The length of the graph of an increasing function with an almost everywhere zero derivative.

1 Answer to problem 1.

Lemma 1 *If A is a set on which f has a null derivative, then $f(A)$ is a null set.*

Proof of lemma: Let $y = f(x)$, with $x \in A$. The inverse function of f has an infinite derivative at $y = f(x)$. This inverse function is, as f, an increasing function. So it has almost everywhere a finite derivative by Lebesgue's theorem. This forces y to stay inside a null set.

Lemma 2 *Let f be continous, increasing defined on $[a, b]$. Then the length of f's graph is bounded below by $b - a$ and $f(b) - f(a)$, and bounded above by $b - a + f(b) - f(a)$.*

Proof of lemma: This length is the upper-bound of

$$\sum_{k=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

with $a = x_0 < x_1 < \cdots < x_n = b$. Each term of the sum is greater than $x_i - x_{i-1}$, greater than $f(x_i) - f(x_{i-1})$, and at most $x_i - x_{i-1} + f(x_i) - f(x_{i-1})$.

Solution of problem 1: We note $c = f(a)$ and $d = f(b)$. By lemma (2), the length of f's graph is not greater than $b - a + d - c$. Now we are looking for a lower bound. We construct a partition of $[a, b]$, with a finite number of intervals, on which the length of the corresponding arc of f is essentially composed of the increase of the x–coordinate, or of the increase of the y–coordinate. This is possible because there is a subset of $[a, b]$ with measure $b - a$ whose image is null.

Let (I_n) be a sequence of disjoint intervals of total length $< \epsilon$, outside of which f has a zero derivative. Let us note $A = [a, b] \setminus \cup_n I_n$, and $J_n = f(I_n)$. $[c, d]$ is the union of $f(A)$ and the $f(I_n)$; by lemma (1), $f(A)$ is null. So the sum of the lengths of the J_n is $d - c$. After renaming we can assume than $\sum_{k=1}^{p} \text{length}(J_k) \geq d - c - \epsilon$. Now, $[a, b] \setminus \cup_{1 \leq k \leq p} I_k$ is a finite union of intervals $(V_n)_{1 \leq n \leq q}$, whose total length is greater then $b - a - \epsilon$. The length of f's graph is the sum of the lengths of f restricted to the I_k, and of f restricted to the V_k. Using (2), f restricted to V_k has length greater than $\text{length}(V_k)$, and f restricted to I_k has length greater than $\text{length}(J_k)$. Summing on all intervals, we get the lower bound

$$\sum_{k=1}^{p} \text{length}(J_k) + \sum_{j=1}^{q} \text{length}(V_j) \geq d - c - \epsilon + b - a - \epsilon.$$

2 Answer to problem 2.

Lemma 3 *If $\beta = \frac{k}{n}$ we have the following upper bound for the binomial coefficient:*

$$\binom{n}{k} \leq \delta(\beta)^n \qquad \textit{with } \delta(\beta) = \frac{1}{\beta^\beta (1 - \beta)^{1-\beta}}$$

and δ is a strictly increasing function of β for $\beta \leq \frac{1}{2}$, and $\delta(1/2) = 2$.

Proof of lemma: We start from

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

which gives, for every $x > 0$, $\log \binom{n}{k} \leq n \log(1+x) - k \log x$. It remains to choose x, minimising the second member. The derivative is $n/(1+x) - k/x$, it is zero for $x = \frac{\beta}{1-\beta}$, and, for this value of x, the second member takes value $-n\left[(1-\beta)\log(1-\beta) - \beta \log(\beta)\right] = \log \delta^n$. The study of the variations of $\delta(\beta)$ is easy.

Now, let us go to the solution of the second problem :

Answer 2 *Let $u, v > 0$, and let L_n defined by*

$$L_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{1 + 4^n u^{2k} v^{2n-2k}}; \tag{1}$$

then

1. *If $u + v > 1$ then $\lim L_n = +\infty$.*
2. *If $u + v < 1$ then $\lim L_n = 1$.*
3. *If $u + v = 1$ with $u = v$ then $\lim L_n = \sqrt{2}$*
4. *If $u + v = 1$ and $u \neq v$ then $\lim L_n = 2$.*

Proof : If we replace the radical in (1) by the lower bound 1 we get at once $L_n \geq 1$. Now

$$\begin{aligned} L_n - (u+v)^n &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{1 + 4^n u^{2k} v^{2n-2}} - \sqrt{4^n u^{2k} v^{2n-2}} \right) \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\sqrt{1 + 4^n u^{2k} v^{2n-2}} + \sqrt{4^n u^{2k} v^{2n-2}}} \\ &\leq \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \quad = 1 \end{aligned}$$

So we get $(u+v)^n \leq L_n \leq (u+v)^n + 1$. Together with $L_n \geq 1$, we deduce points (1) and (2). Point (3) is obvious. There remains point (4). We start with

$$L_n - 1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\sqrt{1 + 4^n u^{2k} v^{2n-2}} + \sqrt{4^n u^{2k} v^{2n-2}}} \tag{2}$$

We can suppose $u < 1/2$. The function $x \mapsto 4^n u^{2x} v^{2n-2x} = (2v)^{2n} \left(\frac{u}{v}\right)^{2x}$, defined on $[0, n]$, is decreasing, from $(2v)^{2n}$ down to $(2u)^{2n}$. It takes value 1 when $\frac{x}{n} = \alpha$, with $\alpha = \dfrac{\log 2v}{\log v - \log u}$. And an easy computation shows that $\alpha < 1/2$. Let γ be such that $\alpha < \gamma < \frac{1}{2}$. We write

$$L_n \geq \frac{1}{2^n} \sum_{k \geq \gamma n} \binom{n}{k} \frac{1}{\sqrt{1 + 4^n u^{2k} v^{2n-2}} + \sqrt{4^n u^{2k} v^{2n-2}}} \tag{3}$$

And, for $k \geq \gamma n$ we have the upper bound

$$4^n u^{2kn} v^{2n-2k} = (2v)^{2n} \left(\frac{u}{v}\right)^{2k} \leq (2v)^n \left(\frac{u}{v}\right)^{2\gamma n} = \left(\frac{u}{v}\right)^{2n(\gamma - \alpha)}$$

and this is converging towards 0, **uniformly in** k, when n goes to infinity. So the fractions in the second member of (3) converge to 1, when n goes to infinity, uniformly in k. Let $\varepsilon > 0$; then for n large enough we have

$$L_n \geq (1-\varepsilon)\frac{1}{2^n}\sum_{k\geq\gamma n}\binom{n}{k} = (1-\epsilon) - (1-\varepsilon)\frac{1}{2^n}\sum_{k<\gamma n}\binom{n}{k}$$

And it remains to show that $2^{-n}\sum_{k<\gamma n}\binom{n}{k}$ tends to 0 with $1/n$. But by lemma (3) there is $\delta < 2$ such that $\binom{n}{k} \leq \delta^n$ for every $k \leq \gamma n$, and

$$\frac{1}{2^n}\sum_{k<\gamma n}\binom{n}{k} \leq \frac{\delta^n}{2^n}\sum_{k<\gamma n} 1 \leq n\frac{\delta^n}{2^n},$$

concluding the proof.