A DISCRETE DUALITY FINITE VOLUME APPROACH TO HODGE DECOMPOSITION AND DIV-CURL PROBLEMS ON ALMOST ARBITRARY TWO-DIMENSIONAL MESHES*

SARAH DELCOURTE[†], KOMLA DOMELEVO[‡], AND PASCAL OMNES[§]

Abstract. We define discrete differential operators such as grad, div and curl, on general two-dimensional non-orthogonal meshes. These discrete operators verify discrete analogues of usual continuous theorems: discrete Green formulae, discrete Hodge decomposition of vector fields, vector curls have a vanishing divergence and gradients have a vanishing curl. We apply these ideas to discretize div-curl systems. We give error estimates based on the reformulation of these systems into equivalent equations for the potentials. Numerical results illustrate the use of the method on several types of meshes, among which degenerating triangular meshes and non-conforming locally refined meshes.

Key words. discrete duality finite volume method, discrete Green formula, discrete Hodge decomposition, discrete differential operators, div-curl equations, arbitrary meshes, non-conforming meshes, degenerating meshes, convergence, error estimates

AMS subject classifications. 35Q60, 65N12, 65N15, 65N30, 78A30

1. Introduction. Discretization schemes which are based on a discrete vector analysis satisfying discrete analogues of the usual continuous theorems lead to robust and efficient approximations of various physical models. Based on finite volume-like formulations, they provide discrete approximations of differential operators such as gradient, divergence and curl.

Such schemes were for example constructed by Hyman, Shashkov and co-workers, initially on logically rectangular grids. We refer to [13, 14] for the construction of the discrete operators and to [15] for the proof of a discrete Hodge decomposition. These schemes were then applied in several different circumstances (see e.g. [16, 17]) and extended to unstructured [5] or even non-conforming grids [19], although on that type of meshes, to our knowledge, no discrete Hodge decomposition has been proved.

Our interests in this paper are related to other schemes based on a discrete vector analysis which were proposed by Nicolaides and co-workers to solve fluid mechanics problems [7], div-curl problems [20, 12] or Maxwell equations [21]. In these works, these so-called covolume schemes are restricted to locally equiangular triangular meshes in the two-dimensional case. Given such a primal triangular mesh, a dual mesh is constructed by joining the circumcenters of adjacent triangles. Thus the edges of the primal and dual meshes are orthogonal. This property will be called in the following "the orthogonality property". The necessity for the mesh to verify this property might be in certain cases a severe restriction, in particular with respect to mesh adaptivity.

In [20], discrete field components are defined normal to the edges of the primal mesh and therefore, thanks to the orthogonality property, along the edges of the dual mesh. This single component is enough to permit the definition of a discrete

*

[†]Commissariat à l'Énergie Atomique de Saclay, DEN-DM2S-SFME, 91191 Gif-sur-Yvette Cedex, FRANCE (sarah.delcourte@cea.fr).

[‡]Mathématiques pour l'Industrie et la Physique, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France (komla@mip.ups-tlse.fr).

[§]Commissariat à l'Énergie Atomique de Saclay, DEN-DM2S-SFME, 91191 Gif-sur-Yvette Cedex, FRANCE (pascal.omnes@cea.fr).

divergence operator on the primal mesh and of a discrete curl operator on the dual mesh. Reciprocally, discrete analogues of the normal (with respect to the edges of the primal mesh) components of the gradients (respectively vector curls) are obtained over the edges with the help of scalar quantities defined at the circumcenters (resp. at the vertices) of the primal cells.

Due to the anisotropy of the media considered in [12], the authors are led to introduce both components of vector fields on the edges of the mesh, which allows them to define discrete divergence and curl operators on both the primal and dual meshes. Nevertheless, they keep on considering only the normal components of the discrete gradient and curl vectors, thus leaving the generalization of [20] incomplete.

In the present work, we extend the covolume ideas of Nicolaides to almost arbitrary two-dimensional meshes, including in particular non-conforming meshes. The only requirement on the mesh is that the dual cells (which are obtained in a different way, see below) form a partition of the domain of computation. These meshes do not necessarily verify the orthogonality property, and we therefore discretize vector fields by their two components over so-called diamond-cells which are quadrilaterals whose vertices are the extremities of primal and associated dual edges. Like in [12], these two field components enable us to define discrete divergence and curl operators both on the primal and dual meshes. Reciprocally, and in contrast to [12], both components of discrete gradient and vector curl operators are defined over the diamond-cells with the help of scalar quantities given on both the primal and dual cells. Together with the definition of appropriate discrete scalar products, we establish that these discrete operators verify discrete properties which are analog to those verified by their continuous counterparts: discrete Green formulae, discrete Hodge decomposition of vector fields, vector curls have a vanishing divergence and gradients have a vanishing curl. These results thus generalize those obtained in [12, 20], with the major novelty that they hold on a much wider class of meshes.

Because of the discrete Green formulae, finite volume schemes based on these ideas have been named "Discrete Duality Finite Volume" (DDFV) schemes in [9] and their use has started with the construction and analysis of a finite volume method for the Laplace equation on almost arbitrary two-dimensional meshes [10]. Then, these ideas have been applied to the discretization of non-linear elliptic equations [2], drift-diffusion and energy-transport models [6] and electro-cardiology problems [22].

In this article, we apply these ideas to the numerical solution of div-curl problems which occur for example in fluid dynamics, electro- and magnetostatics. Using the discrete Hodge decomposition of the discrete unknown vector field, this problem is recast into two discrete Laplace equations for the discrete potentials, just like in the continuous problem. Using results obtained in [10], we prove the convergence of the scheme provided the continuous potentials are smooth enough and under geometrical hypotheses related to the non-degeneracy of the diamond-cells.

This paper is organized as follows: in section 2, we explain the construction of the primal, dual and diamond meshes and we define our notations. In section 3 we construct the discrete differential operators, while section 4 is devoted to the proof of the properties of the discrete operators. Then, we apply these ideas in section 5 to discretize the div-curl problem and obtain error estimates. Several numerical experiments are reported in section 6 and conclusions are drawn in section 7.

2. Definitions and notations. Let Ω be a bounded polygon of $\mathbb{R}^2$, not necessarily simply connected, whose boundary is denoted by Γ. We suppose in addition that the domain has Q holes. Throughout the paper, we shall assume that $Q > 0$, but the results also hold for the case $Q = 0$.

Let Γ_0 denote the exterior boundary of Ω and let Γ_q, with $q \in [1, Q]$, be the interior polygonal boundaries of Ω, so that $\Gamma = \Gamma_0 \bigcup_{q\in[1,Q]} \Gamma_q$.

The domain Ω will be covered by three different meshes whose constructions are similar to those given in [10].

2.1. Construction of the primal mesh. We consider a first partition of Ω (named primal mesh) composed of elements T_i, with $i \in [1, I]$, supposed to be convex polygons. With each element T_i of the mesh is associated a node G_i located inside T_i. This point may be the barycentre of T_i, but this is not necessary. The area of T_i is denoted by $|T_i|$. We shall denote by J the total number of edges of this mesh. Note that in the case of a non-conforming mesh, an edge is any segment whose extremities are nodes of the mesh. We also denote by J^Γ the number of edges which are located on the boundary Γ and we associate with each of these boundary edges its midpoint, also denoted by G_i with $i \in [I + 1, I + J^\Gamma]$. By a slight abuse of notations, we shall write $i \in \Gamma_q$ iff $G_i \in \Gamma_q$.

2.2. Construction of the dual mesh. We denote by S_k, with $k \in [1, K]$, the nodes of the polygons of the primal mesh. To each of these points, we associate a polygon denoted by P_k, obtained by joining the points G_i associated to the elements of the primal mesh (and possibly to the boundary edges) of which S_k is a node. The area of P_k is denoted by $|P_k|$. We shall only consider in the following the cases where the P_ks constitute a second partition of Ω, which we name dual mesh[1]. Figure 2.1 displays an example of a non-conforming primal mesh and its associated dual mesh. Moreover, we suppose that the set $[1, K]$ is ordered so that when S_k is not on Γ, then $k \in [1, K - J^\Gamma]$, and when S_k is on Γ, then $k \in [K - J^\Gamma + 1, K]$. We shall also write $k \in \Gamma_q$ iff $S_k \in \Gamma_q$.

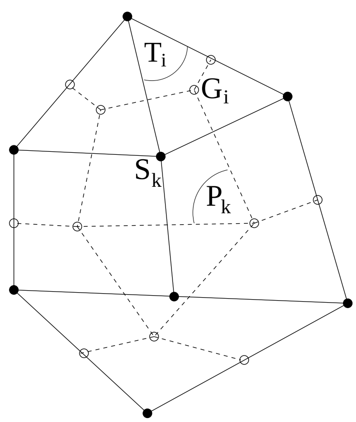

FIG. 2.1. *An example of a primal mesh and its associated dual mesh.*

2.3. Construction of the diamond mesh. With each edge of the primal mesh, denoted by A_j (whose length is $|A_j|$), with $j \in [1, J]$, we associate a quadrilateral named "diamond-cell" and denoted by D_j. When A_j is not on the boundary, this cell is obtained by joining the points $S_{k_1(j)}$ and $S_{k_2(j)}$, which are the two nodes of A_j, with the points $G_{i_1(j)}$ and $G_{i_2(j)}$ associated to the elements of the primal mesh which share

[1] It may happen that the P_ks overlap, as seen on figure 2 of reference [10]

this edge. When A_j is on the boundary Γ, the cell D_j is obtained by joining the two nodes of A_j with the point $G_{i_1(j)}$ associated to the only element of the primal mesh of which A_j is an edge and to the point $G_{i_2(j)}$ associated to A_j (*i.e.*, by convention, $i_2(j)$ is an element of $[I+1, I+J^\Gamma]$ when A_j is located on Γ). The cells D_j constitute a third partition of Ω, which we name "diamond-mesh". The area of the cell D_j is denoted by $|D_j|$. Such cells are displayed on figure 2.2.
Moreover, we suppose that the set $[1, J]$ is ordered so that when A_j is not on Γ, then $j \in [1, J - J^\Gamma]$, and when A_j is on Γ, then $j \in [J - J^\Gamma + 1, J]$. We shall also write $j \in \Gamma_q$ iff $A_j \subset \Gamma_q$.

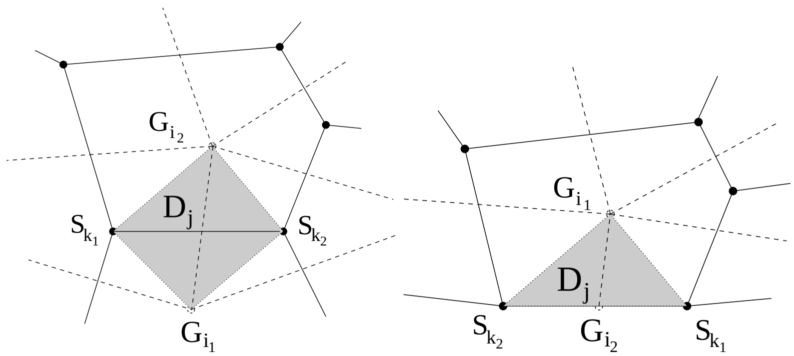

FIG. 2.2. *Examples of diamond-cells.*

2.4. Definitions of geometrical elements. The unit vector normal to A_j is denoted by $\mathbf{n}_j$ and is oriented so that $\mathbf{G_{i_1(j)}G_{i_2(j)}} \cdot \mathbf{n}_j \geq 0$. We further denote by A'_j the segment $[G_{i_1(j)}G_{i_2(j)}]$ (whose length is $|A'_j|$) and by $\mathbf{n}'_j$ the unit vector normal to A'_j oriented so that $\mathbf{S_{k_1(j)}S_{k_2(j)}} \cdot \mathbf{n}'_j \geq 0$.
When $S_k \in \Gamma$ ($k \in [K - J^\Gamma + 1, K]$), we define $\tilde{A}_k$ as the part of the boundary Γ which consists of the union of the halves of the two segments A_j located on Γ and of which S_k is a node, and by $\tilde{\mathbf{n}}_k$ the exterior unit normal vector to $\tilde{A}_k$ (see figure 2.3). We denote by $M_{i_\alpha(j)\,k_\beta(j)}$ the midpoint of the segment $[G_{i_\alpha(j)}S_{k_\beta(j)}]$, for each pair of

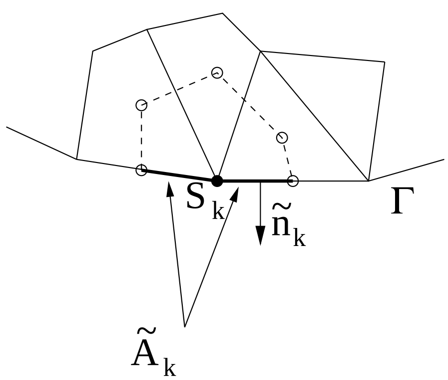

FIG. 2.3. *Definition of $\tilde{A}_k$ and $\tilde{\mathbf{n}}_k$ for the boundary nodes*

integers (α, β) in $\{1; 2\}^2$ (see figure 2.4). We define for each $i \in [1, I]$ the set $\mathcal{V}(i)$ of integers $j \in [1, J]$ such that A_j is an edge of T_i and for each $k \in [1, K]$ the set $\mathcal{E}(k)$ of

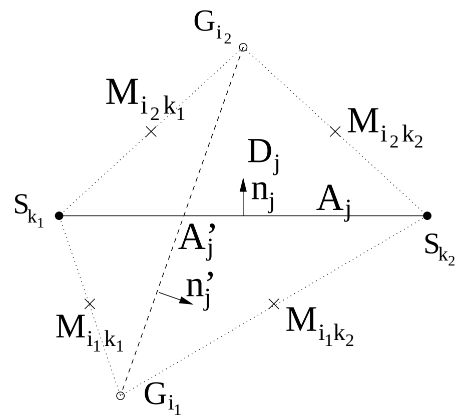

FIG. 2.4. *Notations for the diamond cell.*

integers $j \in [1, J]$ such that S_k is a node of A_j.
We define for each $j \in [1, J]$ and each k such that $j \in \mathcal{E}(k)$ (resp. each i such that $j \in \mathcal{V}(i)$) the real-valued number s'_{jk} (resp. s_{ji}) whose value is $+1$ or -1 whether $\mathbf{n}'_j$ (resp. $\mathbf{n}_j$) points outwards or inwards P_k (resp. T_i). We define $\mathbf{n}'_{jk} := s'_{jk}\mathbf{n}'_j$ (resp. $\mathbf{n}_{ji} := s_{ji}\mathbf{n}_j$) and remark that $\mathbf{n}'_{jk}$ (resp. $\mathbf{n}_{ji}$) always points outwards P_k (resp. T_i).

For $j \in [1, J - J^\Gamma]$, as indicated on figure 2.5, we also denote by $D_{j,1}$ and $D_{j,2}$, the triangles $S_{k_1(j)}G_{i_1(j)}S_{k_2(j)}$ and $S_{k_2(j)}G_{i_2(j)}S_{k_1(j)}$). In the same way, we denote by $D'_{j,1}$ and $D'_{j,2}$, the triangles $G_{i_2(j)}S_{k_1(j)}G_{i_1(j)}$ and $G_{i_1(j)}S_{k_2(j)}G_{i_2(j)}$.

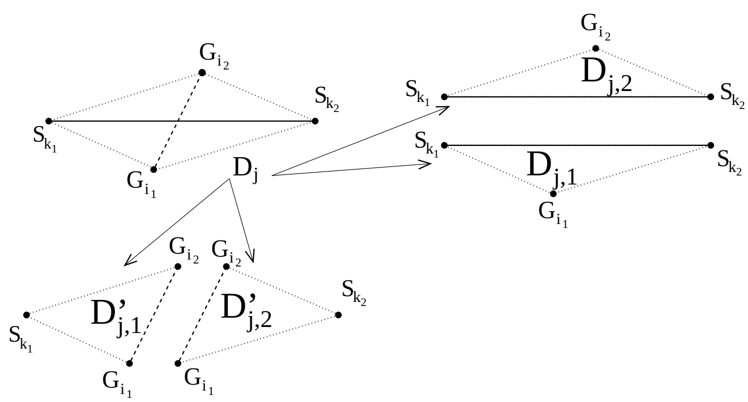

FIG. 2.5. *A diamond-cell may be split into two triangles in two distinct ways.*

The characteristic functions of the cells T_i and P_k will be denoted by θ_i^T and θ_k^P.

2.5. Definitions of discrete and continuous scalar products and norms. As will be seen in the following, we shall associate with each point G_i ($i \in [1, I + J^\Gamma]$) and each vertex S_k ($k \in [1, K]$) discrete values. This leads us to the definition of the following discrete scalar product for all $(\phi, \psi) = \big((\phi_i^T, \phi_k^P), (\psi_i^T, \psi_k^P)\big) \in \big(\mathbb{R}^I \times \mathbb{R}^K\big)^2$

$$(\phi, \psi)_{T,P} := \frac{1}{2}\left(\sum_{i\in[1,I]} |T_i|\, \phi_i^T\, \psi_i^T + \sum_{k\in[1,K]} |P_k|\, \phi_k^P\, \psi_k^P\right) . \qquad (2.1)$$

In the same way, we define a discrete scalar product on the diamond mesh for all $(\mathbf{u},\mathbf{v}) = ((\mathbf{u}_j),(\mathbf{v}_j)) \in \left(\mathbb{R}^2\right)^J \times \left(\mathbb{R}^2\right)^J$

$$(\mathbf{u},\mathbf{v})_D := \sum_{j\in[1,J]} |D_j|\,\mathbf{u}_j\cdot\mathbf{v}_j \tag{2.2}$$

and a discrete scalar product for the traces of $u \in \mathbb{R}^J$ and $\phi \in \mathbb{R}^{I+J^\Gamma}\times\mathbb{R}^K$ on the boundaries Γ_q

$$(u,\phi)_{\Gamma_q,h} := \sum_{j\in\Gamma_q} |A_j|\,u_j \times \frac{1}{4}\left(\phi^P_{k_1(j)} + 2\phi^T_{i_2(j)} + \phi^P_{k_2(j)}\right)$$

and on Γ

$$(u,\phi)_{\Gamma,h} := \sum_{q\in[0,Q]} (u,\phi)_{\Gamma_q,h}\ . \tag{2.3}$$

Further, for any $\phi \in \mathbb{R}^{I+J^\Gamma}\times\mathbb{R}^K$, we define a discrete H^1 semi-norm on the diamond mesh with the help of the discrete gradient operator to be defined below (see Eq. (3.2)):

$$|\phi|_{1,D} := \left(\boldsymbol{\nabla}^D_h\phi, \boldsymbol{\nabla}^D_h\phi\right)^{1/2}_D\ .$$

Finally, H^m is the space of functions v of $\mathrm{L}^2(\Omega)$ whose partial derivatives (in the distributional sense) $\partial^\alpha v$, with $|\alpha| \le m$ all belong to $\mathrm{L}^2(\Omega)$, while $||\cdot||_{m,\Omega}$ is the associated norm. The standard $\mathrm{L}^2(\Omega)$ inner product will be denoted by $(\cdot,\cdot)_\Omega$.

3. Construction of the discrete operators. In this section, we approach the gradient, divergence and curl operators by discrete counterparts. We would like to stress that in two dimensions, a distinction is usually made between the vector curl operator from $\mathbb{R}$ to $\mathbb{R}^2$, defined by $\boldsymbol{\nabla}\times\phi = \left(\frac{\partial\phi}{\partial y}\,,\,-\frac{\partial\phi}{\partial x}\right)^T$ and the scalar curl operator from $\mathbb{R}^2$ to $\mathbb{R}$, defined by $\nabla\times\mathbf{u} = \frac{\partial\mathbf{u}_y}{\partial x} - \frac{\partial\mathbf{u}_x}{\partial y}$.
Figure 3.1 shows the stencils of the different operators and of their combinations: The stencil for the discrete gradient and vector curl operators simply consists of the four corners of the diamond-cell D_j. The stencil for the discrete divergence and scalar curl operators consists of the diamonds associated to the edges of the primal and dual cells. Arrows are displayed on Fig. 3.1 to represent the normal and tengential components of the vector fields associated to the diamonds. The stencils for the discrete laplacian on the primal and dual cells respectively consist of the black and white circles on the left part and on the right part of the figure.

3.1. Construction of the discrete gradient and vector curl operators on the diamond cells. We define the discrete gradient of a function ϕ by its values on the diamond-cells of the mesh. We follow [8, 10] and compute the mean-value of the gradient of any function ϕ on such a cell D_j by the following formula:

$$|D_j|\,\left\langle\boldsymbol{\nabla}\phi_{|D_j}\right\rangle = \int_{D_j}\boldsymbol{\nabla}\phi(\mathbf{x})\,d\mathbf{x} = \int_{\partial D_j}\phi(\xi)\mathbf{n}(\xi)\,d\xi = \sum_{(\alpha,\beta)}\int_{[G_{i_\alpha}S_{k_\beta}]}\phi(\xi)\,\mathbf{n}\,d\xi, \tag{3.1}$$

where $\mathbf{n}(\xi)$ stands for the outward unit normal vector to D_j at point ξ. The integrals in (3.1) can be approximated by the following formula:

$$\int_{[GS]}\phi(\xi)\,d\xi \approx \ell_{GS}\ \frac{[\phi(G)+\phi(S)]}{2}\ ,$$

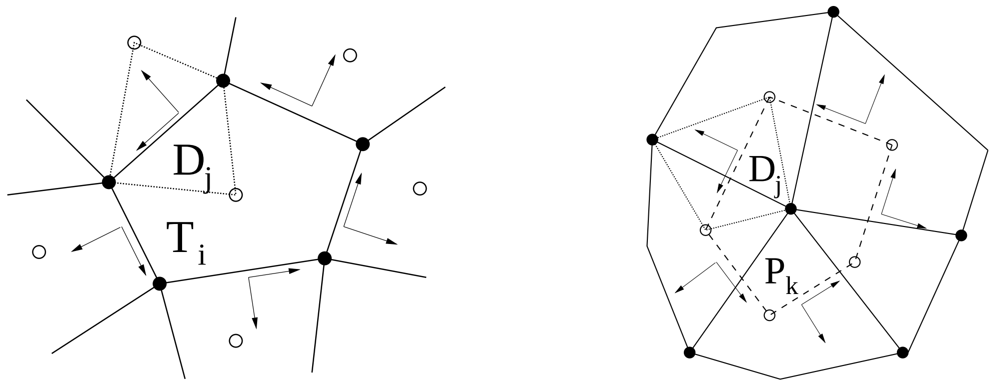

FIG. 3.1. *Stencils for the discrete operators. Left part: primal cell. Right part: dual cell.*

where ℓ_{GS} denotes the length of the segment $[GS]$. Summing the contributions of the different vertices of D_j and using elementary geometrical equalities allows us to give *the definition* of the discrete gradient $\boldsymbol{\nabla}_h^D$ on D_j.

DEFINITION 3.1. *The discrete gradient $\boldsymbol{\nabla}_h^D$ is defined by its values over the diamond-cells D_j:*

$$(\boldsymbol{\nabla}_h^D \phi)_j := \frac{1}{2\,|D_j|}\left\{\left[\phi_{k_2}^P - \phi_{k_1}^P\right]|A'_j|\mathbf{n}'_j + \left[\phi_{i_2}^T - \phi_{i_1}^T\right]|A_j|\mathbf{n}_j\right\}, \tag{3.2}$$

where we set $\phi_{k_\alpha}^P := \phi(S_{k_\alpha})$ and $\phi_{i_\alpha}^T := \phi(G_{i_\alpha})$, for $\alpha \in \{1;2\}$. Note that formula (3.2) is exact for polynomials of degree one. Computing the discrete gradient only requires the values of ϕ at the nodes of the primal and dual meshes. The operator $\boldsymbol{\nabla}_h^D$ thus acts from $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ into $\left(\mathbb{R}^2\right)^J$.

In the same way, we may approach the vector curl operator $\boldsymbol{\nabla} \times \bullet = \left(\frac{\partial \bullet}{\partial y}, -\frac{\partial \bullet}{\partial x}\right)^T$ by a discrete vector curl operator:

DEFINITION 3.2. *The discrete vector curl operator $\boldsymbol{\nabla}_h^D\times$ is defined by its values over the diamond-cells D_j:*

$$(\boldsymbol{\nabla}_h^D \times \phi)_j := -\frac{1}{2\,|D_j|}\left\{\left[\phi_{k_2}^P - \phi_{k_1}^P\right]|A'_j|\boldsymbol{\tau}'_j + \left[\phi_{i_2}^T - \phi_{i_1}^T\right]|A_j|\boldsymbol{\tau}_j\right\}, \tag{3.3}$$

where the unit vectors $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}'_j$ are such that $(\mathbf{n}_j, \boldsymbol{\tau}_j)$ and $(\mathbf{n}'_j, \boldsymbol{\tau}'_j)$ are orthonormal positively oriented bases of $\mathbb{R}^2$.

REMARK 3.3. *In a connected domain, the discrete gradient and vector curl of a given $\phi = ((\phi_i^T), (\phi_k^P))$ vanish if and only if there exist two constants c^T and c^P, such that $\phi_i^T = c^T$ for all i and $\phi_k^P = c^P$ for all k. The fact that c^T and c^P may differ one from the other means that such a ϕ may in general present oscillations. However, in the applications studied in the present work, such oscillations never appear due to information on the mean-value of ϕ (Eq. (4.16) and (5.7d) below), or due to boundary conditions (Eq. (4.17) and (5.8e)).*

3.2. Construction of the discrete divergence and scalar curl operators on the primal and dual meshes. Next, we choose to define the discrete divergence of a vector field $\mathbf{u}$ by its values both on the primal and dual cells of the mesh. A very natural way to do so on the primal cell T_i is to write

$$|T_i|\left\langle \nabla\cdot\mathbf{u}_{|T_i}\right\rangle = \int_{T_i}\nabla\cdot\mathbf{u}(\mathbf{x})\,d\mathbf{x} = \int_{\partial T_i}\mathbf{u}(\xi)\cdot\mathbf{n}(\xi) = \sum_{j\in\mathcal{V}(i)}\int_{A_j}\mathbf{u}(\xi)\cdot\mathbf{n}_{ji}\,,$$

where we recall that $\mathcal{V}(i)$ is the set of integers $j \in [1, J]$ such that A_j is an edge of T_i and that $\mathbf{n}_{ji}$ is the unit vector orthogonal to A_j pointing outward T_i. Supposing that the vector field $\mathbf{u}$ is given by both of the Cartesian components of its discrete values $\mathbf{u}_j$ on the diamond cells D_j, and performing a similar computation over the cells P_k, we obtain *the definition* of the discrete divergence $\nabla_h^T\cdot$ on each T_i and the discrete divergence $\nabla_h^P\cdot$ on each P_k.

Definition 3.4. *The discrete divergence $\nabla_h^{T,P}\cdot := (\nabla_h^T\cdot, \nabla_h^P\cdot)$ is defined by its values over the primal cells T_i and the dual cells P_k:*

$$(3.4)\quad \begin{aligned}(\nabla_h^T\cdot\mathbf{u})_i &:= \frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)}|A_j|\mathbf{u}_j\cdot\mathbf{n}_{ji}\\ (\nabla_h^P\cdot\mathbf{u})_k &:= \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)}|A'_j|\mathbf{u}_j\cdot\mathbf{n}'_{jk} + \sum_{j\in\mathcal{E}(k)\cap[J-J^\Gamma+1,J]}\frac{1}{2}\,|A_j|\mathbf{u}_j\cdot\mathbf{n}_j\right)\,.\end{aligned}$$

Remark that if the node S_k is not on the boundary Γ (*i.e.* if $k \in [1, K - J^\Gamma]$), then the set $\mathcal{E}(k)\cap[J - J^\Gamma + 1, J]$ is empty. On the contrary, if P_k is a boundary dual cell, then the set $\mathcal{E}(k)\cap[J - J^\Gamma + 1, J]$ is composed of the two boundary edges which have S_k as a vertex. In this case, the quantity $\displaystyle\sum_{j\in\mathcal{E}(k)\cap[J-J^\Gamma+1,J]}\frac{1}{2}\,|A_j|\mathbf{u}_j\cdot\mathbf{n}_j$ is an approximation of $\displaystyle\int_{\tilde{A}_k}\mathbf{u}\cdot\tilde{\mathbf{n}}_k(\xi)\,d\xi$ (see figure 2.3).

For a given vector field $\mathbf{u}$, it is easily checked that these formulae are the exact mean-values of $\nabla\cdot\mathbf{u}$ over the primal and the inner dual cells if $\mathbf{u}_j\cdot\mathbf{n}_{ji}$ and $\mathbf{u}_j\cdot\mathbf{n}'_{jk}$ represent the mean-values of $\mathbf{u}\cdot\mathbf{n}_{ji}$ over A_j and of $\mathbf{u}\cdot\mathbf{n}'_{jk}$ over A'_j. The operator $\nabla_h\cdot$ acts from $\left(\mathbb{R}^2\right)^J$ into $\mathbb{R}^I\times\mathbb{R}^K$.

In the same way, we may approach the scalar curl operator $\nabla\times\bullet = \left(\frac{\partial\bullet_y}{\partial x} - \frac{\partial\bullet_x}{\partial y}\right)$ by a discrete scalar curl operator:

Definition 3.5. *The discrete scalar curl operator $\nabla_h^{T,P}\times := (\nabla_h^T\times, \nabla_h^P\times)$ is defined by its values over the primal cells T_i and the dual cells P_k:*

$$(3.5)\quad \begin{aligned}(\nabla_h^T\times\mathbf{u})_i &:= \frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)}|A_j|\mathbf{u}_j\cdot\boldsymbol{\tau}_{ji}\\ (\nabla_h^P\times\mathbf{u})_k &:= \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)}|A'_j|\mathbf{u}_j\cdot\boldsymbol{\tau}'_{jk} + \sum_{j\in\mathcal{E}(k)\cap[J-J^\Gamma+1,J]}\frac{1}{2}\,|A_j|\mathbf{u}_j\cdot\boldsymbol{\tau}_j\right)\,.\end{aligned}$$

4. Properties of the operators.

4.1. Discrete Green formulae. Here, we check that the discrete operators verify some discrete duality principles.

Proposition 4.1. *The following discrete analogues of the Green formulae hold:*

$$(4.1)\quad (\nabla_h^{T,P}\cdot\mathbf{u},\phi)_{T,P} = -(\mathbf{u},\boldsymbol{\nabla}_h^D\phi)_D + (\mathbf{u}\cdot\mathbf{n},\phi)_{\Gamma,h}\,,$$

$$(\nabla_h^{T,P} \times \mathbf{u}, \phi)_{T,P} = (\mathbf{u}, \boldsymbol{\nabla}_h^D \times \phi)_D + (\mathbf{u} \cdot \boldsymbol{\tau}, \phi)_{\Gamma,h} \ , \tag{4.2}$$

for all $\mathbf{u} \in \left(\mathbb{R}^2\right)^J$ *and all* $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, *where the definitions (2.1), (2.2) and (2.3) have been used.*

Proof. The proof of (4.1) may be found in [10] and is based on a discrete summation by parts. The proof of (4.2) follows exactly the same lines. ☐

4.2. Compositions of the discrete operators. The aim of this section is to verify a discrete analogue of the following continuous identities: $\nabla \cdot (\boldsymbol{\nabla}\times) = 0$, $\nabla \times \boldsymbol{\nabla} = 0$ and $\nabla \times \boldsymbol{\nabla}\times = -\nabla \cdot \boldsymbol{\nabla}$. For this, we start with a useful lemma.

Lemma 4.2. *Recall that* s_{ji} *and* s'_{jk} *are defined in section 2.4. Then,*

$$\sum_{j \in \mathcal{V}(i)} s_{ji} \left(\phi^P_{k_2(j)} - \phi^P_{k_1(j)}\right) = 0, \ \ \forall i \in [1, I] \ , \tag{4.3}$$

$$\sum_{j \in \mathcal{E}(k)} s'_{jk} \left(\phi^T_{i_2(j)} - \phi^T_{i_1(j)}\right) = 0, \ \ \forall \, k \in [1, K - J^\Gamma] \ . \tag{4.4}$$

Proof. Let us consider a given primal cell T_i. For each edge A_j of T_i, with $j \in \mathcal{V}(i)$, there are two possibilities for the orientation of $\mathbf{n}_j$ (see figure 4.1): If $\mathbf{n}_j$ is the inward unit normal vector to T_i (case 1), then $s_{ji} = -1$ and $s_{ji}\,(\phi^P_{k_2(j)} - \phi^P_{k_1(j)}) = \phi^P_{k_1(j)} - \phi^P_{k_2(j)}$. If $\mathbf{n}_j$ is the outward unit normal vector to T_i (case 2), then $s_{ji} = 1$ and $s_{ji}\,(\phi^P_{k_2(j)} - \phi^P_{k_1(j)}) = \phi^P_{k_2(j)} - \phi^P_{k_1(j)}$; moreover $S_{k_1}(j)$ and $S_{k_2}(j)$ are swapped. What appears finally is that, whatever the case, the value ϕ^P_k associated to the "left" vertex of the considered edge A_j appears in the sum (4.3) with a positive sign and the value ϕ^P_k associated to the "right" vertex of the considered edge A_j appears in the sum (4.3) with a negative sign. But each ϕ^P_k appears twice in that sum, once as the value associated to the "right" vertex of a given edge, and once as the value associated to the "left" vertex of the following edge, so that these two contributions cancel. This ends the proof of (4.3). The proof of (4.4) follows the same lines. ☐

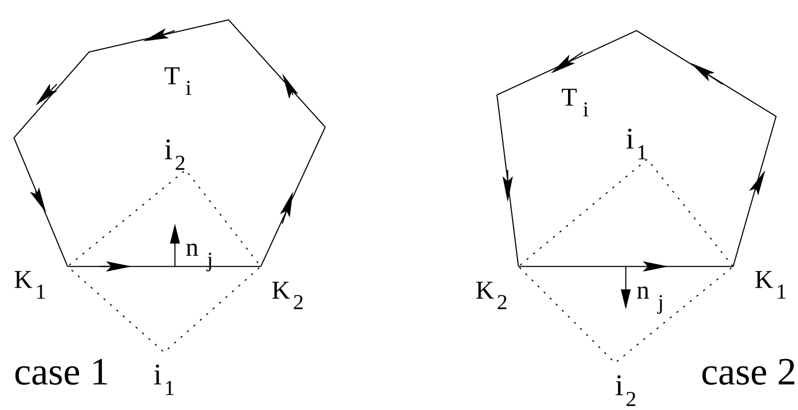

Fig. 4.1. *Two possibilities of orientation for each edge*

Next, the following properties are direct consequences of the computation of the area $|D_j|$:

LEMMA 4.3.

$$\frac{|A_j|\ |A'_j|}{2\,|D_j|}\ \mathbf{n}_j\cdot\boldsymbol{\tau}'_j=1,\ \ \forall j\in[1,J]\,, \tag{4.5}$$

$$\frac{|A_j|\ |A'_j|}{2\,|D_j|}\ \mathbf{n}'_j\cdot\boldsymbol{\tau}_j=-1,\ \ \forall j\in[1,J] \tag{4.6}$$

We may now state the following results

PROPOSITION 4.4. *Given any* $\phi=(\phi_i^T,\phi_k^P)\in\mathbb{R}^{I+J^\Gamma}\times\mathbb{R}^K$, *there holds*

$$\Big(\nabla_h^T\cdot(\boldsymbol{\nabla}_h^D\times\phi)\Big)_i=0,\forall i\in[1,I]\,, \tag{4.7}$$

$$\Big(\nabla_h^P\cdot(\boldsymbol{\nabla}_h^D\times\phi)\Big)_k=0,\forall k\in[1,K-J^\Gamma]\,, \tag{4.8}$$

$$\Big(\nabla_h^T\times(\boldsymbol{\nabla}_h^D\phi)\Big)_i=0,\forall i\in[1,I]\,, \tag{4.9}$$

$$\Big(\nabla_h^P\times(\boldsymbol{\nabla}_h^D\phi)\Big)_k=0,\forall k\in[1,K-J^\Gamma]\,. \tag{4.10}$$

Moreover, on each boundary dual cell P_k ($k\in[K-J^\Gamma+1,K]$)*, (4.8) and (4.10) still hold if there exist for each boundary* Γ_q*, with* $q\in[0,Q]$*, two real numbers* (c_q^T,c_q^P) *such that* $\phi_i^T=c_q^T$ *and* $\phi_k^P=c_q^P$ *uniformly over* Γ_q*.*

Proof. Let us first prove (4.7); combining (3.4), (3.3), and the fact that $\mathbf{n}_{ji}\cdot\boldsymbol{\tau}_j=0$, we get:

$$\begin{aligned}(\nabla_h^T\cdot(\boldsymbol{\nabla}_h^D\times\phi))_i&=\frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)}|A_j|(\boldsymbol{\nabla}_h^D\times\phi)_j\cdot\mathbf{n}_{ji}\\&=-\frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)}\frac{|A_j|\,|A'_j|}{2\,|D_j|}\mathbf{n}_j\cdot\boldsymbol{\tau}'_j\,s_{ji}\left(\phi^P_{k_2(j)}-\phi^P_{k_1(j)}\right),\quad\forall i\in[1,I].\end{aligned}$$

Applying (4.5) and (4.3) successively, we obtain:

$$(\nabla_h^T\cdot(\boldsymbol{\nabla}_h^D\times\phi))_i=0,\quad\forall i\in[1,I].$$

Eq. (4.9) can be proved in a similar way.

Next, for each interior dual cell P_k, with $k\in[1,K-J^\Gamma]$, the set $\mathcal{E}(k)\cap[J-J^\Gamma+1,J]$ is empty, so that (4.8) and (4.10) can be proved like (4.7) and (4.9), using (4.6), (4.4) and the fact that $\mathbf{n}'_{jk}\cdot\boldsymbol{\tau}'_j=0$.

As far as the boundary dual cells P_k are concerned ($k\in[K-J^\Gamma+1,K]$), similar computations show that (see Fig. 4.2 for the notations):

$$(\nabla_h^P\cdot(\boldsymbol{\nabla}_h^D\times\phi))_k=\frac{1}{|P_k|}\left(\phi^T_{I_2}-\phi^T_{I_1}\right)+\frac{1}{2\,|P_k|}\left(\phi^P_{K_1}-\phi^P_{K_2}\right)\,. \tag{4.11}$$

If all ϕ_i^T are equal to the same constant c_q^T over Γ_q and if all ϕ_k^P are equal to the same constant c_q^P over Γ_q, then $\phi^T_{I_2}=\phi^T_{I_1}$ and $\phi^P_{K_1}=\phi^P_{K_2}$ so that

$$(\nabla_h^P\cdot(\boldsymbol{\nabla}_h^D\times\phi))_k=0,$$

for the boundary dual cells, and (4.10) for the boundary dual cells is proved in a similar way. ∎

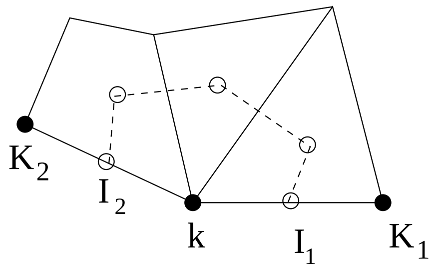

FIG. 4.2. *Notations for the boundary dual cells*

PROPOSITION 4.5. *The following equalities hold*

(4.12)
$$(\nabla_h^T \times \boldsymbol{\nabla}_h^D \times \phi)_i = -(\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \phi)_i, \quad \forall i \in [1, I]$$
$$(\nabla_h^P \times \boldsymbol{\nabla}_h^D \times \phi)_k = -(\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \phi)_k, \quad \forall k \in [1, K];.$$

Proof. These formulae follow immediately from the definitions (3.2), (3.3), (3.4) and (3.5) and from the equality $\boldsymbol{\tau}_j \cdot \boldsymbol{\tau}'_j = \mathbf{n}_j \cdot \mathbf{n}'_j, \quad \forall j \in [1, J]$. □

4.3. Hodge's decomposition. In the continuous case, the Hodge decomposition for non simply connected domains reads:

(4.13)
$$(L^2)^2 = \boldsymbol{\nabla} V \overset{\perp}{\oplus} \boldsymbol{\nabla} \times W \ ,$$

with $V = \{\phi \in H^1 : \int_\Omega \phi = 0\}$ and $W = \{\psi \in H^1 : \psi_{|\Gamma_0} = 0, \quad \psi_{|\Gamma_q} = c_q, \forall q \in [1, Q]\}$. To prove an analogous property in the discrete case, we rely on the following result:

LEMMA 4.6 (Euler's Formula). *For a non simply connected bidimensional domain covered by a mesh with I elements, K vertices, J edges and Q holes, there holds:*

(4.14)
$$I + K = J + 1 - Q \ .$$

We may now state the following discrete Hodge decomposition:

THEOREM 4.7. *Let $(\mathbf{u}_j)_{j\in[1,J]}$ be a discrete vector field defined by its values on the diamond-cells D_j. There exist unique $\phi = (\phi_i^T, \phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$, $\psi = (\psi_i^T, \psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ and $(c_q^T, c_q^P)_{q\in[1,Q]}$ such that:*

(4.15)
$$\mathbf{u}_j = (\boldsymbol{\nabla}_h^D \phi)_j + (\boldsymbol{\nabla}_h^D \times \psi)_j, \quad \forall j \in [1, J] \ ,$$

(4.16)
$$\sum_{i\in[1,I]} |T_i| \phi_i^T = \sum_{k\in[1,K]} |P_k| \phi_k^P = 0 \ ,$$

(4.17)
$$\psi_i^T = 0 \ , \ \forall i \in \Gamma_0 \quad , \quad \psi_k^P = 0 \ , \ \forall k \in \Gamma_0 \ ,$$

and

(4.18)
$$\forall q \in [1, Q] \ , \quad \psi_i^T = c_q^T \ , \ \forall i \in \Gamma_q \quad , \quad \psi_k^P = c_q^P \ , \ \forall k \in \Gamma_q \ .$$

Moreover, the decomposition (4.15) is orthogonal.

Proof. There are $2(I+K+J^\Gamma)+2Q$ unknowns corresponding to (ϕ_i^T,ϕ_k^P) and (ψ_i^T,ψ_k^P) and to the constants (c_q^T,c_q^P). On the other hand, $2J$ equations are given by (4.15), while (4.17) and (4.18) provide with $2J^\Gamma$ constraints. Finally, (4.16) gives two supplementary equalities, so that the total number of equations is $2J+2+2J^\Gamma$. Consequently, according to (4.14), there are as many equations as unknowns. Therefore, existence and uniqueness of the decomposition are equivalent, and we shall prove uniqueness through injectivity.

Proving the orthogonality of $(\boldsymbol{\nabla}_h^D\phi)$ and $(\boldsymbol{\nabla}_h^D\times\psi)$ for any (ϕ,ψ) verifying (4.17) and (4.18) amounts to showing $(\boldsymbol{\nabla}_h^D\times\psi,\boldsymbol{\nabla}_h^D\phi)_D=0$. Thanks to (4.1), there holds

$$(\boldsymbol{\nabla}_h^D\times\psi,\boldsymbol{\nabla}_h^D\phi)_D=-(\nabla_h^{T,P}\cdot\boldsymbol{\nabla}_h^D\times\psi,\phi)_{T,P}+(\boldsymbol{\nabla}_h^D\times\psi\cdot\mathbf{n},\phi)_{\Gamma,h}\ .$$

Next, thanks to Prop. 4.4, $\nabla_h^{T,P}\cdot\boldsymbol{\nabla}_h^D\times\psi$ vanishes on all primal and inner dual cells. Because ψ verifies (4.17) and (4.18), we infer from Prop. 4.4, that $\nabla_h^{T,P}\cdot\boldsymbol{\nabla}_h^D\times\psi$ also vanishes on the boundary dual cells. Finally, according to (3.3), we have

$$(\boldsymbol{\nabla}_h^D\times\psi)_j\cdot\mathbf{n}_j=-\frac{1}{2\,|D_j|}\left(\psi_{k_2}^P-\psi_{k_1}^P\right)|A_j^{'}|\boldsymbol{\tau}_j^{'}\cdot\mathbf{n}_j\ ,$$

which also vanishes on the boundary because of (4.17) and (4.18). Thus, orthogonality is proved. In order to prove injectivity, we suppose $\mathbf{u}_j=0$, $\forall j\in[1,J]$:

$$0=(\boldsymbol{\nabla}_h^D\phi)_j+(\boldsymbol{\nabla}_h^D\times\psi)_j\ ,\ \forall j\in[1,J]\ . \tag{4.19}$$

We carry out the scalar product of (4.19) with $|D_j|\,(\boldsymbol{\nabla}_h^D\phi)_j$ and sum over $j\in[1,J]$:

$$0=(\boldsymbol{\nabla}_h^D\phi,\boldsymbol{\nabla}_h^D\phi)_D+(\boldsymbol{\nabla}_h^D\times\psi,\boldsymbol{\nabla}_h^D\phi)_D\ . \tag{4.20}$$

Thanks to the orthogonality proved above, Eq. (4.20) implies that $(\boldsymbol{\nabla}_h^D\phi,\boldsymbol{\nabla}_h^D\phi)_D=\sum_{j\in[1,J]}|D_j||(\boldsymbol{\nabla}_h^D\phi)_j|^2=0$, so that $(\boldsymbol{\nabla}_h^D\phi)_j=0$ for all j. Since the domain is connected, there exist two real constants α and β such that $\phi_k^P=\alpha$, $\forall k\in[1,K]$ and $\phi_i^T=\beta$, $\forall i\in[1,I+J^\Gamma]$. Equation (4.16) implies that these two constants vanish, so that

$$\phi_i^T=0,\ \ \forall i\in[1,I+J^\Gamma]\ \text{ and }\ \phi_k^P=0,\ \ \forall k\in[1,K]\ .$$

Consequently, (4.19) is equivalent to $(\boldsymbol{\nabla}_h^D\times\psi)_j=0$, $\forall j\in[1,J]$. Since the domain is connected, there exist two real constants α and β such as $\psi_k^P=\alpha$, $\forall\,k\in[1,K]$ and $\psi_i^T=\beta$, $\forall\,i\in[1,I+J^\Gamma]$. As $\psi=0$ over Γ_0 these two constants vanish and

$$\psi_i^T=0,\ \ \forall i\in[1,I+J^\Gamma]\ \text{ and }\ \psi_k^P=0,\ \ \forall k\in[1,K]\ .$$

□

REMARK 4.8. *Formulae (4.16) are discrete analogues (respectively stated on the primal mesh and on the dual mesh) of the condition* $\int_\Omega\phi=0$ *that appears in the definition of the space* V *in (4.13), while formulae (4.17) and (4.18) are discrete analogues of the boundary conditions that appear in the definition of* W*.*

5. Numerical solution of the div-curl problem for non simply connected domains.

5.1. Discretization of the div-curl problem with normal boundary conditions. We are interested in the approximation of the following continuous problem: given f, g, σ, $(k_q)_{q\in[1,Q]}$, find $\mathbf{u}$ such that

$$\left\{\begin{array}{l} \nabla\cdot\mathbf{u}=f \ \text{ in } \ \Omega\ ,\\ \nabla\times\mathbf{u}=g \ \text{ in } \ \Omega\ ,\\ \mathbf{u}\cdot\mathbf{n}=\sigma \ \text{ on } \ \Gamma\ ,\\ \int_{\Gamma_q}\mathbf{u}\cdot\boldsymbol{\tau}=k_q,\quad \forall q\in[1,Q]\ . \end{array}\right. \tag{5.1}$$

A necessary condition for the existence of a solution to (5.1) is given by the formula:

$$\int_\Omega f(\mathbf{x})d\mathbf{x}=\int_\Gamma \sigma(\xi)\ d\xi\ . \tag{5.2}$$

We discretize the solution of this problem by a vector field $(\mathbf{u}_j)_{j\in[1,J]}$ defined by its values over the diamond-cells of the mesh. Using the discrete differential operators defined in section 3, and following [12], we write the following discrete equations:

$$\left(\nabla_h^T\cdot\mathbf{u}\right)_i=f_i^T,\quad \forall i\in[1,I]\ , \tag{5.3a}$$
$$\left(\nabla_h^P\cdot\mathbf{u}\right)_k=f_k^P,\quad \forall k\in[1,K]\ , \tag{5.3b}$$
$$\left(\nabla_h^T\times\mathbf{u}\right)_i=g_i^T,\quad \forall i\in[1,I]\ , \tag{5.3c}$$
$$\left(\nabla_h^P\times\mathbf{u}\right)_k=g_k^P,\quad \forall k\in[1,K-J^\Gamma]\ , \tag{5.3d}$$
$$\mathbf{u}_j\cdot\mathbf{n}_j=\sigma_j,\quad \forall j\in[J-J^\Gamma+1,J]\ , \tag{5.3e}$$
$$(\mathbf{u}\cdot\boldsymbol{\tau},1)_{\Gamma_q,h}=k_q,\quad \forall q\in[1,Q]\ , \tag{5.3f}$$
$$\sum_{k\in\Gamma_q}|P_k|\ \ (\nabla_h^P\times\mathbf{u})_k=\sum_{k\in\Gamma_q}|P_k|\ \ g_k^P,\quad \forall q\in[1,Q]\ , \tag{5.3g}$$

where the following definitions have been used

$$f_i^T=\frac{1}{|T_i|}\int_{T_i}f(\mathbf{x})\,d\mathbf{x}\quad \forall i\in[1,I],\ \ f_k^P=\frac{1}{|P_k|}\int_{P_k}f(\mathbf{x})\,d\mathbf{x}\quad \forall k\in[1,K] \tag{5.4}$$
$$g_i^T=\frac{1}{|T_i|}\int_{T_i}g(\mathbf{x})\,d\mathbf{x}\quad \forall i\in[1,I],\ \ g_k^P=\frac{1}{|P_k|}\int_{P_k}g(\mathbf{x})\,d\mathbf{x}\quad \forall k\in[1,K] \tag{5.5}$$
$$\sigma_j=\frac{1}{|A_j|}\int_{A_j}\sigma(\xi)\,d\xi,\ \ \forall j\in[J-J^\Gamma+1,J]\ . \tag{5.6}$$

Using the discrete Hodge decomposition of $(\mathbf{u}_j)_{j\in[1,J]}$, problem (5.3) may be split into two independent problems involving the potentials

PROPOSITION 5.1. *Problem (5.3) can be split into two independent problems: Find* $(\phi_i^T,\phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ *such that*

$$(\nabla_h^T\cdot\boldsymbol{\nabla}_h^D\phi)_i=f_i^T,\quad \forall i\in[1,I] \tag{5.7a}$$
$$(\nabla_h^P\cdot\boldsymbol{\nabla}_h^D\phi)_k=f_k^P,\quad \forall k\in[1,K] \tag{5.7b}$$
$$(\boldsymbol{\nabla}_h^D\phi)_j\cdot\mathbf{n}_j=\sigma_j,\quad \forall j\in[J-J^\Gamma+1,J] \tag{5.7c}$$
$$\sum_{i\in[1,I]}|T_i|\phi_i^T=\sum_{k\in[1,K]}|P_k|\phi_k^P=0 \tag{5.7d}$$

and Find $(\psi_i^T, \psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ *and* $(c_q^T, c_q^P)_{q\in[1,Q]}$ *such that*

$$-(\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \psi)_i = g_i^T, \quad \forall i \in [1, I] \tag{5.8a}$$

$$-(\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k = g_k^P, \quad \forall k \in [1, K - J^\Gamma] \tag{5.8b}$$

$$(\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}, 1)_{\Gamma_q,h} = -k_q, \quad \forall q \in [1, Q] \tag{5.8c}$$

$$-\sum_{k\in\Gamma_q} |P_k| \ (\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k = \sum_{k\in\Gamma_q} |P_k| \ g_k^P, \quad \forall q \in [1, Q] \tag{5.8d}$$

$$\psi_i^T = \psi_k^P = 0, \quad \forall i \in \Gamma_0 \ , \forall k \in \Gamma_0 \ , \tag{5.8e}$$

$$\forall q \in [1, Q] \ , \ \ \psi_i^T = c_q^T \ , \ \forall i \in \Gamma_q \ , \tag{5.8f}$$

$$\forall q \in [1, Q] \ , \ \ \psi_k^P = c_q^P \ , \ \forall k \in \Gamma_q \ . \tag{5.8g}$$

The vector $\mathbf{u}$ *is then reconstructed by*

$$\mathbf{u}_j = (\boldsymbol{\nabla}_h^D \phi)_j + (\boldsymbol{\nabla}_h^D \times \psi)_j \ , \ \forall j \in [1, J] \ . \tag{5.9}$$

Proof. First, the discrete Hodge decomposition of $(\mathbf{u}_j)_{j\in[1,J]}$ shows the existence of $(\phi_i^T, \phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$, $(\psi_i^T, \psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ and $(c_q^T, c_q^P)_{q\in[1,Q]}$ such that (5.9), (5.7d) and (5.8e)-(5.8f)-(5.8g) are verified. Next, (5.7a) is proved using (4.7):

$$f_i^T = (\nabla_h^T \cdot \mathbf{u})_i = (\nabla_h^T \cdot (\boldsymbol{\nabla}_h^D \phi + \boldsymbol{\nabla}_h^D \times \psi))_i = (\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \phi)_i, \forall i \in [1, I].$$

Similarly, using (4.8) and $\psi_i^T = c_q^T$ and $\psi_k^P = c_q^P$, $\forall q \in [0, Q]$, we obtain (5.7b). As far as the boundary conditions are concerned, using (3.3) shows that

$$(\boldsymbol{\nabla}_h^D \times \psi)_j \cdot \mathbf{n}_j = -\frac{1}{2\,|D_j|} (\psi_{k_2} - \psi_{k_1})\, |A_j^{'}| \boldsymbol{\tau}_j^{'} \cdot \mathbf{n}_j \ , \forall j \in [J - J^\Gamma + 1, J]. \tag{5.10}$$

Since $\psi_k^P = c_q^P$, $\forall q \in [0, Q]$, we infer from (5.10)

$$(\boldsymbol{\nabla}_h^D \times \psi)_j \cdot \mathbf{n}_j = 0, \ \ \forall j \in [J - J^\Gamma + 1, J] \ ,$$

so that (5.3e) and (5.9) imply (5.7c). Further, using (5.9), (5.3c)-(5.3d), (4.9), (4.10) and (4.12), we may prove (5.8a)-(5.8b). Moreover, there holds

$$(\boldsymbol{\nabla}_h^D \phi)_j \cdot \boldsymbol{\tau}_j = \frac{1}{2|D_j|} \left(\phi_{k_2(j)}^T - \phi_{k_1(j)}^T \right) |A_j'| \mathbf{n}_j' \cdot \boldsymbol{\tau}_j \ ,$$

so that, using (4.6),

$$(\boldsymbol{\nabla}_h^D \phi \cdot \boldsymbol{\tau}, 1)_{\Gamma_q,h} = \sum_{j\in\Gamma_q} \frac{|A_j|\,|A_j'|}{2\,|D_j|} \mathbf{n}_j' \cdot \boldsymbol{\tau}_j \left(\phi_{k_2(j)}^T - \phi_{k_1(j)}^T \right) = -\sum_{j\in\Gamma_q} \left(\phi_{k_2(j)}^T - \phi_{k_1(j)}^T \right) \ ,$$

which vanishes because Γ_q is a closed contour. Thus, (5.3f) implies (5.8c) because $(\boldsymbol{\nabla}_h^D \times \psi) \cdot \boldsymbol{\tau}_j = -\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}_j$. Finally, a computation similar to that which led to (4.11) shows that

$$(\nabla_h^P \times (\boldsymbol{\nabla}_h^D \phi))_k = \frac{1}{|P_k|} \left(\phi_{I_2}^T - \phi_{I_1}^T \right) + \frac{1}{2\,|P_k|} \left(\phi_{K_1}^P - \phi_{K_2}^P \right) \ .$$

for boundary cells $k \in [K - J^\Gamma + 1, K]$ (see Fig. 4.2 for the notations). Thus, when summing these contributions over a closed contour Γ_q, we obtain

$$\sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times (\boldsymbol{\nabla}_h^D \phi))_k = 0 \ ,$$

so that (5.3g) implies (5.8d). ☐

PROPOSITION 5.2. *Problems (5.7) and (5.8) both have a unique solution.*

Proof. As far as problem (5.7) is concerned, existence and uniqueness of its solution have been proved in [10] if the following discrete equivalent of (5.2) is verified

$$\sum_{i \in [1,I]} |T_i| \ f_i^T = \sum_{k \in [1,K]} |P_k| \ f_k^P = \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \ \sigma_j \ ,$$

which is the case here because thanks to the definitions (5.4) and (5.6) we have

$$\sum_{i \in [1,I]} |T_i| \ f_i^T = \sum_{k \in [1,K]} |P_k| \ f_k^P = \int_\Omega f(\mathbf{x}) \, d\mathbf{x} \ \text{ and } \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \ \sigma_j = \int_\Gamma \sigma(\xi) \, d\xi \ .$$

As far as problem (5.8) is concerned, there are $I + K + J^\Gamma + 2Q$ unknowns, while (5.8a) and (5.8b) respectively provide I and $K - J^\Gamma$ equations. Equations (5.8c) and (5.8d) provide $2Q$ additional relations. Finally, boundary conditions (5.8e)-(5.8f)-(5.8g) provide the last $2J^\Gamma$ equations. Since there are as many equations as unknowns, it suffices to check the injectivity of the system. Let us set $g_i^T = g_k^P = k_q = 0$ in system (5.8) and compute the following discrete scalar product $(\nabla_h^{T,P} \cdot \boldsymbol{\nabla}_h^D \psi, \psi)_{T,P}$ (see (2.1) for the definition). In this scalar product, the sum over the indices $i \in [1, I]$ and the sum over the indices $k \in [1, K - J^\Gamma]$ vanish respectively because of (5.8a) and (5.8b). Further, due to (5.8e), the contributions of the indices $k \in \Gamma_0$ also vanish, so that

$$(\nabla_h^{T,P} \cdot \boldsymbol{\nabla}_h^D \psi, \psi)_{T,P} = \frac{1}{2} \sum_{q \in [1,Q]} \sum_{k \in \Gamma_q} |P_k| \ (\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k \ \psi_k^P \ .$$

Further, (5.8g) implies that

$$(\nabla_h^{T,P} \cdot \boldsymbol{\nabla}_h^D \psi, \psi)_{T,P} = \frac{1}{2} \sum_{q \in [1,Q]} c_q^P \sum_{k \in \Gamma_q} |P_k| \ (\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k \ ,$$

which vanishes due to (5.8d). Thanks to the discrete Green formula (4.2), there holds

$$(5.11) \qquad (\nabla_h^{T,P} \cdot \boldsymbol{\nabla}_h^D \psi, \psi)_{T,P} = -(\boldsymbol{\nabla}_h^D \psi, \boldsymbol{\nabla}_h^D \psi)_D + (\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}, \psi)_{\Gamma,h} = 0 \ .$$

Now, due to boundary conditions (5.8e)-(5.8f)-(5.8g), we may write

$$(5.12) \qquad (\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}, \psi)_{\Gamma,h} = \sum_{q \in [1,Q]} \frac{c_q^T + c_q^P}{2} \ (\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}, 1)_{\Gamma_q,h} \ ,$$

which vanishes thanks to (5.8c). Thus, (5.11), (5.12) and definition (2.2) imply that

$$(\boldsymbol{\nabla}_h^D \psi, \boldsymbol{\nabla}_h^D \psi)_D = \sum_{j \in [1,J]} |D_j| |\boldsymbol{\nabla}_h^D \psi|^2 = 0 \ .$$

Consequently, just like at the end of the proof of Theorem 4.7, we infer that

$$\psi_i^T = 0, \ \ \forall i \in [1, I + J^\Gamma] \ \text{ and } \ \psi_k^P = 0, \ \ \forall k \in [1, K] \ ,$$

which proves uniqueness and thus existence. ☐

5.2. The div-curl problem with tangential boundary conditions. We consider the following continuous problem: given f, g, σ, $(k_q)_{q\in[1,Q]}$, find $\mathbf{u}$ such that:

$$\begin{cases} \nabla\cdot\mathbf{u} = f \text{ in } \Omega\,,\\ \nabla\times\mathbf{u} = g \text{ in } \Omega\,,\\ \mathbf{u}\cdot\boldsymbol{\tau} = \sigma \text{ on } \Gamma\,,\\ \int_{\Gamma_q}\mathbf{u}\cdot\mathbf{n} = k_q,\ \ \forall q\in[1,Q]\,. \end{cases}$$

A necessary condition for the existence of a solution to this system is given by Green's formula: $\int_\Omega g(\mathbf{x})d\mathbf{x} = \int_\Gamma \sigma(\xi)\,d\xi$. This problem is discretized like in section 5.1 by a vector field $(\mathbf{u}_j)_{j\in[1,J]}$ defined by its values over the diamond-cells. Using the discrete differential operators defined in section 3, we write the following discrete equations:

$$\begin{cases} \left(\nabla_h^T\cdot\mathbf{u}\right)_i &=& f_i^T, \quad \forall i\in[1,I]\,,\\ \left(\nabla_h^P\cdot\mathbf{u}\right)_k &=& f_k^P, \quad \forall k\in[1,K-J^\Gamma]\,,\\ \left(\nabla_h^T\times\mathbf{u}\right)_i &=& g_i^T, \quad \forall i\in[1,I]\,,\\ \left(\nabla_h^P\times\mathbf{u}\right)_k &=& g_k^P, \quad \forall k\in[1,K]\,,\\ \mathbf{u}_j\cdot\boldsymbol{\tau}_j &=& \sigma_j, \quad \forall j\in[J-J^\Gamma+1,J]\,,\\ (\mathbf{u}\cdot\mathbf{n},1)_{\Gamma_q,h} &=& k_q, \quad \forall q\in[1,Q]\,,\\ \sum_{k\in\Gamma_q}|P_k|\ \ (\nabla_h^P\cdot\mathbf{u})_k &=& \sum_{k\in\Gamma_q}|P_k|\ \ f_k^P, \quad \forall q\in[1,Q]\,. \end{cases} \tag{5.13}$$

Existence and uniqueness of the solution of (5.13) are proved similarly to section 5.1; the main difference is that the Hodge decomposition is modified in the following way

THEOREM 5.3. *Let $(\mathbf{u}_j)_{j\in[1,J]}$ be a discrete vector field defined by its values on the diamond-cells D_j. There exist unique $\phi = (\phi_i^T,\phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$, $\psi = (\psi_i^T,\psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ and $(c_q^T,c_q^P)_{q\in[1,Q]}$ such that:*

$$\mathbf{u}_j = (\boldsymbol{\nabla}_h^D\psi)_j + (\boldsymbol{\nabla}_h^D\times\phi)_j, \quad \forall j\in[1,J]\,, \tag{5.14}$$

$$\sum_{i\in[1,I]}|T_i|\phi_i^T = \sum_{k\in[1,K]}|P_k|\phi_k^P = 0\,,$$

$$\psi_i^T = 0\,,\ \forall i\in\Gamma_0 \quad , \quad \psi_k^P = 0\,,\ \forall k\in\Gamma_0\,,$$

and

$$\forall q\in[1,Q]\,,\qquad \psi_i^T = c_q^T\,,\ \forall i\in\Gamma_q \quad , \quad \psi_k^P = c_q^P\,,\ \forall k\in\Gamma_q\,.$$

Moreover, the decomposition (5.14) is orthogonal.

Further, problem (5.13) decouples into two independent sub-problems involving the potentials

PROPOSITION 5.4. *Problem (5.13) can be split into two independent problems: Find $(\phi_i^T,\phi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ such that*

$$\begin{cases} -(\nabla_h^T\cdot\boldsymbol{\nabla}_h^D\phi)_i &=& g_i^T, \quad \forall i\in[1,I]\,,\\ -(\nabla_h^P\cdot\boldsymbol{\nabla}_h^D\phi)_k &=& g_k^P, \quad \forall k\in[1,K]\,,\\ -(\boldsymbol{\nabla}_h^D\phi)_j\cdot\mathbf{n}_j &=& \sigma_j, \quad \forall j\in[J-J^\Gamma+1,J]\,,\\ \sum_{i\in[1,I]}|T_i|\phi_i^T &=& \sum_{k\in[1,K]}|P_k|\phi_k^P = 0 \end{cases}$$

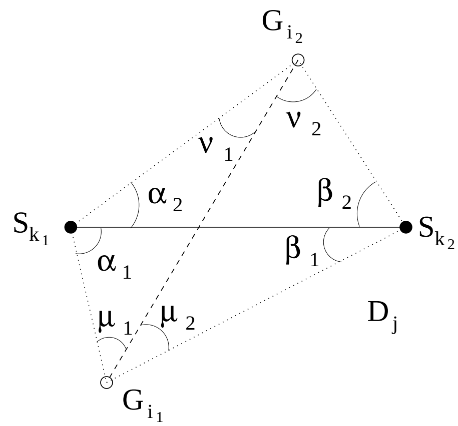

FIG. 5.1. *Notations for the paragraph 5.3*

and Find $(\psi_i^T, \psi_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ *and* $(c_q^T, c_q^P)_{q\in[1,Q]}$

$$\left\{\begin{array}{rcl}
(\nabla_h^T \cdot \boldsymbol{\nabla}_h^D \psi)_i & = & f_i^T, \quad \forall i \in [1,I]\ , \\
(\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k & = & f_k^P, \quad \forall k \in [1,K-J^\Gamma]\ , \\
(\boldsymbol{\nabla}_h^D \psi \cdot \mathbf{n}, 1)_{\Gamma_q} & = & k_q, \quad \forall q \in [1,Q]\ , \\
\sum_{k\in\Gamma_q} |P_k| \ \ (\nabla_h^P \cdot \boldsymbol{\nabla}_h^D \psi)_k & = & \sum_{k\in\Gamma_q} |P_k| \ \ f_k^P, \quad \forall q \in [1,Q]\ , \\
\psi_i^T = \psi_k^P & = & 0, \quad \forall i \in \Gamma_0\ , \forall k \in \Gamma_0\ , \\
\forall q \in [1,Q]\ , \ \ \psi_i^T & = & c_q^T\ , \ \forall i \in \Gamma_q\ , \\
\forall q \in [1,Q]\ , \ \ \psi_k^P & = & c_q^P\ , \ \forall k \in \Gamma_q\ .
\end{array}\right.$$

The vector $\mathbf{u}$ *is then reconstructed by*

$$\mathbf{u}_j = (\boldsymbol{\nabla}_h^D \psi)_j + (\boldsymbol{\nabla}_h^D \times \phi)_j \forall j \in [1,J]\ .$$

5.3. Error estimate for the div-curl problem. Unlike in [20], we shall derive estimates for the potentials involved in the Hodge decomposition of $\mathbf{u}$; indeed we shall rely on similar estimates which have been obtained in [10]. For the sake of simplicity, we shall restrict ourselves to the case where all diamond-cells are convex; the case of non-convex diamond-cells requires additional hypotheses similar to those given in [10]. We shall obtain error estimates under the following hypothesis (see Fig. 2.5 and Fig. 5.1 for the notations)

HYPOTHESIS 5.5. *There exists an angle* τ^**, strictly lower than* π *and independent of the mesh, such that :*
1. For any interior *diamond-cell* D_j*, the smallest in the maximum angle of the couple of triangles* $(D_{j,1}, D_{j,2})$ or *in the maximum angle of the couple of triangles* $(D'_{j,1}, D'_{j,2})$ *is bounded by* τ^**:*

$$\min\left(\max(\alpha_1, \beta_1, \mu_1+\mu_2, \alpha_2, \beta_2, \nu_1+\nu_2), \max(\mu_1, \nu_1, \alpha_1+\alpha_2, \mu_2, \nu_2, \beta_1+\beta_2)\right) \le \tau^*$$

2. The greatest angle of any boundary *cell* D_j *is bounded by the angle* τ^**.*

Obtaining error estimates usually relies on regularity assumptions on the solution of the problem. In order to apply results given in [10], we shall assume regularity of the potentials given by the following proposition

PROPOSITION 5.6. *Let (f,g,σ) belong to $\mathrm{L}^2(\Omega)^2 \times \mathrm{H}^{1/2}(\Gamma)$ and let $(k_q)_{q\in[1,Q]}$ be a set of given real numbers; let $\hat{\mathbf{u}}$ be the exact solution of problem (5.1). Then, there exist $\hat\phi$ and $\hat\psi$ both in $\mathrm{H}^1(\Omega)$ and a set of real numbers $(C_q)_{q\in[1,Q]}$ such that*

$$\hat{\mathbf{u}} = \boldsymbol{\nabla}\hat\phi + \boldsymbol{\nabla}\times\hat\psi \ ,$$

where $\hat\phi$ is the solution of

$$\left\{\begin{array}{l} \Delta\hat\phi = \nabla\cdot\hat{\mathbf{u}} = f \ \ in \ \ \Omega\,, \\ \boldsymbol{\nabla}\hat\phi\cdot\mathbf{n} = \hat{\mathbf{u}}\cdot\mathbf{n} = \sigma \ \ on \ \ \Gamma\,, \\ \int_\Omega \hat\phi = 0\ , \end{array}\right. \tag{5.15}$$

and $\hat\psi$ is the solution of

$$\left\{\begin{array}{l} -\Delta\hat\psi = \nabla\times\hat{\mathbf{u}} = g \ \ in \ \ \Omega\,, \\ \hat\psi_{|\Gamma_0} = 0 \ \ ; \ \ \hat\psi_{|\Gamma_q} = C_q \ \ \forall q\in[1,Q] \\ \int_{\Gamma_q} \boldsymbol{\nabla}\hat\psi\cdot n = -k_q\ . \end{array}\right. \tag{5.16}$$

Proof. The Hodge decomposition of $\hat{\mathbf{u}}$ and the determination of $\hat\phi$ and $\hat\psi$ through (5.15) and (5.16) are direct consequences of [11, Theorem 3.2 and Corollary 3.1]. ☐

HYPOTHESIS 5.7. *We suppose that the potentials $\hat\phi$ and $\hat\psi$ given by proposition 5.6 belong to* $\mathrm{H}^2(\Omega)$.

We remark that due to reentrant corners related to the internal polygonal boundaries Γ_q, the H^2 regularity of the potentials is not a consequence of the regularity of the data (f,g,σ).

Obviously, we may relate the L^2 error between the solution $\hat{\mathbf{u}}$ of (5.1) and the discrete solution $(\mathbf{u}_j)_{j\in[1,J]}$ of (5.3) to the errors between the solutions $\hat\phi$ and $\hat\psi$ of (5.15) and (5.16) and the discrete solutions (ϕ_i^T,ϕ_k^P) and (ψ_i^T,ψ_k^P) defined in Proposition 5.1 respectively by (5.7) and (5.8). Indeed:

$$\begin{aligned} \sum_{j\in[1,J]}\int_{D_j}|\mathbf{u}_j-\hat{\mathbf{u}}(\mathbf{x})|^2\,d\mathbf{x} \leq 2\Bigg(&\sum_{j\in[1,J]}\int_{D_j}\left|(\boldsymbol{\nabla}_h^D\phi)_j-\boldsymbol{\nabla}\hat\phi(\mathbf{x})\right|^2 d\mathbf{x} \\ &+\sum_{j\in[1,J]}\int_{D_j}\left|(\boldsymbol{\nabla}_h^D\psi)_j-\boldsymbol{\nabla}\hat\psi(\mathbf{x})\right|^2 d\mathbf{x}\Bigg)\ . \end{aligned} \tag{5.17}$$

5.3.1. Equivalent Finite Element formulations for the potentials. In order to evaluate the errors on the potentials, we follow [10] and rewrite (5.7) and (5.8) in terms of equivalent (non-conforming) finite element formulations. Recalling that the points $M_{i_\alpha(j)\,k_\beta(j)}$ are illustrated on figure 2.4, we construct the following functions:

PROPOSITION 5.8. *Let $(\phi_i^T,\phi_k^P)\in\mathbb{R}^{I+J^\Gamma}\times\mathbb{R}^K$ be given; there exists a function ϕ_h defined by*

$$\begin{aligned} &(\phi_h)_{|D_j}\in P^1(D_j)\ , &&\forall j\in[1,J]\ , \\ &\phi_h(M_{i_\alpha(j)\,k_\beta(j)}) = \frac{1}{2}(\phi^T_{i_\alpha(j)}+\phi^P_{k_\beta(j)})\ , \forall j\in[1,J]\ , &&\forall(\alpha\,,\,\beta)\in\{1;2\}^2\ . \end{aligned} \tag{5.18}$$

Moreover, we have the following essential property:

$$(\boldsymbol{\nabla}\phi_h)_{|D_j} = (\boldsymbol{\nabla}_h^D\phi)_j \ \ \forall j\in[1,J]\ . \tag{5.19}$$

Proof. The proof is given in [10]. We recall that the definition of ϕ_h through the four equalities contained in (5.18) is possible because $(M_{i_1k_1}M_{i_1k_2}M_{i_2k_2}M_{i_2k_1})$ is a parallelogram and $\phi_h(M_{i_1k_1}) + \phi_h(M_{i_2k_2}) = \phi_h(M_{i_1k_2}) + \phi_h(M_{i_2k_1})$. ☐

DEFINITION 5.9. *We shall denote by L the linear operator which associates ϕ_h, defined by Proposition 5.8, to a given $(\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$. Further, the solution of (5.7) is in the following space*

$$V_N := \left\{ (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K \ / \ \sum_{i\in[1,I]} |T_i|\phi_i^T = \sum_{k\in[1,K]} |P_k|\phi_k^P = 0 \right\} .$$

The solution of (5.8) is in the following space

$$V_D := \Big\{ (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K \ / \ \phi_i^T = \phi_k^P = 0 \, \forall\, i \in \Gamma_0 \ \ \forall\, k \in \Gamma_0 \ \text{ and}$$
$$\exists\, (c_{q,\phi}^T, c_{q,\phi}^P) \in (\mathbb{R}^2)^Q \ \text{ s.t. } \ \phi_i^T = c_{q,\phi}^T \, \forall\, i \in \Gamma_q, \ \text{and}\, \phi_k^P = c_{q,\phi}^P \,\forall\, k \in \Gamma_q \ \ \forall\, q \in [1,Q] \Big\} .$$

REMARK 5.10. *It is easily proved that the linear operator L introduced in Definition 5.9 is injective over V_N and over V_D. Thus, for any Φ_h in $L(V_N)$ or in $L(V_D)$, there exists a unique $\Phi = (\Phi_i^T, \Phi_k^P)$ in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, either in V_N or in V_D such that $\Phi_h = L(\Phi)$. The values (Φ_i^T, Φ_k^P) are used in the definitions of Φ_h^* and $\tilde{\Phi}_h$ associated to Φ_h respectively by (5.22) and (5.23).*

With these definitions, we may state the following result

PROPOSITION 5.11. *Problem (5.7) amounts to finding $\phi_h \in L(V_N)$, such that,*

$$a_h(\phi_h, \Phi_h) = \ell_N(\Phi_h) \ \ ,\forall \ \Phi_h \in L(V_N) \tag{5.20}$$

with

$$\begin{aligned} a_h(\phi_h, \Phi_h) &:= \sum_{j\in[1,J]} \int_{D_j} \boldsymbol{\nabla}\phi_h \cdot \boldsymbol{\nabla}\Phi_h\,(\mathbf{x}) d\mathbf{x}\,, \\ \ell_N(\Phi_h) &:= -\int_\Omega f\Phi_h^*(\mathbf{x})d\mathbf{x} \ + \int_\Gamma \sigma\,\tilde{\Phi}_h(\xi)\,d\xi\ , \end{aligned} \tag{5.21}$$

where Φ_h^* is defined over Ω by

$$\Phi_h^*(\mathbf{x}) := \frac{1}{2}\left(\sum_{i\in[1,I]} \Phi_i^T\theta_i^T(\mathbf{x}) + \sum_{k\in[1,K]} \Phi_k^P\theta_k^P(\mathbf{x}) \right) \tag{5.22}$$

and $\tilde{\Phi}_h$ is defined over Γ by

$$\tilde{\Phi}_h(\xi) = \sum_{j\in[1,J]} \frac{1}{4}\left(\Phi_{k_1(j)}^P + 2\Phi_{i_2(j)}^T + \Phi_{k_2(j)}^P \right)\theta_j^\Gamma(\xi)\ , \tag{5.23}$$

where we recall that θ_i^T, θ_k^P and θ_j^Γ are respectively the characteristic functions of the cells T_i, P_k and of the boundary edge A_j.

Proof. Let us suppose that $\phi \in V_N$ is the solution of (5.7); then multiplying the first equation by $\frac{1}{2}|T_i|\Phi_i^T$, the second equation by $\frac{1}{2}|P_k|\Phi_k^P$, and summing over all $i \in [1,I]$ and all $k \in [1,K]$ yields

$$(\nabla_h^{T,P} \cdot \boldsymbol{\nabla}_h^D \phi, \Phi)_{T,P} = (f, \Phi)_{T,P}\ . \tag{5.24}$$

Thanks to the discrete Green formula (4.1), we may write the left-hand-side of (5.24) in the following way:

$$-(\nabla_h^D\phi,\nabla_h^D\Phi)_D+(\nabla_h^D\phi\cdot\mathbf{n},\Phi)_{\Gamma,h} = -\sum_{j\in[1,J]}|D_j|\,(\nabla_h^D\phi)_j\cdot(\nabla_h^D\Phi)_j + \sum_{j\in[J-J^\Gamma+1,J]}|A_j|\,(\nabla_h^D\phi)_j\cdot\mathbf{n}_j\times\frac{1}{4}\left(\Phi^P_{k_1(j)}+2\Phi^T_{i_2(j)}+\Phi^P_{k_2(j)}\right).$$

Next, thanks to (5.19), and because $(\nabla_h^D\phi)_j\cdot(\nabla_h^D\Phi)_j$ is a constant over D_j, we may write

$$-\sum_{j\in[1,J]}|D_j|\,(\nabla_h^D\phi)_j\cdot(\nabla_h^D\Phi)_j=-\sum_{j\in[1,J]}\int_{D_j}\nabla\phi_h\cdot\nabla\Phi_h\,(\mathbf{x})d\mathbf{x}\,.$$

Moreover, according to the boundary conditions given by (5.7c),

$$|A_j|\,(\nabla_h^D\phi)_j\cdot\mathbf{n}_j=|A_j|\,\sigma_j=\int_{A_j}\sigma(\xi)d\xi\,,$$

so that

$$|A_j|\,(\nabla_h^D\phi)_j\cdot\mathbf{n}_j\times\frac{1}{4}\left(\Phi^P_{k_1}+2\Phi^T_{i_2}+\Phi^P_{k_2}\right)=\int_{A_j}\sigma\,\left(\tilde\Phi_h\right)_{|A_j}(\xi)\,d\xi\,.$$

Finally, the left-hand-side of (5.24) is equal to

$$-a_h(\phi_h,\Phi_h)+\int_\Gamma\sigma\,\tilde\Phi_h(\xi)\,d\xi\,.$$

By Eq. (5.4), and because $\Phi_i^T\theta_i^T(\mathbf{x})_{|T_i}=\Phi_i^T$ and $\Phi_k^P\theta_k^P(\mathbf{x})_{|P_k}=\Phi_k^P$, the right-hand side of (5.24) is equal to:

$$\int_\Omega f(\mathbf{x})\,\frac{1}{2}\left(\sum_{i\in[1,I]}\Phi_i^T\theta_i^T(\mathbf{x})+\sum_{k\in[1,K]}\Phi_k^P\theta_k^P(\mathbf{x})\right)d\mathbf{x}\,,$$

which ends this part of the proof.

Conversely, let $\phi_h\in L(V_N)$ satisfy (5.20) for all $\Phi_h\in L(V_N)$; then $\phi=L^{-1}(\phi_h)$ satisfies (5.7d) by definition of V_N. Further, we prove that the boundary condition (5.7c) is verified along each boundary edge $j_0\in[J-J^\Gamma+1,J]$ by considering its corresponding basis element $\Phi_0\in V_N$ defined by (recall that the index $i_2(j_0)$ is associated to the unknown located at the center of the segment A_{j_0})

$$\forall i\in[1,I+J^\Gamma]\,,\;(\Phi_0)_i^T=\delta_i^{i_2(j_0)}\;\text{ and }\;\forall k\in[1,K]\,,\;(\Phi_0)_k^P=0\,.$$

Then, defining $(\Phi_0)_h=L(\Phi_0)$, we obviously have the following properties

$$(\nabla(\Phi_0)_h)_{|D_j}=0\;\text{ if }\;j\neq j_0\quad\text{and}\quad(\nabla(\Phi_0)_h)_{|D_{j_0}}=\frac{1}{2\,|D_{j_0}|}\,|A_{j_0}|\,\mathbf{n}_{j_0}$$

and

$$(\Phi_0)_h^*(\mathbf{x})=0\;\;\forall\mathbf{x}\in\Omega\quad\text{and}\quad(\tilde\Phi_0)_h(\xi)=\frac{1}{2}\theta^\Gamma_{j_0}(\xi)\;\;\forall\xi\in\Gamma\,.$$

We thus have

$$\sum_{j\in[1,J]}\int_{D_j}\boldsymbol{\nabla}\phi_h\cdot\boldsymbol{\nabla}(\Phi_0)_h\,(\mathbf{x})d\mathbf{x}=\frac{1}{2}\,|A_{j_0}|\,(\boldsymbol{\nabla}\phi_h)_{|D_{j_0}}\cdot\mathbf{n}_{j_0}=\frac{1}{2}\,|A_{j_0}|\,(\boldsymbol{\nabla}_h^D\phi)_{j_0}\cdot\mathbf{n}_{j_0}$$

and

$$-\int_\Omega f(\Phi_0)_h^*(\mathbf{x})d\mathbf{x}\;+\int_\Gamma\sigma\,(\tilde\Phi_0)_h(\xi)\,d\xi\;=\int_{A_{j_0}}\frac{1}{2}\sigma(\xi)\,d\xi\;=\frac{1}{2}\,|A_{j_0}|\,\sigma_{j_0}\;.$$

Finally, writing (5.20) for $(\Phi_0)_h$ proves that ϕ satisfies the boundary condition:

$$(\boldsymbol{\nabla}_h^D\phi)_{j_0}\cdot\mathbf{n}_{j_0}=\sigma_{j_0}\;,\;\forall\,j_0\in[J-J^\Gamma+1,J]\;.$$

Next, in order to prove (5.7a) for any primal cell $i_0\in[1,I]$, we consider its corresponding basis element $\Phi_1\in V_N$ defined by

$$\forall i\in[1,I+J^\Gamma]\;,\;(\Phi_1)_i^T=\delta_i^{i_0}-\frac{|T_{i_0}|}{|\Omega|}\;\text{ and }\;\forall k\in[1,K]\;,\;(\Phi_1)_k^P=0\;.$$

Then, defining $(\Phi_1)_h=L(\Phi_1)$ and according to (5.20), we may write

$$\sum_{j\in[1,J]}\int_{D_j}\boldsymbol{\nabla}\phi_h\cdot\boldsymbol{\nabla}(\Phi_1)_h\,(\mathbf{x})d\mathbf{x}=-\int_\Omega f(\Phi_1)_h^*(\mathbf{x})d\mathbf{x}\;+\int_\Gamma\sigma\,(\tilde\Phi_1)_h(\xi)\,d\xi\;. \tag{5.25}$$

To evaluate the left-hand-side of (5.25), we consider $\Phi\in\mathbb{R}^{I+J^\Gamma}\times\mathbb{R}^K$ such that

$$\forall i\in[1,I+J^\Gamma]\;,\;\Phi_i^T=\delta_i^{i_0}\;\text{ and }\;\forall k\in[1,K]\;,\;\Phi_k^P=0\;.$$

Note that $\Phi\notin V_N$ but that its discrete gradient (see (3.2)) obviously equals that of Φ_1. Thanks to this equality and to (5.19), we have

$$\sum_{j\in[1,J]}\int_{D_j}\boldsymbol{\nabla}\phi_h\cdot\boldsymbol{\nabla}(\Phi_1)_h\,(\mathbf{x})d\mathbf{x}=(\boldsymbol{\nabla}_h^D\phi,\boldsymbol{\nabla}_h^D\Phi_1)_D=(\boldsymbol{\nabla}_h^D\phi,\boldsymbol{\nabla}_h^D\Phi)_D,$$

which, in turn, can be transformed thanks to (4.1) into

$$-(\nabla_h^{T,P}\cdot\boldsymbol{\nabla}_h^D\phi,\Phi)_{T,P}+(\boldsymbol{\nabla}_h^D\phi\cdot\mathbf{n},\Phi)_{\Gamma,h}\;.$$

Thanks to the definition of Φ, this quantity reduces to the contribution of i_0, which proves that the left-hand-side of (5.25) may be written

$$\sum_{j\in[1,J]}\int_{D_j}\boldsymbol{\nabla}\phi_h\cdot\boldsymbol{\nabla}(\Phi_1)_h\,(\mathbf{x})d\mathbf{x}=-\frac{1}{2}\,|T_{i_0}|\,(\nabla_h^T\cdot\boldsymbol{\nabla}_h^D\phi)_{i_0}\;. \tag{5.26}$$

Next, we compute the right-hand-side of (5.25)

$$\begin{aligned}-\int_\Omega f(\Phi_1)_h^*(\mathbf{x})d\mathbf{x}&=-\frac{1}{2}\sum_{i\in[1,I]}\int_{T_i}\left(\delta_i^{i_0}-\frac{|T_{i_0}|}{|\Omega|}\right)f(\mathbf{x})\,d\mathbf{x}\\&=-\frac{1}{2}\int_{T_{i_0}}f(\mathbf{x})\,d\mathbf{x}+\frac{1}{2}\frac{|T_{i_0}|}{|\Omega|}\int_\Omega f(\mathbf{x})\,d\mathbf{x}\;;\\\int_\Gamma\sigma\,(\tilde\Phi_1)_h(\xi)\,d\xi&=\sum_{j\in[J-J^\Gamma+1,J]}\int_{A_j}\sigma(\xi)\,\frac{1}{4}\left(-2\frac{|T_{i_0}|}{|\Omega|}\right)\,d\xi=-\frac{1}{2}\frac{|T_{i_0}|}{|\Omega|}\int_\Gamma\sigma(\xi)d\xi\;,\end{aligned}$$

so that the right-hand-side of (5.25) equals

$$-\frac{1}{2}\int_{T_{i_0}} f(\mathbf{x})\,d\mathbf{x} + \frac{1}{2}\frac{|T_{i_0}|}{|\Omega|}\int_\Omega f(\mathbf{x})\,d\mathbf{x} - \frac{1}{2}\frac{|T_{i_0}|}{|\Omega|}\int_\Gamma \sigma(\xi)d\xi\ .$$

Because of (5.2), the last two terms in the previous sum cancel and we get

$$-\int_\Omega f(\Phi_1)^*_h(\mathbf{x})d\mathbf{x} + \int_\Gamma \sigma\,(\tilde{\Phi}_1)_h(\xi)\,d\xi = -\frac{1}{2}\int_{T_{i_0}} f(\mathbf{x})\,d\mathbf{x} = -\frac{1}{2}\,|T_{i_0}|\,f^T_{i_0}\ . \tag{5.27}$$

Comparing (5.25), (5.26) and (5.27), we infer that

$$(\nabla^T_h\cdot\boldsymbol{\nabla}^D_h\phi)_{i_0} = f^T_{i_0}\ .$$

In a similar way, we can prove (5.7b) for any dual cell $k_0\in[1,K]$ by considering its corresponding basis element $\Phi_2\in V_N$, defined by

$$\forall i\in[1,I+J^\Gamma]\ ,\ (\Phi_2)^T_i = 0\ \text{ and }\ \forall k\in[1,K]\ ,\ (\Phi_2)^P_k = \delta^{k_0}_k - \frac{|P_{k_0}|}{|\Omega|}\ \ ,$$

which ends the proof of the equivalence. □

Proposition 5.12. *Problem (5.8) is equivalent to finding $\psi_h\in L(V_D)$, such that $\forall\Psi_h\in L(V_D)$,*

$$a_h(\psi_h,\Psi_h) = \ell_D(\Psi_h) \tag{5.28}$$

with

$$\ell_D(\Psi_h) := \int_\Omega g\Psi^*_h(\mathbf{x})d\mathbf{x}\ -\sum_{q\in[1,Q]} k_q\left(\frac{c^T_{q,\Psi}+c^P_{q,\Psi}}{2}\right)\ .$$

Proof. Let us suppose that $\psi\in V_D$ is the solution of (5.8); then we may compute the following discrete scalar product

$$\begin{aligned}-(\nabla^{T,P}_h\cdot\boldsymbol{\nabla}^D_h\psi,\Psi)_{T,P} = &-\frac{1}{2}\sum_{i\in[1,I]}|T_i|(\nabla^T_h\cdot\boldsymbol{\nabla}^D_h\psi)_i\Psi^T_i\\ &-\frac{1}{2}\sum_{k\in[1,K-J^\Gamma]}|P_k|(\nabla^P_h\cdot\boldsymbol{\nabla}^D_h\psi)_k\Psi^P_k\\ &-\frac{1}{2}\sum_{k\in[K-J^\Gamma+1,K]}|P_k|(\nabla^P_h\cdot\boldsymbol{\nabla}^D_h\psi)_k\Psi^P_k\ .\end{aligned} \tag{5.29}$$

Due to (5.8a)-(5.8b), the sum of the first two terms on the right-hand-side of (5.29) equals

$$\frac{1}{2}\sum_{i\in[1,I]}|T_i|g^T_i\Psi^T_i + \frac{1}{2}\sum_{k\in[1,K-J^\Gamma]}|P_k|g^P_k\Psi^P_k\ .$$

Next, using the fact that Ψ^P is equal to a constant $c^P_{q,\Psi}$ over each Γ_q and vanishes over Γ_0, we may write, according to (5.8d)

$$-\sum_{k\in[K-J^\Gamma+1,K]}|P_k|(\nabla^P_h\cdot\boldsymbol{\nabla}^D_h\psi)_k\Psi^P_k = -\sum_{q\in[1,Q]}c^P_{q,\Psi}\sum_{k\in\Gamma_q}|P_k|(\nabla^P_h\cdot\boldsymbol{\nabla}^D_h\psi)_k =$$

$$\sum_{q\in[1,Q]}c^P_{q,\Psi}\sum_{k\in\Gamma_q}|P_k|g^P_k = \sum_{k\in[K-J^\Gamma+1,K]}|P_k|g^P_k\Psi^P_k\ .$$

Finally, (5.29) may be rewritten in the following way

$$-(\nabla_h^{T,P}\cdot\boldsymbol{\nabla}_h^D\psi,\Psi)_{T,P}=(g,\Psi)_{T,P}\ . \tag{5.30}$$

Using the discrete Green formula (4.1), the left-hand-side of (5.30) is equal to

$$(\boldsymbol{\nabla}_h^D\psi,\boldsymbol{\nabla}_h^D\Psi)_D-(\boldsymbol{\nabla}_h^D\psi\cdot\mathbf{n},\Psi)_{\Gamma,h}\ .$$

Like previously, the first of these terms equals $a_h(\psi_h,\Psi_h)$. Next, using the fact that Ψ^P (respectively Ψ^T) is equal to a constant $c^P_{q,\Psi}$ (resp. $c^T_{q,\Psi}$) over each Γ_q and vanishes over Γ_0, and using (5.8c), there holds

$$(\boldsymbol{\nabla}_h^D\psi\cdot\mathbf{n},\Psi)_{\Gamma,h}=\sum_{q\in[1,Q]}\left(\frac{c^T_{q,\Psi}+c^P_{q,\Psi}}{2}\right)\sum_{\Gamma_q}(\boldsymbol{\nabla}_h^D\psi)_j\cdot\mathbf{n}_j=-\sum_{q\in[1,Q]}k_q\left(\frac{c^T_{q,\Psi}+c^P_{q,\Psi}}{2}\right),$$

which shows that the left-hand-side of (5.30) is equal to

$$a_h(\psi_h,\Psi_h)+\sum_{q\in[1,Q]}k_q\left(\frac{c^T_{q,\Psi}+c^P_{q,\Psi}}{2}\right)\ .$$

This ends this part of the proof.

Conversely, if $\psi_h\in L(V_D)$ satisfies (5.28) for all $\Psi_h\in L(V_D)$, then $\psi=L^{-1}(\psi_h)$ verifies (5.8e), (5.8f) and (5.8g) by definition of V_D. Next, in order to prove (5.8a) for any primal cell $i_0\in[1,I]$, let us consider its associated basis element $\Psi_1\in V_D$ defined through

$$(\Psi_1)^T_i=\delta_i^{i_0},\ \ \forall\, i\in[1,I+J^\Gamma]\ \text{ and }\ (\Psi_1)^P_k=0,\ \ \forall\, k\in[1,K]\ .$$

Applying (5.28) for $\Psi_h=L(\Psi_1)$ and using (5.19), (4.1) and (5.5) shows that (5.8a) is verified for the considered $i_0\in[1,I]$. Equality (5.8b) can be proved in the same way for any dual cell $k_0\in[1,K-J^\Gamma]$ by considering its associated basis element $\Psi_2\in V_D$ defined through

$$(\Psi_2)^T_i=0,\ \ \forall\, i\in[1,I+J^\Gamma]\ \text{ and }\ (\Psi_2)^P_k=\delta_k^{k_0},\ \ \forall\, k\in[1,K]\ .$$

Next, let us consider an internal boundary Γ_{q_0} with $q_0\in[1,Q]$ and let us consider $\Psi_3\in V_D$ which vanishes everywhere but on Γ_{q_0}, where it has a constant value:

$$(\Psi_3)^T_i=(\Psi_3)^P_k=0,\ \forall\, i\in[1,I],\ \forall\, k\in[1,K]\text{ and }(\Psi_3)^T_i=\delta_q^{q_0},\ \forall\, i\in\Gamma_q,\ \forall\, q\in[0,Q].$$

Applying (5.28) for $\Psi_h=L(\Psi_3)$ and using (5.19) and (4.1) shows that (5.8c) is verified for the considered $q_0\in[1,Q]$. In the same way, we prove (5.8d) for a given $q_0\in[1,Q]$ by choosing $\Psi_4\in V_D$ defined through

$$(\Psi_4)^T_i=0,\ \forall\, i\in[1,I]\ ,\ (\Psi_4)^P_k=0,\ \forall\, k\in[1,K-J^\Gamma]\ ,$$
$$(\Psi_4)^T_i=\delta_q^{q_0}\ ,\forall\, i\in\Gamma_q\text{ and }(\Psi_4)^P_k=-\delta_q^{q_0},\ \forall\, k\in\Gamma_q,\ \forall\, q\in[0,Q]\ .$$

This ends the proof of Prop. 5.12. ◻

5.3.2. Error estimates for the potentials. We may now turn to error estimates for the potentials $\hat{\phi}$ and $\hat{\psi}$. First, given the equivalent finite element formulation stated by Prop. 5.11 (respectively Prop. 5.12), we may study the numerical error concerning $\hat{\phi}$ (resp. $\hat{\psi}$) in a traditional way by noting that a_h acts on $\mathrm{H}^1 + L(V_N)$ (resp. $\mathrm{H}^1 + L(V_D)$), on which we define $|x|_{1,h} := \sqrt{a_h(x,x)}$, and by using the so-called "Strang second lemma" [24]:

$$|\hat{\phi} - \phi_h|_{1,h} \le 2 \inf_{\omega_h \in L(V_N)} |\hat{\phi} - \omega_h|_{1,h} + \sup_{\omega_h \in L(V_N)} \frac{|a_h(\hat{\phi}, \omega_h) - \ell_N(\omega_h)|}{|\omega_h|_{1,h}} \,. \tag{5.31}$$

and

$$|\hat{\psi} - \psi_h|_{1,h} \le 2 \inf_{\omega_h \in L(V_D)} |\hat{\psi} - \omega_h|_{1,h} + \sup_{\omega_h \in L(V_D)} \frac{|a_h(\hat{\psi}, \omega_h) - \ell_D(\omega_h)|}{|\omega_h|_{1,h}} \,. \tag{5.32}$$

The first term in (5.31) and (5.32) is named "interpolation error", while the second is called "consistency error".

Interpolation error for $\hat{\phi}$. We start with

PROPOSITION 5.13. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant $C(\tau^*)$ depending only on τ^* such that*

$$\inf_{\omega_h \in L(V_N)} |\hat{\phi} - \omega_h|_{1,h} \le C(\tau^*)\, h\, ||\hat{\phi}||_{2,\Omega} \,. \tag{5.33}$$

Proof. Consider the pointwise projection of the exact solution onto $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$:

$$\forall i \in [1, I+J^\Gamma]\,,\ (\Pi\hat{\phi})_i^T = \hat{\phi}(G_i) \quad \text{and} \quad \forall k \in [1,K]\,,\ (\Pi\hat{\phi})_k^P = \hat{\phi}(S_k)\,.$$

Then, this element is itself projected onto V_N in the following way:

$$\forall i \in [1, I+J^\Gamma],\ (\tilde{\Pi}\hat{\phi})_i^T = (\Pi\hat{\phi})_i^T - \frac{\sum_{i\in[1,I]} |T_i| (\Pi\hat{\phi})_i^T}{|\Omega|}$$
$$\forall k \in [1,K],\ (\tilde{\Pi}\hat{\phi})_k^P = (\Pi\hat{\phi})_k^P - \frac{\sum_{k\in[1,K]} |P_k| (\Pi\hat{\phi})_k^P}{|\Omega|}\,.$$

Obviously, $\tilde{\Pi}\hat{\phi}$ and $\Pi\hat{\phi}$ have the same discrete gradient so that the interpolation error in (5.33) is bounded in the following way

$$\inf_{\omega_h \in L(V_N)} |\hat{\phi} - \omega_h|_{1,h} \le |\hat{\phi} - L(\tilde{\Pi}\hat{\phi})|_{1,h} = |\hat{\phi} - L(\Pi\hat{\phi})|_{1,h}\,.$$

Finally, an upper bound for $|\hat{\phi} - L(\Pi\hat{\phi})|_{1,h}$ has been given in [10] and is based on the relation between $L(\Pi\hat{\phi})$ and the standard Lagrange P^1 interpolants on the pairs $(D_{j,1}, D_{j,2})$ and $(D'_{j,1}, D'_{j,2})$. It leads to the estimation (5.33). Hypothesis 5.5 is here to ensure that the so-called maximum angle condition [3, 18] is verified for at least one of the pairs of triangles $(D_{j,1}, D_{j,2})$ or $(D'_{j,1}, D'_{j,2})$. ☐

Consistency error for $\hat{\phi}$. Let $\omega_h = L(\omega)$. Thanks to (5.21), we start by writing

$$a_h(\hat{\phi},\omega_h) - \ell_N(\omega_h) = \left[a_h(\hat{\phi},\omega_h) + (f,\omega_h)_\Omega - \int_\Gamma \sigma\, \tilde{\omega}_h(\xi)\, d\xi\right] - (f,\omega_h - \omega_h^*)_\Omega. \tag{5.34}$$

The last term in (5.34) can be bounded by the following lemma:

LEMMA 5.14. *If all diamond-cells are convex, there exists a constant C independent of the grid such that*

$$|(f,\omega_h-\omega_h^*)_\Omega| \le Ch||f||_{0,\Omega}|\omega_h|_{1,h}\ . \tag{5.35}$$

Proof. The proof is identical to that given in [10] for homogeneous Dirichlet conditions. □

Then, we follow [10] with a slight modification due to non-homogeneous Neumann boundary conditions. We divide each *interior* diamond-cell D_j (with $j \in [1, J - J^\Gamma]$) either into $D_{j,1} \cup D_{j,2}$, or into $D'_{j,1} \cup D'_{j,2}$ (see figure 2.5). Note that this choice is local to D_j and does not influence the choice which can be made for the division of $D_{j'}$, for $j' \neq j$. Boundary diamond-cells are such that $D_{j,1} = D_j$ and $D_{j,2} = \emptyset$ and will never be split into $D'_{j,1} \cup D'_{j,2}$. To simplify notations, we shall write $\mathcal{T}_{j,\alpha}$ to represent either $D_{j,\alpha}$ or $D'_{j,\alpha}$. Further, we define $RT(\nabla\hat{\phi})$, the Raviart-Thomas interpolation of $\nabla\hat{\phi}$ on each $\mathcal{T}_{j,\alpha}$ (see [23]) by

$$RT(\nabla\hat{\phi})_{|\mathcal{T}_{j,\alpha}} \in (P_0(\mathcal{T}_{j,\alpha}))^2 \oplus \begin{pmatrix} x \\ y \end{pmatrix} P_0(\mathcal{T}_{j,\alpha}) \quad \text{and} \quad \int_s RT(\nabla\hat{\phi})\cdot\mathbf{n}\, d\xi = \int_s \nabla\hat{\phi}.\mathbf{n}\, d\xi$$

for any edge s of $\mathcal{T}_{j,\alpha}$ whose normal exterior unit vector is denoted by $\mathbf{n}$. We can state the following lemma

LEMMA 5.15. *Let $\hat{\phi}$ be the solution of (5.15) and let $\omega_h \in L(V_N)$. Denote by $\langle\omega_h\rangle_{j,\alpha}$ the average value of ω_h over $\mathcal{T}_{j,\alpha}$. Then, if all diamond-cells are convex*

$$\begin{aligned} &a_h(\hat{\phi},\omega_h) + (f,\omega_h)_\Omega - \int_\Gamma \sigma\,\tilde{\omega}_h(\xi)\,d\xi = \\ &\sum_{j\in[1,J]}\sum_{\alpha=1}^{2}\int_{\mathcal{T}_{j,\alpha}} \left[(\nabla\hat{\phi} - RT(\nabla\hat{\phi}))\cdot\nabla\omega_h - f\ \left(\langle\omega_h\rangle_{j,\alpha} - \omega_h\right)\right]\, d\mathbf{x}\ . \end{aligned} \tag{5.36}$$

Proof. By definition, $RT(\nabla\hat{\phi})\cdot\mathbf{n}$ is a constant on each edge of $\mathcal{T}_{j,\alpha}$. In addition, on two neighboring triangles $\mathcal{T}_{j,\alpha}$, the values of $RT(\nabla\hat{\phi})\cdot\mathbf{n}$ on both sides of their common edge are opposite one to the other, because of the orientation of the normal vector $\mathbf{n}$. By noting $\mathcal{S}$ the set of all the edges of all the $\mathcal{T}_{j,\alpha}$ and $\mathbf{n}$ the normal unit vector to an edge s in $\mathcal{S}$, and $[\omega_h]_s$ the jump of ω_h through s, then

$$\begin{aligned} \sum_{j\in[1,J]}\sum_{\alpha=1}^{2}\int_{\partial\mathcal{T}_{j,\alpha}} RT(\nabla\hat{\phi})\cdot\mathbf{n}\,\omega_h\,d\xi &= \sum_{s\in\mathcal{S},\ s\not\subset\Gamma} RT(\nabla\hat{\phi})\cdot\mathbf{n}\int_s[\omega_h]_s\,d\xi \\ &\quad + \sum_{s\in\mathcal{S},\ s\subset\Gamma} RT(\nabla\hat{\phi})\cdot\mathbf{n}\int_s\omega_h\,d\xi\ . \end{aligned} \tag{5.37}$$

Since ω_h is in $L(V_N)$, then $[\omega_h]_s$ is a polynomial of degree one, which vanishes at the midpoint of s (by construction of the functions of $L(V_N)$). Its integral on s is thus null.

Further, there is an obvious one to one correspondence between a given $s \in \mathcal{S},\ s \subset \Gamma$ and some boundary edge A_j, with $j \in [J - J^\Gamma + 1, J]$ because boundary diamond-cells are such that $D_j = D_{j,1} = \mathcal{T}_{j,\alpha}$, with $\alpha = 1$. Therefore, for such $s \in \mathcal{S},\ s \subset \Gamma$, there exists a unique $j \in [J - J^\Gamma + 1, J]$ such that

$$RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \mathbf{n} = \frac{1}{|A_j|} \int_{A_j} RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \mathbf{n}_j = \frac{1}{|A_j|} \int_{A_j} \boldsymbol{\nabla}\hat{\phi} \cdot \mathbf{n}_j = \frac{1}{|A_j|} \int_{A_j} \sigma(\xi)\, d\xi \ .$$

Further, on this A_j, the function ω_h is a polynomial of degree one, whose integral is easy to compute:

$$\int_s \omega_h \, d\xi = \frac{|A_j|}{4} \left(\omega^P_{k_1} + 2\omega^T_{i_2} + \omega^P_{k_2} \right)$$

Recalling the definition (5.23) of the piecewise constant function $\tilde{\omega}_h$, we may write

$$\sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \int_{\partial \mathcal{T}_{j,\alpha}} RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \mathbf{n}\, \omega_h \, d\xi = \sum_{s \in \mathcal{S},\ s \subset \Gamma} RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \mathbf{n} \int_s \omega_h \, d\xi = \int_\Gamma \sigma \tilde{\omega}_h(\xi) d\xi \ .$$

But we may also write the above equality in the following way

$$\sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \left(\int_{\mathcal{T}_{j,\alpha}} \nabla \cdot (RT(\boldsymbol{\nabla}\hat{\phi}))\, \omega_h \, d\mathbf{x} + \int_{\mathcal{T}_{j,\alpha}} RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \boldsymbol{\nabla}\omega_h \, d\mathbf{x} \right) = \int_\Gamma \sigma \tilde{\omega}_h(\xi) d\xi \ .$$

By subtracting this equality from $a_h(\hat{\phi}, \omega_h)$, we obtain

$$\begin{aligned} a_h(\hat{\phi}, \omega_h) - \int_\Gamma \sigma \tilde{\omega}_h(\xi) d\xi = & \sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \int_{\mathcal{T}_{j,\alpha}} (\boldsymbol{\nabla}\hat{\phi} - RT(\boldsymbol{\nabla}\hat{\phi})) \cdot \boldsymbol{\nabla}\omega_h \, d\mathbf{x} \\ & - \sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \int_{\mathcal{T}_{j,\alpha}} \nabla \cdot (RT(\boldsymbol{\nabla}\hat{\phi}))\, \omega_h \, d\mathbf{x} \, . \end{aligned} \tag{5.38}$$

Let us note $\langle \omega_h \rangle_{j,\alpha}$ the mean value of ω_h on $\mathcal{T}_{j,\alpha}$. Since $\nabla \cdot (RT(\boldsymbol{\nabla}\hat{\phi}))$ is by construction a constant on $\mathcal{T}_{j,\alpha}$, we may write the following series of equalities

$$\begin{aligned} & \int_{\mathcal{T}_{j,\alpha}} \nabla \cdot (RT(\boldsymbol{\nabla}\hat{\phi}))\, \omega_h \, d\mathbf{x} = \langle \omega_h \rangle_{j,\alpha} \int_{\mathcal{T}_{j,\alpha}} \nabla \cdot (RT(\boldsymbol{\nabla}\hat{\phi}))\, d\mathbf{x} \ = \\ & \langle \omega_h \rangle_{j,\alpha} \int_{\partial \mathcal{T}_{j,\alpha}} RT(\boldsymbol{\nabla}\hat{\phi}) \cdot \mathbf{n} \, d\xi = \langle \omega_h \rangle_{j,\alpha} \int_{\partial \mathcal{T}_{j,\alpha}} \boldsymbol{\nabla}\hat{\phi} \cdot \mathbf{n} \, d\xi = \\ & \langle \omega_h \rangle_{j,\alpha} \int_{\mathcal{T}_{j,\alpha}} \Delta \hat{\phi} \, d\mathbf{x} = \langle \omega_h \rangle_{j,\alpha} \int_{\mathcal{T}_{j,\alpha}} f \, d\mathbf{x} \ . \end{aligned} \tag{5.39}$$

Equality (5.36) follows from (5.38) and (5.39). ▯

The first term in the right-hand side of (5.34) can be bounded by the following lemma

Lemma 5.16. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant C independent of the grid such that*

$$\left| a_h(\hat{\phi}, \omega_h) + (f, \omega_h)_\Omega - \int_\Gamma \sigma \tilde{\omega}_h(\xi)\, d\xi \right| \le C \frac{h}{\sin \tau^*} |\omega_h|_{1,h} \left(||f||_{0,\Omega} + ||\hat{\phi}||_{2,\Omega} \right) . \tag{5.40}$$

Proof. By virtue of lemma 5.15, bounding the left-hand-side of (5.40) amounts to bounding the right-hand-side of (5.36). This was performed in [10]. There again, hypothesis 5.5 is here to ensure the maximum angle condition needed by the Raviart-Thomas interpolation of $\boldsymbol{\nabla}\hat{\phi}$, see [1]. ◻

We end the consistency error estimation with

PROPOSITION 5.17. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant C, independent of the grid such that*

$$\sup_{\omega_h \in L(V_N)} \frac{|a_h(\hat{\phi}, \omega_h) - \ell_N(\omega_h)|}{|\omega_h|_{1,h}} \leq C \frac{h}{\sin \tau^*} \left(||f||_{0,\Omega} + ||\hat{\phi}||_{2,\Omega} \right) . \tag{5.41}$$

Proof. The result follows from (5.34), (5.35) and (5.40). ◻

Interpolation error for $\hat{\psi}$. Next, given the equivalent finite element formulation stated by Prop. 5.12, we may study the numerical error concerning ψ in a very analogous way: The interpolation error is bounded by choosing $\omega_h = L(\Pi\hat{\psi})$ with $\Pi\hat{\psi} \in V_D$ defined by

$$\forall i \in [1, I + J^\Gamma] \,, \;\; (\Pi\hat{\psi})_i^T = \hat{\psi}(G_i) \quad \text{and} \quad \forall k \in [1, K] \,, \;\; (\Pi\hat{\psi})_k^P = \hat{\psi}(S_k)$$

and we obtain a result analogous to (5.33):

PROPOSITION 5.18. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant $C(\tau^*)$ depending only on τ^* such that*

$$\inf_{\omega_h \in L(V_D)} |\hat{\psi} - \omega_h|_{1,h} \leq C(\tau^*) \, h \, ||\hat{\psi}||_{2,\Omega}. \tag{5.42}$$

Consistency error for $\hat{\psi}$. Concerning the consistency error, we may prove a result analogous to Eq. (5.36):

LEMMA 5.19. *Let $\hat{\psi}$ be the solution of (5.16) and let $\omega_h \in L(V_D)$. Then, if all diamond-cells are convex*

$$\begin{aligned} & a_h(\hat{\psi}, \omega_h) - (g, \omega_h)_\Omega + \sum_{q \in [1,Q]} k_q \left(\frac{c_{q,\omega}^T + c_{q,\omega}^P}{2} \right) = \\ & \sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \int_{\mathcal{T}_{j,\alpha}} \left[(\boldsymbol{\nabla}\hat{\psi} - RT(\boldsymbol{\nabla}\hat{\psi})) \cdot \boldsymbol{\nabla}\omega_h + g \, \left(\langle \omega_h \rangle_{j,\alpha} - \omega_h \right) \right] \, d\mathbf{x} \, . \end{aligned} \tag{5.43}$$

Proof. We first write for $\hat{\psi}$ an equality analogous to Eq. (5.37). For the same reasons as in the proof of lemma 5.15, this amounts to evaluating the boundary part:

$$\sum_{j \in [1,J]} \sum_{\alpha=1}^{2} \int_{\partial\mathcal{T}_{j,\alpha}} RT(\boldsymbol{\nabla}\hat{\psi}) \cdot \mathbf{n} \, \omega_h \, d\xi = \sum_{q \in [1,Q]} \sum_{j \in \Gamma_q} RT(\boldsymbol{\nabla}\hat{\psi}) \cdot \mathbf{n}_j \int_{A_j} \omega_h \, d\xi =$$

$$\sum_{q \in [1,Q]} \left(\frac{c_{q,\omega}^T + c_{q,\omega}^P}{2} \right) \sum_{j \in \Gamma_q} |A_j| RT(\boldsymbol{\nabla}\hat{\psi}) \cdot \mathbf{n}_j = \sum_{q \in [1,Q]} \left(\frac{c_{q,\omega}^T + c_{q,\omega}^P}{2} \right) \sum_{j \in \Gamma_q} \int_{A_j} \boldsymbol{\nabla}\hat{\psi} \cdot \mathbf{n}_j =$$

$$\sum_{q \in [1,Q]} \left(\frac{c_{q,\omega}^T + c_{q,\omega}^P}{2} \right) \int_{\Gamma_q} \boldsymbol{\nabla}\hat{\psi} \cdot \mathbf{n}_j = - \sum_{q \in [1,Q]} k_q \left(\frac{c_{q,\omega}^T + c_{q,\omega}^P}{2} \right) \, .$$

The end of the proof of (5.43) follows exactly the same lines as that of (5.36) and is thus skipped. □

Next, bounding the right-hand side of (5.43) is performed like in [10] and we obtain a result analogous to (5.41)

PROPOSITION 5.20. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant C, independent of the grid such that*

$$\sup_{\omega_h \in L(V_D)} \frac{|a_h(\hat{\psi}, \omega_h) - \ell_D(\omega_h)|}{|\omega_h|_{1,h}} \leq C \frac{h}{\sin \tau^*} \left(||g||_{0,\Omega} + ||\hat{\psi}||_{2,\Omega} \right) . \tag{5.44}$$

In conclusion of paragraph 5.3.2, estimates (5.31), (5.33) and (5.41) on the one hand and (5.32), (5.42) and (5.44) on the other hand allow us to state the following theorem:

THEOREM 5.21. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant $C(\tau^*)$ independent of the grid such that*

$$|\hat{\phi} - \phi_h|_{1,h} \leq C(\tau^*) h \left(||f||_{0,\Omega} + ||\hat{\phi}||_{2,\Omega} \right) . \tag{5.45}$$

and

$$|\hat{\psi} - \psi_h|_{1,h} \leq C(\tau^*) h \left(||g||_{0,\Omega} + ||\hat{\psi}||_{2,\Omega} \right) . \tag{5.46}$$

To conclude paragraph 5.3, Th. 5.21, along with (5.17) and (5.19) lead to

THEOREM 5.22. *If all diamond-cells are convex and under hypotheses 5.5 and 5.7, there exists a constant $C(\tau^*)$ independent of the grid such that*

$$\left(\sum_{j \in [1,J]} \int_{D_j} |\mathbf{u}_j - \hat{\mathbf{u}}(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \leq C(\tau^*) h \left(||f||_{0,\Omega} + ||g||_{0,\Omega} + ||\hat{\phi}||_{2,\Omega} + ||\hat{\psi}||_{2,\Omega} \right) .$$

6. Numerical results. We test the finite volume method over different types of meshes and we define the discrete relative L^2 error by:

$$e^2(h) := \frac{\sum_j |D_j| \, |\mathbf{u} - \Pi\hat{\mathbf{u}}|_j^2}{\sum_j |D_j| \, |\Pi\hat{\mathbf{u}}|_j^2} ,$$

where $(\Pi\hat{\mathbf{u}})_j$ is the value of the exact solution at the barycenter of D_j (noted B_j):

$$\forall j \in [1, J] , (\Pi\hat{\mathbf{u}})_j = \hat{\mathbf{u}}(B_j).$$

For the first three families of meshes (triangular unstructured, non-conforming, degenerating triangular), the domain of computation is the unit square $\Omega = [0;1] \times [0;1]$. We choose the data f, g and the boundary conditions so that the analytical solution is given by

$$\hat{\mathbf{u}}(x,y) = \begin{pmatrix} \exp(x) \cos(\pi y) + \pi \sin(\pi x) \cos(\pi y) \\ -\pi \exp(x) \sin(\pi y) - \pi \cos(\pi x) \sin(\pi y) \end{pmatrix}.$$

This means that the exact potentials are given by

$$\hat{\phi}(x,y) = \exp(x) \cos(\pi y) \quad \text{and} \quad \hat{\psi}(x,y) = \sin(\pi x) \sin(\pi y).$$

In addition, we always choose the points G_i associated to the control volumes of the primal mesh to be the barycenters of the cell T_i.

6.1. Unstructured meshes. First of all, we consider a family of six unstructured grids made up of increasingly small triangles. The first two of these grids are represented on the left and central parts of figure 6.1. The numerical errors in the discrete L^2 norm are presented in logarithmic scale on the right part of figure 6.1, on which we also plotted a straight line of slope 1. We remark, as proved previously, a first-order convergence of the presented scheme.

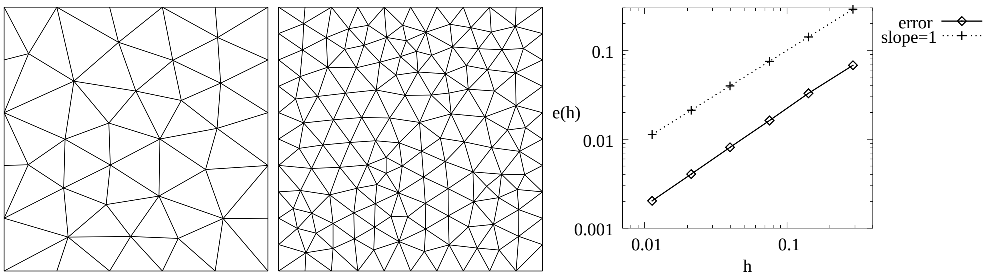

Fig. 6.1. *Unstructured triangular meshes.*

6.2. Non-conforming meshes. Next, we consider the non-conforming family of meshes constructed in the following way. Let n be a non-zero integer. We split Ω into $(2^n + 1) \times (2^n + 1)$ identical squares. Then, every other square is itself divided into $2^n \times 2^n$ identical sub-squares. We choose $n \in [1; 4]_N$. The left and central parts of figure 6.2 display the first two of these meshes. Of course, this family of meshes is not of practical use, but constitutes in our opinion a good choice in order to test the applicability of the presented method on arbitrarily locally refined non-conforming meshes. A zoom on the most distorted diamond cell for this type of mesh (with $n = 2$) is displayed on figure 6.3. Comparing this figure with Fig. 5.1, we infer that

$$\max(\alpha_1, \beta_1, \mu_1 + \mu_2, \alpha_2, \beta_2, \nu_1 + \nu_2) = \beta_2 ,$$

which is always lower than $\frac{3\pi}{4}$ for all values of n. Moreover, it is easily checked that the maximum angle of every boundary diamond-cell equals $\frac{\pi}{2}$, so that this family of meshes satisfies hypothesis 5.5 with an angle $\tau^* = \frac{3\pi}{4}$. The discrete L^2 error is displayed in logarithmic scale on the right part of figure 6.2, together with a reference straight line with a slope equal to one. We observe, on this family of non-conforming, locally refined meshes, a first-order convergence in the discrete L^2 norm.

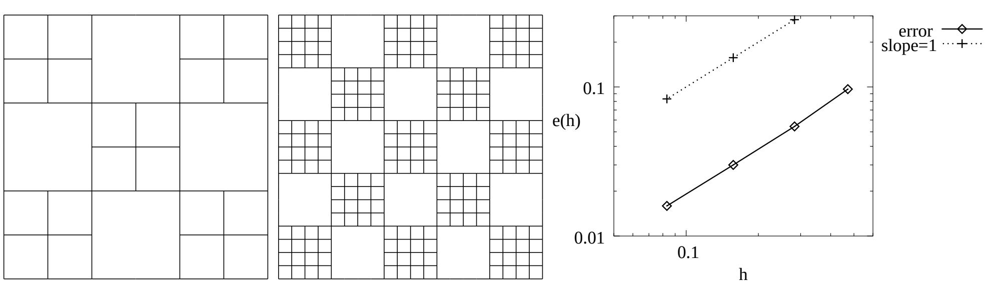

Fig. 6.2. *Non-conforming square meshes.*

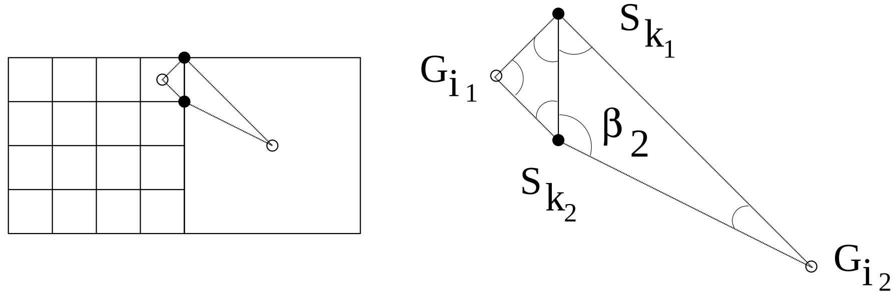

FIG. 6.3. *Zoom on a diamond-cell for the locally refined meshes with $n = 2$.*

6.3. Degenerating meshes. The third family is made up of grids of increasingly flat triangles built in the following way. Let n be a non-zero integer. We divide Ω into 4^n horizontal stripes of the same height and we divide each one of these stripes into similar triangles (except those at both ends) so that there are 2^n bases of triangles in the width of a stripe and we choose $n \in [1;6]_N$. The left and central parts of figure 6.4 represent the first two of these grids. The numerical errors in the L^2 norm are presented in logarithmic scale on the right part of figure 6.4, as well as a straight line of slope 1.5. Although such a family of meshes does not verify Hyp. 5.5 (due to boundary diamond-cells), we observe a superconvergence of the method in this case, which is due to the fact, as shown in [10], that almost all diamond-cells (except those at the boundary) are parallelograms.

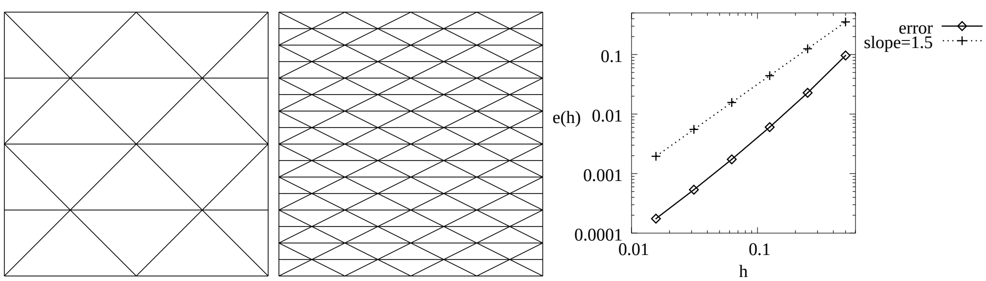

FIG. 6.4. *Degenerating triangular meshes.*

6.4. Non simply connected domains. Here, the domain of computation is $\Omega = [0,1]^2 \setminus [1/3,2/3]^2$ and the data and boundary conditions are chosen so that the analytic solution is given by

$$\hat{\mathbf{u}}(x,y) = \begin{pmatrix} \exp(x)\,\cos(\pi y) + 3\pi\,\sin(3\pi x)\,\cos(3\pi y) \\ -\pi\,\exp(x)\,\sin(\pi y) - 3\pi\,\cos(3\pi x)\,\sin(3\pi y) \end{pmatrix}.$$

This means that the exact potentials are given by

$$\hat{\phi}(x,y) = \exp(x)\,\cos(\pi y) \quad \text{and} \quad \hat{\psi}(x,y) = \sin(3\pi x)\,\sin(3\pi y)\,.$$

We compute the numerical solution on a family of five increasingly fine triangular meshes. The first two of the meshes are displayed on the left and central parts of figure 6.5. The numerical errors in the L^2 norm are presented in logarithmic scale on the right part of figure 6.5, as well as a straight line of slope 1. We observe the first order convergence of the scheme on this type of non-convex meshes when the solution is regular enough, which is not the case of the last example.

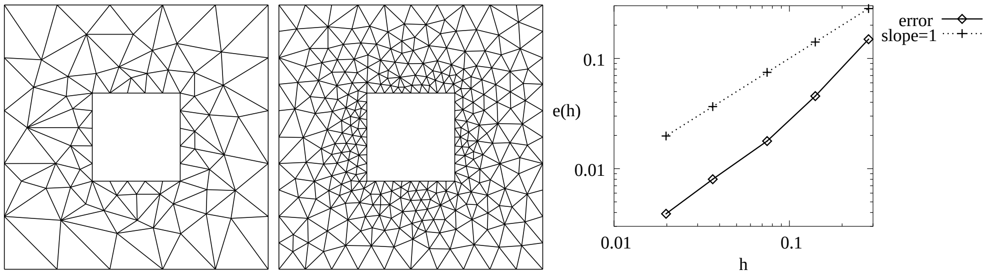

FIG. 6.5. *Non simply connected meshes.*

6.5. Non-convex domains and less regular solutions. Here, the domain of computation is $\Omega =]-1/2;1/2[^2 \backslash]0;1/2[^2$ and the data and boundary conditions are chosen so that the analytic solution, expressed in polar coordinates centered on $(0,0)$, is given by

$$\hat{\mathbf{u}}(r,\theta) = \boldsymbol{\nabla}(r^{2/3}\cos(\frac{2}{3}\theta)) \ ,$$

that is to say $\hat{\phi}(r,\theta) = r^{2/3}\cos(\frac{2}{3}\theta)$ and $\hat{\psi} = 0$. Note that $\hat{\phi}$ is still in H^1 but not in H^2, so that the error estimate derived in section 5.3 is not valid. More precisely, $\hat{\phi} \in (\mathrm{H}^{1+\mathrm{s}}(\Omega))^2$ with $\mathrm{s} < 2/3$. We use a family of five unstructured triangular grids. The first two meshes of this family are displayed on the left and central parts of figure 6.6, while the error curve in the discrete L^2 norm is shown on the right part of figure 6.6, together with a reference line of slope 2/3. The order of convergence of the scheme seems to be 2/3 in this case, like that obtained in [4].

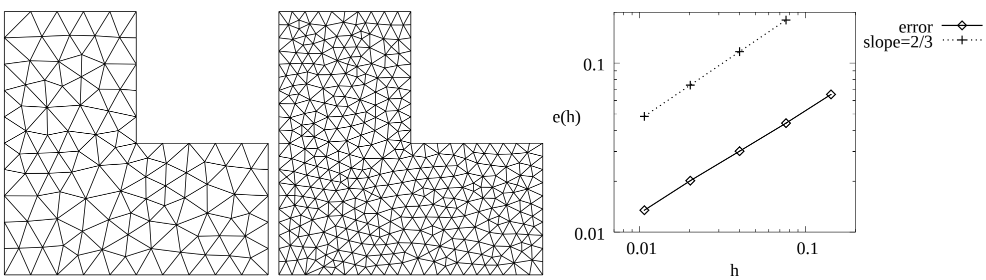

FIG. 6.6. *Non-convex meshes*

7. Conclusion. We have proposed new discretizations of differential operators such as divergence, gradient and curl on almost arbitrary two-dimensional meshes. These discrete operators verify discrete properties analogue to their continuous counterparts. We have applied these ideas to approximate the solution of two-dimensional div-curl problems and we have given error estimations for the resulting scheme. Finally, we have demonstrated the possibilities of the method by providing a series of numerical tests. Extensions of these ideas to problems with inhomogeneous and/or anisotropic and/or discontinuous coefficients and to the discretization of Stokes-like problems are currently being investigated.

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