Preconditioning Navier-Stokes Problem Discretized by Discrete Duality Finite Volume schemes [1]

Sarah Delcourte * — **Delphine Jennequin** **

** NACHOS Project, INRIA Sophia-Antipolis, France, Sarah.Delcourte@inria.fr*
*** DASSAULT SYSTEM, Suresnes, France, Delphine.Jennequin@3ds.com*

ABSTRACT. *We focus on the Discrete Duality Finite Volume (DDFV) method whose particularity is to allow the use of unstructured or nonconforming meshes. We discretize the non-linear Navier-Stokes problem, using the rotational formulation of the convection term, associated with the Bernoulli pressure. With an iterative algorithm, we are led to solve a saddle-point problem at each iteration. We give a particular interest to this linear problem by testing some preconditioners issued from finite elements, which we adapt to the DDFV method.*

KEYWORDS: *finite volumes, preconditioners, saddle-point, Navier-Stokes equations*

1. Introduction

Let be Ω an open bounded connected domain of $\mathbb{R}^2$ with a Lipschitz boundary denoted by Γ. We consider the numerical resolution of the bidimensional stationnary Navier-Stokes equations: given $\mathbf{f}$, find $(\mathbf{u}, p)$ such that

$$\left\{\begin{array}{rcl} -\nu\boldsymbol{\Delta}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}p & = & \mathbf{f} \ \text{ in } \Omega, \\ \nabla\cdot\mathbf{u} & = & 0 \ \text{ in } \Omega, \\ \mathbf{u} & = & 0 \ \text{ on } \Gamma, \\ \int_\Omega p(\mathbf{x})\,d\mathbf{x} & = & 0. \end{array}\right. \qquad [1]$$

The discretization of the Navier-Stokes equations by finite volume schemes has attracted interest these last years but classical finite volume methods work on meshes with orthogonal constraints like rectangular grids (see Harlow & Welch [HAR 65]) or so-called "admissible meshes" (see [EYM 00, Def. 9.1]), which can be seen as a generalization of Delaunay-Voronoi meshes.

[1] This work was performed when the authors were at the CEA Saclay, DANS/DM2S/SFME/LMPE, Gif sur Yvette, France.

In what follows, we focus on the DDFV approach described in [DOM 05] for the Laplace equation. The main interest of this staggered finite volume method is that it applies on almost all meshes (unstructured and non-conforming meshes) without any orthogonality constraint. However, the extension of this method to three-dimensional meshes need some adaptation. Here, we present the DDFV scheme for fluid dynamics equations and this one can be seen as a generalization of Nicolaides' scheme [NIC 95] developed on Delaunay-Voronoi meshes.

This paper is organized as follows: in section 2, we present the construction of the primal, dual and diamond meshes. Then, we define discrete gradient, divergence and curl operators on these meshes. In section 3, we focus on the Navier-Stokes equations and present its discretization. Section 4 is devoted to the description of several kinds of preconditioners adapted to the DDFV method and some numerical comparisons of these solvers are given.

2. Definitions and notations

We consider a first partition of Ω (named primal mesh) composed of elements T_i, with $i \in [1, I]$, supposed to be convex polygons.

Further, we denote by S_k, with $k \in [1, K]$, the nodes of the polygons of the primal mesh. With each of these points, we associate a polygon denoted by P_k, obtained by joining the centers of gravity G_i associated to the elements of the primal mesh (and possibly to midpoints of the boundary sides) of which S_k is a vertex to the midpoints of the edges of which S_k is an extremity. The P_ks constitute a second partition of Ω, referenced as dual mesh. Figure 1(a) displays an example of a primal mesh and its associated dual mesh.

With each edge of the primal mesh, denoted by $A_j = [S_{k_1(j)} S_{k_2(j)}]$, with $j \in [1, J]$, we associate a quadrilateral named "diamond cell" and denoted by D_j. When A_j is not on the boundary, this cell is obtained by joining the points $S_{k_1(j)}$ and $S_{k_2(j)}$, which are the two nodes of A_j, with the gravity centers $G_{i_1(j)}$ and $G_{i_2(j)}$ of the elements of the primal mesh sharing this side. When A_j is on the boundary Γ, the cell D_j is obtained by joining the two nodes of A_j with the point $G_{i_1(j)}$ associated with the only element of the primal mesh of which A_j is a side. The cells D_j constitute a third partition of Ω, named "diamond-mesh". Such cells are displayed in Figures 1(b) and 1(c). The unit normal vector to A_j and $A'_j = [G_{i_1(j)} G_{i_2(j)}]$ are respectively denoted by $\mathbf{n}_j$ and $\mathbf{n}'_j$. More precisely, $\mathbf{n}_{ji}$ points outward T_i while $\mathbf{n}'_{jk}$ points outward P_k. At last, the area of the cells T_i, P_k and D_j is denoted by $|T_i|$, $|P_k|$ and $|D_j|$.

Definition 1 *Given any $\phi = (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, the discrete gradient $\boldsymbol{\nabla}_h^D$ is defined by its values over the diamond cells D_j:*

$$(\boldsymbol{\nabla}_h^D \phi)_j := \frac{1}{2\,|D_j|} \Big\{ \Big[\phi_{k_2}^P - \phi_{k_1}^P\Big] \, |A'_j| \, \mathbf{n}'_j + \Big[\phi_{i_2}^T - \phi_{i_1}^T\Big] \, |A_j| \, \mathbf{n}_j \Big\}. \qquad [2]$$

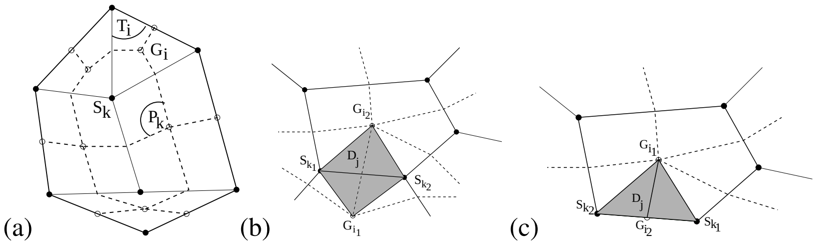

Figure 1. *(a) An example of a primal mesh and its associated dual mesh. (b) An inner diamond cell. (c) A boundary diamond cell.*

In the very same way, we may approach the vector curl operator $\boldsymbol{\nabla}\times\bullet = \left(\frac{\partial\bullet}{\partial y}\,,\,-\frac{\partial\bullet}{\partial x}\right)^T$ *by a discrete vector curl operator* $\boldsymbol{\nabla}_h^D\times$ *on the diamond cells:*

$$(\boldsymbol{\nabla}_h^D \times \phi)_j := -\frac{1}{2\,|D_j|}\left\{\left[\phi_{k_2}^P - \phi_{k_1}^P\right]|A_j'|\,\tau_j' + \left[\phi_{i_2}^T - \phi_{i_1}^T\right]|A_j|\,\tau_j\right\}, \qquad [3]$$

where the unit vectors τ_j *and* τ_j' *are such that* $(\mathbf{n}_j, \tau_j)$ *and* $(\mathbf{n}_j', \tau_j')$ *are orthogonal positively oriented bases of* $\mathbb{R}^2$*.*

Definition 2 *Given any* $\mathbf{u} = (\mathbf{u}_j) \in \mathbb{R}^{2J}$*, the discrete divergence* $\nabla_h^{T,P}\cdot := (\nabla_h^T\cdot, \nabla_h^P\cdot)$ *is defined by its values over the primal cells* T_i *and the dual cells* P_k*:*

$$\begin{aligned}(\nabla_h^T\cdot\mathbf{u})_i &:= \frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)}|A_j|\,\mathbf{u}_j\cdot\mathbf{n}_{ji}\;,\\ (\nabla_h^P\cdot\mathbf{u})_k &:= \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)}|A_j'|\,\mathbf{u}_j\cdot\mathbf{n}_j' + \sum_{j\in\mathcal{E}(k)\cap[J-J^\Gamma+1,J]}\frac{1}{2}\,|A_j|\,\mathbf{u}_j\cdot\mathbf{n}_j\right)\;.\end{aligned} \qquad [4]$$

In the very same way, we may approach the scalar curl operator $\nabla\times\bullet = \left(\frac{\partial\bullet_y}{\partial x} - \frac{\partial\bullet_x}{\partial y}\right)$ *by a discrete scalar curl operator* $\nabla_h^{T,P}\times := (\nabla_h^T\times, \nabla_h^P\times)$ *replacing the normal unit vector* $\mathbf{n}$ *by the tangential unit vector* τ *in [4].*

Definition 3 *If* $(\phi,\psi) \in \left(\mathbb{R}^I\times\mathbb{R}^K\right)^2$ *and* $(\mathbf{u},\mathbf{v}) \in \left(\mathbb{R}^{2J}\right)^2$*, then we define the following scalar products:*

$$(\mathbf{u},\mathbf{v})_D := \sum_{j\in[1,J]}|D_j|\,\mathbf{u}_j\cdot\mathbf{v}_j, \qquad [5]$$

$$(\phi,\psi)_{T,P} := \frac{1}{2}\left(\sum_{i\in[1,I]}|T_i|\,\phi_i^T\,\psi_i^T + \sum_{k\in[1,K]}|P_k|\,\phi_k^P\,\psi_k^P\right)\,. \qquad [6]$$

We also define the trace of $u \in \mathbb{R}^J$ and $\phi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ on the boundary Γ by

$$(u,\phi)_{\Gamma,h} := \sum_{j\in\Gamma} |A_j|\, u_j \times \frac{1}{4}\left(\phi^P_{k_1(j)} + 2\phi^T_{i_2(j)} + \phi^P_{k_2(j)}\right). \qquad [7]$$

Proposition 1 *The following discrete analogues of the Green formulae hold:*

$$(\nabla_h^{T,P} \cdot \mathbf{u}, \phi)_{T,P} = -(\mathbf{u}, \boldsymbol{\nabla}_h^D \phi)_D + (\mathbf{u}\cdot\mathbf{n}, \phi)_{\Gamma,h}\ , \qquad [8]$$

$$(\nabla_h^{T,P} \times \mathbf{u}, \phi)_{T,P} = (\mathbf{u}, \boldsymbol{\nabla}_h^D \times \phi)_D + (\mathbf{u}\cdot\tau, \phi)_{\Gamma,h}\ , \qquad [9]$$

for all $\mathbf{u} \in \left(\mathbb{R}^J\right)^2$ and all $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$.

3. Discretization of the Navier-Stokes equations

We are interested in the approximation of non-linear problem [1]. For continuous operators, $-\boldsymbol{\Delta}\mathbf{u}$ can be rewritten as $-\boldsymbol{\Delta}\mathbf{u} = \boldsymbol{\nabla} \times \nabla \times \mathbf{u} - \boldsymbol{\nabla}\nabla\cdot\mathbf{u}$. On the other hand, to avoid a problem of definition of the convective term on staggered meshes, we use the rotational formulation of $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ which reads:

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = (\nabla\times\mathbf{u})\,\mathbf{u}\times\mathbf{e}_z + \boldsymbol{\nabla}\left(\frac{\mathbf{u}^2}{2}\right), \qquad [10]$$

where $\mathbf{u}\times\mathbf{e}_z = (-\mathbf{u}_y, \mathbf{u}_x)^T$ with $\mathbf{u}_x$ and $\mathbf{u}_y$ the two components of $\mathbf{u}$, and we introduce the Bernoulli pressure: $\pi = p + \dfrac{\mathbf{u}^2}{2}$. At last, in order to ensure the uniqueness of π, we set $\int_\Omega \pi(\mathbf{x})\,d\mathbf{x} = 0$.

With an iterative process to solve the non-linearity (the fixed-point method for example), we are led to solve the following linear system, called Oseen equations: given $\mathbf{f}$ and $\mathbf{u}_G$, find $(\mathbf{u},\pi)$ such that

$$\left\{\begin{array}{rcl} -\nu\ [\boldsymbol{\nabla}\times\nabla\times\mathbf{u} - \boldsymbol{\nabla}\nabla\cdot\mathbf{u}] + (\nabla\times\mathbf{u}_G)\,\mathbf{u}\times\mathbf{e}_z + \boldsymbol{\nabla}\pi &=& \mathbf{f}\ \text{ in }\Omega,\\ \nabla\cdot\mathbf{u} &=& 0\ \text{ in }\Omega,\\ \mathbf{u} &=& 0\ \text{ on }\Gamma,\\ \int_\Omega \pi(\mathbf{x})\,d\mathbf{x} &=& 0. \end{array}\right. \qquad [11]$$

Hypothesis 1 *We assume that each boundary primal cell has only one edge which belongs to the boundary Γ.*

We look for the approximation $(\mathbf{u}_j)_{j\in[1,J]}$ of the velocity $\mathbf{u}$ on the diamond cells and the approximation $(\pi_i^T)_{i\in[1,I]}$, $(\pi_k^P)_{k\in[1,K]}$ of the Bernoulli pressure π on the primal and dual cells respectively.

We discretize the first equation of [11] on the interior diamond cells and the second equation of [11] both on the primal and dual cells. Then, the boundary condition $\mathbf{u} = 0$ is discretized on the boundary diamond cells while the condition of vanishing mean pressure is discretized on the primal and dual cells.

We shall suppose that the locations of the values of $\mathbf{u}_G$ are the same as those of $\mathbf{u}$, that is on the diamond cells. Therefore, we may easily calculate $\nabla \times \mathbf{u}_G$ on the primal and dual cells according to the discrete operator $\nabla_h^{T,P} \times$. However, since the first equation in [11] is discretized on diamond cells, we shall use the following quadrature formula to calculate $\nabla \times \mathbf{u}_G$ over any D_j:

$$(\nabla \times \mathbf{u}_G)_{|D_j} \approx \frac{(\nabla_h^T \times \mathbf{u}_G)_{i_1} + (\nabla_h^T \times \mathbf{u}_G)_{i_2} + (\nabla_h^P \times \mathbf{u}_G)_{k_1} + (\nabla_h^P \times \mathbf{u}_G)_{k_2}}{4}. \quad [12]$$

Then, for all diamond cells, we set:

$$-\left[\boldsymbol{\Delta}_h^D \mathbf{u}\right]_j = (\boldsymbol{\nabla}_h^D \times \nabla_h^{T,P} \times \mathbf{u})_j - (\boldsymbol{\nabla}_h^D \nabla_h^{T,P} \cdot \mathbf{u})_j.$$

Now, we can discretize the continuous problem [11] by the following system:

$$\begin{aligned} -\nu \left[\boldsymbol{\Delta}_h^D \mathbf{u}\right]_j + (\nabla \times \mathbf{u}_G)_{|D_j} \mathbf{u}_j \times \mathbf{e}_z + (\boldsymbol{\nabla}_h^D \pi)_j &= \mathbf{f}_j^D, \quad \forall D_j \notin \Gamma, && [13a] \\ (\nabla_h^{T,P} \cdot \mathbf{u})_{i,k} &= 0, \quad \forall T_i, \forall P_k, && [13b] \\ \mathbf{u}_j &= \mathbf{f}_j^D, \quad \forall D_j \in \Gamma, && [13c] \\ \sum_{i \in [1,I]} |T_i|\, \pi_i^T = \sum_{k \in [1,K]} |P_k|\, \pi_k^P &= 0, && [13d] \end{aligned}$$

where we have set $\mathbf{f}_j^D = \frac{1}{|D_j|} \int_{D_j} \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \ \forall j \notin \Gamma$ and $\mathbf{f}_j^D = 0, \ \forall j \in \Gamma$.

Property 1 *For any vector $\mathbf{u}_j$, the convection term of problem [13] satisfies:*

$$(\nabla \times \mathbf{u}_G)_{|D_j} (\mathbf{u}_j \times \mathbf{e}_z) \cdot \mathbf{u}_j = 0, \ \forall j \in [1, J - J^\Gamma]. \quad [14]$$

This property allows us to state the following proposition.

Proposition 2 *Under Hyp. 1, the solution $((\mathbf{u}_j)_{j \in [1,J]}, (\pi_i^T)_{i \in [1,I]}, (\pi_k^P)_{k \in [1,K]})$ of [13] exists and is unique.*

Proof 1 *Equations [13a] and [13c] provide $2J$ equations and as many velocity unknowns. Then, Eq. [13b] provides $I + K$ equations and as many pressure unknowns. However, Eqs. [13b] and [13c] are not independent, which is balanced by the 2 Eqs. of [13d]. Then, we have as many independent equations as unknowns. It remains to show the injectivity of the system. Multiplying [13a] by $\mathbf{u}_j$ such that $\mathbf{u}_j = 0 \ \forall Dj \in \Gamma$ and using property 1, it follows that:*

$$\nu\, (\boldsymbol{\nabla}_h^D \times \nabla_h^{T,P} \times \mathbf{u}, \mathbf{u})_D + (\boldsymbol{\nabla}_h^D \pi, \mathbf{u})_D = 0. \quad [15]$$

Applying discrete Green formulae [8] and [9] to the previous line, Eqs. [13b] and [13c] imply that $(\nabla_h^{T,P} \times \mathbf{u})_{i,k} = 0 \ \forall Ti, \forall P_k$. Thus, we obtain an homogeneous Div-Curl problem and it follows from [DEL 07b] that

$$\mathbf{u}_j = 0, \, \forall j \in [1, J]. \quad [16]$$

Since $\mathbf{u}_j = 0$, Eq. [13a] shows that $(\boldsymbol{\nabla}_h^D \pi)_j = 0$, for all interior diamond cells, which implies, according to [2], that $\pi^T_{i_{2(j)}} = \pi^T_{i_{1(j)}}$ and $\pi^P_{k_{2(j)}} = \pi^P_{k_{1(j)}}$, $\forall D_j \notin \Gamma$. Since the domain is connected, any pair of primal cells may be joined by a finite number of interior dual edges. Concerning the boundary nodes S_k, Hyp. 1 is here to ensure that all these nodes are vertices of at least one interior diamond-cell. Thus, all the p_i^T (resp. p_k^P) are equal to the same constant c^T (resp. c^P). Finally, using [13d], we conclude that:

$$\forall T_i,\ \pi_i^T = 0 \textit{ and } \forall P_k,\ \pi_k^P = 0. \qquad [17]$$

Once the $(\mathbf{u}_j)_{j\in[1,J]}$ and the $(\pi_i^T, \pi_k^P)_{i\in[1,I],k\in[1,K]}$ have been calculated, we can easily deduce the approximation $(p_i^T, p_k^P)_{i\in[1,I],k\in[1,K]}$ of p thanks to the formula $p = \pi - \frac{\mathbf{u}^2}{2}$ and using quadrature formulae to define the velocity $\mathbf{u}$ on the primal and dual cells.

Many numerical convergence results are detailed about this scheme in [DEL 07a] on unstructured and non-conforming meshes by comparison to analytical solutions. On strongly non-conforming meshes, the DDFV scheme seems to converge with an order one for the velocity $\mathbf{u}$, and an order 0.5 for the pressure p and the vorticity $\omega = \nabla \times \mathbf{u}$. For unstructured meshes, we observe at best a second order of convergence for the velocity, the pressure and the vorticity.

4. Preconditionners and numerical results

4.1. *Preconditioners*

When the Navier-Stokes equations are solved by a fixed-point type method, we must solve a linear system at each non-linear iteration, which takes the form of a saddle point-problem and can be solved by an Uzawa method:

$$\left\{ \begin{array}{rcl} A\mathbf{u} + B^T p &=& \mathbf{f}\,, \\ -BA^{-1}B^T p &=& -BA^{-1}\mathbf{f}\,, \end{array} \right. \qquad [18]$$

which requires a preconditioner for the Schur complement $S = -BA^{-1}B^T$. Many efficient preconditioners are known for saddle-point problems arising from finite-element discretizations, but their adaptation to matrices issued from DDFV discretizations is not trivial. For example, the preconditioners based on formal commutators described in [KAY 02] are not well defined on staggered grids. On the other hand, the preconditioner proposed by Olshanskii and Vassilevski [OLS 07] for the rotational formulation does not fit the discretization by the DDFV method and the reason is unclear.

In what follows, we focus on two kinds of preconditioners $\tilde{S}^{-1}$ for the Schur complement $S = -BA^{-1}B^T$ which can be applied to our problem:

– the SIMPLE preconditioner: $\tilde{S}^{-1} = -(B\hat{A}^{-1}B^T)^{-1}$ where $\hat{A}$ is an approximation of A (for example, the diagonal matrix of A).

– approximate commutators (called BFBt method, which is the notation introduced by Elman [ELM 99]):

$$\tilde{S}^{-1} = -(BM_2^{-1}B^T)^{-1}(BM_2^{-1}AM_2^{-1}B^T)(BM_2^{-1}B^T)^{-1},$$

where the main candidates for M_2 are the identity matrix, the diagonal of A or the diagonal velocity mass-matrix X defined on the diamond cells and whose elements are the weights $|D_j|$. Note that this preconditioner is more expensive than SIMPLE because we need to invert $BM_2^{-1}B^T$ twice whereas for the SIMPLE preconditioner, we need to invert $B\hat{A}^{-1}B^T$ only one time.

For the BFBt method, Elman [ELM 99] shows that, for the MAC finite difference scheme with Dirichlet boundary conditions, constant $\mathbf{u}_G$ or mildly ν-dependent variable $\mathbf{u}_G$, the iteration count of the Krylov method is independent of ν and increases in proportion to $h^{-1/2}$.

4.2. *Numerical results*

The experiments are done in Fortran 90 with PETSC and MUMPS libraries. The meshes are unstructured triangulations obtained with Emc2. The domain size is $[0,1]\times[0,1]$. The test case to be considered is a lid-driven cavity type problem with $\mathbf{u}_G = \left(\ 2(2y-1)(1-(2x-1)^2)\ , -2(2x-1)(1-(2y-1)^2)\ \right)^T$ and with boundary conditions $\mathbf{u}(x,1) = (1,0)^T$ and $\mathbf{u}(x,y) = 0$ elsewhere. The right-hand side is supposed to vanish.

Mesh size h	Preconditioner	$\nu = 1$	$\nu = 10^{-1}$	$\nu = 10^{-2}$	$\nu = 10^{-3}$
0.0753	$M_2 = I$	11	12	27	275
	$M_2 = X$	10	12	32	440
	$M_2 = \text{diag}(A)$	12	13	27	178
	SIMPLE	84	94	94	293
0.0398	$M_2 = I$	17	19	36	207
	$M_2 = X$	15	15	42	NC
	$M_2 = \text{diag}(A)$	17	18	35	223
	SIMPLE	171	175	187	278
0.02125	$M_2 = I$	32	35	44	189
	$M_2 = X$	19	22	73	NC
	$M_2 = \text{diag}(A)$	32	34	40	175
	SIMPLE	294	342	346	394
0.01129	$M_2 = I$	53	58	64	163
	$M_2 = X$	28	32	101	NC
	$M_2 = \text{diag}(A)$	66	61	69	160
	SIMPLE	614	814	909	830

Table 1. *Iteration count for the linear solver.*

The use of an incomplete factorization gives some good results, but using exact solvers for A and $BM_2^{-1}B^T$ produces a better comparison with the results of Elman: indeed, his tests were done in Matlab using the "backslash" solver.

Table 1 shows the iteration count of a Bicgstab preconditioned by the SIMPLE and BFBt preconditioners described in section 4.1. The linear iterations are stopped when the tolerance is smaller than 10^{-8}, and "NC" for "No Convergence" means that the stopping criterion is not satisfied after 5000 iterations.

The SIMPLE iteration numbers present some small variations with the viscosity but grows h^{-1}-linearly. On the other hand, we observe that the iteration count of the BFBt method grows slowly with h^{-1}. The scaling by the velocity mass-matrix X leads to an improvement with $h^{-1/2}$ only for large viscosities (greater than 10^{-2}). The scaling by $\text{diag}(A)$ aims to precondition A and is useful only when the mesh is not sufficiently refined (for example, for $\nu = 10^{-3}$ and $h = 0.0753$). Since the DDFV scheme is a MAC-type scheme, $M_2 = I$ shows a performance almost similar to that of Elman [ELM 99] in the case of a circular vortex.

In summary, the BFBt preconditioner seems to be a very robust black-box solver and it allows us to solve efficiently the 2D linearized Navier-Stokes equations discretized by the DDFV method for the moderate Reynolds number.

5. References

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