\varepsilon\Delta}\right)}_{\text{by cyclicity of }Tr_{L^2}} &\leqslant \underbrace{Tr_{L^2}\left(\Delta^{-1}e^{-2\varepsilon\Delta}V^2\Delta^{-1}e^{-2\varepsilon\Delta}\right)}_{\text{by Cauchy–Schwartz for }\|.\|_{HS}}\\ \leqslant\underbrace{Tr_{L^2}\left(V^2\Delta^{-2}e^{-4\varepsilon\Delta}\right)}_{\text{cyclicity again}} &\leqslant \underbrace{\|V\|^2_{\mathcal{B}(L^2,L^2)}Tr_{L^2}\left(\Delta^{-2}e^{-4\varepsilon\Delta}\right)}_{\text{Hölder}}\leqslant\|V\|^2_{L^\infty(M)}Tr_{L^2}\left(\Delta^{-2}e^{-4\varepsilon\Delta}\right)\end{aligned}$$

where in the last inequality, we used the fact that $\|V\|_{L^\infty(M)}=\|V\|_{\mathcal{B}(L^2,L^2)}$.

3.0.2. *Relating to functional determinants.* Now we observe that expanding the log as a power series and $Tr_{L^2}\left(\left(e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}\right)^k\right)=Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)$ by cyclicity of the L^2–trace yields :

$$\begin{aligned}&\exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I+\lambda e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}\right)-\lambda e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}\right)\right)\\ =\;&\exp\left(\frac{1}{2}\sum_{k=2}^\infty\frac{(-1)^k\lambda^k}{k}Tr_{L^2}\left(\left(e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}\right)^k\right)\right)=\exp\left(\frac{1}{2}\sum_{k=2}^\infty\frac{(-1)^k\lambda^k}{k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)\right)\\ =\;&\exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)-\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)\right).\end{aligned}$$

Then by Lemma 7.1, there is an explicit relation connecting Fredholm determinants $\det_F$, Gohberg–Krein's determinants $\det_2$ and functional traces (see also [46, p. 212]), this relation reads :

$$\begin{aligned}&\exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)-\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)\right)\\ =\ &\det{}_F\left(I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{-\frac{1}{2}}\exp(\frac{\lambda}{2}Tr_{L^2}(e^{-2\varepsilon\Delta}\Delta^{-1}V))=\det{}_2\left(I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{-\frac{1}{2}}\end{aligned}$$

which follows immediately from the properties of Gohberg–Krein's determinants $\det_2$.

For the moment, for every $\varepsilon>0$ and $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|e^{-2\varepsilon\Delta}\Delta^{-1}\|_{HS}}$, we obtained the relation

$$\boxed{\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv\right)\right)=\det{}_2\left(I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{-\frac{1}{2}}} \tag{3.2}$$

relating the partition function of the regularized Wick square and the Gohberg–Krein determinant for some regularized operator $I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V$ and where both sides can be expanded as convergent power series in λ provided $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|e^{-2\varepsilon\Delta}\Delta^{-1}\|_{HS}}$. For fixed $\varepsilon>0$, by analytic continuation property of Gohberg–Krein's determinant, both sides of equation 3.2 extend as entire functions of $\lambda\in\mathbb{C}$.

3.0.3. *The limit $\varepsilon\to 0^+$.* The goal of this short paragraph is to study the limit of the Fredholm operator $I+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}V$ when $\varepsilon\to 0^+$. We will say that a pseudodifferential A belongs to $\Psi^{+0}_{1,0}(M)$ if $A\in\Psi^s_{1,0}(M)$ for all $s>0$.

Lemma 3.2 (Microlocal convergence of heat operator)**.** *Let $e^{-t\Delta}$ be the heat operator. Then we have the convergence $e^{-t\Delta}\underset{t\to 0^+}{\to} Id$ in $\Psi^{+0}_{1,0}(M)$.*

Proof. For every real number s, a symbol $p\in S^s_{1,0}(\mathbb{R})$ iff p is in $C^\infty(\mathbb{R})$ and $|\partial^j_\xi p(\xi)|\leqslant C_j(1+|\xi|)^{s-j}$ [47, Lemm 1.2 p. 295] for every $j\in\mathbb{N}$. Observe that the function $p_t:\xi\in\mathbb{R}\mapsto e^{-t|\xi|^2}$ defines a family $(p_t)_{t\in[0,+\infty)}$ of symbols in $S^0_{1,0}(\mathbb{R})$ such that $p_t\underset{t\to 0}{\to}1$ in $S^{+0}_{1,0}(\mathbb{R})$. Indeed, for $k\in\mathbb{N}$ and for t in some compact interval $[0,a], a>0$, we find by direct computation that : $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}|\leqslant C(1+|\xi|)^k\sum_{0\leqslant l\leqslant\frac{k}{2}}t^{k-l}|\xi|^{k-2l}e^{-t\xi^2}$ where the constant C depends only on k.

When $|\xi|\geqslant a$, the function $t\in[0,+\infty)\mapsto(t^{k-l}\xi^{k-2l})e^{-t\xi^2}$ goes to 0 when $t=0,t\to+\infty$ and reaches its maximum when $\frac{d}{dt}\left((t^{k-l}\xi^{k-2l})e^{-t\xi^2}\right)=((k-l)t^{k-l-1}\xi^{k-2l}-t^{k-l}\xi^{k-2l+2})e^{-t\xi^2}=((k-l)-t\xi^2)t^{k-l-1}\xi^{k-2l}e^{-t\xi^2}=0$ for $t=\frac{k-l}{\xi^2}$. Hence when $|\xi|\geqslant a$,

$$\sup_{t\in[0,a]}(1+|\xi|)^k|(t^{k-l}\xi^{k-2l})|e^{-t\xi^2}\leqslant(k-l)^{k-l}(1+|\xi|)^k|\xi|^{-k}\leqslant(k-l)^{k-l}(1+a^{-k})^k.$$

On the other hand, if $|\xi|\leqslant a$, $t\in[0,a]$, we find that $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}|\leqslant C(1+a)^k\sum_{0\leqslant l\leqslant\frac{k}{2}}a^{2k-3l}$.

Therefore, we showed that $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}|\leqslant C_k$ uniformly on $t\in[0,a]$, hence $p_t\in S^0_{1,0}$ uniformly on $t\in[0,a]$. We also have for all $\delta,u>0$, $t\leqslant\delta^{1+2u}$ implies that $\sup_\xi|(1+|\xi|)^{-u}(e^{-t\xi^2}-1)|\leqslant\delta$ which means that $\sup_\xi|(1+|\xi|)^{-u}(e^{-t\xi^2}-1)|\to 0$ when $t\to 0^+$ which implies the convergence $p_t \to 1$ in $S^{+0}_{1,0}$. By a result of Strichartz [47, Thm 1.3 p. 296],

$$p_t(\sqrt{\Delta}) = e^{-t\Delta} \underset{t\to 0^+}{\to} Id \text{ in } \Psi^{+0}_{1,0}(M). \tag{3.3}$$

□

3.0.4. *No counterterms for $d \leqslant 3$.* We now discuss the case of dimension $d = (2,3)$ where we show that the regularized partition function converges when $\varepsilon \to 0^+$ and we do not need to subtract local counterterms. By composition of pseudodifferential operators, we find that $e^{-2\varepsilon\Delta}\Delta^{-1}V \underset{\varepsilon\to 0}{\to} \Delta^{-1}V$ in the space $\Psi^{-2+0}(M)$ of pseudodifferential operators of order $-2+\varepsilon, \forall \varepsilon > 0$ which implies that the convergence occurs in the ideal $\mathcal{I}_2$ of Hilbert–Schmidt operators by [49, Prop B 21]. By continuity of Gohberg–Krein's determinant on the ideal $\mathcal{I}_2$ i.e. of the map $H \in \mathcal{I}_2 \mapsto \det_2(I+H)$ [68, Thm 9.2], we find that

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) : \phi_\varepsilon^2(x) : dv\right)\right) = \det{}_2\left(I + \lambda\Delta^{-1}V\right)^{-\frac{1}{2}}. \tag{3.4}$$

By Lemma 7.1, the function $\lambda \mapsto \det_2\left(I+\lambda\Delta^{-1}V\right)$ has an analytic continuation to the complex plane as an entire function whose zeroes is exactly the set $\{\lambda \in \mathbb{C} \text{ s.t. } \ker\left(I+\lambda\Delta^{-1}V\right) \neq \{0\}\}$. Thus, we find that the divisor of Z_g^{-2} coincides with the subset $\{\lambda \text{ s.t. } z\lambda = -1, z \in \sigma(\Delta^{-1}V)\} \subset \mathbb{C}$ hence when $V = 1$, the partition function Z_g determines the spectrum $\sigma(\Delta)$ of the Laplace–Beltrami operator Δ.

3.0.5. *A remark when V is an indicator function.* In this case $\dim(M) = 2$, Ω is some smooth domain, $V = 1_\Omega\psi$, $\psi \in C^\infty(M)$ and we no longer work with smooth potentials. We want to make sense of the partition function $\mathbb{E}\left(\exp(-\frac{\lambda}{2}\int_M V(x) : \phi^2(x) : d\sigma(x))\right)$. In fact, we just regularize the potential, set $V_\varepsilon = e^{-\varepsilon\Delta}V$, then the above result tells us that

$$\lim_{\varepsilon_2\to 0} \mathbb{E}\left(\exp(-\frac{\lambda}{2}\int_M V_\varepsilon(x) : \phi^2_{\varepsilon_2}(x) : d\sigma(x))\right) = \det{}_2\left(Id + \Delta^{-1}V_\varepsilon\right)^{-\frac{1}{2}}$$

So we just need to verify that the family $(\Delta^{-1}V_\varepsilon)_{\varepsilon\in(0,1]}$ converges to $\Delta^{-1}V$ in the ideal of Hilbert–Schmidt operators. We just need to verify that:

Lemma 3.3. *The operator $\Delta^{-1}V = \Delta^{-1}1_\Omega\psi \in \mathcal{B}(L^2, L^2)$ is Hilbert–Schmidt and the sequence $(\Delta^{-1}V_\varepsilon)_{\varepsilon\in(0,1]}$ is convergent for the Hilbert–Schmidt norm.*

Assume ψ is nonnegative and by the Markovian interpretation of the heat kernel $1_\Omega\psi \geqslant 0 \implies e^{-\varepsilon\Delta}(1_\Omega\psi) \geqslant 0$ for all $\varepsilon > 0$. We use the classical bound on the Green function $|G(x,y)| \leqslant C|\log(\mathbf{d}(x,y))|$ for all $(x,y) \in M\times M$ near the diagonal and $\mathbf{d}$ is the Riemannian distance function, G is smooth outside the diagonal. Hence G belongs to all $L^p(M\times M)$ for $p \in [1,+\infty)$. Therefore $\|\Delta^{-1}V\|^2_{HS} = |\int_{M\times M} G^2(x,y)V(x)V(y)| \leqslant \|G\|^2_{L^4(M\times M)}\|V\otimes V\|_{L^2(M\times M)} = \|G\|^2_{L^4(M\times M)}\|V\|^2_{L^2(M)}$.

And also when $V_\varepsilon \to V$ in $L^2(M)$, by Cauchy–Schwartz

$$\|\Delta^{-1}(V-V_\varepsilon)\|^2_{HS} = |\int_{M\times M} G^2(x,y)(V-V_\varepsilon)(x)(V-V_\varepsilon)(y)| \leqslant \|G\|^2_{L^4(M\times M)}\|V-V_\varepsilon\|^2_{L^2(M)} \to 0,$$

which concludes the proof that one can define the partition function $\mathbb{E}\left(\exp(-\frac{\lambda}{2}\int_M V(x):\phi^2(x):d\sigma(x))\right)$ for an indicator function V.

3.1. **Explicit counterterms in dimension $d\leqslant 4$.** In what follows, for a separable Hilbert space H and every integer $p\geqslant 1$, we shall denote by $\mathcal{I}_p(H)$ the Schatten ideal of operators whose p-th power is trace class. When there is no ambiguity on H, we will shortly write $\mathcal{I}_p$. The space $\mathcal{I}_p(H)$ is endowed with the Schatten norm $\|.\|_{\mathcal{I}_p}$. $\mathcal{I}_1(H),\mathcal{I}_2(H)$ are the usual ideals of trace class and Hilbert–Schmidt operators respectively.

When $d=(2,3)$, Δ^{-1} is only Hilbert–Schmidt but not trace class and we only need the Wick renormalization to renormalize the partition function. This is exactly what Gohberg–Krein's renormalized determinant $\det_2$ is doing. When $d=4$, for $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|e^{-2\varepsilon\Delta}\Delta^{-1}\|_{HS}}$, we start again from the series expansion :

$$\log\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv\right)\right)=\frac{\lambda^2}{4}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right)$$
$$+\quad\frac{1}{2}\sum_{k=3}^\infty\frac{(-1)^k\lambda^k}{k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)$$

where we need to renormalize $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right)$ since for all $k\geqslant 3$, equation 3.3 implies $\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\underset{\varepsilon\rightarrow 0^+}{\rightarrow}\left(\Delta^{-1}V\right)^k\in\Psi^{-2k}(M)\subset\Psi^{-6}(M)$ which are trace class. We shall use pseudodifferential calculus to extract the singular part of this term. The extraction of the singular part would be easy if we considered the term $Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)$ using the asymptotic expansion of the heat kernel. But as usual, the difficulty lies in the fact that operators do not commute hence $\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\neq e^{-4\varepsilon\Delta}\Delta^{-2}V^2$. The trick is to arrange the term $\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2$ to produce a commutator term which is trace class :

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right)\quad=\quad Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)+Tr_{L^2}\left(\underbrace{e^{-2\varepsilon\Delta}\Delta^{-1}[V,e^{-2\varepsilon\Delta}\Delta^{-1}]V}_{\in\Psi^{-5}\text{ hence trace class}}\right),$$

where the family of heat operators $(e^{-\varepsilon\Delta})_{\varepsilon\in[0,1]}$ is bounded in $\Psi^0(M)$ by equation 3.3, the commutator term $[V,e^{-2\varepsilon\Delta}\Delta^{-1}]$ is therefore bounded in $\Psi^{-3}(M)$ **uniformly** in the parameter $\varepsilon\in[0,1]$ [48, p. 14]. By composition in the pseudodifferential calculus and properties of the commutator of pseudodifferential operators, we thus find that $e^{-2\varepsilon\Delta}\Delta^{-1}[V,e^{-2\varepsilon\Delta}\Delta^{-1}]V\in\Psi^{-5}(M)$, uniformly in $\varepsilon\in[0,1]$ and is therefore of trace class by Proposition [49, Prop B 21] since we are in dimension $d=4$, uniformly in the small parameter $\varepsilon\in[0,1]$. So we found that $Tr_{L^2}\left(\left(\Delta^{-1}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)^2\right)=Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)+\mathcal{O}(1)$, the singular part of $Tr_{L^2}\left(\left(\Delta^{-1}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)^2\right)$ coincides with that of $Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)$. Then the singular part of $Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)$ is easily extracted using the heat kernel asymptotic expansion [50]. First by [19, Proposition 3.3 p. 12] based on the functional calculus of the Laplace operator and Mellin transform, we have $\Delta^{-2}=\frac{1}{\Gamma(2)}\int_0^\infty(e^{-t\Delta}-\Pi)tdt$ where both sides are defined a priori as elements in $\mathcal{B}(L^2(M),L^2(M))$ and Π is the orthogonal projector on $\ker(\Delta)$

i.e. constant functions. Hence $e^{-4\varepsilon\Delta}\Delta^{-2}V^2=\frac{1}{\Gamma(2)}\int_0^\infty e^{-(t+4\varepsilon)\Delta}(Id-\Pi)V^2tdt$. We use the notation $\mathcal{O}(1)$ to refer to something which is bounded when $\varepsilon\to 0^+$. First we have the decomposition :

$$Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)=\int_0^1 Tr_{L^2}\left(e^{-(4\varepsilon+t)\Delta}V^2\right)tdt$$
$$+\underbrace{\int_0^1 Tr_{L^2}\left(-\Pi V^2\right)tdt+\int_1^\infty Tr_{L^2}\left(e^{-(t+4\varepsilon)\Delta}(Id-\Pi)V^2\right)tdt}_{\mathcal{O}(1)}$$

since $e^{-(t+4\varepsilon)\Delta}(-\Pi)=-\Pi$ and the integral $\int_1^\infty Tr_{L^2}\left(e^{-(t+4\varepsilon)\Delta}(Id-\Pi)V^2\right)tdt$ converges uniformly in $\varepsilon\to 0$ by exponential decay in t of $Tr_{L^2}\left(e^{-(t+4\varepsilon)\Delta}(Id-\Pi)V^2\right)$. The proof of the exponential decay follows [19, 7.3 Proof of Lemma 4.1] and is a consequence of the **spectral gap** for $e^{-(t+4\varepsilon)\Delta}(Id-\Pi)$. Then we may use the asymptotic expansion of the heat kernel [50, Thm 2.30] to study the term $e^{-(t+4\varepsilon)\Delta}(x,x)=\frac{1}{(4\pi(t+4\varepsilon))^2}+\mathcal{O}((t+4\varepsilon)^{-1})$. This yields :

$$\begin{aligned}
&Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)=\int_0^1 Tr_{L^2}\left(e^{-(4\varepsilon+t)\Delta}V^2\right)tdt+\mathcal{O}(1)\\
=\ &\frac{1}{(4\pi)^2}\int_0^1\frac{t}{(4\varepsilon+t)^2}dt\int_M V^2(x)dv+\mathcal{O}(1)=\frac{\int_M V^2(x)dv}{16\pi^2}\int_{4\varepsilon}^{1+4\varepsilon}(u^{-1}-4\varepsilon u^{-2})du+\mathcal{O}(1)\\
=\ &\frac{-\log(\varepsilon)\int_M V^2(x)dv}{16\pi^2}+\mathcal{O}(1)=\frac{\int_M V^2(x)dv}{16\pi^2}|\log(\varepsilon)|+\mathcal{O}(1).
\end{aligned}$$

We conclude by the observation that for $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|\Delta^{-1}\|_{\mathcal{I}_3}}$, $Z_g(\lambda)$

$$\begin{aligned}
=\ &\lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv-\frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\right)\\
=\ &\lim_{\varepsilon\to 0^+}\exp\left(\underbrace{\frac{\lambda^2}{4}Tr_{L^2}\left(\left(\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^2\right)-\frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|}_{\mathcal{O}(1)}+\sum_{k=3}^\infty\frac{(-1)^k\lambda^k}{2k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)\right)\\
=\ &\lim_{\varepsilon\to 0^+}\exp\left(\frac{\lambda^2}{4}Tr_{L^2}\left(\left(\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^2\right)-\frac{\lambda^2\int_M V^2(x)dv}{64\pi^2}|\log(\varepsilon)|\right)\det{}_3\left(I+\lambda\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^{-\frac{1}{2}}\\
=\ &e^{P(\lambda)}\det{}_3\left(I+\lambda\Delta^{-1}V\right)^{-\frac{1}{2}}
\end{aligned}$$

where we recognized Gohberg–Krein's renormalized determinant $\det_3$. By the properties of $\det_3$ recalled in Lemma 7.1, the expression on the r.h.s. converges when expanded as power series in λ provided $|\lambda|<\frac{1}{\|V\|_{L^\infty(M)}\|\Delta^{-1}\|_{\mathcal{I}_3}}$ since $\Delta^{-1}e^{-2\varepsilon\Delta}V\underset{\varepsilon\to 0^+}{\to}\Delta^{-1}V\in\Psi^{-2}(M)$ hence in the Schatten ideal $\mathcal{I}_3$ and P is a polynomial of degree 2.

Now we conclude similarly as for dimension $d=(2,3)$, by Lemma 7.1, $\det_3\left(I+\lambda\Delta^{-1}V\right)$ has analytic continuation as an entire function in $\lambda\in\mathbb{C}$ which vanishes with multiplicity on the set $-\sigma(\Delta^{-1}V)$ which implies that Z_g determines $\sigma(\Delta)$ when $V=1$.

3.1.1. *Conclusion of the proof.* The proof of identity

$$Z_g(\lambda, V) = \exp\left(P(\lambda) + \sum_{n>\frac{d}{2}} \frac{(-1)^n c_n(g,V)\lambda^n}{2n}\right) \tag{3.5}$$

follows immediately from the fact that for $n > \frac{d}{2}$, composition in the pseudodifferential calculus implies that $(\Delta^{-1}V)^n \in \Psi^{-2n}(M)$ is trace class hence the integrals

$$c_n(g,V) = \int_{M^n} t_n(x_1, \dots, x_n)V(x_1)\dots V(x_n)dv_n$$

are convergent and equal to $Tr_{L^2}\left((\Delta^{-1}V)^n\right)$ and P vanishes if the dimension $d \leqslant 3$. The conclusion follows from the relation of Gohberg–Krein's determinants $\det_p$ with functional traces summarized in Lemma 7.1.

3.2. **Relation with zeta regularization.** To the authors knowledge, it seems that zeta regularization first appeared in the physics litterature in the celebrated article of Dowker–Critchley [6] [6] to calculate the effective Lagrangian and vacuum expectation value of the stress energy momentum tensor $\langle T^{\mu\nu}\rangle$ of a scalar field in de Sitter space. Then it was popularized by Hawking [40] since it gives a mean to regularize Euclidean quantum fields partition function on curved spaces [7]. The spectral zeta regularization relies on the analytic continuation of the spectral zeta function $\zeta_{\Delta+V}(s) = \sum_{\lambda\in\sigma(\Delta+V)\setminus\{0\}} \lambda^{-s}$ which has meromorphic continuation to the complex plane without poles at $s = 0$ and the zeta determinant which is supposed to regularize the partition function $\int[d\varphi]\exp(-\frac{S_0(\varphi)}{2})$ reads $\det_\zeta(\Delta + V) = \exp(-\zeta'(0))$. However it is not the only regularization technique for the partition function, it does not have a straightforward probabilistic interpretation in terms of the GFF and it does not have an obvious relation to loop spaces. In fact, it follows from [45] that there is a connection between zeta regularization and the one from the present paper which reads in dimension $d = 2, 3$: $\det_\zeta(\Delta + V)^{-\frac{1}{2}} = \mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V : \phi^2(x) : d\sigma(x)\right)\right)e^{P(\lambda V)}$ where P is an affine continuous map in $V \in C^\infty(M)$ which classifies renormalization ambiguities of the partition function. When $d = 4$, one has a similar factorized form where the ambiguity polynomial P has degree 2 in V.

4. Proof of Corollary 2.2.

Let us prove the equivalence of claims 1),2),3) in Corollary 2.2. In this paragraph, we shall denote the Laplace-Beltrami operator of the respective metrics $g_i, i = 1, 2$ by $\Delta_{g_i}, i = 1, 2$ to stress the dependence in the metric.

- Let us first show that 1) $\implies$ 3) namely if some infinite number of Feynman amplitudes coincide then the metrics are isospectral. Set $[\frac{d}{2}] = \sup_{k\leqslant \frac{d}{2}, k\in\mathbb{Z}} k$. Observe

[6] We would like to warmly thank the referee for this reference since we thought the first use in physics of zeta regularization was due to Hawking.

[7] In global analysis, zeta determinants appeared in the work of Ray–Singer on the analytic torsion [39] which was understood by A. Schwarz [12] as the partition function of the abelian BF theory

that arguing as in the proof of Proposition 2, when $|\lambda| < \frac{1}{\|\Delta^{-1}\|_{[\frac{d}{2}]+1}}$, the series $\exp\left(\sum_{n>\frac{d}{2}} \frac{(-1)^{n+1}c_n(g_i)\lambda^n}{n}\right), i=1,2$ converges absolutely to the Gohberg–Krein determinant $\det_{[\frac{d}{2}]+1}\left(Id+\lambda\Delta_{g_i}^{-1}\right)$. So the coincidence of Feynman amplitudes $c_n(g_1)=c_n(g_2), \forall n>\frac{d}{2}$ implies the equality $\det_{[\frac{d}{2}]+1}\left(Id+\lambda\Delta_{g_1}^{-1}\right)=\det_{[\frac{d}{2}]+1}\left(Id+\lambda\Delta_{g_2}^{-1}\right)$ as entire functions by analytic continuation. Hence (g_1,g_2) are isospectral by the properties of the zeros of $\det_{[\frac{d}{2}]+1}$.

- Assume 3) namely that (M_1,g_1) and (M_2,g_2) are isospectral. Our goal is to show how to recover the partition function Z_g from the spectrum. In proposition 2, we established the relation $Z_g(\lambda)=\det_2\left(Id+\lambda\Delta^{-1}\right)^{-\frac{1}{2}}$ when $d\leqslant 3, |\lambda|<\frac{1}{\|\Delta^{-1}\|_{HS}}$ and $Z_g(\lambda)=\lim_{\varepsilon\to 0^+}\exp\left(-\frac{\lambda^2\mathrm{Vol}_g(M)}{64\pi^2}|\log(\varepsilon)|\right)\det_2\left(Id+\lambda e^{-2\varepsilon\Delta}\Delta^{-1}\right)^{-\frac{1}{2}}=e^{P(\lambda)}\det_3\left(Id+\lambda\Delta^{-1}\right)^{-\frac{1}{2}}$ when $d=4, |\lambda|<\frac{1}{\|\Delta^{-1}\|_3}$ and where P is some polynomial of degree 2. The r.h.s of both equalities are **purely spectral** since :
  (1) by Lemma 7.1, the Gohberg–Krein determinants can be expressed in terms of $Tr_{L^2}\left(\Delta^{-n}\right)=\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\lambda^{-n}=c_n(g), \forall n>\frac{d}{2}$ by Lidskii's Theorem,
  (2) for the $d=4$ case, we use the fact that $\mathrm{Vol}_g(M)$ is spectral [8] and $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta\right)^{-n}\right)=\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}e^{-2n\varepsilon\lambda}\lambda^{-n}$,
  which imply both 3) $\implies$ 2) and 3) $\implies$ 1).
- Finally, assume 2) namely that the partition functions are equal $Z_{g_1}=Z_{g_2}$. The equality implies $\det_{[\frac{d}{2}]+1}\left(Id+\lambda\Delta_{g_1}^{-1}\right)=\det_{[\frac{d}{2}]+1}\left(Id+\lambda\Delta_{g_2}^{-1}\right)$ by the formula relating the partition functions and Gohberg–Krein's determinants hence (g_1,g_2) are isospectral and 2) $\implies$ 1) again by $c_n(g)=Tr_{L^2}\left(\Delta^{-n}\right)=\sum_{\lambda\in\sigma(\Delta)\setminus\{0\}}\lambda^{-n}$.

We proved the desired equivalences.

Now the main implication of Corollary 2.2 is a consequence of the deep Theorem of Colin de Verdière [4, 5], Duistermaat–Guillemin [51, Thm 4.5 p. 60] relating the spectrum of the Laplacian and the length spectrum in negative curvature. We recall, in the particular case of metrics with negative curvature :

Theorem 3 (Trace formula)**.** *Let (M,g) be a smooth compact Riemannian manifold with negative sectional curvatures and Δ the Laplace-Beltrami operator. Then the spectrum $\sigma(\Delta)$ determines the non marked length spectrum by the* ***trace formula*** *:*

$$2Re\left(\sum_{\lambda\in\sigma(\Delta)} e^{i\sqrt{\lambda}t}\right)=\sum_\gamma \frac{\ell_\gamma}{m_\gamma|\det\left(I-P_\gamma\right)|^{\frac{1}{2}}}\delta\left(t-\ell_\gamma\right)+L^1_{loc}, \tag{4.1}$$

[8] which follows from Weyl's law.

[9] *where ℓ_γ, m_γ are the period and multiplicity of the orbit γ and P_γ is the Poincaré return map. Furthermore, the singularities of the wave trace equals the length spectrum :*

$$\textit{singular support}\left(2Re\left(\sum_{\lambda\in\sigma(\Delta)} e^{i\sqrt{\lambda}t}\right)\right)=\{\ell_\gamma|[\gamma]\in\pi_1(M)\} \tag{4.2}$$

which implies the Laplace spectrum $\sigma(\Delta)$ determines the length spectrum of (M,g).

For geodesic flows in negative curvature, the set of periods forms a discrete subset of $\mathbb{R}_{>0}$ hence each period is isolated and the corresponding periodic orbits are isolated and in finite number. In that case, [52, Theorem 3 p. 495]) gives a leading term for the real part of the distributional flat trace $2Re\left(Tr^\flat(U(.))\right)\in\mathcal{D}'(\mathbb{R}_{>0})$ of the wave propagator $U(t)=e^{it\sqrt{\Delta}}$ of the form :

$$2Re\left(Tr^\flat(U(t))\right)=\sum_{[\gamma]\in\pi_1(M)}\frac{i^{-\sigma_\gamma}\ell_\gamma}{m_\gamma|\det(I-P_\gamma)|^{\frac{1}{2}}}\delta(t-\ell_\gamma)+L^1_{\text{loc}}.$$

The flat trace $Tr^\flat(U(t))$ of the wave propagator $U(t)$, also called distributional trace of $U(t)$, is defined to be the integral of the Schwartz kernel of $U(t)$ on the diagonal: $t\mapsto Tr^\flat(U(t))=\int_M K_t(x,x)dv(x)$ and it is a **distribution in the variable** t. This formula holds true for every geodesic flow whose periodic orbits are countable (form a discrete set) and such that each periodic orbit is **nondegenerate** in the sense the Poincaré map P_γ is hyperbolic. In case the metric has negative curvature, each closed geodesic make a non-zero contribution to the singular support of U since the Maslov index $\sigma_\gamma=0$ for all γ as noted in [53, Coro 1.1 p. 73].

The identity

$$S_{EH}(g)=Res|_{s=\frac{d}{2}-1}\sum_{\lambda,Z_g(\lambda)^{-2}=0}\lambda^{-s}$$

follows immediately from the spectral interpretation of the integral of the scalar curvature (Einstein–Hilbert action) [57, Thm 6.1 p. 26]. Let us briefly recall the principle of this derivation. The first heat invariant of the scalar Laplacian is directly related to the scalar curvature, for $Re(s)>\frac{d}{2}$, the sum $\sum_{\lambda\in\sigma(\Delta),\lambda>0}\lambda^{-s}$ converges by Weyl's law and coincides with $Tr_{L^2}(\Delta^{-s})$. By the heat kernel expansion, the trace $Tr_{L^2}(\Delta^{-s})$ admits an analytic continuation as a meromorphic function whose poles at $s=\frac{d}{2}-1$ are related to the first heat invariant.

5. Proof of Theorem 2.

Let us explain the central ideas in the proof of Theorem 2. Recall that we denoted by N some smooth finite dimensional submanifold of $\mathcal{G}\subset\mathcal{R}(M)$ such that $\partial N\subset\partial\mathcal{G}$. When $d\leqslant 3$, our goal is to show that given such N, the probability distribution of the random variable $\int_M:\phi^2(x):dv$ determines a **finite number** of elements in $N\cap\mathcal{R}(M)_{\leqslant-\varepsilon}\cap\mathcal{G}_{\geqslant\varepsilon}$. First, the probability distribution of the random variable $\int_M:\phi^2(x):dv$ determines the

[9] which is an equality in the sense of distributions in $\mathcal{D}'(\mathbb{R}_{>0})$

moments $\mathbb{E}\left(\left(\int_M :\phi^2(x): dv\right)^k\right), k \geqslant 2$ of $\int_M : \phi^2(x) : dv$, hence the partition function $Z_g(\lambda)$ whose zeroes give the spectrum $\sigma(\Delta)$ of the Laplace operator by Proposition 2. Then the length spectrum $\mathcal{P}(g)$ of M is recovered from the Laplace spectrum using the trace formula of Duistermaat–Guillemin. This is the content of subsubsections 5.0.1 and 5.0.2. When $d = 4$, the discussion is simpler since we are directly given the partition function Z_g which determines the Laplace spectrum by Theorem 2.

Now it remains to show that in some finite dimensional submanifold N in $\mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ for $d \leqslant 3$ and $\mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]} \cap \mathcal{G}_{>\varepsilon}$ when $d = 4$, the Laplace spectrum together with the length spectrum determine a **finite number** of isometry types. This follows from Proposition 5.1 and let us explain informally the intuition behind this result. Near every isometry class $[g_0]$ in N, one should think that there is some neighborhood $U \subset N$ of $[g_0]$ such that the map

$$\text{Isometry class of metric } [g] \in U \subset \tilde{N} \longmapsto \text{length spectrum } \mathcal{P}([g]) \subset \mathbb{R}_{>0}$$

is **injective**. In fact in our proof, we do not deal directly with this nonlinear map (nonlinear in the metric g) but with its linearization which is the X-ray transform.

Thinking in terms of representatives instead of isometry class, it means that in some neighborhood of every metric $g_0 \in N$, two metrics $(g_1, g_2) \in N^2$ with the same length spectrum must be isometric. Then the finiteness follows from the compactness properties of isospectral metrics and finite dimensionality of N. Note that for simplicity of exposition, we prove Proposition 5.1 by a contradiction argument but the reader should keep in mind the intuitive picture explained above.

5.0.1. *Existence of Wick square as random variable.* We use Proposition 3.1 on Gaussian measures. In dimension $d = (2, 3)$, the operator $\Delta^{-\frac{1}{2}} V \Delta^{-\frac{1}{2}} \in \Psi^{-2}(M)$ is Hilbert–Schmidt and therefore the Wick renormalized functional $\int_M V : \phi^2(x) : dv$ is a **well–defined random variable** in all $L^p(\mathcal{D}'(M), \mu), p \in [2, +\infty)$ where μ is the Gaussian Free Field measure on $\mathcal{D}'(M)$ with covariance Δ^{-1}. From now on, we let $V = 1 \in C^\infty(M)$.

5.0.2. *Spectrum of Δ and probability distribution of the Wick square $\int_M : \phi^2(x) : dv$.* Furthermore, the probability distribution of the random variable $\int_M : \phi^2(x) : dv$, more precisely its moments are related to the partition function $Z_g(\lambda)$ by the observation that the series

$$Z_g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{2^n n!} \mathbb{E}\left(\left(\int_M : \phi^2(x) : dv\right)^n\right)$$

converges absolutely for $|\lambda| < \frac{1}{\|\Delta^{-1}\|_{HS}}$ where $\|.\|_{HS}$ is the Hilbert–Schmidt norm. Therefore by Proposition 1, the **probability distribution of $\int_M : \phi^2(x) : dv$ determines Z_g and its zeroes, hence the spectrum of Δ.**

To prove the second claim of Theorem 1, we need to recall some results on compactness of isospectral metrics which will also be useful for the proof of the first claim of Theorem 1.

5.0.3. *Compactness of isospectral metrics.* A main ingredient of our proof of Theorem 1 is compactness of the space of isospectral metrics. Note that two isospectral Riemannian surfaces (M_1, g_1) and (M_2, g_2) have the same genus since the second heat invariant $a_1 =$

$\frac{1}{6}\int_M \mathfrak{K}_g$ is a spectral invariant and is proportional to the integral of the scalar curvature $\mathfrak{K}_g$ on M. Therefore, a_1 determines the Euler characteristic thus the genus of M by Gauss–Bonnet. We start by the compactness result of Osgood–Philips–Sarnak [63] which deals with isospectral families surfaces.

Theorem 4 (Compactness for $d = 2$)**.** *An isospectral set of isometry classes of metrics on a closed surface is sequentially compact in the C^∞–topology.*

A general compactness result for isospectral metrics is no longer true in dimension $d = 3$, we need some further assumptions on the metric. For $d = 3$, we shall use the celebrated result of Brooks-Petersen–Perry [53, Thm 0.2 p. 68] and Anderson [58, Thm 0.1 p. 700].

Theorem 5 (Compactness for $d = 3$)**.** *The space of smooth compact isospectral* 3*-manifolds* (M, g) *for which the length of the shortest closed geodesic is bounded from below*

$$\ell_M \geqslant \ell > 0 \tag{5.1}$$

is compact in the C^∞ topology. It follows that there are only finitely many diffeomorphism types of isospectral 3*-manifolds which satisfy 5.1.*

Let us explain the meaning of the above statement in practice. Let $(M_i, g_i)_i$ denotes a sequence of isospectral smooth compact 3–manifolds without boundary whose shortest closed geodesic has length bounded from below. Then there is a finite number of manifolds $(M'_1, \dots, M'_k)$ and on each M'_j a compact family of metrics $\mathcal{M}'_j$ such that each of the manifolds M_j is diffeomorphic to one of the M'_i and isometric to an element of $\mathcal{M}'_i$.

In dimension $d = 4$, we use the following Theorem by Zhou [71, Thm 1.1 p. 188] which requires an additional assumption on sectional curvatures :

Theorem 6 (Compactness for $d = 4$)**.** *On a given smooth compact manifold M, the set of isospectral metrics whose sectional curvatures are bounded in some compact interval is* ***compact*** *for the C^∞ topology. If (M_i, g_i) is a family of isospectral manifolds of dimension* 4 *with negative sectional curvatures bounded in some compact interval, then (M_i, g_i) contains only* ***finitely many diffeomorphism types****.*

We would like to sketch the ideas behind the compactness results and we refer to the original papers for additional details. Set n some positive integer and some real parameters $C, \delta, v > 0$. We denote by $\mathcal{M}(n, \delta, v, C)$ the set of n–dimensional manifolds with bounded sectional curvature $|K| \leqslant C$, a lower bound on the injectivity radius $\text{inj}(M) \geqslant \delta > 0$ and an upper bound on the volume $\text{Vol}(M) \leqslant v$. Consider isospectral metrics $I^n(C)_{<0}$ whose sectional curvature $|K|$ is bounded by some fixed constant C and whose sectional curvature is negative. Isospectrality and negative curvature ensures that Riemannian manifolds in $I^n(C)_{<0}$ have **fixed volume** since the volume is a spectral invariant by Weyl's law. The **length ℓ_1 of the shortest closed geodesic is fixed** using Duistermaat–Guillemin's trace formula. Therefore the injectivity radius of every Riemannian manifolds in $I^n(C)_{<0}$ is uniformly bounded from below using the inequality due to Klingenberger [66, p. 78] :

$$\text{inj}(M) \geqslant \inf\left(\frac{\pi}{\sqrt{C}}, \frac{1}{2}\ell_1\right). \tag{5.2}$$

Hence $I^n(C)_{<0} \subset \mathcal{M}(n, \delta, v, C)$ for some real parameters $C, \delta, v > 0$ and by the Cheeger finiteness Theorem in the version of Peters [65, Corollary 3.8 p. 9] [66, p. 77], we find that :

Theorem 7 (Cheeger finiteness for diffeomorphism types)**.** *The Riemannian manifolds in $\mathcal{M}(n, \delta, v, C)$ hence in $I^n(C)_{<0}$ have **finite number of diffeomorphism types**.*

To explain the compactness, we recall that fixing the spectrum of Δ fixes the heat coefficients. Using some results of Gilkey on the structure of heat coefficients [71, Thm 2.1 p. 189, Lemm 3.1 p. 193], Zhou proves that for all $g \in I^n(C)_{<0}$, its curvature $R(g)$ is bounded in all Sobolev norms $W^{k,2}(g)$ of order k where the Sobolev norms are also defined using the same metric $g \in I^n(C)_{<0}$ [71, Lemma 3.2 p. 193]. Then by some geometric properties of Sobolev constants, we use Sobolev inequalities to control the C^k norms of $R(g)$ in terms of the $W^{k',2}(g), k' \in \mathbb{N}$ **uniformly in the metric** $g \in I^n(C)_{<0}$. So we converted global integrated informations on the metric and curvature into pointwise bounds on the curvature. The conclusion now follows from the C^k version of Cheeger–Gromov compactness Theorem [58, p. 701] [71, Thm 2.2 p. 190] [67, Thm A' p. 27] :

Theorem 8 (Cheeger–Gromov C^k compactness)**.** *Fix a positive integer k and $\alpha \in (0,1)$. The space of n–dimensional Riemannian manifolds s.t. $\|\nabla^j R\|_{C^0} \leqslant C$, $\forall j \leqslant k$, $Vol(M) \geqslant v > 0$ and $diam_M \leqslant D$ is precompact in the $C^{k+1,\alpha}$ topology. More precisely*

1. *for fixed M, given any $\alpha < 1$ and any sequence of metrics $(g_i)_i$ on M satisfying the above bounds, we can extract a convergent subsequence in the Hölder $C^{k+1,\alpha'}$ topology for all $\alpha' < \alpha$ to a limit metric g of Hölder regularity $C^{k+1,\alpha}$.*
2. *For any sequence (M_i, g_i) satisfying the above bounds, there is a subsequence which converges **in the Lipschitz topology** to a limit metric of Hölder regularity $C^{k+1,\alpha}$ for any $0 < \alpha < 1$.*

We use the fact that for Riemannian manifolds with metric $g \in I^n(C)_{<0}$, the assumptions of the above Theorem are satisfied for every k, which explains the compactness result of Theorem 6.

5.0.4. *Consequence of the compactness result and proof of the second claim of Theorem 2.* A sequence (M_i, g_i) of Riemannian manifolds of negative curvature s.t. $\int_M : \phi^2(x) : dv$ has fixed probability distribution is in fact an isospectral sequence of Riemannian manifolds. But since the Laplace spectrum determines the length spectrum, the sequence (M_i, g_i) of Riemannian manifolds is isospectral and along this sequence the geodesics of shortest length has fixed length $\ell > 0$ hence the sequence (M_i, g_i) is precompact in the sense of Anderson [58] and by Theorem 5, there exists a subsequence such that M_i has **fixed diffeomorphism type** and $g_i \to g$ to some metric g in the C^∞ topology which is the second claim from Theorem 2. The discussion for $d = 4$ is similar. A sequence $(M_i, g_i)_i$ of Riemannian manifold whose partition function Z_g is given, is isospectral and the condition that the sectional curvature is in some bounded interval $[-\varepsilon^{-1}, -\varepsilon]$ implies that the sequence $(M_i, g_i)_i$ satisfies the assumptions of Theorem 6. Then the conclusion follows.

5.0.5. *Rigidity in negative curvature.* Up to now, we have proved that the probability distribution of $\int_M : \phi^2(x) : dv$ determines the Laplace spectrum. Recall $\mathcal{R}(M)_{\leqslant -\varepsilon}$ denotes the set of isometry classes of metrics whose sectional curvatures are bounded from above by $-\varepsilon$. To conclude the proof of claim 1) from Theorem 2, it remains to show that :

Proposition 5.1. *Let M be a smooth closed compact manifold and N be some finite dimensional submanifold in $\mathcal{G} \subset \mathcal{R}(M)$ such that $\partial N \subset \partial \mathcal{G}$. For all $\varepsilon > 0$, the set of isospectral metrics*

- *in $N \cap \mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ when $d \leqslant 3$,*
- *in $N \cap \mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$ when $d = 4$,*

is finite.

We prove Proposition 5.1 by giving a simple adaptation of a result due to Sarnak [30] in dimension 2 and Sharafutdinov [59] for hyperbolic metrics that for a finite dimensional manifold of metrics of negative curvature, there are only a **finite number of isospectral metrics**.

6. Proof of Proposition 5.1.

In the next subsection, we introduce the geometrical tools needed to prove Proposition 5.1.

6.0.1. *Convergence in the space of metrics.* Let us recall the notion of convergence in the moduli space $\mathcal{R}(M)$. We work on a smooth closed compact manifold M of dimension $d = 2, 3$. The convergence of isometry classes $[g_n] \to [g]$ means that there is a sequence of representatives $g_n \to g$ in the C^∞ topology for 2-tensors.

6.0.2. *Symmetric tensors on Riemannian manifolds and a Hodge type decomposition of metrics.* In the sequel, for any smooth vector bundle $E \mapsto M$, we shall use the notation $C^{k,\alpha}(E), k \in \mathbb{N}, \alpha \in (0,1)$, $H^s(E), s \in \mathbb{R}$, $C^0(E)$ to denote sections of E of Hölder regularity $C^{k,\alpha}$, Sobolev H^s and C^0 respectively. Consider a Riemannian manifold (M, g) and denote by $d\lambda$ the Liouville metric on SM. Consider the space of symmetric covariant m–tensor denoted by $S^mT^*M \subset T^mT^*M$ where T^mT^*M are the covariant m–tensors on M. We will denote by σ the natural symmetrization operator acting on sections of T^mT^*M [60, p. 1267-1268] whose image are sections of S^mT^*M. The metric g on TM defines a canonical vertical metric on T^mT^*M which induces a canonical L^2 structure on the space of smooth sections $C^\infty(T^mT^*M)$. The Liouville measure $d\lambda$ also induces an L^2 structure on $C^\infty(SM)$. Then there is a map denoted by π_m^* going from $C^0(S^mT^*M)$ to $C^0(SM)$ which identifies a symmetric m-tensor with a function on SM : for $(x; v) \in SM$, $f \in C^0(S^mT^*M)$ we have $\pi_m^* f(x, v) = f(x; v, \ldots, v)$ whose formal adjoint π_{m*} with respect to the two L^2 structures defined above is defined as :

$$\langle \pi_m^* f, u \rangle_{L^2(SM)} = \langle f, \pi_{m*} u \rangle_{L^2(S^*T^*M)} . \tag{6.1}$$

If ∇ denotes the Levi–Civita connection, we define an operator $D_g = \sigma \circ \nabla : C^\infty(S^mT^*M) \longmapsto C^\infty\left(S^{m+1}T^*M\right)$. Its formal adjoint w.r.t. the L^2 scalar product on S^mT^*M reads $-D_g^* =$

$-Tr\,(D_g) = -Tr\,(\nabla)$ where the trace is taken w.r.t. the first two factors:

$$Tr\,(\nabla u)_{\alpha_1\ldots\alpha_{m-1}} = \nabla^{\beta} u_{\beta\alpha_1\ldots\alpha_{m-1}}$$

where we sum over the repeated index β. There is an explicit relation between the operator D_g and the generator $X \in C^\infty(SM)$ of the geodesic flow acting by Lie derivative [60] [61, Prop 3.10 p. 28] :

$$X\pi_m^* = \pi_{m+1}^* D_g. \tag{6.2}$$

A fundamental result for inverse problems in negative curvature is some kind of Hodge type decomposition of metrics due to Croke–Sharafutdinov [60, Thm 2.2 p. 1269] [61, Thm 3.8 p. 26] :

Theorem 9. *Let (M, g) be a compact Riemannian manifold s.t. the geodesic flow on SM has at least one dense geodesic and $k \geqslant 1$ an integer. Then every symmetric 2–tensor $T \in H^k\left(S^2T^*M\right)$ admits the following* ***unique decomposition***

$$T = T^s + D_g\theta,\ \ D_g^*T^s = 0 \tag{6.3}$$

where $T^s \in \ker(D_g^*) \cap H^k\left(S^2T^*M\right)$ is called the ***solenoidal part*** *w.r.t. g of the tensor* [10] *and $D_g\theta = \sigma\nabla\theta$ is the* ***potential part*** *w.r.t. g where $\theta \in H^{k+1}(T^*M)$ is a 1–form, ∇ is the covariant derivative w.r.t. g and σ is the symmetrization operator.*

The uniqueness of the decomposition and $C^\infty = \bigcap_{k+1}^\infty H^k$ implies the above Theorem holds true for C^∞ tensors. Geometrically, a consequence of the above Theorem is that the tangent space $T_g\mathbf{Met}(M) \simeq C^\infty(S^2T^*M)$ to any metric g whose geodesic flow admits at least one dense orbit, there is a decomposition of the form :

$$T_g\mathbf{Met}(M) = \text{solenoidal tensors for } g \oplus \text{potential tensors for } g.$$

6.1. **The geometry of $\mathbf{Met}(M)$ and a slice Theorem.** In the next Theorem, we shall examine the consequences of the above Hodge type decomposition for the geometry of $\mathbf{Met}(M)$. In some sense, it shows that the space $g + \ker\left(D_g^*\right)$ of **perturbations of g which are solenoidal w.r.t.** g is transverse, near g, to the orbits of $\mathbf{Diff}(M)$ and is therefore a local slice to the orbit of $\mathbf{Diff}(M)$ near g. In fact, such result was proved by Ebin in his thesis in the ILH setting as described in subsubsection 2.0.2 but the slice Theorem we present here is more adapted to negatively curved metrics. A consequence of the Hodge type decomposition in the space of metrics from Theorem 9 is the following [17, Lemma 4.1](see also [62, Thm 2.1] for the boundary case) :

Theorem 10. *[Croke–Dairbekov–Sharafutdinov, Guillarmou–Lefeuvre] Let M be a compact manifold. For any smooth metric g_0 whose geodesic flow has one dense orbit, for every integer $k \in \mathbb{N}, k \geqslant 2$ and real number $\alpha \in (0,1)$, there exists a neighborhood $\mathcal{U}$ of g_0 in $C^{k,\alpha}(S^2T^*M)$ such that for any $g \in \mathcal{U}$, there is a $C^{k,\alpha}$ metric $g' = \Phi^*g$ isometric to g where Φ is a diffeomorphism of regularity $C^{k+1,\alpha}$ such that $g' - g_0$ is* ***solenoidal*** *w.r.t g_0, moreover the map $\Psi\ : g \in \mathcal{U} \subset C^{k,\alpha}(S^2T^*M) \mapsto g' \in C^{k,\alpha}(S^2T^*M)$ is* ***smooth***.

In particular, the above Theorem holds true for g_0 with negative sectional curvatures.

[10] also called divergence free part

Intuitively, the picture one should have in mind is that in the space $\mathbf{Met}(M)$ of metrics (viewed as an open cone of the space of 2–tensors hence as a Fréchet manifold), the tangent space $T_{g_0}\mathbf{Met}(M)$ to g_0 admits the decomposition :

$$T_{g_0}\mathbf{Met}(M) = \underbrace{\ker D^*_{g_0}}_{\text{solenoidal tensors for } g_0} \oplus \underbrace{\mathrm{Im} D_{g_0}}_{\text{potential tensors for } g_0}. \tag{6.4}$$

The space of potential tensors for g_0 is precisely the tangent part to the orbit through g_0 of the action of the group of diffeomorphisms which is $T_{g_0}(\mathbf{Diff}(M).g_0)$. Hence starting from g_0 and adding a small solenoidal part exactly means moving in the transversal direction to the orbits of $\mathbf{Diff}(M)$ which means after projection that we are moving in the quotient space $\mathcal{R}(M) = \mathbf{Met}(M)/\mathbf{Diff}(M)$. Since the above Theorem is proved using Banach fixed point, the metric g' isometric to g is only known to belong to some Hölder space $C^{k,\alpha}$ and not necessarily C^∞ but this is sufficient for our purpose since the index $k \geqslant 2$.

6.2. **Injectivity of the X–ray transform.** Periodic orbits γ of the vector field $X \in C^\infty(T(SM))$ which generates the geodesic flow of g on SM are defined as continuous maps :

$$\gamma : t \in [0, T_\gamma] \longmapsto (\gamma(t), \dot\gamma(t)) \in SM \tag{6.5}$$

where γ is parametrized at unit speed. The closed geodesic γ defines a **distribution** in $\mathcal{D}'(SM)$, denoted by δ_γ, as follows :

$$\langle \delta_\gamma, f\rangle = \int_0^{T_\gamma} f(\gamma(t), \dot\gamma(t))\, dt. \tag{6.6}$$

Recall that any symmetric m-tensor $f \in C^0(S^mT^*M)$, can be lifted as a function on the sphere bundle SM by π_m^* defined in subsubsection 6.0.2. It follows that the distributions δ_γ act on symmetric m-tensor in $C^0(S^mT^*M)$ as follows :

$$\langle \delta_\gamma, f\rangle = \int_0^{T_\gamma} (\pi_m^* f)(\gamma(t), \dot\gamma(t))\, dt. \tag{6.7}$$

Now let us recall some important properties for periodic geodesics on manifolds with negative curvature :

Proposition 6.1. *Let (M, g) be a compact Riemannian manifold s.t. g has negative sectional cuvatures. Denote by $\pi_1(M)$ the free homotopy classes of loops* [11] *in M. Then :*

- *the geodesic flow has the Anosov property in particular it has one everywhere dense geodesic* [15, Thm 17.6.2]*,*
- *the periodic geodesics of g in SM are in $1-1$ correspondence with free homotopy classes of loops in M* [16, Thm 3.8.14 p. 357]*,*
- *each geodesic γ is the unique **minimizer of the length functional** among C^1 loops in the free homotopy class $[\gamma] \in \pi_1(M)$* [16, Thm 3.8.14 p. 357].

By considering the collection of all maps $(\delta_\gamma)_{[\gamma]\in\pi_1(M)}$, for all periodic geodesics, we can define the X-ray transform.

[11] The word free means that the loops are not based.

Definition 6.2 (X-ray transform)**.** *A metric g with negative curvature being fixed, the X-ray transform is a linear map defined as :*

$$I_2 : f \in C^0\left(S^2T^*M\right) \longmapsto \left(\langle \delta_\gamma, f\rangle = \int_0^{T_\gamma} f\left(\gamma(t), \dot{\gamma}(t)\right) dt\right)_{[\gamma]\in\pi_1(M)} \tag{6.8}$$

which maps 2-tensors to sequences indexed by the free homotopy classes $\pi_1(M)$ of closed loops in M. The map I_2 depends on the chosen metric g since geodesics of g explicitely enter in the definition of I_2.

Note that the above X-ray transform is well–defined for **continuous** tensors hence for every tensors of high enough Sobolev regularity $s > \frac{\dim(M)}{2}$ or Hölder regularity $C^{k,\alpha}$ for $k \in \mathbb{N}, \alpha \in (0,1)$.

Before we discuss the injectivity of the X-ray transform, we need to introduce some geometric formalism needed in the formulation of energy identities following [69, section 2]. The tangent bundle to SM admits a direct orthogonal decomposition :

$$T(SM) = \mathbb{V} \perp \mathbb{H} \perp \mathbb{R}X,$$

where $\mathbb{H}$ is the horizontal bundle, $\mathbb{V}$ is the vertical bundle, X is the vector field generating the geodesic flow and SM is endowed with the Sasaki metric $\widehat{g}$ induced from the metric g on the base manifold M. The Levi-Civita connection $\widehat{\nabla}$ for the Sasaki metric $\widehat{g}$ on SM admits the following decomposition :

$$\forall u \in C^\infty(SM), \widehat{\nabla} u = \nabla^v u + \nabla^h u + (Xu)\, X \tag{6.9}$$

where $\nabla^{v,h}$ are the respective vertical and horizontal connections (the orthogonal projection of the connection $\widehat{\nabla}$ on the vertical and horizontal bundles), i.e. $\nabla^v u \in \mathbb{V}, \nabla^h u \in \mathbb{H}$.

An important result about I_2 reads :

Theorem 11 (Injectivity of the X ray transform)**.** *Let $k \in \mathbb{N}, k \geqslant 2, \alpha \in (0,1)$. Let g be a smooth metric with negative curvature. Then the X-ray transform I_2 defined above restricted to solenoidal tensors in* $\ker(D_g^*)$ *of Hölder regularity $C^{k,\alpha}$ is* ***injective****.*

The above result was proved in the C^∞ case by Croke–Sharafutdinov and is well–known in Hölder regularity $C^{k,\alpha}$ although we could not find a reference. In the present work, we shall need the injectivity of I_2 acting on g–solenoidal tensors of regularity $C^{k,\alpha}$ due to the loss of regularity caused by applying the slice Theorem 10. We slightly adapt the proof following notes of Lefeuvre which are themselves heavily based on the original proof of [60].

Proof. The proof in [60] relies on the following ingredients :

(1) The Hodge like decomposition 6.4 which is well–defined in every Sobolev regularity $H^s, s \in \mathbb{R}$ or Hölder regularity $C^{k,\alpha}$ [12].

[12] since it relies on ellipticity of D_g and pseudodifferential calculus

(2) The Pestov identity

$$\|\nabla^v Xu\|^2 \geqslant \|X\nabla^v u\|^2 + d\|Xu\|^2 \tag{6.10}$$

which is well–defined for Sobolev tensors in $H^s(S^2T^*M)$ for $s \geqslant 2$ and another energy identity if $X^2u \in H^0\left(S^{m+1}T^*M\right)$ then

$$\|X\nabla^v u\|^2 - \|\nabla^v Xu\|^2 = \|\nabla^h u\|^2 + ((m+d)(m+1)-d)\|Xu\|^2 + \|\nabla^v Xu\|^2 \tag{6.11}$$

which is also valid for Sobolev tensors in $H^s(S^2T^*M)$ for $s \geqslant 2$. In particular, both identities are valid for tensors u of regularity at least $C^2 \subset H^2$ hence for tensors of Hölder regularity $C^{k,\alpha}$ $k \in \mathbb{N}, k \geqslant 2, \alpha \in (0,1)$.

(3) The Livsic Theorem in regularity C^k for all $k \in \mathbb{N}$ due to De La Llave–Marco–Moriyon [70, Remark p. 578] which states that for any function $\tilde{f} \in C^k(SM)$ s.t. $\forall[\gamma] \in \pi_1(M)$, $\langle \delta_\gamma, f\rangle = 0$, there exists $u \in C^k(SM)$ s.t. $\tilde{f} = Xu$.

Fix $k \geqslant 2$ so that we can use the energy identities. The proof is almost verbatim the one of Croke–Sharafutdinov [60, Thm 1.3] using a C^k version of Livsic Theorem instead of the C^∞ version. Assume $f \in C^{k,\alpha}(S^2T^*M)$ is solenoidal and that $I_2f = 0$. Then by the inclusion $C^{k,\alpha} \subset C^k$ and the C^k version of Livsic Theorem, there exists $u \in C^k(SM)$ s.t. $\pi_2^*f = Xu$. Thus $X^2u = X\pi_2^*f = \pi_3^*D_gf$ by equation 6.2 since D_gf is trace free because $D_g^*f = 0 = Tr(D_gf)$. Combining the above energy identities yields :

$$0 \geqslant \underbrace{-d\|Xu\|^2 \geqslant \|\nabla_X\nabla^v u\|^2 - \|\nabla^v\nabla_X u\|^2}_{\text{by Pestov}} \underbrace{= \|\nabla^h u\|^2 + ((2+d)(2+1)-d)\|Xu\|^2 + \|\nabla^v Xu\|^2 \geqslant 0}_{\text{by the second energy identity}}$$

which implies that $Xu = 0$. □

In the sequel, for every negatively curved metric g, we denote by $\ell_g(\gamma)$ the length of the unique closed geodesic γ given a class $[\gamma] \in \pi_1(M)$.

There is a natural map from moduli space of metrics of negative curvature to periods

$$[g] \in \tilde{N} \in \mathcal{R}(M)_{<0} \longmapsto \mathcal{P}([g]) = \{\ell_g(\gamma); [\gamma] \in \pi_1(M)\} \subset \mathbb{R}_{>0}. \tag{6.12}$$

The assignment $g \mapsto \ell_g(\gamma)$ depends nonlinearly on the metric g. However a classical observation which can be found in [13] is the relation between the differential of the length function and the X–ray transform, for any metric $g \in \mathbf{Met}(M)$ and symmetric 2–tensor h, the differential of ℓ at g in the direction h reads :

$$D\ell_g(\gamma)(h) = \frac{1}{2}I_2(h)_{[\gamma]}. \tag{6.13}$$

Therefore, one should think about the X–ray transform I_2 as a linearized version of the length function and the injectivity of I_2 reflects the injectivity of the nonlinear map 6.12.

After these rather long geometric preparations, we can proceed to prove Proposition 5.1.

6.3. **Proof of Proposition 5.1 by a contradiction argument.** In dimension $d \leqslant 3$ (resp $d = 4$), we assume by contradiction that the set of isospectral metrics in $N \cap \mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ (resp $N \cap \mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$) has an infinite number of classes. Therefore, we assume there exists an infinite sequence $(g'_n)_n$ of smooth isospectral metrics on M whose isometry classes $([g'_n])_n$ are 2 by 2 distinct in $N \cap \mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ (resp $N \cap \mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$). So if $(g'_n)_n$ is a sequence of isospectral metrics of negative curvature $\leqslant -\varepsilon$ (resp in $[-\varepsilon^{-1}, -\varepsilon]$), the above compactness Theorems 4, 5, 6 tell us that we may extract a subsequence such that $g'_n \to g$ in the C^∞–topology for some metric g of negative curvature $\leqslant -\varepsilon$ (resp in $[-\varepsilon^{-1}, -\varepsilon]$). It is important to note that the limit metric g has no isometry group since its class $[g]$ belongs to $\mathcal{G}_{\geqslant \varepsilon}$ and $[g]$ belongs to N since $N \cap \mathcal{R}(M)_{\leqslant -\varepsilon} \cap \mathcal{G}_{\geqslant \varepsilon}$ (resp $N \cap \mathcal{R}(M)_{[-\varepsilon^{-1}, -\varepsilon]} \cap \mathcal{G}_{\geqslant \varepsilon}$) is **closed** from the condition $\partial N \subset \partial \mathcal{G}$. In dimension 3, we can apply the compactness Theorem 5 since the spectrum determines the length of the shortest closed geodesic by Theorem 3.

Now since g has negative curvature, we can make use of the slice Theorem 10 and produce a new sequence $(g_n)_n$ of metrics with the following properties :

Corollary 6.3. *There exists a sequence of metrics* $(g_n = \Psi(g'_n))_n$ *of regularity* $C^{k,\alpha}, k \geqslant 2, \alpha \in (0,1)$ *such that* $[g_n] = [g'_n], \forall n \in \mathbb{N}$*, the difference* $\varepsilon_n = g_n - g \in C^{k,\alpha}(S^2T^*M) \to 0$ *is* ***solenoidal*** *w.r.t.* g*, the metrics* g_n *all have the* ***same length spectrum****.*

It is important to note that it is no longer a priori true that the sequence $(g_n)_n$ is made of smooth metrics. The **solenoidal property** will be very important in the sequel since we shall use the injectivity of the X-ray transform for solenoidal tensors w.r.t. g. We assume by contradiction that the sequence of metrics g_n is non stationary, which means that the sequence $\varepsilon_n = g_n - g_0$ never vanishes for every n and $\varepsilon_n \to 0$ in $C^{k,\alpha}(S^2T^*M)$. By Proposition 6.1, for each metric g_n and for every class $[\gamma] \in \pi_1(M)$, we shall denote by γ_n (resp γ) the unique geodesic representative of $[\gamma]$ in $\pi_1(M)$ for the metric g_n (resp for the metric g).

For each closed curve γ in SM, we define a Radon measure $\delta_\gamma \in \mathcal{D}'(SM)$ by equation (6.7) in subsection 6.2. Recall that a sequence μ_n of Radon measures on SM is said to weakly–$*$ converge to μ if for every continuous $\varphi \in C^0(SM)$, $\mu_n(\varphi) \to \mu(\varphi)$ when $n \to +\infty$. By Proposition 7.2 proved in the appendix, we have the weak–$*$ convergence $\delta_{\gamma_n} \to \delta_\gamma$ of the Radon measures on SM. By the convergence of metrics $g_n \to g$, for every free homotopy class $[\gamma]$ in $\pi_1(M)$, for every n, we have the convergence $\ell_{g_n}(\gamma_n) \to \ell_g(\gamma)$ by [64, Lemma 4.1 p. 11].

6.3.1. *Inequalities satisfied by* ε_n. From the fact that the metrics are isospectral and the length spectrum is discrete, we deduce that $\ell_{g_n}(\gamma_n) = \ell_g(\gamma)$ for every $n \geqslant N_\gamma$ where the integer N_γ depends on $[\gamma] \in \pi_1(M)$. By equation (6.7) defining the Radon measures $\delta_\gamma \in \mathcal{D}'(SM)$ carried by closed curves γ, the length of the curve γ for the metric g is defined as $\ell_g(\gamma) = \delta_\gamma(g)$. The metrics g_n have negative sectional curvatures hence by Proposition 6.1, every closed geodesic γ_n is minimizing for g_n in the class $[\gamma] \in \pi_1(M)$ which implies the inequality $\delta_{\gamma_n}(g_n) \leqslant \delta_\gamma(g_n)$. But for $n \geqslant N_\gamma$, we find that $\delta_{\gamma_n}(g_n) = \delta_\gamma(g_0)$ from which we deduce the inequality $\delta_\gamma(g_0) \leqslant \delta_\gamma(g_n)$ which implies

$$\delta_\gamma(\varepsilon_n) \geqslant 0, \forall n \geqslant N_\gamma. \tag{6.14}$$

Conversely since γ minimizes the length for g_0 we have a reverse inequality $\delta_{\gamma_n}(g_n) = \delta_\gamma(g_0) \leqslant \delta_{\gamma_n}(g_0)$ which implies the second inequality :

$$\delta_{\gamma_n}(\varepsilon_n) \leqslant 0, \forall n \geqslant N_\gamma. \tag{6.15}$$

6.3.2. *Another compactness argument and conclusion of the proof of Proposition 5.1.* Now we would like to know if we can extract a subsequence from $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty} \in C^{k,\alpha}(S^2T^*M)$ **with non trivial limit** so that we obtain inequalities on the X-ray transform which are independent of n. We denote by $\mathbf{Met}^{k,\alpha}(M)$ the set of Riemannian metrics of Hölder regularity $C^{k,\alpha}$. The assumption in corollary 6.3 means that there exists an abstract smooth submanifold $\tilde{N}$ of the same dimension as N near g which contains the sequence $(g_n)_n$ and a C^∞- map $\iota : \tilde{N} \longmapsto \mathbf{Met}^{k,\alpha}(M)$ such that $\pi \circ \iota(\tilde{N}) = N$ near $[g]$ where $\pi : \mathbf{Met}^{k,\alpha}(M) \mapsto \mathcal{R}(M)$ is the natural projection induced by the quotient map. One should think of $\tilde{N}$ as some kind of cover of N near $[g]$.

Choose any smooth Riemannian metric $\tilde{g}$ on the finite dimensional submanifold $\tilde{N}$ and denote by v_n a sequence of tangent vectors in $T_{g_0}\tilde{N}$ such that $\iota\left(\exp_{g_0}(v_n)\right) = g_0 + \frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}$ where exp is the Riemannian exponential map induced by the metric $\tilde{g}$. Since the exponential map $v \in T_{g_0}\tilde{N} \mapsto \exp_{g_0}(v)$ is a diffeomorphism near the origin whose differential at 0 is the identity, we may find that the norm of the sequence v_n is equivalent to the distance dist $\left(g_0 + \frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}, g_0\right)$. Since $\tilde{N}$ has finite dimension and the sequence $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}$ has sup norm 1, the sequence of tangent vectors v_n is contained in some closed bounded subset of $T_{g_0}\tilde{N}$ which avoids 0. Then by compactness of closed bounded subsets in **finite dimension**, we can extract a subsequence of $(v_n)_n$ s.t. $v_n \to v_\infty \neq 0 \in T_{g_0}\tilde{N}$. So along this subsequence, $\frac{\varepsilon_n}{\|\varepsilon_n\|}$ has a nontrivial limit $u = \exp_{g_0}(v_\infty) \in C^{k,\alpha}(S^2T^*M)$. Hence, up to extracting a subsequence, we may assume that $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty} \underset{n\to\infty}{\longrightarrow} u \in C^{k,\alpha}(S^2T^*M)$ in the $C^{k,\alpha}$ topology where $u \neq 0$ and $\|u\|_\infty = 1$.

Passing to the limit in both inequalities 6.14 and 6.15 and using the fact that the Radon measures δ_{γ_n} weakly–$*$ converges to δ_γ, we find that the limit u satisfies $I_2(u)_\gamma = \delta_\gamma(u) \geqslant 0$ and $I_2(u)_\gamma = \delta_\gamma(u) \leqslant 0$ hence for any free homotopy class $[\gamma] \in \pi_1(M)$, $I_2(u)_\gamma = \delta_\gamma(u) = 0$. The above means that u is a 2–tensor which belongs to the kernel of the linear map I_2. But since u is solenoidal w.r.t. g and the restriction of I_2 to solenoidal tensors w.r.t. g is **injective** by Theorem 11, we conclude that $u = 0$ which contradicts $u \neq 0$ in $C^{k,\alpha}(S^2T^*M)$.

7. Appendix.

7.1. **Gohberg–Krein's determinants.** Set $p = [\frac{d}{2}] + 1$ and let A belong to the Schatten ideal $\mathcal{I}_p$. Following [68, chapter 9], we shall summarize the main properties of Gohberg–Krein's determinants and their relation with functional traces :

Lemma 7.1 (Gohberg–Krein's determinants and functional traces)**.** *For all $A \in \mathcal{I}_p$, the Gohberg–Krein determinant* $\det_p(1 + zA)$ *is an **entire function** in $z \in \mathbb{C}$ and is related to* *traces $Tr(A^n)$ for $n > \frac{d}{2}$ by the following formulas :*

$$\boxed{det_p(1+zA) = \exp\left(\sum_{n=p}^{\infty} \frac{(-1)^{n+1}z^n}{n} Tr(A^n)\right) = \prod_k \left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}(-1)^n n^{-1}\lambda_k(A)^n\right)\right]}$$

where the series $\exp\left(\sum_{n=p}^{\infty} \frac{(-1)^{n+1}z^n}{n} Tr(A^n)\right)$ converges when $|z| < \|A\|_{\mathcal{I}_p}$ and the infinite product vanishes exactly when $z\lambda_k(A) = -1$ with multiplicity.

7.2. **Convergence of Radon measures corresponding to closed geodesics.** The goal of this paragraph is to show that if $g_n \mapsto g$ in the metrics of negative curvature, then for every free homotopy class $[\gamma] \in \pi_1(M)$, denote by γ_n (resp γ) the unique corresponding sequence of closed geodesic for g_n (resp g), the sequence of Radon measures δ_{γ_n} weak–$*$ converges to δ_γ. We shall use the structural stability result of Anosov flows in the version of De La Llave–Marco–Moriyon [70, Thm A.2 p. 598].

Theorem 12 (Structural stability)**.** *Let (M, g) be a Riemannian manifold of negative curvature and set $\mathcal{M} = SM$ to be the sphere bundle of M. We denote by $X \in C^1(T\mathcal{M})$ the geodesic vector field of the metric g and by $C^0_X(\mathcal{M}, \mathcal{M})$ the space of homeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ which are C^1 along integral curves of X and $C^0(\mathcal{M})$ denotes continuous functions on $\mathcal{M}$. Then there exists a C^1 neighborhood $\mathcal{U}$ of X, a submanifold $\mathcal{N} \subset C^0_X(\mathcal{M}, \mathcal{M})$ and a C^1 map :*

$$S : \mathcal{U} \longmapsto \mathcal{N} \times C^0(\mathcal{M}) \tag{7.1}$$

$$Y \longmapsto (\Phi_Y, h_Y) \tag{7.2}$$

satisfying the structure equation :

$$\boxed{(\Phi_Y^{-1*}h_Y)Y = \Phi_{Y*}X} \tag{7.3}$$

where $(\Phi_X, h_X) = (Id, 1) \in C^0_X(\mathcal{M}, \mathcal{M}) \times C^0(M)$.

The equation 7.3 follows from [70, equation (e) p. 592]

$$D\Phi_Y(x,v)(X(x,v)) = h_Y(x,v)\,Y(\Phi_Y(x,v)), \tag{7.4}$$

this implies that $D\Phi_Y\left(\Phi_Y^{-1}(x,v)\right)\left(X(\Phi_Y^{-1}(x,v))\right) = h_Y\left(\Phi_Y^{-1}(x,v)\right)Y(x,v)$ hence $\Phi_{Y*}X = \left(\Phi_Y^{-1*}h_Y\right)Y$. The above equation means that flows in a neighborhood $\mathcal{U}$ of X are conjugated to the flow generated by X up to reparametrization of time, more precisely let $\varphi_Y^t : \mathcal{M} \mapsto \mathcal{M}$ denotes the flow generated by $Y \in \mathcal{U} \subset C^1(T\mathcal{M})$, then there exists $\tau_Y \in C^0(\mathbb{R} \times \mathcal{M})$ s.t. :

$$\varphi_Y^t(x,v) = \Phi_Y \circ \varphi_X^{\tau_Y(t,x,v)} \circ \Phi_Y^{-1}(x,v) \tag{7.5}$$

where $\tau_Y(t,x,v) \to t$ in $C^0([0,T] \times \mathcal{M})$ for all $T > 0$ when $Y \to X$ in $C^1(T\mathcal{M})$.

A corollary of the above result

Proposition 7.2 (Convergence result for Radon measures.)**.** *Under the assumptions of Theorem 12. Let $(g_n)_n$ be a sequence of metrics of negative curvature which converges to g in the C^2 topology. We denote by $X \in C^1(T\mathcal{M})$ (resp $X_n \in C^1(T\mathcal{M})$) the geodesic vector field of the metric g (resp g_n).*

Then X_n is a sequence of vector fields which converges to X in $C^1(\mathcal{M})$ where for every free homotopy class $[\gamma] \in \pi_1(M)$, there exists $N_\gamma \in \mathbb{N}$ and a unique subsequence of periodic orbits $(\gamma_n)_{n\geqslant N_\gamma}$ of the vector field X_n which converges to a periodic orbit γ of X. The corresponding Radon measures $\delta_{\gamma_n}, n \geqslant N_\gamma$ will weak–$*$ converge to the limit Radon measure δ_γ.

In particular for every 2–tensor $h \in C^0(S^2T^*M)$, recall $\pi_2^* : C^0(S^2T^*M) \mapsto C^0(SM)$, then

$$\delta_{\gamma_n}(\pi_2^* h) \to \delta_\gamma(\pi_2^* h).$$

Proof. Let $f \in C^0(SM)$ be a continuous test function. Denote by φ_n^t (resp φ^t) the flow generated by X_n (resp X) on SM. By definition $\delta_{\gamma_n}(f) = \int_0^{\ell_{g_n}(\gamma_n)} f \circ \varphi_n^t(x_n, v_n) dt$ for any $(x_n, v_n) \in \gamma_n$. The existence of the sequence $\gamma_n \to \gamma$ is a simple consequence of structural stability. Let $\Phi_n \in C^0_X(M, M)$ denotes the sequence of homeomorphisms conjugating the two flows whose existence comes from Theorem 12 :

$$\varphi_n^t(x, v) = \Phi_n \circ \varphi^{\tau_n(t,x,v)} \circ \Phi_n^{-1}(x, v)$$

where $\tau_n(t, x, v) \to t$ uniformly on $[0, T] \times SM$ for all $T > 0$ and $\Phi_n \to Id$ in $C^0(SM)$. Therefore for every (x, v) on the periodic orbit γ, the sequence $(x_n, v_n) = \Phi_n(x, v)$ lies in the periodic orbit γ_n by structural stability and converges to (x, v). It follows that when $n \to +\infty$,

$$\delta_{\gamma_n}(f) = \int_0^{\ell_{g_n}(\gamma_n)} f \circ \varphi_n^t(x_n, v_n) dt = \int_0^{\ell_{g_n}(\gamma_n)} f \circ \Phi_n \circ \varphi^{\tau_n(t,x_n,v_n)}(x, v) dt \underset{n\to+\infty}{\to} \int_0^{\ell_g(\gamma)} f \circ \varphi^t(x, v) dt$$

by dominated convergence and since the periods $\ell_{g_n}(\gamma_n) \underset{n\to+\infty}{\to} \ell_g(\gamma)$ converge [64, Lemma 4.1 p. 11] and $\frac{1}{2}\ell_g(\gamma) \leqslant \ell_{g_n}(\gamma_n) \leqslant 2\ell_g(\gamma)$ for all $n \geqslant N_\gamma$ [64, Remark 3]. It follows that the sequence of Radon measures δ_{γ_n} will weak–$*$ converge to the limit Radon measures δ_γ. □

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