Arithmetic properties of Apéry-like numbers

É. Delaygue

Abstract

We provide lower bounds for p-adic valuations of multisums of factorial ratios which satisfy an Apéry-like recurrence relation: these include Apéry, Domb, Franel numbers, the numbers of abelian squares over a finite alphabet, and constant terms of powers of certain Laurent polynomials. In particular, we prove Beukers' conjectures on the p-adic valuation of Apéry numbers. Furthermore, we give an effective criterion for a sequence of factorial ratios to satisfy the p-Lucas property for almost all primes p.

1. Introduction

1.1 Classical results of Lucas and Kummer

It is a well-known result of Lucas [Lu78] that, for all nonnegative integers m,n and all primes p, we have

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p, \tag{1.1}$$

where $m = m_0 + m_1 p + \cdots + m_k p^k$ and $n = n_0 + n_1 p + \cdots + n_k p^k$ are the base p expansions of m and n.

In particular, a prime p divides the binomial $\binom{m}{n}$ if, and only if there is $0 \leqslant i \leqslant k$ such that $m_i < n_i$. Precisely, Kummer proved in [Ku52] that, for all nonnegative integers $m \geqslant n$, the p-adic valuation ([1]) of the binomial $\binom{m}{n}$ is the number of carries which occur when n is added to $m-n$ in base p. As a consequence, we have

$$\binom{m}{n} \in p^{\alpha}\mathbb{Z}, \quad \text{where} \quad \alpha = \#\left\{0 \leqslant i \leqslant k \,:\, \binom{m_i}{n_i} = 0\right\}. \tag{1.2}$$

In this article, we show that many sequences $(A(n))_{n\geqslant 0}$ of Apéry-like numbers satisfy congruences similar to (1.1), that is, for all nonnegative integers n and all primes p, we have

$$A(n) \equiv \prod_{i=0}^{k} A(n_i) \mod p,$$

where $n = n_0 + n_1 p + \cdots + n_k p^k$ is the base p expansion of n. Furthermore, we prove that an analogue of (1.2) holds for those numbers, that is

$$A(n) \in p^{\alpha}\mathbb{Z}, \quad \text{where } \alpha = \#\big\{0 \leqslant i \leqslant k \,:\, A(n_i) \equiv 0 \mod p\big\},$$

2010 Mathematics Subject Classification Primary 11B50; Secondary 11B65 05A10
Keywords: Apéry numbers, Constant terms of powers of Laurent polynomials, p-Lucas property, Congruences
This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No 648132. Research partially supported by the project Holonomix (PEPS CNRS INS2I 2012).

[1]The p-adic valuation of an integer m is the maximum integer β such that p^{β} divides m.

which proves Beukers' conjectures on the p-adic valuation of Apéry numbers.

1.2 Beukers' conjectures on Apéry numbers

For all nonnegative integers n, we set

$$A_1(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A_2(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

These sequences were used in 1979 by Apéry in his proofs of the irrationality of $\zeta(3)$ and $\zeta(2)$ (see [Ap79]). In the 1980's, several congruences satisfied by these sequences were proved (see for example [Be85], [Be87], [CCC80], [Ge82], [Mi83]). In particular, Gessel proved in [Ge82] that A_1 satisfies the p-Lucas property for all prime numbers p, that is, for any prime p, all v in $\{0, \dots, p-1\}$ and all nonnegative integers n, we have

$$A_1(v + np) \equiv A_1(v)A_1(n) \mod p.$$

Thereby, if $n = n_0 + n_1 p + \cdots + n_N p^N$ is the base p expansion of n, then we obtain

$$A_1(n) \equiv A_1(n_0) \cdots A_1(n_N) \mod p. \tag{1.3}$$

In particular, p divides $A_1(n)$ if, and only if there exists k in $\{0, \dots, N\}$ such that p divides $A_1(n_k)$. Beukers stated in [Be86] two conjectures, when $p = 5$ or 11, which generalize this property ([2]). Before stating these conjectures, we observe that the set of all v in $\{0, \dots, 4\}$ (respectively v in $\{0, \dots, 10\}$) satisfying $A_1(v) \equiv 0 \mod 5$ (respectively $A_1(v) \equiv 0 \mod 11$) is $\{1, 3\}$ (respectively $\{5\}$).

Conjecture A Beukers, [Be86]. *Let n be a nonnegative integer whose base 5 expansion is $n = n_0 + n_1 5 + \cdots + n_N 5^N$. Let α be the number of k in $\{0, \dots, N\}$ such that $n_k = 1$ or 3. Then 5^α divides $A_1(n)$.*

Conjecture B Beukers, [Be86]. *Let n be a nonnegative integer whose base 11 expansion is $n = n_0 + n_1 11 + \cdots + n_N 11^N$. Let α be the number of k in $\{0, \dots, N\}$ such that $n_k = 5$. Then 11^α divides $A_1(n)$.*

Similarly, Sequence A_2 satisfies the p-Lucas property for all primes p. Furthermore, Beukers and Stienstra proved in [BS85] that, if $p \equiv 3 \mod 4$, then $A_2\left(\frac{p-1}{2}\right) \equiv 0 \mod p$, and Beukers stated in [Be86] the following conjecture.

Conjecture C. *Let p be a prime number satisfying $p \equiv 3 \mod 4$. Let n be a nonnegative integer whose base p expansion is $n = n_0 + n_1 p + \cdots + n_N p^N$. Let α be the number of k in $\{0, \dots, N\}$ such that $n_k = \frac{p-1}{2}$. Then p^α divides $A_2(n)$.*

Conjectures A-C have been extended to generalized Apéry numbers and any prime p by Deutsch and Sagan in [DS06, Conjecture 5.13] but this extension is false for at least one generalization of Apéry numbers. Indeed, a counterexample is given by

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^3,$$

since $A(1) = 9 \equiv 0 \mod 3$ but $A(4) = A(1+3) = 1152501$ is not divisible by 3^2.

[2] If p is 2, 3 or 7, then for all v in $\{0, \dots, p-1\}$, $A_1(v)$ is coprime to p so that, according to (1.3), for all nonnegative integers n, $A_1(n)$ is coprime to p.

The main aim of this article is to prove Theorem 1, stated in Section 1.4, which confirms and generalizes Conjectures A-C. First, we introduce some notations which we use throughout this article.

1.3 Notations

In order to study arithmetic properties of sums of products of binomial coefficients, such as Apéry numbers, we first study families, indexed by $\mathbb{N}^d$, of ratios of factorials of linear forms with integer coefficients. For example, we will obtain congruences for $A_1(n)$ by studying the factorial ratios

$$\frac{(2n_1+n_2)!^2}{n_1!^4 n_2!^2},\quad (n_1,n_2\in\mathbb{N}),$$

as we have the useful formula

$$A_1(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{n_1+n_2=n}\frac{(2n_1+n_2)!^2}{n_1!^4 n_2!^2}.$$

Let d be a positive integer. Given tuples of vectors in $\mathbb{N}^d$, $e=(\mathbf{e}_1,\dots,\mathbf{e}_u)$ and $f=(\mathbf{f}_1,\dots,\mathbf{f}_v)$, we shall prove congruences for the factorial ratios

$$\mathcal{Q}_{e,f}(\mathbf{n}):=\frac{\prod_{i=1}^{u}(\mathbf{e}_i\cdot\mathbf{n})!}{\prod_{i=1}^{v}(\mathbf{f}_i\cdot\mathbf{n})!},\quad (\mathbf{n}\in\mathbb{N}^d),$$

to deduce arithmetic properties of the numbers ([3])

$$\mathfrak{S}_{e,f}(n):=\sum_{\mathbf{n}\in\mathbb{N}^d,|\mathbf{n}|=n}\mathcal{Q}_{e,f}(\mathbf{n}),\quad (n\in\mathbb{N}).\tag{1.4}$$

Here $\cdot$ denotes the standard scalar product on $\mathbb{R}^d$ and $|\mathbf{n}|=n_1+\cdots+n_d$ if $\mathbf{n}=(n_1,\dots,n_d)$. For example, we obtain that $\mathfrak{S}_{e,f}(n)=A_1(n)$ with the tuples

$$e=((2,1),(2,1))\quad\text{and}\quad f=((1,0),(1,0),(1,0),(1,0),(0,1),(0,1)).$$

Because of the summation in (1.4), it is usually difficult to study arithmetic properties of $\mathfrak{S}_{e,f}(n)$, however we will show that, in many interesting cases, we can transfer the p-Lucas property from $\mathcal{Q}_{e,f}(\mathbf{n})$ to $\mathfrak{S}_{e,f}(n)$. To that purpose, we define the p-Lucas property for families of p-adic integers indexed by $\mathbb{N}^d$. For all primes p, we write $\mathbb{Z}_p$ for the ring of p-adic integers.

If $A=(A(\mathbf{n}))_{\mathbf{n}\in\mathbb{N}^d}$ is a $\mathbb{Z}_p$-valued family, then we say that A satisfies *the p-Lucas property* if, for all vectors $\mathbf{v}$ in $\{0,\dots,p-1\}^d$ and $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$A(\mathbf{v}+\mathbf{n}p)\equiv A(\mathbf{v})A(\mathbf{n})\mod p\mathbb{Z}_p.\tag{1.5}$$

If $\mathbf{n}$ is nonzero, then we say that $\mathbf{n}=\mathbf{n}_0+\mathbf{n}_1p+\cdots+\mathbf{n}_Np^N$ is the base p expansion of $\mathbf{n}$ if, for all i in $\{0,\dots,N\}$, we have $\mathbf{n}_i\in\{0,\dots,p-1\}^d$, and $\mathbf{n}_N\neq\mathbf{0}$, where $\mathbf{0}:=(0,\dots,0)$. Hence, if A satisfies the p-Lucas property, then we have

$$A(\mathbf{n})\equiv A(\mathbf{n}_0)\cdots A(\mathbf{n}_N)\mod p\mathbb{Z}_p.$$

We write $\mathcal{Z}_p(A)$ for the set of all vectors $\mathbf{v}$ in $\{0,\dots,p-1\}^d$ such that $A(\mathbf{v})$ belongs to $p\mathbb{Z}_p$. Hence $A(\mathbf{n})$ is in $p\mathbb{Z}_p$ if, and only if at least one $\mathbf{n}_i$, $0\leqslant i\leqslant N$, belongs to $\mathcal{Z}_p(A)$. To state our generalization of Conjectures A–C we define the following counting function. For every nonzero

[3] We also provide a proof of Beukers' conjectures which directly uses congruences for Apéry numbers due to their representation as constant terms of powers of Laurent polynomials.

vector $\mathbf{n}$ in $\mathbb{N}^d$ whose base p expansion is $\mathbf{n} = \mathbf{n}_0 + \mathbf{n}_1 p + \cdots + \mathbf{n}_N p^N$, we write $\alpha_p(A, \mathbf{n})$ for the number of i in $\{0, \dots, N\}$ such that $\mathbf{n}_i \in \mathcal{Z}_p(A)$, and we set $\alpha_p(A, \mathbf{0}) = 0$. Thereby, to prove Conjectures A-C, it is enough to show that $A_i(n) \in p^{\alpha_p(A_i, n)}\mathbb{Z}$ with $i = 1, p = 5$ or 11 and $i = 2$, $p \equiv 3 \mod 4$.

Our generalization of Beukers' conjectures will apply to sequences $\mathfrak{S}_{e,f}$ restricted to the following two conditions.

The first condition (the r-admissibility) ensures that we can transfer the p-Lucas property from $\mathcal{Q}_{e,f}(\mathbf{n})$ to $\mathfrak{S}_{e,f}(n)$. If $\mathbf{m} = (m_1, \dots, m_d)$ and $\mathbf{n} = (n_1, \dots, n_d)$ belong to $\mathbb{R}^d$, then we write $\mathbf{m} \geqslant \mathbf{n}$ if, for all i in $\{1, \dots, d\}$, we have $m_i \geqslant n_i$. Furthermore, we set $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^d$ and we write $\mathbf{1}_k$ for the vector in $\mathbb{N}^d$, all of whose coordinates equal zero except the k-th which is 1. Let $\mathcal{S} := \{1 \leqslant i \leqslant u \,:\, \mathbf{e}_i \geqslant \mathbf{1}\}$. For every positive integer r, we say that e is *r-admissible* if

$$\#\mathcal{S} + \min_{1 \leqslant k \leqslant d} \#\{1 \leqslant i \leqslant u \,:\, i \notin \mathcal{S} \text{ and } \mathbf{e}_i \geqslant d\mathbf{1}_k\} \geqslant r.$$

We will use this definition with $r = 1$ or 2. In the case of the Apéry numbers $A_1(n)$, we study the family $\mathcal{Q}_{e,f}$ with the tuple $e = ((2,1),(2,1))$ so that $\#\mathcal{S} = 2$ and e is 2-admissible. As another example, we will also prove a result similar to Beukers' conjectures for the sequence

$$A_6(n) := \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

We can write

$$A_6(n) = \sum_{n_1+n_2=n} \frac{(n_1+n_2)!(2n_1)!(2n_2)!}{n_1!^3 n_2!^3},$$

so that $A_6(n) = \mathfrak{S}_{e,f}(n)$ with $e = ((1,1),(2,0),(0,2))$. In this case, we have $d = 2$, $\#\mathcal{S} = 1$ but e is also 2-admissible because for $k = 1$ or 2 we have $\#\{2 \leqslant i \leqslant 3 \,:\, \mathbf{e}_i \geqslant 2\mathbf{1}_k\} = 1$.

The second condition is of differential type. To apply our main result, we need the generating series of $(\mathfrak{S}_{e,f}(n))_{n \geqslant 0}$ to be annihilated by a differential operator of a special form that we describe below. We set $\theta := z\frac{d}{dz}$ and we say that a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z,\theta]$ is of *type I* if there is a nonnegative integer q such that:

- $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^q P_q(\theta)$ with $P_k(X) \in \mathbb{Z}_p[X]$ for $0 \leqslant k \leqslant q$;
- $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$;
- for all k in $\{2, \dots, q\}$, we have $P_k(X) \in \prod_{i=1}^{k-1}(X+i)^2\mathbb{Z}_p[X]$.

We say that a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z,\theta]$ is of *type II* if

- $\mathcal{L} = P_0(\theta) + zP_1(\theta) + z^2P_2(\theta)$ with $P_k(X) \in \mathbb{Z}_p[X]$ for $0 \leqslant k \leqslant 2$;
- $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$;
- $P_2(X) \in (X+1)\mathbb{Z}_p[X]$.

For example, the generating series of $(A_1(n))_{n \geqslant 0}$ is annihilated by the differential operator

$$\mathcal{L}_1 = \theta^3 - z(34\theta^3 + 51\theta^2 + 27\theta + 5) + z^2(\theta+1)^3,$$

which is of type I for every prime p. We will also prove a result similar to Beukers' conjectures for the numbers

$$A_5(n) = \sum_{k=0}^{n} \binom{n}{k}^4.$$

The generating series of A_5 is annihilated by the differential operator

$$\mathcal{L}_5 = \theta^3 - z2(2\theta+1)(3\theta^2+3\theta+1) - z^2 4(\theta+1)(4\theta+5)(4\theta+3),$$

which is of type II for every prime p.

Our main result confirms Conjectures A–C, and also provides surprising similar properties for some deformations of Apéry-like numbers. For example, while proving that, for every prime p and all nonnegative integers n, we have

$$A_1(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in p^{\alpha_p(A_1,n)}\mathbb{Z},$$

we will also show that, for every nonnegative integer a, we have

$$\sum_{k=0}^{n} k^a \binom{n}{k}^2 \binom{n+k}{k}^2 \in p^{\alpha_p(A_1,n)-1}\mathbb{Z}.$$

More generally, we will obtain congruences for deformations $\mathfrak{S}_{e,f}^g$ of the sequences $\mathfrak{S}_{e,f}$ defined as follows. For any prime p, we write $\mathfrak{F}_p^d$ for the set of all functions $g : \mathbb{N}^d \to \mathbb{Z}_p$ such that, for all nonnegative integers K, there exists a sequence $(P_{K,k})_{k\geqslant 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to g on $\{0,\dots,K\}^d$. For all tuples e and f of vectors in $\mathbb{N}^d$, all $g \in \mathfrak{F}_p^d$ and all nonnegative integers m, we set

$$\mathfrak{S}_{e,f}^g(m) := \sum_{\mathbf{n}\in\mathbb{N}^d, |\mathbf{n}|=m} \mathcal{Q}_{e,f}(\mathbf{n})g(\mathbf{n}).$$

1.4 Main results

In the rest of the article, if $e = (\mathbf{e}_1,\dots,\mathbf{e}_u)$ is a tuple of vectors in $\mathbb{N}^d$, then we set $|e| := \mathbf{e}_1+\cdots+\mathbf{e}_u$. The main result of this article is the following.

Theorem 1. *Let e and $f = (\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$ be two disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$, for all i in $\{1,\dots,v\}$, k_i is in $\{1,\dots,d\}$, and e is 2-admissible. Let p be a fixed prime. Assume that the generating series of $\mathfrak{S}_{e,f}$ is annihilated by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(\mathfrak{S}_{e,f})$.*

Then, for all nonnegative integers n and all functions g in $\mathfrak{F}_p^d$, we have

$$\mathfrak{S}_{e,f}(n) \in p^{\alpha_p(\mathfrak{S}_{e,f},n)}\mathbb{Z} \quad \text{and} \quad \mathfrak{S}_{e,f}^g(n) \in p^{\alpha_p(\mathfrak{S}_{e,f},n)-1}\mathbb{Z}_p.$$

In Section 1.6, we show that Theorem 1 applies to many classical sequences. In particular, Theorem 1 implies Conjectures A-C. Indeed, we have $A_1 = \mathfrak{S}_{e_1,f_1}$ and $A_2 = \mathfrak{S}_{e_2,f_2}$ with $d=2$,

$$e_1 = ((2,1),(2,1)) \quad \text{and} \quad f_1 = ((1,0),(1,0),(1,0),(1,0),(0,1),(0,1)),$$

and

$$e_2 = ((2,1),(1,1)) \quad \text{and} \quad f_2 = ((1,0),(1,0),(1,0),(0,1),(0,1)).$$

Furthermore, it is well known that f_{A_1}, respectively f_{A_2}, is annihilated by the differential operator $\mathcal{L}_1$, respectively $\mathcal{L}_2$, defined by

$$\mathcal{L}_1 = \theta^3 - z(34\theta^3+51\theta^2+27\theta+5) + z^2(\theta+1)^3$$

and

$$\mathcal{L}_2 = \theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta + 1)^2.$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are of type I for all primes p, the conditions of Theorem 1 are satisfied by A_1 and A_2, and Conjectures A-C hold. In addition, for all primes p and all nonnegative integers n and a, we obtain that

$$\sum_{k=0}^{n} k^a \binom{n}{k}^2 \binom{n+k}{k}^2 \in p^{\alpha_p(A_1,n)-1}\mathbb{Z} \quad \text{and} \quad \sum_{k=0}^{n} k^a \binom{n}{k}^2 \binom{n+k}{k} \in p^{\alpha_p(A_2,n)-1}\mathbb{Z}.$$

We provide a similar result which applies to the constant terms of powers of certain Laurent polynomials. Consider a Laurent polynomial

$$\Lambda(\mathbf{x}) = \sum_{i=1}^{k} \alpha_i \mathbf{x}^{\mathbf{a}_i} \in \mathbb{Z}_p[x_1^{\pm}, \dots, x_d^{\pm}],$$

where $\mathbf{a}_i \in \mathbb{Z}^d$ and $\alpha_i \neq 0$ for i in $\{1, \dots, k\}$. Recall that the Newton polyhedron of Λ is the convex hull of $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ in $\mathbb{R}^d$. Hence we have the following result.

THEOREM 2. *Let p be a fixed prime. Let $\Lambda(\mathbf{x}) \in \mathbb{Z}_p[x_1^{\pm}, \dots, x_d^{\pm}]$ be a Laurent polynomial, and consider the sequence of the constant terms of powers of Λ defined, for all nonnegative integers n, by*

$$A(n) := \left[\Lambda(\mathbf{x})^n\right]_{\mathbf{0}}.$$

Assume that the Newton polyhedron of Λ contains the origin as its only interior integral point, and that f_A is annihilated by a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z, \theta]$ such that at least one of the following conditions holds:

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p - 1 \in \mathscr{Z}_p(A)$.*

Then, for all nonnegative integers n, we have

$$A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}_p.$$

For example, Theorem 2 applies to Apéry numbers A_1 thanks to the following formula of Lairez [Lai13]:

$$A_1(n) = \left[\left(\frac{(1+z)(yz+z+1)(1+x)(xy+x+y)}{xyz}\right)^n\right]_{(0,0,0)}.$$

By a result of Samol and van Straten [SvS15], if $\Lambda(\mathbf{x}) \in \mathbb{Z}_p[x_1^{\pm}, \dots, x_d^{\pm}]$ contains the origin as its only interior integral point, then $\left([\Lambda(\mathbf{x})^n]_{\mathbf{0}}\right)_{n \geqslant 0}$ satisfies the p-Lucas property, which is essential for the proof of Theorem 2. Likewise, the proof of Theorem 1 rests on the fact that $\mathfrak{S}_{e,f}$ satisfies the p-Lucas property when $|e| = |f|$, e is 2-admissible and $f = (\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_v})$. Since those results deal with multisums of factorial ratios, it seems natural to study similar arithmetic properties for simpler numbers such as families of factorial ratios. To that purpose, we prove Theorem 3 below which gives an effective criterion for $\mathcal{Q}_{e,f}$ to satisfy the p-Lucas property for almost all primes p ([4]). Furthermore, Theorem 3 shows that if $A := \mathcal{Q}_{e,f}$ satisfies the p-Lucas property for almost all

[4]Throughout this article, we say that an assertion $\mathcal{A}_p$ is true for almost all primes p if it holds for all but finitely many primes p.

primes p, then, for all nonnegative integers n and all primes p, we have $A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}$.

To state this result, we introduce some additional notations. For all tuples e and f of vectors in $\mathbb{N}^d$, we write $\Delta_{e,f}$ for Landau's function defined, for all $\mathbf{x}$ in $\mathbb{R}^d$, by

$$\Delta_{e,f}(\mathbf{x}) := \sum_{i=1}^{u} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor - \sum_{i=1}^{v} \lfloor \mathbf{f}_i \cdot \mathbf{x} \rfloor \in \mathbb{Z},$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Therefore, according to Landau's criterion [Lan85] and a precision of the author [De13], we have the following dichotomy.

- If, for all $\mathbf{x}$ in $[0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) \geqslant 0$, then $\mathcal{Q}_{e,f}$ is a family of integers;
- if there exists $\mathbf{x}$ in $[0,1]^d$ such that $\Delta_{e,f}(\mathbf{x}) \leqslant -1$, then there are only finitely many primes p such that $\mathcal{Q}_{e,f}$ is a family of p-adic integers.

In the rest of the article, we write $\mathcal{D}_{e,f}$ for the semi-algebraic set of all $\mathbf{x}$ in $[0,1)^d$ such that there exists a component $\mathbf{d}$ of e or f satisfying $\mathbf{d} \cdot \mathbf{x} \geqslant 1$. Observe that $\Delta_{e,f}$ vanishes on the nonempty set $[0,1)^d \setminus \mathcal{D}_{e,f}$.

Theorem 3. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers. Then we have the following dichotomy.*

(i) *If $|e| = |f|$ and if, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geqslant 1$, then for all primes p, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property;*

(ii) *if $|e| \neq |f|$ or if there exists $\mathbf{x}$ in $\mathcal{D}_{e,f}$ such that $\Delta_{e,f}(x) = 0$, then there are only finitely many primes p such that $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property.*

Furthermore, if $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p, then, for all $\mathbf{n}$ in $\mathbb{N}^d$ and every prime p, we have

$$\mathcal{Q}_{e,f}(\mathbf{n}) \in p^{\alpha_p(\mathcal{Q}_{e,f},\mathbf{n})}\mathbb{Z}.$$

Remark. Theorem 3 implies that $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p if and only if all Taylor coefficients at the origin of the associated mirror maps $z_{e,f,k}$, $1 \leqslant k \leqslant d$, are integers (see Theorems 1 and 3 in [De13]). Indeed, if $\Delta_{e,f}$ is nonnegative on $[0,1]^d$ and if $|e| \neq |f|$, then there exists k in $\{1,\dots,d\}$ such that the kth component of $|e|$ is greater than the kth component of $|f|$.

Coster proved in [Co88] results similar to Theorems 1-3 for the coefficients of certain algebraic power series. Namely, given a prime $p \geqslant 3$, integers $a_1,\dots,a_{p-1}$, and a sequence A such that

$$f_A(z) = (1 + a_1 z + \cdots + a_{p-1} z^{p-1})^{\frac{1}{1-p}},$$

Coster proved that, for all nonnegative integers n, we have

$$v_p\big(A(n)\big) \geqslant \left\lfloor \frac{\alpha_p(A,n)+1}{2} \right\rfloor.$$

1.5 Auxiliary results

The proof of Theorem 1 rests on three important results. The first one is stated rather formally but we believe that it may be useful to study results similar to Beukers' conjectures for other sequences. Throughout this article, if $(A(n))_{n\geqslant 0}$ is a sequence taking its values in $\mathbb{Z}$ or $\mathbb{Z}_p$, then, for all negative integers n, we set $A(n) := 0$.

PROPOSITION 1. *Let p be a fixed prime and A a $\mathbb{Z}_p$-valued sequence satisfying the p-Lucas property with $A(0)$ in $\mathbb{Z}_p^\times$. Let $\mathfrak{A}$ be the $\mathbb{Z}_p$-module spanned by A. Assume that*

(*a*) *there exists a set $\mathfrak{B}$ of $\mathbb{Z}_p$-valued sequences with $\mathfrak{A} \subset \mathfrak{B}$ such that, for all B in $\mathfrak{B}$, all v in $\{0, \dots, p-1\}$ and all positive integers n, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k \geqslant 0}$, B_k in $\mathfrak{B}$, such that*

$$B(v+np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k);$$

(*b*) *$f_A(z)$ is annihilated by a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z, \theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(A)$.*

Then, for all B in $\mathfrak{B}$ and all nonnegative integers n, we have

$$A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}_p \quad \text{and} \quad B(n) \in p^{\alpha_p(A,n)-1}\mathbb{Z}_p.$$

We will apply Proposition 1 with $A = \mathfrak{S}_{e,f}$ for some tuples e and f satisfying the conditions of Theorem 1 for a fixed prime p. Then we will choose the set $\mathfrak{B}$ to be the set of the deformations $\mathfrak{S}_{e,f}^g$ for g in $\mathfrak{F}_p^d$. Taking g to be a constant in $\mathbb{Z}_p$ shows that the set $\mathfrak{B}$ contains the $\mathbb{Z}_p$-module $\mathfrak{A}$ spanned by A. The main difficulty in this article is to show, by p-adic techniques, that Assertion (a) in Proposition 1 holds with these choices. In particular, we shall prove and use several times the following result.

PROPOSITION 2. *Let p be a fixed prime. We write Γ_p for the p-adic Gamma function. Then, there exists a function g in $\mathfrak{F}_p^2$ such that, for all nonnegative integers n and m, we have*

$$\frac{\Gamma_p((m+n)p)}{\Gamma_p(mp)\Gamma_p(np)} = 1 + g(m,n)p.$$

Our proof of Theorem 2 does not use Proposition 1 but rests on the beautiful result of Mellit and Vlasenko [MV16, Lemma 1] which gives useful congruences modulo powers of p for some constant terms of powers of Laurent polynomials. In this case, the p-adic difficulties are hidden in the result of Mellit and Vlasenko.

Finally, we give a general result to prove the p-Lucas property for many sums of products of binomial coefficients. We recall that a tuple $e = (\mathbf{e}_1, \dots, \mathbf{e}_u)$ of vectors in $\mathbb{N}^d$ is 1-admissible if either $\mathbf{e}_i \geqslant \mathbf{1}$ for some i, or if, for every k in $\{1, \dots, d\}$, we have $\mathbf{e}_i \geqslant d\mathbf{1}_k$ for some i.

PROPOSITION 3. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$ and, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, $\Delta_{e,f}(\mathbf{x}) \geqslant 1$. Assume that e is 1-admissible. Then, $\mathfrak{S}_{e,f}$ is integer-valued and satisfies the p-Lucas property for all primes p.*

1.6 Application of Theorem 1

By applying Theorem 1, we obtain results similar to Conjectures A-C for numbers satisfying Apéry-like recurrence relations which we list below. Characters in brackets in the last column of the following table form the sequence number in the Online Encyclopedia of Integer Sequences [OEIS13].

Sequence	$\mathcal{Q}_{e,f}(n_1,n_2)$	$\mathcal{L}$	Reference
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$	$\frac{(2n_1+n_2)!^2}{n_1!^4 n_2!^2}$	[AZ06, (γ)]	Apéry numbers (A005259)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$	$\frac{(2n_1+n_2)!(n_1+n_2)!}{n_1!^3 n_2!^2}$	[Za09, **D**]	Apéry numbers (A005258)
$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$	$\frac{(n_1+n_2)!^2}{n_1!^2 n_2!^2}$	type I	Central binomial coefficients (A000984)
$\sum_{k=0}^{n}\binom{n}{k}^3$	$\frac{(n_1+n_2)!^3}{n_1!^3 n_2!^3}$	[Za09, **A**]	Franel numbers (A000172)
$\sum_{k=0}^{n}\binom{n}{k}^4$	$\frac{(n_1+n_2)!^4}{n_1!^4 n_2!^4}$	[Fr94],[Fr95]	(A005260)
$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!(2n_1)!(2n_2)!}{n_1!^3 n_2!^3}$	[AZ06, (d)]	(A081085)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$	$\frac{(n_1+n_2)!^2(2n_1)!}{n_1!^4 n_2!^2}$	[Za09, **C**]	Number of abelian squares of length $2n$ over an alphabet with 3 letters (A002893)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!^2(2n_1)!(2n_2)!}{n_1!^4 n_2!^4}$	[AZ06, (α)]	Domb numbers (A002895)
$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2$	$\frac{(2n_1)!^2(2n_2)!^2}{n_1!^4 n_2!^4}$	[AZ06, (β)]	(A036917)

All differential operators listed in the above table are of type I for all primes p, except the one associated with $A_5(n) := \sum_{k=0}^{n}\binom{n}{k}^4$ which reads

$$\mathcal{L}_5 = \theta^3 - z2(2\theta+1)(3\theta^2+3\theta+1) - z^2 4(\theta+1)(4\theta+5)(4\theta+3).$$

Hence $\mathcal{L}_5$ is of type II for all primes p. By a result of Calkin [Ca98, Proposition 3], for all primes p, we have $A_5(p-1)\equiv 0 \mod p$, *i. e.* $p-1$ is in $\mathcal{Z}_p(A_5)$. Thus we can apply Theorem 1 to A_5.

Observe that the generating function of the central binomial coefficients is annihilated by the differential operator $\mathcal{L}=\theta - z(4\theta+2)$ which is of type I for all primes p.

We set $A_6(n) := \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$. In 1885, Catalan gave in [Ca85] a recurrence relation for the Catalan-Larcombe-French sequence $2^n A_6(n)$ from which we deduce a recurrence relation for $A_6(n)$ (see also Case (d) in [AZ06]). According to this relation, $A_6(n)$ is also Sequence **E** in Zagier's list [Za09], that is

$$A_6(n) = \sum_{k=0}^{\lfloor n/2\rfloor} 4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2.$$

Furthermore, according to [RS09], Domb numbers $A_8(n) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$ are also the numbers of abelian squares of length $2n$ over an alphabet with 4 letters.

Now we consider the numbers $C_i(n)$ of abelian squares of length $2n$ over an alphabet with i

letters which, for all positive integers $i \geqslant 2$, satisfy (see [RS09])

$$C_i(n) = \sum_{\substack{k_1+\cdots+k_i=n\\ k_1,\dots,k_i\in\mathbb{N}}} \left(\frac{n!}{k_1!\cdots k_i!}\right)^2.$$

According to [BNSW11], $C_i(n)$ is also the $2n$-th moment of the distance to the origin after i steps traveled by a walk in the plane with unit steps in random directions.

To apply Theorem 1 to C_i, it suffices to show that its generating series f_{C_i} is annihilated by a differential operator of type I for all primes p. Indeed, by Proposition 1 and Theorem 2 in [BNSW11], for all $j \geqslant 2$, $C_j(n)$ satisfies the recurrence relation of order $\lceil j/2 \rceil$ with polynomial coefficients of degree $j-1$:

$$n^{j-1}C_j(n) + \sum_{i\geqslant 1}\left(n^{j-1}\sum_{\alpha_1,\dots,\alpha_i}\prod_{k=1}^{i}(-\alpha_k)(j+1-\alpha_k)\left(\frac{n-k}{n-k+1}\right)^{\alpha_k-1}\right)C_j(n-i) = 0, \quad (1.6)$$

where the sum is over all sequences of positive integers $\alpha_1,\dots,\alpha_i$ satisfying $\alpha_k \leqslant j$ and $\alpha_{k+1} \leqslant \alpha_k - 2$. We consider $i \geqslant 2$ and i positive integers $\alpha_1,\dots,\alpha_i \leqslant j$ satisfying $\alpha_{k+1} \leqslant \alpha_k - 2$. We have

$$n^{j-1}\prod_{k=1}^{i}\left(\frac{n-k}{n-k+1}\right)^{\alpha_k-1} = \frac{n^{j-1}}{n^{\alpha_1-1}}\left(\prod_{k=1}^{i-1}(n-k)^{\alpha_k-\alpha_{k+1}}\right)(n-i)^{\alpha_i-1},$$

with $j-\alpha_1 \geqslant 0$, $\alpha_k - \alpha_{k+1} \geqslant 2$ and $\alpha_i - 1 \geqslant 0$. Then, $f_{C_j}(z)$ is annihilated by a differential operator $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^q P_q(\theta)$ with $P_0(\theta) = \theta^{j-1}$ and, for all $i \geqslant 2$,

$$P_i(\theta) \in \prod_{k=1}^{i-1}(\theta+i-k)^2\mathbb{Z}[\theta] \subset \prod_{k=1}^{i-1}(\theta+k)^2\mathbb{Z}[\theta],$$

so that $\mathcal{L}$ is of type I for all primes p, as expected.

1.7 Structure of the article

In Section 2, we use several results of [De13] to prove Theorem 3. Section 3 is devoted to the proofs of Theorem 2 and Proposition 1. In particular, we prove Lemma 1 which points out the role played by differential operators in our proofs. In Section 4, we prove Theorem 1 by applying Proposition 1 to $\mathfrak{S}_{e,f}$. It is the most technical part of this article.

2. Proof of Theorem 3

First, we prove that if $|e| = |f|$, then, for all primes p, all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$ and all $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} \in \frac{\prod_{i=1}^{u}\prod_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}}{j}\right)}(1+p\mathbb{Z}_p). \quad (2.1)$$

Indeed, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} = \frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)}\cdot\frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})}.$$

Since $|e| = |f|$, we can apply [De13, Lemma 7] ([5]) with $\mathbf{c} = \mathbf{0}$, $\mathbf{m} = \mathbf{n}$ and $s = 0$ which yields

$$\frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})} \in 1 + p\mathbb{Z}_p.$$

Furthermore, we have

$$\begin{aligned}
\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)} &= \frac{1}{\mathcal{Q}_{e,f}(\mathbf{a})} \frac{\prod_{i=1}^{u}\prod_{j=1}^{\mathbf{e}_i\cdot\mathbf{a}}(j+\mathbf{e}_i\cdot\mathbf{n}p)}{\prod_{i=1}^{v}\prod_{j=1}^{\mathbf{f}_i\cdot\mathbf{a}}(j+\mathbf{f}_i\cdot\mathbf{n}p)} \\
&= \frac{\prod_{i=1}^{u}\prod_{j=1}^{\mathbf{e}_i\cdot\mathbf{a}}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}p}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\mathbf{f}_i\cdot\mathbf{a}}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}p}{j}\right)} \\
&\in \frac{\prod_{i=1}^{u}\prod_{j=1}^{\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\lfloor\mathbf{f}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}}{j}\right)}(1+p\mathbb{Z}_p),
\end{aligned}$$

because, if p does not divide j, then $1 + (\mathbf{e}_i \cdot \mathbf{n}p)/j$ belongs to $1 + p\mathbb{Z}_p$. This finishes the proof of (2.1).

Now we prove Assertion (i) in Theorem 3. Let p be a fixed prime number. It is well known that, for all nonnegative integers n, we have

$$v_p(n!) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{p^\ell} \right\rfloor.$$

We remind the reader that the Landau function $\Delta_{e,f}$ is defined by

$$\Delta_{e,f}(\mathbf{x}) = \sum_{i=1}^{u} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor - \sum_{j=1}^{v} \lfloor \mathbf{f}_j \cdot \mathbf{x} \rfloor, \quad (\mathbf{x} \in \mathbb{R}^d).$$

Thus, for all vectors $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\frac{\mathbf{n}}{p^\ell}\right).$$

Fix $\mathbf{n}$ in $\mathbb{N}^d$ and $\mathbf{a}$ in $\{0, \ldots, p-1\}^d$. Let $\{\cdot\}$ denote the fractional part function. For any vector of real numbers $\mathbf{x} = (x_1, \ldots, x_d)$, we set $\{\mathbf{x}\} := (\{x_1\}, \ldots, \{x_d\})$. Since $|e| = |f|$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\left\{\frac{\mathbf{a}+\mathbf{n}p}{p^\ell}\right\}\right) \geqslant \Delta_{e,f}\left(\frac{\mathbf{a}}{p}\right),$$

because $\Delta_{e,f}$ is nonnegative on $[0,1]^d$. By assumption, if $\mathbf{x}$ belongs to $\mathcal{D}_{e,f}$, then $\Delta_{e,f}(\mathbf{x}) \geqslant 1$. On the one hand, if $\mathbf{a}/p$ is in $\mathcal{D}_{e,f}$, then both $\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)$ and $\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})$ are congruent to 0 modulo p. On the other hand, if $\mathbf{a}/p$ is not in $\mathcal{D}_{e,f}$, then by definition, for all $\mathbf{d}$ in e or f, we have $\lfloor \mathbf{d} \cdot \mathbf{a}/p \rfloor = 0$ so that (2.1) yields

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p) \equiv \mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}) \mod p\mathbb{Z}_p,$$

as expected. This proves Assertion (i) in Theorem 3.

[5] The proof of this lemma uses a lemma of Lang which contains an error. Fortunately, Lemma 7 remains true. Details of this correction are presented in [DRR13, Section 2.4].

Now we prove Assertion (ii) in Theorem 3. If $\mathbf{m}=(m_1,\dots,m_d)$ is a vector in $\mathbb{R}^d$ and $k\in\{1,\dots,d\}$, then we set $\mathbf{m}^{(k)}:=m_k$. If $|e|\neq|f|$ then, since $\Delta_{e,f}$ is nonnegative on $[0,1]^d$, there exists k in $\{1,\dots,d\}$ such that $|e|^{(k)}-|f|^{(k)}=\Delta_{e,f}(\mathbf{1}_k)\geqslant 1$. Thereby, for almost all primes p, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{1}_k+\mathbf{1}_kp)\big)=\sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\frac{\mathbf{1}_k+\mathbf{1}_kp}{p^\ell}\right)\geqslant\Delta_{e,f}\left(\frac{\mathbf{1}_k}{p}+\mathbf{1}_k\right)\geqslant 1,$$

but $v_p(\mathcal{Q}_{e,f}(\mathbf{1}_k))=0$ so that $\mathcal{Q}_{e,f}$ does not satisfy the p-Lucas property.

Throughout the rest of this proof, we assume that $|e|=|f|$. According to Section 7.3.2 in [De13], there exist k in $\{1,\dots,d\}$ and a rational fraction $R(X)$ in $\mathbb{Q}(X)$, $R(X)\neq 1$, such that, for all large enough prime numbers p, we can choose $\mathbf{a}_p$ in $\{0,\dots,p-1\}^d$ satisfying $\mathcal{Q}_{e,f}(\mathbf{a}_p)\in\mathbb{Z}_p^\times$, and such that, for all nonnegative integers n, we have (see [De13, (7.10)])

$$\mathcal{Q}_{e,f}(\mathbf{a}_p+\mathbf{1}_knp)\in R(n)\mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_kn)(1+p\mathbb{Z}_p).$$

We fix a nonnegative integer n satisfying $R(n)\neq 1$. For almost all primes p, the numbers $R(n)$, $\mathcal{Q}_{e,f}(\mathbf{1}_kn)$ and $\mathcal{Q}_{e,f}(\mathbf{a}_p)$ are invertible in $\mathbb{Z}_p$, and $R(n)\not\equiv 1\mod p\mathbb{Z}_p$. Thus, we obtain

$$\mathcal{Q}_{e,f}(\mathbf{a}_p+\mathbf{1}_knp)\not\equiv\mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_kn)\mod p\mathbb{Z}_p,$$

which finishes the proof of Assertion (ii) in Theorem 3.

Now we assume that $|e|=|f|$ and that, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x})\geqslant 1$. Hence, for every prime p, we have

$$\mathcal{Z}_p(\mathcal{Q}_{e,f})=\big\{\mathbf{v}\in\{0,\dots,p-1\}^d\,:\,\mathbf{v}/p\in\mathcal{D}_{e,f}\big\}.$$

Furthermore, if $\mathbf{v}/p$ belongs to $\mathcal{D}_{e,f}$, then, for all positive integers N and all vectors $\mathbf{a}_0,\dots,\mathbf{a}_{N-1}$ in $\{0,\dots,p-1\}^d$, we have

$$\left\{\frac{\mathbf{a}_0+\mathbf{a}_1p+\cdots+\mathbf{a}_{N-1}p^{N-1}+\mathbf{v}p^N}{p^{N+1}}\right\}=\frac{\mathbf{a}_0+\mathbf{a}_1p+\cdots+\mathbf{a}_{N-1}p^{N-1}+\mathbf{v}p^N}{p^{N+1}}\geqslant\frac{\mathbf{v}}{p},$$

so that

$$\left\{\frac{\mathbf{a}_0+\mathbf{a}_1p+\cdots+\mathbf{a}_{N-1}p^{N-1}+\mathbf{v}p^N}{p^{N+1}}\right\}\in\mathcal{D}_{e,f}.$$

Hence, for every $\mathbf{n}$ in $\mathbb{N}^d$, $\mathbf{n}=\sum_{k=0}^{\infty}\mathbf{n}_kp^k$ with $\mathbf{n}_k\in\{0,\dots,p-1\}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big)=\sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\left\{\frac{\sum_{k=0}^{\ell-1}\mathbf{n}_kp^k}{p^\ell}\right\}\right)\geqslant\alpha_p(\mathcal{Q}_{e,f},\mathbf{n}),$$

and Theorem 3 is proved.

3. Proofs of Theorem 2 and Proposition 1

3.1 Induction *via* Apéry-like recurrence relations

In this section, we fix a prime p. We remind the reader that if A is a $\mathbb{Z}_p$-valued sequence, then $\mathcal{Z}_p(A)$ is the set of the digits $v\in\{0,\dots,p-1\}$ such that $A(v)\in p\mathbb{Z}_p$. If n is a nonnegative integer whose base p expansion is $n=n_0+n_1p+\cdots+n_Np^N$, then $\alpha_p(A,n)$ is the number of i in $\{0,\dots,N\}$ such that n_i belongs to $\mathcal{Z}_p(A)$.

If A is a $\mathbb{Z}_p$-valued sequence, then, for all nonnegative integers r, we write $\mathcal{U}_A(r)$ for the assertion "For all $n, i \in \mathbb{N}$, $i \leqslant r$, if $\alpha_p(A,n) \geqslant i$, then $A(n) \in p^i\mathbb{Z}_p$". As a first step, we shall prove the following result.

Lemma 1. *Let A be a $\mathbb{Z}_p$-valued sequence satisfying the p-Lucas property with $A(0)$ in $\mathbb{Z}_p^\times$. Assume that the generating series of A is annihilated by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(A)$.*

Let r be a nonnegative integer such that $\mathcal{U}_A(r)$ holds. Then, for all n_0 in $\mathcal{Z}_p(A)$ and all nonnegative integers m satisfying $\alpha_p(A,m) \geqslant r$, we have

$$A(n_0 + mp) \in p^{r+1}\mathbb{Z}_p.$$

Proof. Since A satisfies the p-Lucas property, we can assume that r is nonzero. The generating series of A is annihilated by a differential operator $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^qP_q(\theta)$ with $P_k(X)$ in $\mathbb{Z}_p[X]$ and $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$. Thus, for every nonnegative integer n, we have

$$\sum_{k=0}^{q} P_k(n-k)A(n-k) = 0. \tag{3.1}$$

We fix a nonnegative integer m satisfying $\alpha_p(A,m) \geqslant r$. In particular, since r is nonzero and $A(0)$ is invertible in $\mathbb{Z}_p$, we have $m \geqslant 1$. Furthermore, for all v in $\{0,\dots,p-1\}$, we also have $\alpha_p(A, v+mp) \geqslant r$. According to $\mathcal{U}_A(r)$, we obtain that, for all v in $\{0,\dots,p-1\}$, $A(v+mp)$ belongs to $p^r\mathbb{Z}_p$ so that $A(v+mp) =: \beta(v,m)p^r$, with $\beta(v,m) \in \mathbb{Z}_p$.

By (3.1), for all v in $\{q,\dots,p-1\}$, we have

$$\begin{aligned} 0 = \sum_{k=0}^{q} P_k(v-k+mp)A(v-k+mp) &= p^r \sum_{k=0}^{q} P_k(v-k+mp)\beta(v-k,m) \\ &\equiv p^r \sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p, \end{aligned}$$

because, for all polynomials P in $\mathbb{Z}_p[X]$ and all integers a and c, we have $P(a+cp) \equiv P(a) \mod p\mathbb{Z}_p$. Thus, for all v in $\{q,\dots,p-1\}$, we obtain

$$\sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.2}$$

We claim that if v is in $\{1,\dots,q-1\}$, then, for all k in $\{v+1,\dots,q\}$, we have

$$P_k(v+mp-k)A(v+mp-k) \in p^{r+1}\mathbb{Z}_p. \tag{3.3}$$

Indeed, on the one hand, if $\mathcal{L}$ is of type II, then we have $q = 2$ and $P_2(X)$ belongs to $(X+1)\mathbb{Z}_p[X]$ which yields

$$P_2(-1+mp)A(-1+mp) \in pA\big(p-1+(m-1)p\big)\mathbb{Z}_p.$$

Since 0 is not in $\mathcal{Z}_p(A)$, we have $\alpha_p(A,m-1) \geqslant r-1$ which, together with $p-1 \in \mathcal{Z}_p(A)$, leads to

$$\alpha_p\big(A, p-1+(m-1)p\big) \geqslant r.$$

According to $\mathcal{U}_A(r)$, we obtain that $pA(p-1+(m-1)p)$ is in $p^{r+1}\mathbb{Z}_p$, as expected. On the other hand, if $\mathcal{L}$ is of type I, then for all v in $\{1,\dots,q-1\}$ and all k in $\{v+1,\dots,q\}$, we have

$$P_k(X)\in\prod_{i=1}^{k-1}(X+i)^2\mathbb{Z}_p[X],$$

so that

$$v_p\big(P_k(v+mp-k)\big)\geqslant v_p\left(\prod_{i=1}^{k-1}(v+mp-k+i)^2\right).$$

Writing $k-v=a+bp$ with a in $\{0,\dots,p-1\}$ and b in $\mathbb{N}$, we obtain $k-1\geqslant a+bp$ so that

$$v_p\left(\prod_{i=1}^{k-1}(mp+i-a-bp)\right)\geqslant\begin{cases}b & \text{if } a=0;\\ b+1 & \text{if } a\geqslant 1.\end{cases},$$

which yields

$$v_p\big(P_k(v+mp-k)\big)\geqslant\begin{cases}2b & \text{if } a=0;\\ 2b+2 & \text{if } a\geqslant 1.\end{cases}.$$

Thus to prove (3.3), it is enough to show that

$$A(v+mp-k)\in\begin{cases}p^{r+1-2b}\mathbb{Z}_p & \text{if } a=0;\\ p^{r-1-2b}\mathbb{Z}_p & \text{if } a\geqslant 1.\end{cases}. \tag{3.4}$$

By definition of a and b, we have $v+mp-k=-a+(m-b)p$ with a in $\{0,\dots,p-1\}$. If $-a+(m-b)p$ is negative, then $A(v+mp-k)=0$ and (3.4) holds. By assumption, we have $\alpha_p(A,m)\geqslant r$ and $0\notin\mathcal{Z}_p(A)$. Hence, if $m-b$ is nonnegative, then we have $\alpha_p(A,m-b)\geqslant r-b$. Thus, we have either $a=0$ and $\alpha_p(A,v+mp-k)\geqslant r-b$, or $a,m-b\geqslant 1$ and

$$\alpha_p(A,v+mp-k)=\alpha_p\big(A,p-a+(m-b-1)p\big)\geqslant r-b-1.$$

Hence Assertion $\mathcal{U}_A(r)$ yields

$$A(v+mp-k)\in\begin{cases}p^{r-b}\mathbb{Z}_p & \text{if } a=0;\\ p^{r-1-b}\mathbb{Z}_p & \text{if } a\geqslant 1.\end{cases}.$$

If $a=0$, then $b\geqslant 1$ and $-b\geqslant 1-2b$ so that (3.4) holds and (3.3) is proved.

By (3.3), for all nonnegative integers v satisfying $1\leqslant v\leqslant\min(q-1,p-1)$, we have

$$\begin{aligned}0&=\sum_{k=0}^{q}P_k(v-k+mp)A(v-k+mp)\\&\equiv\sum_{k=0}^{v}P_k(v-k+mp)A(v-k+mp)\mod p^{r+1}\mathbb{Z}_p\\&\equiv p^r\sum_{k=0}^{v}P_k(v-k+mp)\beta(v-k,m)\mod p^{r+1}\mathbb{Z}_p\\&\equiv p^r\sum_{k=0}^{v}P_k(v-k)\beta(v-k,m)\mod p^{r+1}\mathbb{Z}_p.\end{aligned}$$

Thus, for all nonnegative integers v satisfying $1 \leqslant v \leqslant \min(q-1, p-1)$, we have

$$\sum_{k=0}^{v} P_k(v-k)\beta(v-k, m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.5}$$

Both sequences $(\beta(v,m))_{0 \leqslant v \leqslant p-1}$ and $(A(v))_{0 \leqslant v \leqslant p-1}$ satisfy Equations (3.2) and (3.5). Furthermore, for all v in $\{1, \dots, p-1\}$, $P_0(v)$ and $A(0)$ are invertible in $\mathbb{Z}_p$. Hence there exists $\gamma(m)$ in $\{0, \dots, p-1\}$ such that, for all v in $\{0, \dots, p-1\}$, we have $\beta(v,m) \equiv A(v)\gamma(m) \mod p\mathbb{Z}_p$ so that

$$A(v+mp) \equiv A(v)\gamma(m)p^r \mod p^{r+1}\mathbb{Z}_p.$$

Since n_0 is in $\mathcal{Z}_p(A)$, we have $A(n_0) \in p\mathbb{Z}_p$ so that $A(n_0+mp)$ belongs to $p^{r+1}\mathbb{Z}_p$ and Lemma 1 is proved. $\square$

3.2 Proof of Theorem 2

Let p be a fixed prime number. For every positive integer n, we set $\ell(n) := \lfloor \log_p(n) \rfloor + 1$ the length of the expansion of n to the base p, and $\ell(0) := 1$. For all nonnegative integers $n_1, \dots, n_r$, we set

$$n_1 * \cdots * n_r := n_1 + n_2 p^{\ell(n_1)} + \cdots + n_r p^{\ell(n_1)+\cdots+\ell(n_{r-1})},$$

so that the expansion of $n_1 * \cdots * n_r$ to the base p is the concatenation of the respective expansions of $n_1, \dots, n_r$. Then, by a result of Mellit and Vlasenko [MV16, Lemma 1], there exists a $\mathbb{Z}_p$-valued sequence $(c_n)_{n \geqslant 0}$ such that, for all positive integers n, we have

$$A(n) = \sum_{\substack{n_1 * \cdots * n_r = n \\ 1 \leqslant r \leqslant \ell(n),\, n_r > 0}} c_{n_1} \cdots c_{n_r} \quad \text{and} \quad c_n \equiv 0 \mod p^{\ell(n)-1}\mathbb{Z}_p. \tag{3.6}$$

For every nonnegative integer r, we write $\mathcal{U}(r)$ for the assertion: "For all $n, i \in \mathbb{N}$, $i \leqslant r$, if $\alpha_p(A,n) \geqslant i$, then $A(n), c_n \in p^i\mathbb{Z}_p$". To prove Theorem 2, it suffices to show that, for all nonnegative integers r, Assertion $\mathcal{U}(r)$ holds.

First we prove $\mathcal{U}(1)$. By Theorem 1 in [MV16], A satisfies the p-Lucas property. In addition, if v is in $\mathcal{Z}_p(A)$, then v is nonzero because $A(0) = 1$, and by (3.6) we have $c_v = A(v) \in p\mathbb{Z}_p$. Now, if a nonnegative integer n satisfies $\ell(n) = 2$ and $\alpha_p(A,n) \geqslant 1$, then Equation (3.6) yields $A(n) \equiv c_n \mod p\mathbb{Z}_p$, so that c_n is in $p\mathbb{Z}_p$. Hence, by induction on $\ell(n)$, we obtain that, for all nonnegative integers n satisfying $\alpha_p(A,n) \geqslant 1$, c_n belongs to $p\mathbb{Z}_p$, so that $\mathcal{U}(1)$ holds.

Let r be a positive integer such that $\mathcal{U}(r)$ holds. We shall prove that $\mathcal{U}(r+1)$ is true. For all positive integers M, we write $\mathcal{U}_M(r+1)$ for the assertion:

"For all $n, i \in \mathbb{N}$, $n \leqslant M$, $i \leqslant r+1$, if $\alpha_p(A,n) \geqslant i$, then $A(n), c_n \in p^i\mathbb{Z}_p$".

Hence $\mathcal{U}_M(r+1)$ is true if $\ell(M) \leqslant r$. Let M be a positive integer such that $\mathcal{U}_M(r+1)$ holds. We shall prove $\mathcal{U}_{M+1}(r+1)$. By Assertions $\mathcal{U}(r)$ and $\mathcal{U}_M(r+1)$, it suffices to prove that if $\alpha_p(A, M+1)$ is greater than r, then $A(M+1)$ and c_{M+1} belong to $p^{r+1}\mathbb{Z}_p$. In the rest of the proof, we assume that $\alpha_p(A, M+1)$ is greater than r.

If u and $n_1, \dots, n_u$ are nonnegative integers satisfying $2 \leqslant u \leqslant \ell(M+1)$ and $n_1 * \cdots * n_u = M+1$ with $n_u > 0$, then, for all i in $\{1, \dots, u\}$, we have $n_i \leqslant M$ and

$$\alpha_p(A, n_1) + \cdots + \alpha_p(A, n_u) = \alpha_p(A, M+1) \geqslant r+1.$$

Then there exist a positive integer k, and integers $1 \leqslant a_1 < \cdots < a_k \leqslant u$ and $1 \leqslant i_1, \dots, i_k \leqslant r+1$ such that $\alpha_p(A, n_{a_j}) \geqslant i_j$ and $i_1 + \cdots + i_k \geqslant r+1$. Thereby, Assertion $\mathcal{U}_M(r+1)$ yields $c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p$, so that

$$\sum_{\substack{n_1 * \cdots * n_u = M+1 \\ 2 \leqslant u \leqslant \ell(M+1),\, n_u > 0}} c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p.$$

By (3.6), we obtain

$$A(M+1) \equiv c_{M+1} \mod p^{r+1}\mathbb{Z}_p \quad \text{and} \quad c_{M+1} \equiv 0 \mod p^{\ell(M+1)-1}\mathbb{Z}_p.$$

Hence it suffices to consider the case $\ell(M+1) = r+1$. In particular, we have $M+1 = v+mp$ where v is in $\mathcal{Z}_p(A)$ and m is a nonnegative integer satisfying $\alpha_p(A, m) = r$. Since $\mathcal{U}(r)$ holds, Lemma 1 yields $A(M+1) \in p^{r+1}\mathbb{Z}_p$. Thus we also have $c_{M+1} \in p^{r+1}\mathbb{Z}_p$ and Assertion $\mathcal{U}_{M+1}(r+1)$ holds. This finishes the proof of $\mathcal{U}(r+1)$ so that of Theorem 2. □

3.3 Proof of Proposition 1

Let p be a prime and A a $\mathbb{Z}_p$-valued sequence satisfying the hypotheses of Proposition 1. For every nonnegative integer n, we write $\alpha(n)$, respectively $\mathcal{Z}$, as a shorthand for $\alpha_p(A, n)$, respectively for $\mathcal{Z}_p(A)$. For every nonnegative integer r, we define Assertions

$$\mathcal{U}(r): \text{“For all } n, i \in \mathbb{N},\ i \leqslant r, \text{ if } \alpha(n) \geqslant i, \text{ then } A(n) \in p^i\mathbb{Z}_p.\text{”},$$

and

$$\mathcal{V}(r): \text{“For all } n, i \in \mathbb{N},\ i \leqslant r, \text{ and all } B \in \mathfrak{B}, \text{ if } \alpha(n) \geqslant i, \text{ then } B(n) \in p^{i-1}\mathbb{Z}_p\text{”}.$$

To prove Proposition 1, we have to show that, for all nonnegative integers r, Assertions $\mathcal{U}(r)$ and $\mathcal{V}(r)$ are true. We shall prove those assertions by induction on r.

Observe that Assertions $\mathcal{U}(0)$, $\mathcal{V}(0)$ and $\mathcal{V}(1)$ are trivial. Furthermore, since A satisfies the p-Lucas property, Assertion $\mathcal{U}(1)$ holds. Let r_0 be a fixed positive integer, $r_0 \geqslant 2$, such that Assertions $\mathcal{U}(r_0 - 1)$ and $\mathcal{V}(r_0 - 1)$ are true. First, we prove Assertion $\mathcal{V}(r_0)$.

Let B in $\mathfrak{B}$ and m in $\mathbb{N}$ be such that $\alpha(m) \geqslant r_0$. We write $m = v+np$ with v in $\{0, \dots, p-1\}$. Since $r_0 \geqslant 2$ and 0 does not belong to $\mathcal{Z}$, we have $n \geqslant 1$ and, by Assertion (a) in Proposition 1, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k \geqslant 0}$, with B_k in $\mathfrak{B}$, such that

$$B(v+np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k). \tag{3.7}$$

In addition, we have $\alpha(n) \geqslant r_0 - 1$ and, since 0 is not in $\mathcal{Z}$, we have $\alpha(n-1) \geqslant r_0 - 2$. By induction, for all nonnegative integers k satisfying $k \leqslant n$, we have $\alpha(n-k) \geqslant r_0 - 1 - k$. Thus, by (3.7) in combination with $\mathcal{U}(r_0 - 1)$ and $\mathcal{V}(r_0 - 1)$, we obtain

$$A'(n) \in p^{r_0 - 1}\mathbb{Z} \quad \text{and} \quad p^{k+1} B_k(n-k) \in p^{k+1+r_0-2-k}\mathbb{Z}_p \subset p^{r_0-1}\mathbb{Z}_p,$$

so that $B(v+np)$ belongs to $p^{r_0-1}\mathbb{Z}_p$ and $\mathcal{V}(r_0)$ is true.

Now we prove Assertion $\mathcal{U}(r_0)$. We write $\mathcal{U}_N(r_0)$ for the assertion:

“For all $n, i \in \mathbb{N}$, $n \leqslant N$, $i \leqslant r_0$, if $\alpha(n) \geqslant i$, then $A(n) \in p^i\mathbb{Z}_p$”.

We shall prove $\mathcal{U}_N(r_0)$ by induction on N. Assertion $\mathcal{U}_1(r_0)$ holds. Let N be a positive integer such that $\mathcal{U}_N(r_0)$ is true. Let $n := n_0 + mp \leqslant N+1$ with n_0 in $\{0,\dots,p-1\}$ and m in $\mathbb{N}$. We can assume that $\alpha(n) \geqslant r_0$.

If n_0 is in $\mathcal{Z}$, then we have $\alpha(m) \geqslant r_0 - 1$ and, by Lemma 1, we obtain that $A(n)$ belongs to $p^{r_0}\mathbb{Z}_p$ as expected. If n_0 is not in $\mathcal{Z}$, then we have $\alpha(m) \geqslant r_0$. By Assertion (a) in Proposition 1, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\geqslant 0}$ with B_k in $\mathfrak{B}$ such that

$$A(n) = A'(m) + \sum_{k=0}^{\infty} p^{k+1} B_k(m-k).$$

We have $m \leqslant N$, $\alpha(m) \geqslant r_0$ and $\alpha(m-k) \geqslant r_0 - k$, hence, by Assertions $\mathcal{U}_N(r_0)$ and $\mathcal{V}(r_0)$, we obtain that $A(n)$ belongs to $p^{r_0}\mathbb{Z}_p$. This finishes the induction on N and proves $\mathcal{U}(r_0)$. Therefore, by induction on r_0, Proposition 1 is proved. □

4. Proof of Theorem 1

To prove Theorem 1, we shall apply Proposition 1 to $\mathfrak{S}_{e,f}$. As a first step, we prove that this sequence satisfies the p-Lucas property.

Proof of Proposition 3. For all $\mathbf{x}$ in $[0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) = \Delta_{e,f}(\{\mathbf{x}\}) \geqslant 0$ so that, by Landau's criterion, $\mathcal{Q}_{e,f}$ is integer-valued. Let p be a fixed prime, v in $\{0,\dots,p-1\}$ and n a nonnegative integer. We have

$$\mathfrak{S}_{e,f}(v+np) = \sum_{\substack{k_1+\cdots+k_d=v+np\\ k_i\in\mathbb{N}}} \mathcal{Q}_{e,f}(k_1,\dots,k_d).$$

Write $k_i = a_i + m_i p$ with a_i in $\{0,\dots,p-1\}$ and m_i in $\mathbb{N}$. If $a_1+\cdots+a_d \neq v$, then we have $a_1+\cdots+a_d \geqslant p$ and there exists i in $\{1,\dots,d\}$ such that $a_i \geqslant p/d$. Write $\mathbf{a} = (a_1,\dots,a_d)$ so that $\mathbf{1}\cdot\mathbf{a}/p \geqslant 1$ and $d\mathbf{1}_i\cdot\mathbf{a}/p \geqslant 1$. Since $e = (\mathbf{e}_1,\dots,\mathbf{e}_u)$ is 1-admissible, there exists a j in $\{1,\dots,u\}$ such that either $\mathbf{e}_j \geqslant \mathbf{1}$ or $\mathbf{e}_j \geqslant d\mathbf{1}_i$. Hence $\mathbf{e}_j\cdot\mathbf{a}/p \geqslant 1$ and $\mathbf{a}/p$ belongs to $\mathcal{D}_{e,f}$ so that $\Delta_{e,f}(\mathbf{a}/p) \geqslant 1$ and $\mathcal{Q}_{e,f}(k_1,\dots,k_d)$ is in $p\mathbb{Z}_p$. In addition, by Theorem 3, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p. Hence we obtain

$$\begin{aligned}
\mathfrak{S}_{e,f}(v+np) &\equiv \sum_{\substack{a_1+\cdots+a_d=v\\ 0\leqslant a_i\leqslant p-1}} \sum_{\substack{m_1+\cdots+m_d=n\\ m_i\in\mathbb{N}}} \mathcal{Q}_{e,f}(a_1+m_1p,\dots,a_d+m_dp) \mod p\mathbb{Z}_p\\
&\equiv \sum_{\substack{a_1+\cdots+a_d=v\\ 0\leqslant a_i\leqslant p-1}} \sum_{\substack{m_1+\cdots+m_d=n\\ m_i\in\mathbb{N}}} \mathcal{Q}_{e,f}(a_1,\dots,a_d)\mathcal{Q}_{e,f}(m_1,\dots,m_d) \mod p\mathbb{Z}_p\\
&\equiv \mathfrak{S}_{e,f}(v)\mathfrak{S}_{e,f}(n) \mod p\mathbb{Z}_p.
\end{aligned}$$

This finishes the proof of Proposition 3. □

If e is 2-admissible then e is also 1-admissible. Furthermore, if $f = (\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$, then, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have

$$\Delta_{e,f}(\mathbf{x}) = \sum_{i=1}^{u} \lfloor \mathbf{e}_i\cdot\mathbf{x}\rfloor \geqslant 1.$$

Hence, if e and f satisfy the conditions of Theorem 1, then Proposition 3 implies that, for all primes p, $\mathfrak{S}_{e,f}$ has the p-Lucas property and $\mathfrak{S}_{e,f}(0)=1$ is invertible in $\mathbb{Z}_p$. Thereby, to prove

Theorem 1, it remains to prove that $\mathfrak{S}_{e,f}$ satisfies Condition (a) in Proposition 1 with the set

$$\mathfrak{B}=\big\{\mathfrak{S}_{e,f}^{g}\;:\;g\in\mathfrak{F}_p^d\big\}.$$

First we prove that some special functions belong to $\mathfrak{F}_p^1$.

4.1 Special functions in $\mathfrak{F}_p^1$

For all primes p, we write $|\cdot|_p$ for the ultrametric norm on $\mathbb{Q}_p$ (the field of p-adic numbers) defined by $|a|_p:=p^{-v_p(a)}$. Note that $(\mathbb{Z}_p,|\cdot|_p)$ is a compact space. Furthermore, if $(c_n)_{n\geqslant 0}$ is a $\mathbb{Z}_p$-valued sequence, then $\sum_{n=0}^{\infty}c_n$ is convergent in $(\mathbb{Z}_p,|\cdot|_p)$ if and only if $|c_n|_p$ tends to 0 as n tends to infinity. In addition, if $\sum_{n=0}^{\infty}c_n$ converges, then $(c_n)_{n\in\mathbb{N}}$ is a summable family in $(\mathbb{Z}_p,|\cdot|_p)$.

In the rest of the article, for all primes p and all positive integers k, we set $\Psi_{p,k,0}(0)=1$, $\Psi_{p,k,i}(0)=0$ for $i\geqslant 1$ and, for all nonnegative integers i and m, $m\geqslant 1$, we set

$$\Psi_{p,k,i}(m):=(-1)^i\sigma_{m,i}\left(\frac{1}{k},\frac{1}{k+p},\dots,\frac{1}{k+(m-1)p}\right),$$

where $\sigma_{m,i}$ is the i-th elementary symmetric polynomial of m variables. Let us remind the reader that, for all nonnegative integers m and i satisfying $i>m\geqslant 1$, we have $\sigma_{m,i}=0$.

The aim of this section is to prove that, for all primes p, all k in $\{1,\dots,p-1\}$ and all nonnegative integers i, we have

$$i!\Psi_{p,k,i}\in\mathfrak{F}_p^1, \tag{4.1}$$

that is, for every nonnegative integer M, there exists a sequence of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to $i!\Psi_{p,k,i}$ on $\{0,\dots,M\}$.

Proof of (4.1). Throughout this proof, we fix a prime number p and an integer k in $\{1,\dots,p-1\}$. Furthermore, for all nonnegative integers i, we use Ψ_i as a shorthand for $\Psi_{p,k,i}$ and $\mathbb{N}_{\geqslant i}$ as a shorthand for the set of integers larger than or equal to i. We shall prove (4.1) by induction on i. To that end, for all nonnegative integers i, we write $\mathcal{A}_i$ for the following assertion:

> "There exists a sequence $(T_{i,r})_{r\geqslant 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $i!\Psi_i$ on $\mathbb{N}$".

First, observe that, for all nonnegative integers m, we have $\Psi_0(m)=1$, so that Assertion $\mathcal{A}_0$ is true. Let i be a fixed positive integer such that assertions $\mathcal{A}_0,\dots,\mathcal{A}_{i-1}$ are true. According to the Newton-Girard formulas, for all integers $m\geqslant i$, we have

$$i(-1)^i\sigma_{m,i}(X_1,\dots,X_m)=-\sum_{t=1}^{i}(-1)^{i-t}\sigma_{m,i-t}(X_1,\dots,X_m)\Lambda_t(X_1,\dots,X_m),$$

where $\Lambda_t(X_1,\dots,X_m):=X_1^t+\cdots+X_m^t$. Thereby, for all integers $m\geqslant i$, we have

$$i\Psi_i(m)=-\sum_{t=1}^{i}\Psi_{i-t}(m)\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right). \tag{4.2}$$

For all nonnegative integers j and t, we have

$$\frac{1}{(k+jp)^t}=\frac{1}{k^t}\frac{1}{(1+\frac{j}{k}p)^t}=\frac{1}{k^t}+\sum_{s=1}^{\infty}\frac{(-1)^s}{k^t}\binom{t-1+s}{s}\left(\frac{j}{k}\right)^s p^s, \tag{4.3}$$

where the right hand side of (4.3) is a convergent series in $(\mathbb{Z}_p, |\cdot|_p)$ because k is invertible in $\mathbb{Z}_p$. Therefore, we obtain that

$$\begin{aligned}\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right) &= \frac{m}{k^t}+\sum_{j=0}^{m-1}\sum_{s=1}^{\infty}\frac{(-1)^s}{k^t}\binom{t-1+s}{s}\left(\frac{j}{k}\right)^s p^s\\ &= \frac{m}{k^t}+\sum_{s=1}^{\infty}\frac{(-1)^s}{k^{t+s}}\binom{t-1+s}{s}p^s\left(\sum_{j=0}^{m-1}j^s\right).\end{aligned}\tag{4.4}$$

According to Faulhaber's formula (see [CG96]), for all positive integers s, we have

$$p^s\sum_{j=0}^{m-1}j^s=\sum_{c=1}^{s+1}(-1)^{s+1-c}\binom{s+1}{c}p^s\frac{B_{s+1-c}}{s+1}(m-1)^c,$$

where B_k is the k-th first Bernoulli number. For all positive integers s and t, we set $R_{0,t}(X) := X/k^t$ and

$$R_{s,t}(X):=\frac{1}{k^{t+s}}\binom{t-1+s}{s}\sum_{c=1}^{s+1}(-1)^{1-c}\binom{s+1}{c}p^s\frac{B_{s+1-c}}{s+1}(X-1)^c,$$

so that

$$\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right)=\sum_{s=0}^{\infty}R_{s,t}(m).$$

In the rest of this article, for all polynomials $P(X)=\sum_{n=0}^{N}a_nX^n$ in $\mathbb{Z}_p[X]$, we set

$$\|P\|_p:=\max\left\{|a_n|_p\,:\,0\leqslant n\leqslant N\right\}.$$

We claim that, for all nonnegative integers s and t, $t\geqslant 1$, we have

$$R_{s,t}(X)\in\mathbb{Z}_p[X],\quad \|R_{s,t}\|_p\underset{s\to\infty}{\longrightarrow}0\quad\text{and}\quad R_{s,t}(0)=0.\tag{4.5}$$

Indeed, on the one hand, if $p=2$ and $s=1$, then we have

$$R_{1,t}(X)=\frac{-t}{k^{t+1}}\big(X-1+(X-1)^2\big)\in X\mathbb{Z}_2[X].$$

On the other hand, if $p\geqslant 3$ or $s\geqslant 2$, then we have $p^s>s+1$ so that $v_p(s+1)\leqslant s-1$. Furthermore, according to the von Staudt-Clausen theorem, we have $v_p(B_{s+1-c})\geqslant -1$. Thus, the coefficients of $R_{s,t}(X)$ belong to $\mathbb{Z}_p$. To be more precise, we have $v_p(s+1)\leqslant\log_p(s+1)$, so that $\|R_{s,t}\|_p\underset{s\to\infty}{\longrightarrow}0$ as expected. In addition, we have

$$\begin{aligned}R_{s,t}(0)&=-\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{c=1}^{s+1}\binom{s+1}{c}B_{s+1-c}\\&=-\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{d=0}^{s}\binom{s+1}{d}B_d=0,\end{aligned}$$

where we used the well known relation satisfied by the Bernoulli numbers

$$\sum_{d=0}^{s}\binom{s+1}{d}B_d=0,\quad(s\geqslant 1).$$

According to $\mathcal{A}_0, \dots, \mathcal{A}_{i-1}$, for all j in $\{0, \dots, i-1\}$, there exists a sequence $(T_{j,r})_{r\geqslant 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $j!\Psi_j$ on $\mathbb{N}$. According to (4.2) and (4.5), for all nonnegative integers N, there exists S_N in $\mathbb{N}$ such that, for all $r \geqslant S_N$ and all $m \geqslant i$, we have

$$i!\Psi_i(m) \equiv -\sum_{t=1}^{i} \frac{(i-1)!}{(i-t)!} T_{i-t,r}(m) \sum_{s=0}^{r} R_{s,t}(m) \mod p^N\mathbb{Z}_p.$$

Thus, the sequence $(T_{i,r})_{r\geqslant 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$, defined by

$$T_{i,r}(x) := -\sum_{t=1}^{i} \frac{(i-1)!}{(i-t)!} T_{i-t,r}(x) \sum_{s=0}^{r} R_{s,t}(x), \quad (x, r \in \mathbb{N}), \tag{4.6}$$

converges uniformly to $i!\Psi_i$ on $\mathbb{N}_{\geqslant i}$. To prove $\mathcal{A}_i$, it suffices to show that, for all m in $\{0, \dots, i-1\}$, we have

$$T_{i,r}(m) \underset{r\to\infty}{\longrightarrow} 0. \tag{4.7}$$

Observe that Equations (4.6) and (4.5) lead to $T_{i,r}(0) = 0$. In particular, if $i = 1$, then (4.7) holds. Now we assume that $i \geqslant 2$. For all $m \geqslant 2$, we have

$$\begin{aligned}
\sum_{j=0}^{m} \Psi_j(m) X^j &= \prod_{w=0}^{m-1} \left(1 - \frac{X}{k+wp}\right) \\
&= \left(1 - \frac{X}{k+(m-1)p}\right) \prod_{w=0}^{m-2} \left(1 - \frac{X}{k+wp}\right) \\
&= \left(1 - \frac{X}{k+(m-1)p}\right) \sum_{j=0}^{m-1} \Psi_j(m-1) X^j.
\end{aligned}$$

Thereby, for all j in $\{1, \dots, m-1\}$, we obtain that

$$\Psi_j(m) = \Psi_j(m-1) - \frac{\Psi_{j-1}(m-1)}{k+(m-1)p},$$

with

$$\frac{1}{k+(m-1)p} = \sum_{s=0}^{\infty} \frac{(-1)^s}{k^{s+1}} p^s (m-1)^s.$$

Thus, there exists a sequence $(U_r)_{r\geqslant 0}$ of polynomials with coefficients in $\mathbb{Z}_p$ such that, for all positive integers N, there exists a nonnegative integer S_N such that, for all $r \geqslant S_N$ and all $m \geqslant i+1$, we have

$$T_{i,r}(m) \equiv T_{i,r}(m-1) - T_{i-1,r}(m-1) U_r(m-1) \mod p^N\mathbb{Z}_p. \tag{4.8}$$

But, if $V_1(X)$ and $V_2(X)$ are polynomials with coefficients in $\mathbb{Z}_p$ and if there exists a nonnegative integer a such that, for all $m \geqslant a$, we have $V_1(m) \equiv V_2(m) \mod p^N\mathbb{Z}_p$, then, for all integers n, we have $V_1(n) \equiv V_2(n) \mod p^N\mathbb{Z}_p$. Indeed, let n be an integer, there exists a nonnegative integer v such that $n + vp^N \geqslant a$. Thus, we obtain that

$$V_1(n) \equiv V_1(n+vp^N) \equiv V_2(n+vp^N) \equiv V_2(n) \mod p^N\mathbb{Z}_p.$$

In particular, Equation (4.8) also holds for all positive integers m.

Furthermore, according to $\mathcal{A}_{i-1}$, for all m in $\{0,\dots,i-2\}$, $T_{i-1,r}(m)$ tends to zero as r tends to infinity. Thus, for all positive integers N, there exists a nonnegative integer S_N such that, for all $r \geqslant S_N$ and all m in $\{1,\dots,i-1\}$, we have

$$T_{i,r}(m) \equiv T_{i,r}(m-1) \mod p^N\mathbb{Z}_p.$$

Since $T_{i,r}(0)=0$, we obtain that $T_{i,r}(m)\equiv 0 \mod p^N\mathbb{Z}_p$ for all m in $\{0,\dots,i-1\}$ and $r \geqslant S_N$, so that (4.7) holds. This finishes the induction on i and proves (4.1). $\square$

4.2 On the p-adic Gamma function

For every prime p, we write Γ_p for the p-adic Gamma function, so that, for all nonnegative integers n, we have

$$\Gamma_p(n) = (-1)^n \prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{n-1} \lambda.$$

The aim of this section is to prove Proposition 2.

Proof of Proposition 2. Let p be a fixed prime number. For all nonnegative integers n and m, we have

$$\begin{aligned}
\frac{\Gamma_p\big((m+n)p\big)}{\Gamma_p(mp)\Gamma_p(np)} &= \left(\prod_{\substack{\lambda=np\\ p\nmid\lambda}}^{(m+n)p} \lambda\right) \Big/ \left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} \lambda\right)\\
&= \left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} (np+\lambda)\right) \Big/ \left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} \lambda\right)\\
&= \prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} \left(1+\frac{np}{\lambda}\right). \qquad (4.9)
\end{aligned}$$

Let $X, T_1, \dots, T_m$ be $m+1$ variables. Then, we have

$$\prod_{j=1}^{m}(X-T_j) = X^m + \sum_{i=1}^{\infty}(-1)^i\sigma_{m,i}(T_1,\dots,T_m)X^{m-i}.$$

Therefore, we obtain that

$$\begin{aligned}
\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} \left(1+\frac{np}{\lambda}\right) &= \prod_{k=1}^{p-1}\prod_{\omega=0}^{m-1}\left(1+\frac{np}{k+\omega p}\right)\\
&= \prod_{k=1}^{p-1}\left(1+\sum_{i=1}^{\infty}(-1)^i\sigma_{m,i}\left(\frac{-np}{k},\cdots,\frac{-np}{k+(m-1)p}\right)\right)\\
&= \prod_{k=1}^{p-1}\left(1+\sum_{i=1}^{\infty}(-1)^i n^i p^i \Psi_{p,k,i}(m)\right). \qquad (4.10)
\end{aligned}$$

Let k in $\{1,\dots,p-1\}$ be fixed. By (4.1), for all positive integers i, there exists a sequence $(P_{i,\ell})_{\ell\geqslant 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to $i!\Psi_{p,k,i}$. We fix a nonnegative integer K. For all positive integers N, we set

$$f_N(x,y):=1+\sum_{i=1}^{K+1}(-1)^i x^i\frac{p^i}{i!}P_{i,N}(y).$$

If n and m belong to $\{0,\dots,K\}$, then we have

$$1+\sum_{i=1}^{\infty}(-1)^i n^i p^i\Psi_{p,k,i}(m)-f_N(n,m)=\sum_{i=1}^{K+1}(-1)^i n^i\frac{p^i}{i!}\big(i!\Psi_{p,k,i}(m)-P_{i,N}(m)\big)\underset{N\to\infty}{\longrightarrow}0.$$

Furthermore, we have $f_N(x,y)\in 1+p\mathbb{Z}_p[x,y]$. Indeed, if $i=i_0+i_1p+\cdots+i_ap^a$ with i_j in $\{0,\dots,p-1\}$, then we set $\mathfrak{s}_p(i):=i_0+\cdots+i_a$ so that, for all positive integers i, we have

$$i-v_p(i!)=i-\frac{i-\mathfrak{s}_p(i)}{p-1}=\frac{i(p-2)+\mathfrak{s}_p(i)}{p-1}>0.$$

Hence, by (4.10), we obtain that there exists a function g in $\mathfrak{F}_p^2$ such that, for all nonnegative integers n and m, we have

$$\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\Big(1+\frac{np}{\lambda}\Big)=1+g(n,m)p,$$

which, together with (4.9), finishes the proof of Proposition 2. □

4.3 Last step in the proof of Theorem 1

Let e and $f=(\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$ be two disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e|=|f|$, for all i in $\{1,\dots,v\}$, k_i is in $\{1,\dots,d\}$, and e is 2-admissible. Let p be a fixed prime and $\mathfrak{A}$ the $\mathbb{Z}_p$-module spanned by $\mathfrak{S}_{e,f}$. We set $\mathfrak{B}:=\{\mathfrak{S}_{e,f}^g\,:\,g\in\mathfrak{F}_p^d\}$ which is obviously constituted of $\mathbb{Z}_p$-valued sequences and contains $\mathfrak{A}$. To finish the proof of Theorem 1, we shall prove that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) in Proposition 1. Hence we have to show that, for all B in $\mathfrak{B}$, all v in $\{0,\dots,p-1\}$ and all positive integers n, there exists A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\geqslant 0}$, B_k in $\mathfrak{B}$, such that

$$B(v+np)=A'(n)+\sum_{k=0}^{\infty}p^{k+1}B_k(n-k). \tag{4.11}$$

Let g be a fixed function in $\mathfrak{F}_p^d$, that is a function $g:\mathbb{N}^d\to\mathbb{Z}_p$ such that, for all nonnegative integers K, there exists a sequence of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to g on $\{0,\dots,K\}^d$. In the rest of the proof, we write $\mathbb{Z}_p+p\mathfrak{F}_p^d$ for the set of functions of the form $\alpha+ph$ where α is a constant in $\mathbb{Z}_p$ and h belongs to $\mathfrak{F}_p^d$. Observe that $\mathbb{Z}_p+p\mathfrak{F}_p^d$ is a ring. We consider the sequence $B:=\mathfrak{S}_{e,f}^g$. Let $\mathbf{a}$ be in $\{0,\dots,p-1\}^d$ and $\mathbf{m}$ in $\mathbb{N}^d$. First we shall prove that, for every $\mathbf{a}$ in $\{0,\dots,p-1\}^d$ there exists a function $\tau_{\mathbf{a}}$ in $\mathbb{Z}_p+p\mathfrak{F}_p^d$ such that, for all v in $\{0,\dots,p-1\}$ and n in $\mathbb{N}$, we have

$$\mathfrak{S}_{e,f}^g(v+np)=\sum_{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)}\sum_{\substack{\mathbf{m}\geqslant\mathbf{0}\\ |\mathbf{a}+\mathbf{m}p|=v+np}}\mathcal{Q}_{e,f}(\mathbf{m})\tau_{\mathbf{a}}(\mathbf{m}). \tag{4.12}$$

To that end, we express $\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p)$ as a product of $\mathcal{Q}_{e,f}(\mathbf{m})$ and elements of $\mathbb{Z}_p+p\mathfrak{F}_p^d$. We have

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p)=\frac{\prod_{i=1}^{u}(\mathbf{e}_i\cdot\mathbf{m}p)!\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^{v}(\mathbf{f}_i\cdot\mathbf{m}p)!\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)}.$$

For every nonnegative integer n, we have

$$\frac{(np)!}{n!} = p^n(-1)^{np}\Gamma_p(np),$$

so that we have

$$\frac{\prod_{i=1}^u(\mathbf{e}_i\cdot\mathbf{m}p)!}{\prod_{i=1}^v(\mathbf{f}_i\cdot\mathbf{m}p)!} = p^{(|e|-|f|)\cdot\mathbf{m}}\mathcal{Q}_{e,f}(\mathbf{m})\frac{\prod_{i=1}^u(-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^v(-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)}.$$

Furthermore, we have

$$\frac{\prod_{i=1}^u\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^v\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)} = \frac{\prod_{i=1}^u\prod_{k=1,p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^v\prod_{k=1,p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)}\cdot p^{\Delta_{e,f}(\mathbf{a}/p)}\frac{\prod_{i=1}^u\prod_{k=1}^{\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k)}{\prod_{i=1}^v\prod_{k=1}^{\lfloor\mathbf{f}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{f}_i\cdot\mathbf{m}+k)}.$$

Since $|e|=|f|$, we obtain that

$$\frac{\prod_{i=1}^u(-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^v(-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)} = \frac{\prod_{i=1}^u\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^v\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)}.$$

Let $\alpha_1,\dots,\alpha_d$ be nonnegative integers with $\alpha_{i_0}\geqslant 1$ for some i_0 in $\{1,\dots,d\}$. By Proposition 2, there exists a function h in $\mathfrak{F}_p^d$ such that, for all nonnegative integers $m_1,\dots,m_d$, we have

$$\frac{\Gamma_p\big((\alpha_1m_1+\cdots+\alpha_dm_d)p\big)}{\Gamma_p\big((\alpha_1m_1+\cdots+(\alpha_{i_0}-1)m_{i_0}+\cdots+\alpha_dm_d)p\big)\Gamma_p(m_{i_0}p)} = 1+h(m_1,\dots,m_d)p.$$

Hence, there exists a function h' in $\mathfrak{F}_p^d$ such that, for all nonnegative integers $m_1,\dots,m_d$, we have

$$\frac{\Gamma_p\big((\alpha_1m_1+\cdots+\alpha_dm_d)p\big)}{\Gamma_p(m_1p)^{\alpha_1}\cdots\Gamma_p(m_dp)^{\alpha_d}} = 1+h'(m_1,\dots,m_d)p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, there exists g' in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^u\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^v\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)} = 1+g'(\mathbf{m})p.$$

Furthermore, if k is an integer coprime to p, and $\mathbf{d}$ a vector in $\mathbb{N}^d$, then for every $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{1}{\mathbf{d}\cdot\mathbf{m}p+k} = \sum_{s=0}^{\infty}(-1)^s\frac{(\mathbf{d}\cdot\mathbf{m})^s}{k^{s+1}}p^s,$$

so that there is a function g'' in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{1}{\mathbf{d}\cdot\mathbf{m}p+k} = \frac{1}{k}+g''(\mathbf{m})p.$$

Hence, for all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, there exist a p-adic integer $\lambda_{\mathbf{a}}$ and a function $g_{\mathbf{a}}$ in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^u\prod_{k=1,p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^v\prod_{k=1,p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)} = \lambda_{\mathbf{a}}+g_{\mathbf{a}}(\mathbf{m})p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, for all i in $\{1,\dots,v\}$, we have $\lfloor\mathbf{f}_i\cdot\mathbf{a}/p\rfloor=0$. Thereby, for all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, there exists a function $h_{\mathbf{a}}$ in $\mathbb{Z}_p+p\mathfrak{F}_p^d$, such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p)=\mathcal{Q}_{e,f}(\mathbf{m})h_{\mathbf{a}}(\mathbf{m})p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k).$$

Furthermore, we have either $\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor=0$ for all i, or $\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor\geqslant 1$ for some i and so $\Delta_{e,f}(\mathbf{a}/p)\geqslant 1$. In both cases, we obtain that

$$\mathbf{m}\mapsto p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k)\in\mathbb{Z}_p+p\mathfrak{F}_p^d.$$

Let g be a function in $\mathfrak{F}_p^d$. For all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, the function $\mathbf{m}\mapsto g(\mathbf{a}+\mathbf{m}p)$ belongs to $\mathbb{Z}_p+p\mathfrak{F}_p^d$. For all $\mathbf{m}$ in $\mathbb{N}^d$, we set

$$\tau_{\mathbf{a}}(\mathbf{m}):=g(\mathbf{a}+\mathbf{m}p)h_{\mathbf{a}}(\mathbf{m})p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k),$$

so that $\tau_{\mathbf{a}}\in\mathbb{Z}_p+p\mathfrak{F}_p^d$. Therefore, for all v in $\{0,\dots,p-1\}$ and n in $\mathbb{N}$, we have

$$\begin{aligned}\mathfrak{S}_{e,f}^g(v+np)&=\sum_{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)}\ \sum_{\substack{\mathbf{m}\geqslant\mathbf{0}\\|\mathbf{a}+\mathbf{m}p|=v+np}}g(\mathbf{a}+\mathbf{m}p)\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p)\\&=\sum_{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)}\ \sum_{\substack{\mathbf{m}\geqslant\mathbf{0}\\|\mathbf{a}+\mathbf{m}p|=v+np}}\mathcal{Q}_{e,f}(\mathbf{m})\tau_{\mathbf{a}}(\mathbf{m}),\end{aligned}$$

which proves Equation (4.12).

Now if $|\mathbf{a}+\mathbf{m}p|=v+np$, then we have $|\mathbf{a}|=v+jp$ with

$$0\leqslant j\leqslant\min\left(n,\left\lfloor\frac{d(p-1)-v}{p}\right\rfloor\right)=:M.$$

Furthermore, we have $\lfloor|\mathbf{a}|/p\rfloor=j$ and there is k in $\{1,\dots,d\}$ such that $\mathbf{a}^{(k)}\geqslant(v+jp)/d$. Since e is 2-admissible, there are $1\leqslant i_1<i_2\leqslant u$ such that $\mathbf{e}_{i_1}\cdot\mathbf{a}/p\geqslant j$ and $\mathbf{e}_{i_2}\cdot\mathbf{a}/p\geqslant j$. Hence we obtain that

$$\Delta_{e,f}(\mathbf{a}/p)=\sum_{i=1}^{u}\left\lfloor\frac{\mathbf{e}_i\cdot\mathbf{a}}{p}\right\rfloor\geqslant 2j,$$

because f is constituted by vectors $\mathbf{1}_k$. In particular, there is $\tau'_{\mathbf{a}}$ in $\mathfrak{F}_p^d$ such that $\tau_{\mathbf{a}}=p^{2j}\tau'_{\mathbf{a}}$. Hence we have

$$\mathfrak{S}_{e,f}^g(v+np)=\sum_{\substack{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)\\|\mathbf{a}|=v}}\ \sum_{|\mathbf{m}|=n}\mathcal{Q}_{e,f}(\mathbf{m})\tau_{\mathbf{a}}(\mathbf{m})+\sum_{j=1}^{M}p^{2j}\sum_{\substack{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)\\|\mathbf{a}|=v+jp}}\ \sum_{|\mathbf{m}|=n-j}\mathcal{Q}_{e,f}(\mathbf{m})\tau'_{\mathbf{a}}(\mathbf{m}).$$

For every $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, we write $\tau_{\mathbf{a}}=\alpha_{\mathbf{a}}+p\beta_{\mathbf{a}}$ where $\alpha_{\mathbf{a}}$ is a constant in $\mathbb{Z}_p$ and $\beta_{\mathbf{a}}$ is a function in $\mathfrak{F}_p^d$. We set

$$\alpha:=\sum_{\substack{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)\\|\mathbf{a}|=v}}\alpha_{\mathbf{a}}\in\mathbb{Z}_p\quad\text{and}\quad\beta:=\sum_{\substack{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)\\|\mathbf{a}|=v}}\beta_{\mathbf{a}}\in\mathfrak{F}_p^d.$$

Finally, for every j in $\{1,\dots,M\}$, we set

$$\gamma_j := \sum_{\substack{\mathbf{0}\leqslant\mathbf{a}\leqslant\mathbf{1}(p-1)\\ |\mathbf{a}|=v+jp}} \tau'_{\mathbf{a}} \in \mathfrak{F}_p^d.$$

Hence we obtain that

$$\mathfrak{S}_{e,f}^{g}(v+np)=\alpha\mathfrak{S}_{e,f}(n)+p\mathfrak{S}_{e,f}^{\beta}(n)+\sum_{j=1}^{M}p^{2j}\mathfrak{S}_{e,f}^{\gamma_j}(n-j),$$

where $\alpha\mathfrak{S}_{e,f}\in\mathfrak{A}$, $\mathfrak{S}_{e,f}^{\beta}\in\mathfrak{B}$ and $\mathfrak{S}_{e,f}^{\gamma_j}\in\mathfrak{B}$. For every j, $1\leqslant j\leqslant M$, we have $2j\geqslant j+1$ so that there exist A' in $\mathfrak{A}$ and a sequence $(B_j)_{j\geqslant 0}$, with B_j in $\mathfrak{B}$, such that

$$\mathfrak{S}_{e,f}^{g}(v+np)=A'(n)+pB_0(n)+\sum_{j=1}^{\infty}p^{j+1}B_j(n-j).$$

This shows that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) in Proposition 1, so that Theorem 1 is proved. □

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É. Delaygue delaygue@math.univ-lyon1.fr
Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 Novembre 1918, 69622 Villeurbanne cedex, France.