Discrete-Duality Finite Volume Method for Second Order Elliptic Problems

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ABSTRACT. *This paper deals with applications of the "Discrete-Duality Finite Volume" approach to a variety of elliptic problems. This is a new finite volume method, based on the derivation of discrete operators obeying a Discrete-Duality principle. An appropriate choice of the degrees of freedom allows one to use arbitrary meshes. We show that the method is naturally related to finite and mixed finite element methods.*

RÉSUMÉ. *Cet article présente des applications de la méthode de "Dualité Discrète" à une variétés de problèmes elliptiques. Cette nouvelle méthode de volumes finis s'appuie d'une part sur la construction d'opérateurs discrets satisfaisant des propriétés de dualité discrète, et d'autre part sur un choix judicieux des degrés de liberté. Ceci permet de traiter des maillages arbitraires. Nous montrons la méthode est intimement liée aux méthodes d'éléments finis et d'éléments finis mixtes.*

KEYWORDS: *Finite Volume method, Mixed Finite Element method, Arbitrary Meshes, Laplace equation, Div-Curl problem, Hodge Decomposition, Stokes problem.*

MOTS-CLÉS : *Méthode des Volumes Finis, Méthode des Eléments Finis, Maillages Arbitraires, équation de Laplace, problème Div-Curl, Décomposition de Hodge, problème de Stokes.*

1. This text has been produced using GNU TeXmacs (see `http://www.texmacs.org`).

1. Introduction

This paper presents some applications to some elliptic problems of a recent finite volume method called "Discrete-Duality Finite Volume" method (abbr. DDFV method). In order to analyse finite volume schemes for elliptic problems, it is natural to try to recast the finite volume scheme in terms of a variational formulation hopefully close to what is known in the theory of finite element methods. In the case of rectangular meshes, the analogy is rather straightforward in the case of the classical Laplace equation with constant coefficients and Dirichlet or Neumann boundary conditions. Indeed, one obtains in that case classical finite difference or finite element methods. In the case of nonrectangular structured meshes, an analysis was performed in [13]. A new difficulty arises in the case where one deals with triangular meshes or more generally unstructured meshes, and several approaches were used. A connection between finite volume and finite element methods in the case of triangular meshes was first performed in the scope of the Finite Volume Element Methods (see [2, 3]). On the other hand, connections between finite volumes and finite difference techniques were also performed. In its simpler form, it is based on the possibility of approximating normal fluxes by finite differences which may be done in the case of meshes having suitable orthogonality conditions. The analysis of these methods also rely on the possibilty of writing discrete variational formulations and discrete errors estimates (see the review [8]).

In order to deal with general self-adjoint elliptic boundary value problems on general triangulations, Thomas and Trujillo introduced the Mixed Finite Volume Method, based on the relation between mixed finite element formulations and finite volume formulations (see e.g. [11, 12]). In the case of the Laplace problem with Dirichlet and Neumann boundary conditions, these authors considered the mixed problem

$$\left\{ \begin{array}{lcll} \mathbf{p} & = & -\boldsymbol{\nabla} u, & \text{in}\, \Omega \\ \text{div}\mathbf{p} & = & f & \text{in}\, \Omega \\ u & = & 0, & \text{on}\, \partial\Omega \end{array} \right.$$

and derived a – both finite volume and finite element – discrete problem involving the unknowns u and $\mathbf{p}$ as well as primal and dual meshes. They proved the convergence of the method for triangular unstructured meshes or rectangular meshes.

We recently introduced the DDFV method in order to deal with highly stiff Stokes-like problems on rectangular meshes. The aim was to preserve at the discrete level the duality of the discrete constraints despite the presence of strongly varying coefficients, as well as stability properties. One key ingredient in this approach is the use of **discrete integration formulas** based on the choice of control volumes, which allows us to define **discrete integration by parts**. As a consequence, a discrete gradient operator could be defined as the discrete adjoint operator of a given discrete divergence and *vice versa*. Another key ingredient is the choice of the degrees of freedom. In the case of rectangular grids, the DDFV method applied to Stokes-like problems was proved to lead to a Marker And Cell type scheme on staggered grids. The extension of the method to **general unstructured nonconforming** meshes follows the same lines, that is

– Define control volumes and associated unknowns,

– Define discrete integration, and discrete integration by part formulas,

– Define discrete divergence and gradient operators obeying the discrete duality property,

– Assemble the second order elliptic operator.

As will become clear later, one crucial point here is the choice of the degrees of freedom. Not all choices would lead to a well-posed or asymptotically stable problem. In that direction, we follow the work of F. Hermeline (see [9, 10]). In the case of distorted rectangular meshes or related triangular meshes, he introduced a new finite volume approach for elliptic problems with possibly discontinuous coefficients. The scheme he obtains is exactely the DDFV scheme that we derive thanks to the algorithm quoted above. However, his derivation is based on quadrature formulas for both gradient and divergence operators and relatively involved algebra that do not seem to allow a straightforward analysis. On the converse, the application of the DDFV method allows us to perform an analysis of the scheme as well as error estimates. This was done in [7] by emphasizing the relation between the DDFV method and the classical variational formulation for the usual Laplace equation.

After introducing the general notations, we present the Discrete-Duality Finite Volume method applied to the Laplace problem and emphasize its relation with Mixed Finite Element approaches (see [12]). Then, we show how the DDFV method applies to the Div-Curl and Hodge decomposition problems (see [6]), and finally the Stokes problem (see [5]).

2. Meshes and Notations

2.1. *Meshes,Vertices and Centers*

We want to consider general, including nonconforming meshes, made of arbitrary polygons. The DDFV method uses (i) the primal mesh, (ii) the dual mesh, centered around the vertices of the primal mesh, (iii) the diamond mesh, centered around the edges of both the primal and dual meshes, and finally (iv) the sharp mesh, a common subtriangulation of the previous meshes. The primal mesh and all the unknowns or operators attached to it are denoted by a circle in superscript. For example, Ω_h° denotes the primal mesh, Ω_i° denotes a polygon of the primal mesh, and G_i° its center. We shall use similarly $\Omega_h^\star$ or $\Omega_j^\star$ for the dual mesh or cells, and $\Omega_h^\diamondsuit$ or $\Omega_k^\diamondsuit$ for the diamond mesh and cells.

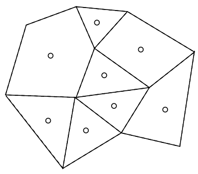

Figure 1. *Primal mesh and control points made of vertices, centers of polygons, and centers of edges.*

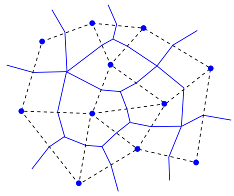

Figure 2. *Dual mesh $\Omega_h^\star$, made of the polygons $\Omega_j^\star$, and whose centers are $S_j^\star$. The primal mesh is in dashed lines.*

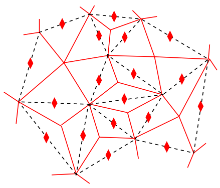

Figure 3. *Diamond mesh $\Omega_h^\diamondsuit$ (and dashed primal cells), made of the polygons $\Omega_k^\diamondsuit$, and whose centers are $D_k^\diamondsuit$.*

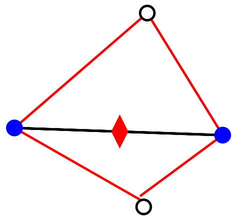

Figure 4. *Zoom on a diamond cell $\Omega_k^\diamondsuit$. This cell is composed of two triangles belonging to the sharp mesh $\Omega_h^\#$.*

One may check that the primal and dual mesh play exactely the same role with respect to the the diamond mesh. Also, all diamond cells are structured objects, quadrilateral, whose vertices are alternatively primal and dual centers. Finally, each diamond cell divides into two triangles, which allows us to define the corresponding subtriangulation $\Omega_h^\#$.

2.2. *Boundaries. Non conforming meshes*

Notice that the possibility of using polygons of arbitrary type implies that the discretisation naturally applies to nonconforming meshes. Indeed, when the edges of two adjacent cells are not identical, it suffices to subdivide the original edges into the smaller edges involving all the vertices of any of the adjacent cells. This allows to take into account locally refined meshes (see e.g. [4, 7]). Finally, there are infinitely many ways of defining a dual mesh given a primal mesh. We use here the median-dual mesh based on the primal centers and the mid-point of the edges. It would be possible to deal with Voronoi dual meshes, which simplifies a lot the DDFV scheme due to the orthogonality properties of this type of discretisation (see the remarks in [10, 7]).

We also need to prescribe some values of $u_h^{\circ\star}$ at on the boundary. For that, we introduce some control points $G_{i'}^\circ$ on the boundary. For each cell Ω_i° with center

G_i°, we define the corresponding $G_{i'}^\circ$ as the center of the boundary edge. Many other choices are allowed by the method, but this is beyond the scope of this presentation.

2.3. *Discrete integration formulas*

Any of the three meshes Ω_h°, $\Omega_h^\star$, and $\Omega_h^\diamondsuit$ is a suitable partition for which we define the usual discrete integration formulas. Also we will note in condensed form $\mathbb{R}_h^{\circ\star} \stackrel{def}{=} \mathbb{R}_h^\circ \times \mathbb{R}_h^\star$, $\Omega_h^{\circ\star} \stackrel{def}{=} \Omega_h^\circ \times \Omega_h^\star$ and accordingly $\langle u_h^{\circ\star}, v_h^{\circ\star}\rangle_{\Omega_h^{\circ\star}} \stackrel{\text{def}}{=} \frac{1}{2}\langle u_h^\circ, v_h^\circ\rangle_{\Omega_h^\circ} + \frac{1}{2}\langle u_h^\star, v_h^\star\rangle_{\Omega_h^\star}$. For example, the space $\mathbb{R}_h^\circ$ denotes the real vectors of size the number of centers G_i° (including those at the boundary as we detail below).

3. The Discrete-Duality Finite Volume Method

3.1. *Discrete Gradient, Divergence, and Trace operators*

CONSTRUCTION OF THE DISCRETE GRADIENT OPERATOR

We start by constructing the gradient operator of the scalar field $u_h^{\circ\star} \in \mathbb{R}_h^{\circ\star}$, defined on $\Omega_h^{\circ\star}$, by using quadrature formulas. Let $\Omega_k^\diamondsuit$ be a diamond cell located inside the domain, that is not a half-diamond cell having a common edge with the boundary of the domain. Then, consider a function $u(\mathbf{x})$ defined on Ω, and its averaged gradient $\langle \boldsymbol{\nabla} u\rangle_k^\diamondsuit$ on $\Omega_k^\diamondsuit$ such that $\mid \Omega_k^\diamondsuit \mid \langle \boldsymbol{\nabla} u\rangle_k^\diamondsuit = \left\langle \boldsymbol{\nabla} u , \mathbb{1}_k^\diamondsuit \right\rangle_\Omega$ where $\mathbb{1}_k^\diamondsuit$ denotes the characteristic function of $\Omega_k^\diamondsuit$ and $\langle \cdot , \cdot \rangle_\Omega$ denotes the usual scalar product of $L^2(\Omega)$. A direct integration leads to the expression of the mean gradient on a given diamond cell $\Omega_k^\diamondsuit = (\mathbf{y}_1^\circ, \mathbf{y}_3^\star, \mathbf{y}_2^\circ, \mathbf{y}_4^\star)$ (these four points are ordered clockwise):

$$\begin{aligned} \mid \Omega_k^\diamondsuit \mid \langle \boldsymbol{\nabla} u\rangle_k^\diamondsuit &= u(\mathbf{y}_1^\circ)\frac{l_{13}\mathbf{n}_{13} + l_{14}\mathbf{n}_{14}}{2} + u(\mathbf{y}_2^\circ)\frac{l_{23}\mathbf{n}_{23} + l_{24}\mathbf{n}_{24}}{2} \\ &\qquad + u(\mathbf{y}_3^\star)\frac{l_{13}\mathbf{n}_{13} + l_{23}\mathbf{n}_{23}}{2} + u(\mathbf{y}_4^\star)\frac{l_{14}\mathbf{n}_{14} + l_{24}\mathbf{n}_{24}}{2}. \end{aligned}$$

We noted $l_{13} = |\mathbf{y}_1^\circ \mathbf{y}_3^\star|$, $\mathbf{n}_{13} = (\mathbf{y}_1^\circ \mathbf{y}_3^\star)^\perp$ and so on. The final expression for the discrete gradient $\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} : \mathbb{R}_h^{\circ\star} \mapsto (\mathbb{R}_h^\diamondsuit)^2$ is written after factorizing with respect to the primal and dual points:

$$(\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star})_k^\diamondsuit = (u(\mathbf{y}_2^\circ) - u(\mathbf{y}_1^\circ))\frac{l_{34}}{2 \mid \Omega_k^\diamondsuit \mid}\mathbf{n}_{43} + \frac{1}{2}(u(\mathbf{y}_4^\star) - u(\mathbf{y}_3^\star))\frac{l_{12}}{2 \mid \Omega_k^\diamondsuit \mid}\mathbf{n}_{12}$$

This quadrature formula is exact for any linear functions $u(\mathbf{x}) = a + \mathbf{b} \cdot \mathbf{x}$, $a \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^2$. Notice finally that in the case where the primal and dual mesh have the orthogonal property $\mathbf{n}_{12} \cdot \mathbf{n}_{34} = 0$ locally in a given diamond cell, then we have $\mid \Omega_k^\diamondsuit \mid = l_{12}l_{34}/2$ and we recover the usual finite difference formulas.

CONSTRUCTION OF THE DISCRETE DIVERGENCE ...

We define now the discrete divergence operator, within the frame of the previous discretisation and degrees of freedom as the **discrete adjoint operator** of minus the discrete gradient operator just constructed. It is important to notice that this discrete integration by parts is made possible thanks to the factorisation performed in the discrete gradient formula. At the continuous level, the duality property of the gradient and divergence operators writes formally as

$$\forall \mathbf{p}, \forall u, \quad \langle \mathrm{div}\mathbf{p} \,, u\rangle_\Omega + \langle \mathbf{p} \,, \boldsymbol{\nabla} u\rangle_\Omega = \langle \mathbf{p}{\cdot}\mathbf{n} \,, u\rangle_{\partial\Omega} = \langle (\gamma_0\mathbf{p}){\cdot}\mathbf{n} \,, \gamma_0 u\rangle_{\partial\Omega} \qquad (1)$$

where γ_0 denotes the trace operator. At the discrete level, let $\mathbf{p}_h^\diamondsuit \in (\mathbb{R}_h^\diamondsuit)^2$ be any test function. First, we consider discrete test functions $\mathbf{p}_h^\diamondsuit$ whose support consist of diamond cells that are **not** located at the boundary. Moreover, we deal first with the contribution of u_h°. For that, we define the partial gradient operators $\boldsymbol{\nabla}_h^{\diamondsuit,\circ}$ and $\boldsymbol{\nabla}_h^{\diamondsuit,\star}$ by $\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star} \overset{\mathrm{def}}{=} \boldsymbol{\nabla}_h^{\diamondsuit,\circ} u_h^\circ + \boldsymbol{\nabla}_h^{\diamondsuit,\star} u_h^\star$ with obvious notations. For any u_h°, we have that

$$\begin{aligned}
\left\langle \mathbf{p}_h^\diamondsuit \,, \boldsymbol{\nabla}_h^{\diamondsuit,\circ} u_h^\circ \right\rangle_{\Omega_h^\diamondsuit} &= \textstyle\sum_{\Omega_k^\diamondsuit} \left((u_h^\circ(\mathbf{y}_2^\circ) - u_h^\circ(\mathbf{y}_1^\circ)) \dfrac{l_{34}}{2 \mid \Omega_k^\diamondsuit \mid} \mathbf{p}_k^\diamondsuit{\cdot}\mathbf{n}_{43}^k \right) \mid \Omega_k^\diamondsuit \mid \\
&= -\frac{1}{2} \sum_{\Omega_h^\diamondsuit} u_h^\circ(\mathbf{y}_2^\circ) l_{34} \mathbf{p}_k^\diamondsuit{\cdot}(-\mathbf{n}_{43}^k) + u_h^\circ(\mathbf{y}_1^\circ) l_{34} \mathbf{p}_k^\diamondsuit{\cdot}\mathbf{n}_{43}^k \\
&= -\frac{1}{2} \sum_{\Omega_i^\circ} \left(u_h^\circ(\mathbf{y}_i^\circ) \sum_{k;\Omega_k^\diamondsuit\cap\Omega_i^\circ\neq\emptyset} \frac{l_{ik}^\star}{\mid \Omega_i^\circ \mid} \mathbf{p}_k^\diamondsuit{\cdot}\mathbf{n}_{ik} \right) \Omega_i^\circ \\
&\overset{\mathrm{def}}{=} -\frac{1}{2} \left\langle u_h^\circ \,, \mathrm{div}_h^{\circ,\diamondsuit} \mathbf{p}_h^\diamondsuit \right\rangle_{\Omega_h^\circ}.
\end{aligned}$$

where $\mathbf{n}_{ik}$ is the outward normal to the cell Ω_i° that belongs to $\Omega_k^\diamondsuit$ and where $l_{ik}^\star = \left| S_{j_1}^\star - S_{j_2}^\star \right|$ is the length of the edge of Ω_i° that belongs to $\Omega_k^\diamondsuit$. The expression above only involves the values of u_h° located at points G_i° in the **interior** of the domain, and does **not** involve the values of u_h° located at points G_i° on the boundary. The last equality in the equation above defines the operator $\mathrm{div}_h^{\circ,\diamondsuit} : (\mathbb{R}_h^\diamondsuit)^2 \to \Omega_h^\circ$. The operator $\mathrm{div}_h^{\star,\diamondsuit} : (\mathbb{R}_h^\diamondsuit)^2 \to \Omega_h^\star$ is defined in a similar way, namely

$$\left(\mathrm{div}_h^{\circ,\diamondsuit} \mathbf{p}_h^\diamondsuit\right)_i^\circ = \sum_{k;\Omega_k^\diamondsuit\cap\Omega_i^\circ\neq\emptyset} \frac{l_{ik}^\star}{\mid \Omega_i^\circ \mid} \mathbf{p}_k^\diamondsuit{\cdot}\mathbf{n}_{ik}, \quad \left(\mathrm{div}_h^{\star,\diamondsuit} \mathbf{p}_h^\diamondsuit\right)_j^\star = \sum_{k;\Omega_k^\diamondsuit\cap\Omega_j^\star\neq\emptyset} \frac{l_{jk}^\circ}{\mid \Omega_j^\star \mid} \mathbf{p}_k^\diamondsuit{\cdot}\mathbf{n}_{jk}$$

... AND TRACE OPERATORS

Let us consider now discrete functions $\mathbf{p}_h^\diamondsuit$ that are not necessarily zero on the boundary half-diamond cells. Mimicking Eq. (1) at the discrete level, allows us to define discrete trace operrators that are consistent (in the sense of the discrete integration formulas) with the discrete gradient and divergence operators. Let us first remind that the complete gradient operator writes as $\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} := \boldsymbol{\nabla}_h^{\diamondsuit,\circ} + \boldsymbol{\nabla}_h^{\diamondsuit,\star}$ and is defined on the spaces $\mathbb{R}_h^{\circ\star} \to (\mathbb{R}_h^{\diamondsuit})^2$. It is natural to introduce its discrete dual operator $\mathrm{div}_h^{\circ\star,\diamondsuit} : (\mathbb{R}_h^{\diamondsuit})^2 \to \mathbb{R}_h^{\circ\star} = \mathbb{R}_h^{\circ} \times \mathbb{R}_h^{\star}$. One should convince himself that we have precisely $\mathrm{div}_h^{\circ\star,\diamondsuit} = (\mathrm{div}_h^{\circ,\diamondsuit}, \mathrm{div}_h^{\star,\diamondsuit}) \in \mathbb{R}_h^{\circ} \times \mathbb{R}_h^{\star} = \mathbb{R}_h^{\circ\star}$, or equivalently (recalling that $\langle u_h^{\circ\star}, v_h^{\circ\star}\rangle_{\Omega_h^{\circ\star}} = \frac{1}{2}\langle u_h^{\circ}, v_h^{\circ}\rangle_{\Omega_h^{\circ}} + \frac{1}{2}\langle u_h^{\star}, v_h^{\star}\rangle_{\Omega_h^{\star}}$),

$$\begin{aligned}
&\left\langle \mathrm{div}_h^{\circ\star,\diamondsuit}\mathbf{p}_h^{\diamondsuit}, u_h^{\circ\star}\right\rangle_{\Omega_h^{\circ\star}} + \left\langle \mathbf{p}_h^{\diamondsuit}, \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}u_h^{\circ\star}\right\rangle_{\Omega_h^{\diamondsuit}} \\
&\quad = \left(\frac{1}{2}\left\langle \mathrm{div}_h^{\circ,\diamondsuit}\mathbf{p}_h^{\diamondsuit}, u_h^{\circ}\right\rangle_{\Omega_h^{\circ}} + \left\langle \mathbf{p}_h^{\diamondsuit}, \boldsymbol{\nabla}_h^{\diamondsuit,\circ}u_h^{\circ}\right\rangle_{\Omega_h^{\diamondsuit}}\right) \\
&\qquad + \left(\frac{1}{2}\left\langle \mathrm{div}_h^{\star,\diamondsuit}\mathbf{p}_h^{\diamondsuit}, u_h^{\star}\right\rangle_{\Omega_h^{\star}} + \left\langle \mathbf{p}_h^{\diamondsuit}, \boldsymbol{\nabla}_h^{\diamondsuit,\star}u_h^{\star}\right\rangle_{\Omega_h^{\diamondsuit}}\right)
\end{aligned} \tag{2}$$

Thanks to the previous construction of the discrete divergence operator, almost all terms cancel except those involving the boundary edges, namely an expression of the form

$$\begin{aligned}
&\left\langle \mathrm{div}_h^{\circ\star,\diamondsuit}\mathbf{p}_h^{\diamondsuit}, u_h^{\circ\star}\right\rangle_{\Omega_h^{\circ\star}} + \left\langle \mathbf{p}_h^{\diamondsuit}, \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}u_h^{\circ\star}\right\rangle_{\Omega_h^{\diamondsuit}} \\
&\quad = \frac{1}{2}\left\langle (\gamma_{0h}^{\diamondsuit}\mathbf{p}_h^{\diamondsuit})\cdot\mathbf{n}_h^{\diamondsuit}, \gamma_{0h}^{\circ}u_h^{\circ}\right\rangle_{\partial\Omega_h^{\circ}} + \frac{1}{2}\left\langle \gamma_{0h}^{\star,\diamondsuit}(\mathbf{p}_h^{\diamondsuit}\cdot\mathbf{n}_h^{\diamondsuit}), \gamma_{0h}^{\star}u_h^{\star}\right\rangle_{\partial\Omega_h^{\star}}.
\end{aligned}$$

Here $\partial\Omega_h^{\circ}$ denotes the set of boundary edges $\partial\Omega_h^{\circ} = \cup_i \partial\Omega_i^{\circ}$, where $\partial\Omega_i^{\circ}$ denotes the boundary edge of Ω_i° (we may assume that there is only one boundary edge for each primal cell). Each boundary edge $\partial\Omega_i^{\circ}$ is also a the boundary edge $\partial\Omega_k^{\diamondsuit}$ of a half-diamond cell, hence $\partial\Omega_h^{\circ} = \partial\Omega_k^{\diamondsuit}$. Finally, for each dual point $S_j^{\star}$ located on the boundary, we define the corresponding boundary element $\partial\Omega_j^{\star}$ as the union of the two half-boundary edges whose $S_j^{\star}$ is an endpointwhere, that is $\partial\Omega_j^{\star} = (\partial\Omega_{i_1(j)}^{\circ} \cup \partial\Omega_{i_2(j)}^{\circ})/2 = (\partial\Omega_{k_1(j)}^{\diamondsuit} \cup \partial\Omega_{k_2(j)}^{\diamondsuit})/2$ with obvious notations. The trace operators write, for all $\partial\Omega_i^{\circ} = \partial\Omega_k^{\diamondsuit}$ and for all $\partial\Omega_j^{\star}$

$$\begin{aligned}
&(\gamma_{0h}^{\circ}u_h^{\circ})_i = u_i^{\circ}, \quad (\gamma_{0h}^{\star}u_h^{\star})_j = u_j^{\star}, \quad (\gamma_h^{\diamondsuit}\mathbf{p}_h^{\diamondsuit})_k = \mathbf{p}_k^{\diamondsuit} \\
&(\gamma_{0h}^{\star,\diamondsuit}(\mathbf{p}_h^{\diamondsuit}\cdot\mathbf{n}_h^{\diamondsuit}))_j = (l_{k_1(j)}\mathbf{p}_{k_1(j)}\cdot\mathbf{n}_{k_1(j)} + l_{k_2(j)}\mathbf{p}_{k_2(j)}\cdot\mathbf{n}_{k_2(j)})/(l_{k_1(j)} + l_{k_2(j)})
\end{aligned} \tag{3}$$

with $l_{k_1(j)} = \left|\partial\Omega_{k_1(j)}\right|$, $l_{k_2(j)} = \left|\partial\Omega_{k_2(j)}\right|$. Notice that the operator $\gamma_{0h}^{\star,\diamondsuit}$ is a suitable average of the trace operator $\gamma_{0h}^{\diamondsuit}$.

3.2. *Discrete-Duality Finite Volume scheme for the Laplace problem*

VARIATIONAL FORMULATION OF THE DDFV METHOD

We may now consider the discrete Laplace problem with Dirichlet boundary conditions and the corresponding variational formulation:

$$\left\{\begin{array}{l} -\Delta_h^{\circ\star,\circ\star} u_h^{\circ\star} = f_h^{\circ\star} \\ \gamma_{0h}^{\circ\star} u_h^{\circ\star} = 0 \end{array}\right. \qquad \left\{\begin{array}{l} \left\langle \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star} , \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} v_h^{\circ\star} \right\rangle_{\Omega_h^{\circ\star}} = \langle f_h^{\circ\star} , v_h^{\circ\star} \rangle_{\Omega_h^{\circ\star}} , \\ \gamma_{0h}^{\circ\star} u_h^{\circ\star} = 0 \end{array}\right. \tag{4}$$

for all $v_h^{\circ\star} \in \mathbb{R}_h^{\circ\star}$, $\gamma_{0h}^{\circ\star} v_h^{\circ\star} = 0$, and where $\Delta_h^{\circ\star,\circ\star}\bullet = \mathrm{div}_h^{\circ\star,\diamondsuit} \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}\bullet$.

MIXED FORMULATION OF THE DDFV METHOD

An alternative approach is to recast the problem in terms of the mixed formulation for the Laplace problem. It is indeed straightforward to rewrite the problem as: find $(u_h^{\circ\star}, \mathbf{p}_h^{\diamondsuit}) \in U_h^{\circ\star} \times \mathbf{P}_h^{\diamondsuit}$ such that

$$\left\{\begin{array}{ll} \mathbf{p}_h^{\diamondsuit} = -\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star}, & \text{in } \Omega_h^{\diamondsuit} \\ \mathrm{div}_h^{\circ\star,\diamondsuit} \mathbf{p}_h^{\diamondsuit} = f_h^{\circ\star}, & \text{in } \Omega_h^{\circ\star} \backslash \partial\Omega_h^{\circ\star} \\ \gamma_{0h}^{\circ\star} u_h^{\circ\star} = 0, & \text{on } \partial\Omega_h^{\circ\star} \end{array}\right.$$

where we noted $\mathbf{P}_h^{\diamondsuit} = \mathbf{Q}_h^{\diamondsuit} = (\mathbb{R}_h^{\diamondsuit})^2$, $U_h^{\circ\star} = V_h^{\circ\star} = \mathbb{R}_h^{\circ\star}$ and $U_{0h}^{\circ\star} = \gamma_{0h}^{\circ\star} u_h^{\circ\star} = 0$. A straightforward discrete integration yields: find $(u_h^{\circ\star}, \mathbf{p}_h^{\diamondsuit}) \in U_{0h}^{\circ\star} \times \mathbf{P}_h^{\diamondsuit}$ such that

$$\left\{\begin{array}{lllll} \left\langle \mathbf{p}_h^{\diamondsuit} , \mathbf{q}_h^{\diamondsuit} \right\rangle_{\Omega_h^{\diamondsuit}} & + \left\langle \mathbf{q}_h^{\diamondsuit} , \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star} \right\rangle_{\Omega_h^{\diamondsuit}} & = & 0, & \forall \mathbf{q}_h^{\diamondsuit} \in \mathbf{Q}_h^{\diamondsuit} \\ & \left\langle \mathrm{div}_h^{\circ\star,\diamondsuit} \mathbf{p}_h^{\diamondsuit} , v_h^{\circ\star} \right\rangle_{\Omega_h^{\circ\star}} & = & \langle f_h^{\circ\star} , v_h^{\circ\star} \rangle_{\Omega_h^{\circ\star}} & \forall v_h^{\circ\star} \in V_h^{\circ\star} \end{array}\right. \tag{5}$$

This is the classical mixed primal-dual formulation for the Laplace equation used by [12]. Notice however that thanks to the discrete-duality property, we can switch **without any quadrature formula** to the equivalent formulation

$$\left\{\begin{array}{lllll} \left\langle \mathbf{p}_h^{\diamondsuit} , \mathbf{q}_h^{\diamondsuit} \right\rangle_{\Omega_h^{\diamondsuit}} & + \left\langle \mathbf{q}_h^{\diamondsuit} , \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star} \right\rangle_{\Omega_h^{\diamondsuit}} & = & 0, & \forall \mathbf{q}_h^{\diamondsuit} \in \mathbf{Q}_h^{\diamondsuit} \\ & - \left\langle \mathbf{p}_h^{\diamondsuit} , \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} v_h^{\circ\star} \right\rangle_{\Omega_h^{\diamondsuit}} & = & \langle f_h^{\circ\star} , v_h^{\circ\star} \rangle_{\Omega_h^{\circ\star}} & \forall v_h^{\circ\star} \in V_h^{\circ\star} \end{array}\right. \tag{6}$$

EXISTENCE AND UNIQUENESS

$\triangleleft$ Using the classical variational formulation (4) of the DDFV method, we have to deal with the discrete Laplace operator $-\Delta_h^{\circ\star,\circ\star} \stackrel{\text{def}}{=} -\mathrm{div}_h^{\circ\star,\diamondsuit} \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}$ defined from $U_{0h}^{\circ\star}$ onto itself. Any function $v_h^{\circ\star} \in \mathrm{Ker}(-\Delta_h^{\circ\star,\circ\star})$ satisfies immediately

$\left\langle \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} v_h^{\circ\star} \,, \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} v_h^{\circ\star} \right\rangle_{\Omega_h^{\diamondsuit}} = 0$, so that we have constant functions $u_h^{\circ} = \mathrm{Cst}^{\circ}$ and $u_h^{\star} = \mathrm{Cst}^{\star}$, and the boundary conditions impose $\mathrm{Cst}^{\circ} = \mathrm{Cst}^{\star} = 0$. This means injectivity, therefore surjectivity, and the discrete problem has a unique solution.

$\triangleleft$ Come now to the mixed variational formulation (5) for the DDFV method. Following the classical procedure of mixed finite element method (see [1]), we denote by $B_h^{\circ\star,\diamondsuit}$ the discrete divergence operator and by $(B^t)_h^{\diamondsuit,\circ\star}$ the discrete gradient operator

$$B_h^{\circ\star,\diamondsuit} \stackrel{\mathrm{def}}{=} \mathrm{div}_h^{\circ\star,\diamondsuit} : \mathbf{Q}_h^{\diamondsuit} \longrightarrow V_{0h}^{\circ\star}, \quad (B^t)_h^{\diamondsuit,\circ\star} \stackrel{\mathrm{def}}{=} -\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} : V_{0h}^{\circ\star} \longrightarrow \mathbf{Q}_h^{\diamondsuit}.$$

The discrete kernels and ranges of these operators are:

$$\begin{array}{lcl}
\mathrm{Ker}\, B_h^{\circ\star,\diamondsuit} & = & \left\{ \mathbf{p}_h^{\diamondsuit} \in \mathbf{P}_h^{\diamondsuit}; \quad \left\langle \mathrm{div}_h^{\circ\star,\diamondsuit} \mathbf{p}_h^{\diamondsuit} \,, v_h^{\circ\star} \right\rangle_{\Omega_h^{\circ\star}} = 0, \quad \forall v_h^{\circ\star} \in V_{0h}^{\circ\star} \right\} \\
\mathrm{Ker} (B^t)_h^{\diamondsuit,\circ\star} & = & \left\{ u_h^{\circ\star} \in U_{0h}^{\circ\star}; \quad \left\langle \mathbf{q}_h^{\diamondsuit} \,, \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star} u_h^{\circ\star} \right\rangle_{\Omega_h^{\circ\star}} = 0, \quad \forall \mathbf{q}_h \in \mathbf{Q}_h^{\diamondsuit} \right\} \\
\mathrm{Im}\, B_h^{\circ\star,\diamondsuit} & = & B_h^{\circ\star,\diamondsuit} \left(\mathbf{P}_h^{\diamondsuit} \right) \quad = \quad \left(\mathrm{Ker} (B^t)_h^{\diamondsuit,\circ\star} \right)^{\perp_h^{\circ\star}} \quad \text{in} \quad V_{0h}^{\circ\star} \\
\mathrm{Im} (B^t)_h^{\diamondsuit,\circ\star} & = & (B^t)_h^{\diamondsuit,\circ\star} (U_{0h}^{\circ\star}) \quad = \quad \left(\mathrm{Ker}\, B_h^{\circ\star,\diamondsuit} \right)^{\perp_h^{\diamondsuit}} \quad \text{in} \quad \mathbf{Q}_h^{\diamondsuit}
\end{array}$$

where $\perp_h^{\circ\star}$ (resp. $\perp_h^{\diamondsuit}$) denote the orthogonal with respect to the scalar product $\langle \cdot \,, \cdot \rangle_{\Omega_h^{\circ\star}}$ (resp. $\langle \cdot \,, \cdot \rangle_{\Omega_h^{\diamondsuit}}$), and where the two equalities involving the orthogonal spaces are straightforward in finite dimension. Clearly, we have $\mathrm{Ker}(B^t)_h^{\diamondsuit,\circ\star} = \{0\}$, so that $\mathrm{Im}\, B_h^{\circ\star,\diamondsuit} = V_h^{\circ}$ from the definition of the lifting of $B_h^{\circ\star,\diamondsuit}$. As a conclusion, there exists $\mathbf{p}_{1h}^{\diamondsuit}$ solution to the second equation of the mixed variational formulation (5). At that stage, $\mathbf{p}_{1h}^{\diamondsuit}$ is unique up to elements of $\mathrm{Ker}\, B_h^{\circ\star,\diamondsuit}$. Let $(\mathbf{p}_{1h}^{\diamondsuit}, u_{1h}^{\circ\star})$ and $(\mathbf{p}_{2h}^{\diamondsuit}, u_{1h}^{\circ\star})$ be two solutions. We have that

$$\forall \mathbf{q}_h^{\diamondsuit} \in \mathbf{Q}_h^{\diamondsuit}, \quad \left\langle \mathbf{p}_{1h}^{\diamondsuit} - \mathbf{p}_{2h}^{\diamondsuit} \,, \mathbf{q}_h^{\diamondsuit} \right\rangle_{\Omega_h^{\diamondsuit}} - \left\langle \mathrm{div}_h^{\circ\star,\diamondsuit} \mathbf{q}_h^{\diamondsuit} \,, u_{1h}^{\circ\star} - u_{2h}^{\circ\star} \right\rangle_{\Omega_h^{\diamondsuit}} = 0$$

where $\mathbf{p}_{1h}^{\diamondsuit} - \mathbf{p}_{2h}^{\diamondsuit} \in \mathrm{Ker}\, B_h^{\circ\star,\diamondsuit}$. Taking for example $\mathbf{q}_h^{\diamondsuit} = \mathbf{p}_{1h}^{\diamondsuit} - \mathbf{p}_{2h}^{\diamondsuit}$ yields trivially the uniqueness of $\mathbf{p}_h^{\diamondsuit}$. Then, we want to characterize $u_h^{\circ\star}$ solution to the first equation of (5), from which we know immediately that $\mathbf{p}_h^{\diamondsuit} \in (\mathrm{Ker}\, B_h^{\circ\star,\diamondsuit})^{\perp_h^{\diamondsuit}}$, but $(\mathrm{Ker}\, B_h^{\circ\star,\diamondsuit})^{\perp_h^{\diamondsuit}} = \mathrm{Im}(B^t)_h^{\diamondsuit,\circ\star}$ yields the existence of $u_h^{\circ\star}$. The uniqueness follows here from the injectivity of $(B^t)_h^{\diamondsuit,\circ\star}$ already proved in the previous paragraph.

3.3. *Convergence Analysis and Error Estimates*

A proof of the convergence of the DDFV method for the Laplace equation based on the classical variational formulation (4) is in [7]. An alternative method of proof is based on the mixed variational formulations (5) or (6). In that case, conforming approximations are obtained by using both ideas of Thomas and Trujillo [12], and by observing the DDFV for the Laplace problems is the superimposition of two coupled discrete Laplace problems such as those investigated in [12].

4. Div–Curl problem and Hodge Decomposition

The DDFV allows us to generalize to general unstructured meshes previous works related to the discretization by finite volume techniques of Div–Curl or corresponding Hodge decomposition problems. More details are in [6].

4.1. *Discrete Curl, Rot and tangential-trace operators*

We introduce here the discrete vectorial (resp. scalar) rotational of a scalar field (resp. vectorial) field. In the continuous case, we have

$$\begin{array}{lcll}\mathbf{rot}\phi & \stackrel{\text{def}}{=} & (-\partial_y\phi, \partial_x\phi) & = \quad (\boldsymbol{\nabla}\phi)^\perp = (\boldsymbol{\nabla}^\perp\phi) \\ \text{curl}\mathbf{u} & \stackrel{\text{def}}{=} & \partial_x u_2 - \partial_y u_1 & \\ \langle\phi\ ,\text{curl}\mathbf{u}\rangle_\Omega - \langle\mathbf{rot}\phi\ ,\mathbf{u}\rangle_\Omega & = & \langle\phi\ ,\mathbf{u}\cdot\tau\rangle_{\partial\Omega} & \end{array}$$

Following the same strategy as before, we may define first the discrete $\mathbf{rot}_h^{\Diamond,\circ\star}$ acting on functions $\phi_h^{\circ\star}$ by using quadrature formulas on the cells $\Omega_k^\Diamond$. We have simply, in terms of the discrete gradient $\boldsymbol{\nabla}_h^{\Diamond,\circ\star}$:

$$\mathbf{rot}_h^{\Diamond,\circ\star} = \left(\boldsymbol{\nabla}_h^{\Diamond,\circ\star}\right)^\perp$$

Then, integrations by part allow us to define on the one hand the discrete operator $\text{curl}_h^{\circ\star,\Diamond}$, as the sums

$$\left(\text{curl}_h^{\circ,\Diamond}\mathbf{p}_h^\Diamond\right)_i^\circ = \sum_{k;\Omega_k^\Diamond\cap\Omega_i^\circ\neq\emptyset}\frac{l_{ik}^\star}{\mid\Omega_i^\circ\mid}\mathbf{p}_k^\Diamond\cdot\mathbf{n}_{ik}^\perp, \quad \left(\text{div}_h^{\star,\Diamond}\mathbf{p}_h^\Diamond\right)_j^\star = \sum_{k;\Omega_k^\Diamond\cap\Omega_j^\star\neq\emptyset}\frac{l_{jk}^\circ}{\mid\Omega_j^\star\mid}\mathbf{p}_k^\Diamond\cdot\mathbf{n}_{jk}^\perp. \tag{7}$$

and on the other hand the tangential trace operators are defined by the constraint that for all $\phi_h^{\circ\star}\in\mathbb{R}_h^{\circ\star}$ and $\mathbf{u}_h^\Diamond\in(\mathbb{R}_h^\Diamond)^2$,

$$\begin{aligned}&\left\langle\phi_h^{\circ\star}\ ,\text{curl}_h^{\circ\star,\Diamond}\mathbf{u}_h^\Diamond\right\rangle_{\Omega_h^{\circ\star}} - \left\langle\mathbf{rot}_h^{\Diamond,\circ\star}\phi_h^{\circ\star}\ ,\mathbf{u}_h^\Diamond\right\rangle_{\Omega_h^\Diamond} \\ &\qquad = \frac{1}{2}\left\langle(\gamma_{0h}^\Diamond\mathbf{u}_h^\Diamond)\cdot\mathbf{n}_h^{\Diamond\perp}\ ,\gamma_{0h}^\circ\phi_h^\circ\right\rangle_{\partial\Omega_h^\circ} + \frac{1}{2}\left\langle\gamma_{0h}^{\star,\Diamond}(\mathbf{p}_h^\Diamond\cdot\mathbf{n}_h^{\Diamond\perp})\ ,\gamma_{0h}^\star\phi_h^\star\right\rangle_{\partial\Omega_h^\star}\end{aligned}$$

Finally, if $\phi_h^{\circ\star}$ has support strictly inside the domain, we have the orthogonality properties

$$\text{div}_h^{\circ\star,\Diamond}(\mathbf{rot}_h^{\Diamond,\circ\star}\phi_h^{\circ\star}) = 0, \quad \text{curl}_h^{\circ\star,\Diamond}(\boldsymbol{\nabla}_h^{\Diamond,\circ\star}\phi_h^{\circ\star}) = 0$$

4.2. *Div-Curl problem and Hodge decomposition*

Thanks to these discrete operators, we can tackle the discrete Div-Curl problem and the discrete Hodge decomposition problem. We have for example the problem

$$\left\{\begin{array}{lcll} \mathrm{div}_h^{\circ\star,\diamondsuit}\mathbf{u}_h^{\diamondsuit} & = & f_h^{\circ\star} & \text{in } \Omega_h^{\circ\star} \\ \mathrm{curl}_h^{\circ\star,\diamondsuit}\mathbf{u}_h^{\diamondsuit} & = & g_h^{\circ\star} & \text{in } \Omega_h^{\circ\star} \\ \mathbf{u}_h^{\diamondsuit}\cdot\mathbf{n}_h^{\diamondsuit} & = & h_h^{\diamondsuit} & \text{on } \partial\Omega_h^{\diamondsuit} \end{array}\right.$$

Classically, by setting $\mathbf{u}_h^{\diamondsuit} = \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}\Phi_h^{\circ\star} + \mathbf{rot}_h^{\diamondsuit,\circ\star}\Psi_h^{\circ\star}$, we can relate this problem to the discrete Hodge decomposition

$$(\mathbb{R}_h^{\diamondsuit})^2 = \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}\left(\mathbb{R}_h^{\circ\star}/(\mathbb{R}\times\mathbb{R})\right) \oplus \mathbf{rot}_h^{\diamondsuit,\circ\star}(\mathbb{R}_h^{\circ\star})$$

This is possible thanks to the orthogonality properties mentionned above. The potential functions $\Phi_h^{\circ\star}$ and $\Psi_h^{\circ\star}$ are solutions to simple Laplace problems with Dirichlet or Neumann boundary conditions which constitute an equivalent formulation to the direct discretisation of the Div–Curl problem. This allows us to find error estimates using the previous results for the Laplace problem. These considerations will be particularly interesting when dealing with Maxwell equations.

5. Stokes Problem

Finally, the DDFV strategy applies to the Stokes problem (please see [5] for more details and numerical examples). A possible discretisation of the Stokes problem thanks to the DDFV method writes

$$\left\{\begin{array}{lclcl} -\mathrm{div}_h^{\diamondsuit,\circ\star}\boldsymbol{\nabla}_h^{\circ\star,\diamondsuit}\mathbf{u}_h^{\circ\star} & + & \boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}p_h^{\diamondsuit} & = & f_h^{\circ\star} \\ & & \mathrm{div}_h^{\diamondsuit,\circ\star}\mathbf{u}_h^{\circ\star} & = & 0 \end{array}\right. , \quad \gamma_{0h}^{\circ\star}\mathbf{u}_h^{\circ\star} = \mathbf{0} \tag{8}$$

Here, the discrete gradient operator $\boldsymbol{\nabla}_h^{\diamondsuit,\circ\star}$ and discrete divergence operator $\mathrm{div}_h^{\circ\star,\diamondsuit}$ associated to the constraint are the opertors introduced in the previous sections. As a consequence, the discrete divergence operator $\mathrm{div}_h^{\diamondsuit,\circ\star}$ and discrete gradient operator $\boldsymbol{\nabla}_h^{\circ\star,\diamondsuit}$ associated to the viscous term are new discrete operators that can be defined following the general DDFV strategy.

6. Conclusion

We have presented in this note a new framework for the derivation of finite volume schemes. On the one hand, we use a Discrete-Duality principle for the derivation of discrete operators. On the other hand, the unknowns are located on different meshes. This choice may be regarded as a generalisation of the staggered grid strategy to general unstructured meshes.

Numerical experiments show that the method is very stable, even for very distorted meshes [9, 10, 7]. The discrete-duality approach allows us to preserve at the discrete level the variational structure of mixed problems, and to perform the error analysis in this framework. The method has been applied to stiff Sotkes problems or Div-Curl problems.

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