GENERALISATIONS OF CAPPARELLI’S AND PRIMC’S IDENTITIES, I:
COLOURED FROBENIUS PARTITIONS AND COMBINATORIAL PROOFS

JEHANNE DOUSSE AND ISAAC KONAN

Abstract. The partition identities of Capparelli and Primc were originally discovered via representation theoretic techniques, and have since then been studied and refined combinatorially, but the question of giving a very broad generalisation remained open. In these two companion papers, we give infinite families of partition identities which generalise Primc’s and Capparelli’s identities, and study their consequences on the theory of crystal bases of the affine Lie algebra $A_{n-1}^{(1)}$.

In this first paper, we focus on combinatorial aspects. We give a $n^2$-coloured generalisation of Primc’s identity by constructing a $n^2 \times n^2$ matrix of difference conditions, Primc’s original identities corresponding to $n = 2$ and $n = 3$. While most coloured partition identities in the literature connect partitions with difference conditions with partitions with congruence conditions, in our case, the natural way to generalise these identities is to relate partitions with difference conditions with coloured Frobenius partitions. This gives a very simple expression for the generating function. With a particular specialisation of the colour variables, our generalisation also yields a partition identity with congruence conditions.

Then, using a bijection from our new generalisation of Primc’s identity, we deduce two families of identities on $(n^2 - 1)$-coloured partitions which generalise Capparelli’s identity, also in terms of coloured Frobenius partitions. The particular case $n = 2$ is Capparelli’s identity and the case $n = 3$ recovers another identity of Primc.

In the second paper, we will focus on crystal theoretic aspects. We will show that the difference conditions we defined in our $n^2$-coloured generalisation of Primc’s identity are actually energy functions for certain $A_{n-1}^{(1)}$ crystals, and use this to derive a non-specialised character formula with positive coefficients for all the irreducible highest weight $U_q(A_{n-1}^{(1)})$-modules of level 1.

1. Introduction and statement of results

1.1. Partition identities from representation theory.

1.1.1. The Rogers-Ramanujan identities. A partition $\lambda$ of a positive integer $n$ is a non-increasing sequence of natural numbers $(\lambda_1, \ldots, \lambda_s)$ whose sum is $n$. The numbers $\lambda_1, \ldots, \lambda_s$ are called the parts of $\lambda$, the number $\ell(\lambda) = s$ is the length of $\lambda$, and $|\lambda| = n$ is the weight of $\lambda$. For example, the partitions of 4 are $4$, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$.

The most famous partition identities are probably the Rogers-Ramanujan identities [RR19]. Using the standard $q$-series notation for $n \in \mathbb{N} \cup \{\infty\}$,

\[ (a; q)_n := (1 - a)(1 -aq) \cdots (1 - aq^{n-1}), \]

they can be stated as follows.

**Theorem 1.1** (Rogers 1894, Ramanujan 1913). Let $i = 0$ or $1$. Then

\[ \sum_{n \geq 0} \frac{q^{n^2 + (1-i)n}}{(q; q)_n} = \frac{1}{(q^{2-i}; q^3)_\infty (q^{3+i}; q^5)_\infty}. \] (1.1)

By interpreting both sides of (1.1) as generating functions for partitions, MacMahon [Mac16] gave the following combinatorial version of the identities.

**Theorem 1.2** (Rogers-Ramanujan identities, partition version). Let $a = 0$ or $1$. For every natural number $n$, the number of partitions of $n$ such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $1 - a$ times is equal to the number of partitions of $n$ into parts congruent to $\pm(1 + a)$ mod 5.
More generally, a partition identity of the Rogers-Ramanujan type is a theorem stating that for all \( n \), the number of partitions of \( n \) satisfying some difference conditions equals the number of partitions of \( n \) satisfying some congruence conditions. Dozens of proofs of these identities have been given, using different techniques, see for example [And84b, Bre83, GM81, Wat29]. But the starting point of our discussion is a representation theoretic proof due to Lepowsky and Wilson [LW84, LW85].

First, Lepowsky and Milne [LM78a, LM78b] noticed that the product side of the Rogers-Ramanujan identities (1.1) multiplied by the “fudge factor” \( 1/(q;q^2)_{\infty} \) is equal to the principal specialisation of the Weyl-Kac character formula for level 3 standard modules of the affine Lie algebra \( A_1^{(1)} \).

Then, Lepowsky and Wilson [LW84, LW85] gave an interpretation of the sum side by constructing a basis of these standard modules using vertex operators. Very roughly, they proceed as follows. They start with a spanning set of the module \( V \), say monomials of the form \( Z_{j_1}^{f_1} \cdots Z_{j_s}^{f_s} \) for \( s, f_1, \ldots, f_s \in \mathbb{N} \). Then by the theory of vertex operators, there are some relations between these monomials, which allows them to reduce the spanning set by removing the monomials containing \( Z_j^2 \) and \( Z_jZ_{j+1} \). The last step is then to prove that this reduced family of monomials is actually free, and therefore a basis of the representation. The connection to Theorem 1.1 is then done by noting that monomials \( Z_{j_1}^{f_1} \cdots Z_{j_s}^{f_s} \) which do not contain \( Z_j^2 \) or \( Z_jZ_{j+1} \) for any \( j \) are in bijection with partitions which do not contain twice the part \( j \) or both the part \( j \) and \( j+1 \) for any \( j \), i.e. partitions with difference at least 2 between consecutive parts.

The theory of vertex operator algebras developed by Lepowsky and Wilson turned out to be very influential: for example, it was used by Frenkel, Lepowsky, and Meurman to construct a natural representation of the Monster finite simple group [FLM88], and was key in the work of Borcherds on vertex algebras and his resolution of the Conway-Norton monstrous moonshine conjecture [Bor92].

1.1.2. Capparelli’s identity. Following Lepowsky and Wilson’s discovery, several other representation theorists studied other Lie algebras or representations at other levels, and discovered new interesting and intricate partition identities, that were previously unknown to the combinatorics community, see for example [Cap93, MP87, MP99, MP01, Nan14, Pri94, PS16, Sil17].

After Lepowsky and Wilson’s work, Capparelli [Cap93] was the first to conjecture a new identity, by studying the level 3 standard modules of the twisted affine Lie algebra \( A_1^{(2)} \). It was first proved combinatorially by Andrews in [And92], then refined by Alladi, Andrews and Gordon in [AAG95] using the method of weighted words, and finally proved by Capparelli [Cap96] and Tamba and Xie [TX95] via representation theoretic techniques. Later, Meurman and Primc [MP99] showed that Capparelli’s identity can also be obtained by studying the \((1,2)\)-specialisation of the character formula for the level 1 modules of \( A_1^{(1)} \). Capparelli’s original identity can be stated as follows.

**Theorem 1.3** (Capparelli’s identity (Andrews 1992)). Let \( C(n) \) denote the number of partitions of \( n \) into parts > 1 such that parts differ by at least 2, and at least 4 unless consecutive parts add up to a multiple of 3. Let \( D(n) \) denote the number of partitions of \( n \) into distinct parts not congruent to \( \pm 1 \) \((\text{mod} \ 6)\). Then for every positive integer \( n \), \( C(n) = D(n) \).

But in this paper, we will mostly be interested in the weighted words version of Theorem 1.3, which was obtained by Alladi, Andrews and Gordon in [AAG95]. The principle of the method of weighted words, introduced by Alladi and Gordon to refine Schur’s identity [AG93], is to give an identity on coloured partitions, which under certain transformations on the coloured partitions, becomes the original identity. We now describe Alladi, Andrews, and Gordon’s refinement of Capparelli’s identity (slightly reformulated by the first author in [Dou18b]).

Consider partitions into natural numbers in three colours, \( a, c, \) and \( d \) (the absence of the colour \( b \) will be made clear in a few paragraphs, when we will mention the connection with Primc’s identity), with the order

\[
1_a < 1_c < 1_d < 2_a < 2_c < 2_d < \cdots ,
\]

satisfying the difference conditions in the matrix

\[
C_2 = \begin{pmatrix}
a & c & d \\
2 & 2 & 2 \\
c & 1 & 1 & 2 \\
d & 0 & 1 & 2 \\
2 & \end{pmatrix},
\]

satisfying the difference conditions in the matrix

\[
C_2 = \begin{pmatrix}
a & c & d \\
2 & 2 & 2 \\
c & 1 & 1 & 2 \\
d & 0 & 1 & 2 \\
2 & \end{pmatrix},
\]

satisfying the difference conditions in the matrix

\[
C_2 = \begin{pmatrix}
a & c & d \\
2 & 2 & 2 \\
c & 1 & 1 & 2 \\
d & 0 & 1 & 2 \\
2 & \end{pmatrix},
\]
where the entry \((x, y)\) gives the minimal difference between consecutive parts of colours \(x\) and \(y\).

The non-dilated version of Capparelli’s identity can be stated as follows.

**Theorem 1.4** (Alladi–Andrews–Gordon 1995). Let \(C_2(n; i, j)\) denote the number of partitions of \(n\) into coloured integers satisfying the difference conditions in matrix \(C_2\), having \(i\) parts coloured \(a\) and \(j\) parts coloured \(d\). We have

\[
\sum_{n, i, j \geq 0} C(n; i, j) a^i d^j q^n = (-q)_{\infty} (-aq; q^2)_{\infty} (-dq; q^2)_{\infty}.
\]

Performing the dilations

\[q \to q^3, \quad a \to aq^{-1}, \quad d \to dq,\]

which correspond to the following transformations on the parts of the partitions

\[k_a \to (3k - 1)a, \quad k_b \to 3k, \quad k_d \to (3k + 1)d,\]

we obtain a refinement of Capparelli’s original identity. Other dilations can lead to infinitely many other (but related) partition identities. Moreover, finding such refinements and non-dilated versions of partition identities can be helpful to find bijective proofs of them. For example, Siladić’s identity [Sil17] was also discovered by using representation theory. Then, based on a non-dilated version of the theorem due to the first author [Dou17b], the second author [Kon19b] was recently able to give a bijective proof and a broad generalisation of the identity. For more on combinatorial refinements of partition identities, see for example [AG93, All97, AAG95, AAB03, CL06, Dou17a, DL18, Dou18a, Dou18b, Kon19a].

1.1.3. **Primc’s identities.** Another way to obtain Rogers-Ramanujan type partition identities using representation theory is the theory of perfect crystals of affine Lie algebras. Much more detail on crystals is given in the second paper [DK19] of this series, but the rough idea is the following. The generating function for partitions with congruence conditions, which is always an infinite product, is still obtained via a specialisation of the Weyl-Kac character formula. The equality with the generating function for partitions with difference conditions is established through the crystal base character formula of Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [KKM92]. This formula expresses, under certain specialisations, the character as the generating function for partitions satisfying difference conditions given by energy matrices of perfect crystals.

The second identity which we study in this paper, due to Primc [Pri99], was obtained that way by studying crystal bases of \(A_1^{(1)}\). The energy matrix of the perfect crystal coming from the tensor product of the vector representation and its dual is given by

\[
P_2 = \begin{pmatrix} a & b & c & d \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.
\]

Let \(P(n; i, j, k, \ell)\) denote the number of partitions of \(n\) into four colours \(a, b, c, d\), with \(i\) (resp. \(j, k, \ell\)) parts coloured \(a\) (resp. \(b, c, d\)), satisfying the difference conditions of the matrix \(P_2\). Then the crystal base character formula and the Weyl-Kac character formula imply that under the dilations

\[k_a \to 2k - 1, \quad k_b \to 2k, \quad k_c \to 2k, \quad k_d \to 2k + 1,\]

the generating function for these coloured partitions becomes \(1/(q; q)_{\infty}\).

**Theorem 1.5** (Primc 1999). We have

\[
\sum_{n, i, j, k, \ell} P(n; i, j, k, \ell) q^{2n-i-j} = \frac{1}{(q; q)_{\infty}}.
\]
By doing the same approach in the affine Lie algebra $A_2^{(1)}$, Primc also gave the following energy matrix (where the naming of the colours comes from our generalisation):

$$
P_3 = \begin{pmatrix}
2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \\
\end{pmatrix}.
$$

(1.6)

**Theorem 1.6** (Primc 1999). Under the dilations

$$
k_{a_{2b_0}} \to 3k - 2, \quad k_{a_{2b_1}} \to 3k - 1, \quad k_{a_{b_0}} \to 3k - 1, \\
k_{a_{2b_1}} \to 3k, \quad k_{a_{b_1}} \to 3k, \quad k_{a_{b_2}} \to 3k, \\
k_{a_{b_0}} \to 3k + 1, \quad k_{a_{b_2}} \to 3k + 1, \quad k_{a_{b_2}} \to 3k + 2,
$$

(1.7)

the generating function for 9-coloured partitions satisfying the difference conditions of (1.6) becomes $1/(q; q)_\infty$.

When seeing these two theorems of Primc, one might find it surprising that the generating function for partitions with such intricate difference conditions simply becomes $1/(q; q)_\infty$, the generating function for unrestricted partitions. However recently, the first author and Lovejoy [DL18] gave a weighted words version of Theorem 1.5.

**Theorem 1.7** (Dousse-Lovejoy 2018, non-dilated version of Primc’s identity). Let $P(n; i, j, k, \ell)$ be defined as above. We have

$$
\sum_{n, i, j, k, \ell} P(n; i, j, k, \ell) q^n a^i b^j c^k d^\ell = \frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q)_\infty (cq; q^2)_\infty}.
$$

Performing the dilations of (1.5) indeed transforms the infinite product above into $1/(q; q)_\infty$. But the theorem above shows that keeping track of all colours except $b$ leads to a much more intricate infinite product as well, and that the extremely simple expression $1/(q; q)_\infty$ appears only because of the particular dilation that Primc considered. Later, the first author [Dou18b] even gave an expression for the generating function for $P(n; i, j, k, \ell)$ keeping track of all the colours, but it can be written as an infinite product only if we do not keep track of the colour $b$.

Thus it is interesting from a combinatorial point of view to see whether a similar phenomenon happens with Theorem 1.6 as well. To do so, we would like to compute the generating function for coloured partitions satisfying the difference conditions (1.6), at the non-dilated level, and keeping track of as many colours as possible. In this paper, not only do we succeed to do this, but we embed both of Primc’s theorems into an infinite family of identities about partitions satisfying difference conditions given by $n^2 \times n^2$ matrices.

Apart from the fact that they can be obtained from the same Lie algebra $A_1^{(1)}$, Capparelli’s and Primc’s identities didn’t seem related from the representation theoretic point of view, as they were obtained in completely different ways, and Capparelli’s identity did not seem related to perfect crystals. However, recently, the first author [Dou18b] gave a bijection between coloured partitions satisfying the difference conditions (1.4) and pairs of partitions $(\lambda, \mu)$, where $\lambda$ is a coloured partition satisfying the difference conditions (1.3), and $\mu$ is a partition coloured $b$. This bijection preserves the total weight, the number of parts, the size of the parts, and the number of parts coloured $a$ and $d$. Therefore, combinatorially, these two identities are very closely related.

We will generalise this bijection to our new generalisation of Primc’s identity and obtain two families of partition identities with difference conditions given by $(n^2 - 1) \times (n^2 - 1)$ matrices, which generalise Capparelli’s identity.
1.2. Statement of results.

1.2.1. The difference conditions generalising Primc’s identity. In this paper, we give a family of partition identities with \( n^2 \) colours which generalises the two identities of Primc, and two families of partition identities with \( n^2 - 1 \) colours which generalise Capparelli’s identity.

In a previous paper [Kon19b], the second author gave a family of identities generalising Siladić’s identity using \( n \) primary colours and \( n^2 \) secondary colours (products of two primary colours), giving \( n^2 + n \) colours in total. In [CL06], Corteel and Lovejoy, gave a family of identities generalising Schur’s theorem, later generalised by the first author to overpartitions [Dou18a]. These generalisations use \( n \) primary colours, and products of at most \( n \) different colours, giving \( 2^n - 1 \) colours in total.

Here, our generalisation will use only secondary colours, so we will have \( n^2 \) colours in total. Let us first define these colours and the corresponding difference conditions. We start with two sequences of symbols \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\), and use them to define two types of colours.

**Definition 1.8.** The **free colours** are the elements of the set \( \{a_i b_i : i \in \mathbb{N}\} \), and the **bound colours** are the elements of the set \( \{a_i b_k : i \neq k, i, k \in \mathbb{N}\} \).

**Remark.** We choose these names because, to obtain our main theorems, we will set \( b_i = a_i^{-1} \) for all \( i \). In that case, the free colours will vanish, while the bound colours will have relations between them.

In this paper, we consider partitions whose parts are coloured in free and bound colours, satisfying some difference conditions. We now define these difference conditions, which generalise those of matrices (1.4) and (1.6) in the two identities of Primc.

**Definition 1.9.** For all \( i, k, i', k' \in \mathbb{N} \), we define the minimal difference \( \Delta \) between a part coloured \( a_i b_k \) and a part coloured \( a_{i'} b_{k'} \) in the following way:

\[
\Delta(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k'),
\]

where \( \chi(prop) \) equals 1 if the proposition \( prop \) is true and 0 otherwise.

We start by observing some basic properties of \( \Delta \) (the proofs, which are straightforward applications of the definition, are left to the reader).

**Property 1.10.** For all \( i, k, i', k' \in \mathbb{N} \), \( \Delta(a_i b_k, a_{i'} b_{k'}) \) belongs to \( \{0, 1, 2\} \).

**Property 1.11.** For all \( i \in \mathbb{N} \), we have \( \Delta(a_i b_i, a_i b_k) = 0 \). In other words, free colours can repeat arbitrarily many times.

**Property 1.12.** For all \( i, k \in \mathbb{N} \) such that \( i \neq k \), we have \( \Delta(a_i b_i, a_k b_k) = 1 \). In other words, a part of a given size cannot appear in two different free colours.

**Property 1.13** (Triangular inequality). Let \( i, k, i', k' \in \mathbb{N} \). For all \( i'', k'' \in \mathbb{N} \), we have

\[
\Delta(a_i b_k, a_{i''} b_{k''}) = \Delta(a_i b_k, a_{i'} b_{k'}) + \Delta(a_{i'} b_{k'}, a_{i''} b_{k''}).
\]

In other words, it is equivalent to say that \( \Delta(a_i b_k, a_{i'} b_{k'}) \) is the minimal difference between parts coloured \( a_i b_k \) and \( a_{i'} b_{k'} \), and that it is the minimal difference between consecutive parts coloured \( a_i b_k \) and \( a_{i'} b_{k'} \).

For every positive integer \( n \), we define \( \mathcal{P}_n \) to be the set of partitions \( \lambda_1 + \cdots + \lambda_s \), where each part has a colour chosen from \( \{a_i b_k : 0 \leq i, k \leq n - 1\} \), satisfying the difference conditions for all \( j \in \{1, \ldots, s-1\} \):

\[
\lambda_j - \lambda_{j+1} \geq \Delta(c(\lambda_j), c(\lambda_{j+1})),
\]

where for all \( j \), \( c(\lambda_j) \) is the colour of part \( \lambda_j \).

To simplify some calculations throughout the paper, we adopt the following convention. If \( c_1, \ldots, c_s \) is the colour sequence of the partition \( \lambda_1 + \cdots + \lambda_s \), we add free colours \( c_0 = c_{s+1} = a_{\infty} b_{\infty} \) to both extremities of the colour sequence. The difference conditions are, for all \( i, k \in \mathbb{N} \),

\[
\Delta(a_{\infty} b_{\infty}, a_i b_k) = \Delta(a_i b_k, a_{\infty} b_{\infty}) = 1,
\]

which is coherent with the definition (1.8) of \( \Delta \). We also assume that \( \lambda_{s+1} = 0 \).

The difference conditions defining \( \mathcal{P}_n \) generalise Primc’s difference conditions matrices \( P_2 \) and \( P_3 \) in (1.4) and (1.6), as we shall see in the next two examples.
Example 1.14. If we set \(a = a_1b_0, b = a_0b_0, c = a_1b_1, d = a_0b_1\), as shown in Table (1.9), then \(P_2\) is exactly the set of partitions with difference conditions (1.4) of Primc’s 4-coloured theorem.

<table>
<thead>
<tr>
<th>(b_i) (\delta)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(b)</td>
<td>(a)</td>
</tr>
<tr>
<td>1</td>
<td>(d)</td>
<td>(c)</td>
</tr>
</tbody>
</table>

(1.9)

For example,

\[
\Delta(a, b) = \Delta(a_1b_0, a_0b_0) = \chi(1 \geq 0) - \chi(1 = 0 = 0) + \chi(0 \leq 0) - \chi(0 = 0 = 0) = 1 - 0 + 1 - 1 = 1.
\]

This is exactly the entry in row \(a\) and column \(b\) in (1.4).

Example 1.15. The set \(P_3\) is exactly the set of partitions with difference conditions (1.6) of Primc’s 9-coloured theorem. For example,

\[
\Delta(a_2b_0, a_2b_1) = \chi(2 \geq 2) - \chi(2 = 0 = 2) + \chi(0 \leq 1) - \chi(0 = 2 = 1) = 1 - 0 + 1 - 0 = 2.
\]

This is exactly the entry in row \(a_2b_0\) and column \(a_2b_1\) in (1.6).

It turns out that the matrix \((\Delta(a_kb_\ell; a_\ell'b_k')_{(k, \ell), (k', \ell') \in \{0, \ldots, n-1\}^2})\) is an energy matrix for the crystal of the tensor product of the vector representation \(B\) of \(A_\nu^{(1)}\) and its dual \(B'\). This will be proved in our second paper [DK19]. Using the formulas for the generating functions proved in this paper, it will allow us to give, for all \(\ell \in \{0, \ldots, n-1\}\), the first explicit expression for the characters \(\mathrm{ch}(L(\Lambda_\ell))\) of the irreducible highest weight modules \(L(\Lambda_\ell)\) as a series in \(\mathbb{Z}[[e^{-\delta}, e^{\pm \alpha_1}, \cdots, e^{\pm \alpha_{n-1}}]]\) with positive coefficients, where the \(\alpha_i\)'s are the simple roots.

1.2.2. The difference conditions generalising Capparelli’s identity. In the previous section, we gave difference conditions which generalise those of Primc’s identities (1.4) and (1.6). In this section, we define two other families of difference conditions which generalise those of Capparelli’s identity (1.3).

For these two generalisations, we still consider partitions whose parts are coloured in free and bound colours, but the free colour \(a_0b_0\) is now forbidden. Let us start with the first family of difference conditions.

Definition 1.16. For \(i, k, i', k' \in \mathbb{N}\), let us define the minimal difference \(\delta(a_ib_k, a_{i'}b_{k'})\) between a part coloured \(a_ib_k\) and a part coloured \(a_{i'}b_{k'}\) in the following way:

\[
\delta(a_ib_k, a_{i'}b_k') = 1 \text{ for all } k \in \mathbb{N}^*, \quad \delta(a_{i}b_k, a_{i}b_{k'}) = 1 \text{ for all } \ell < k, \quad \delta(a_{i}b_k, a_{i'}b_{k'}) = 1 \text{ for all } \ell < k, \quad \delta(a_{i}b_k, a_{i'}b_{k'}) = \Delta(a_{i}b_k, a_{i'}b_{k'}) \text{ in all the other cases.}
\]

(1.10)

Remark. In all the cases where \(\delta \neq \Delta\), then \(\delta = 1\) and \(\Delta = 0\).

For every positive integer \(n\), we define \(C_n\) to be the set of partitions \(\lambda_1 + \cdots + \lambda_s\), where each part has a colour chosen from \(\{a_ib_k : 0 \leq i, k \leq n - 1, (i, k) \neq (0,0)\}\), satisfying the difference conditions for all \(j \in \{1, \ldots, s-1\}\):

\[
\lambda_j - \lambda_{j+1} \geq \delta(c(\lambda_j), c(\lambda_{j+1})).
\]

These difference conditions generalise those of Capparelli’s identity stated in (1.3).

Example 1.17. If we define \(a, c, d\) (omitting \(b = a_0b_0\)) as previously in Table (1.9), then \(C_2\) is exactly the set of partitions with difference conditions (1.3) of Capparelli’s identity. For example,

\[
\delta(c, a) = \delta(a_1b_1, a_1b_0) = 1.
\]
Example 1.18. The set $C_3$ is the set of partitions with difference conditions shown in the following matrix:

\[
C_3 = \begin{pmatrix}
  a_2b_0 & a_2b_1 & a_1b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \\
  a_2b_0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
  a_2b_1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\
  a_1b_0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
  a_2b_2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
  a_1b_1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
  a_0b_1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\
  a_1b_2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
  a_0b_2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
\end{pmatrix}.
\] (1.11)

Let us now turn to the second family of difference conditions.

Definition 1.19. For $i, k, i', k' \in \mathbb{N}$, let us define the minimal difference $\delta'(a_i b_k, a_{i'} b_{k'})$ between a part coloured $a_i b_k$ and a part coloured $a_{i'} b_{k'}$ in the following way:

\[
\delta'(a_i b_k, a_{i'} b_{k'}) = 1 \text{ for all } k \in \mathbb{N}^*, \\
\delta'(a_i b_k, a_{i'} b_{k-1}) = 1 \text{ for all } \ell \geq k \geq 1, \\
\delta'(a_{i-1} b_k, a_{i} b_{k'}) = 1 \text{ for all } \ell \geq k \geq 1, \\
\delta'(a_i b_k, a_{i'} b_{k'}) = \Delta(a_i b_k, a_{i'} b_{k'}) \text{ in all the other cases}.
\]

(1.12)

For every positive integer $n$, we define $C'_n$ to be the set of partitions $\lambda_1 + \cdots + \lambda_s$, where each part has a colour chosen from $\{a_i b_k : 0 \leq i, k \leq m - 1, (i, k) \neq (0,0)\}$, satisfying the difference conditions for all $j \in \{1, \ldots, s-1\}$:

\[
\lambda_j - \lambda_{j+1} \geq \delta'(c(\lambda_j), c(\lambda_{j+1})�).
\]

These difference conditions also generalise those of Capparelli’s identity, as well as those of another partition identity mentioned in Primc’s paper [Pri99].

Example 1.20. Defining the colours $a, c, d$ as before in Table (1.9), $C'_2$ is again exactly the set of partitions with difference conditions of Capparelli’s identity.

Example 1.21. The set $C'_3$ is the set of partitions with difference conditions shown in the following matrix, which appeared in Primc’s paper [Pri99].

\[
C'_3 = \begin{pmatrix}
  a_2b_0 & a_2b_1 & a_1b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \\
  a_2b_0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
  a_2b_1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\
  a_1b_0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
  a_2b_2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
  a_1b_1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\
  a_0b_1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
  a_1b_2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
  a_0b_2 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}.
\] (1.13)

It was proved by Meurman and Primc in [MP01], using basic $A_2^{(1)}$ modules, that after performing the dilations (1.7), the generating function for these partitions becomes $(q; q^3)_\infty^3(q^2; q^3)_\infty^3$.

Recently in [Dou18b], the first author gave a bijection between Primc’s partitions $P_2$ and pairs $(\lambda, \mu)$ where $\lambda \in C_2$ is a Capparelli partition and $\mu$ is a classical partition. This bijection only modifies some free colours, so it preserves the weight, the number of parts, the size of the parts, and the number of appearances of colours $a$ and $d$. In this way, she showed that Capparelli’s identity is very closely related to Primc’s identity and can be deduced from it, even though until then, these two identities seemed unrelated from the representation theoretic point of view.

Here, we generalise this idea and show the following.
Theorem 1.22. For every positive integer \( n \), let \( \mathcal{CC}_n \) (resp. \( \mathcal{CC}'_n \)) denote partition pairs \((\lambda, \mu)\), where \( \lambda \in \mathcal{C}_n \) (resp. \( \mathcal{C}'_n \)) and \( \mu \) is a partition where all parts have colour \( a_0 b_0 \).

There is a bijection between:

- coloured partitions in \( P_n \),
- coloured partition pairs in \( \mathcal{CC}_n \),
- coloured partition pairs in \( \mathcal{CC}'_n \),

This bijection preserves the weight, the number of parts, the size of the parts, and the number of appearances of each bound colour.

Both Capparelli’s identity and Meurman and Primc’s identity with difference conditions (1.13) did not have any apparent connection to the theory of perfect crystals. The bijection between \( P_2 \) and \( \mathcal{CC}_2 \) in [Dou18b] gave an unexpected connection with Primc’s identity and the theory of perfect crystals. The present theorem shows that Meurman and Primc’s identity with difference conditions (1.13) can actually be deduced from Primc’s Theorem 1.6. More generally, through the bijection with the \( P_n \)’s, we related both families of generalisations of Capparelli’s identity to the theory of perfect crystals.

The detailed bijections are given in Section 4.

1.2.3. Coloured Frobenius partitions. Since its discovery, Capparelli’s identity has been one of the most studied partition identities in the literature, see for example [BM15, BU15, DL19, FZ18, KR18, Kur18, Sil04] for articles from the combinatorial point of view.

While the other most important partition identities, such as the Rogers-Ramanujan identities [RR19] and Schur’s theorem [Sch26] have been successfully embedded in large families of identities, such as the Andrews-Gordon identities for Rogers-Ramanujan [And74, Gor65] and Andrews’ theorems for Schur’s theorem [And69, And68], finding such a broad generalisation of Capparelli’s identity was still an open problem.

Here, we solve this problem by giving two different families of identities which generalise Capparelli and a family of identities generalising Primc. Unlike most classical Rogers-Ramanujan type identities, we relate the partitions with difference conditions defined in the previous section to coloured Frobenius partitions. This allows us to find simple and elegant formulations for the generating functions.

Following Andrews [And84a], a Frobenius partition is a two-rowed array

\[
\left( \begin{array}{cccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_s \\
\mu_1 & \mu_2 & \cdots & \mu_s
\end{array} \right),
\]

where \( s \) is a non-negative integer and \( \lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_s \) and \( \mu := \mu_1 + \mu_2 + \cdots + \mu_s \) are two partitions into \( s \) distinct non-negative parts. Frobenius partitions of length \( s \) and weight \( m = s + \sum_{i=1}^{s} \lambda_i + \sum_{i=1}^{s} \mu_i \) are in bijection with partitions of \( n \) whose Durfee square (the largest square fitting in the top-left corner of the Ferrers board of the partition) is of side \( s \). An example can be seen on Figure 1 in the case \( s = 4 \) (where \( \lambda_4 = \mu_4 = 0 \)).

![Figure 1. A Frobenius partition of length 4.](image-url)

The generating function for the number \( F(m) \) of Frobenius partitions of \( m \) is given by

\[
\sum_{m \geq 0} F(m) q^m = [x^0]((-xq; q)_\infty(-x^{-1}; q)_\infty).
\]
Indeed, the product \((-xq;q)_\infty\) generates the partition \(\lambda\) together with the boxes on the diagonal where the power of \(x\) counts the number of parts, \((-x^{-1};q)_\infty\) generates the partition \(\mu\) where the power of \(x^{-1}\) counts the number of parts, and taking the coefficient of \(x^n\) in the above ensures that \(\lambda\) and \(\mu\) have the same number of parts. Using Jacobi’s triple product identity (see, e.g., [And84b]),

\[
(-xq;q)_\infty(-x^{-1};q)_\infty(q;q)_\infty = \sum_{k\in\mathbb{Z}} x^k q^{\frac{k(k+1)}{2}},
\]

we see that the generating function for Frobenius partitions equals \(1/(q;q)_\infty\), the generating function for partitions.

Let us now define the coloured Frobenius partitions which will be related to our coloured partitions with difference conditions. We consider the same families of symbols \((a_i)_{i\in\mathbb{N}}\) and \((b_i)_{i\in\mathbb{N}}\) as in the previous section. We define a \(n^2\)-coloured Frobenius partition to be a Frobenius partition

\[
\left(\begin{array}{cccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_s \\
\mu_1 & \mu_2 & \cdots & \mu_s
\end{array}\right),
\]

where \(\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s\) is a partition into \(s\) distinct non-negative parts, each coloured with some \(a_i\), \(i \in \{0,\ldots, n-1\}\), with the following order

\[
0_{a_{n-1}} < 0_{a_{n-2}} < \cdots < 0_{a_0} < 1_{a_{n-1}} < 1_{a_{n-2}} < \cdots < 1_{a_0} < \cdots,
\]

and \(\mu = \mu_1 + \mu_2 + \cdots + \mu_s\) is a partition into \(s\) distinct non-negative parts, each coloured with some \(b_i\), \(i \in \{0,\ldots, n-1\}\), with the order

\[
0_{b_0} < 0_{b_1} < \cdots < 0_{b_{n-1}} < 1_{b_0} < 1_{b_1} < \cdots < 1_{b_{n-1}} < \cdots.
\]

Let \(\mathcal{F}_n\) denote the set of \(n^2\)-coloured Frobenius partitions. Note that in \(\lambda\) and \(\mu\), a part of a given size can appear in different colours. We define the colour sequence of such a \(n^2\)-coloured Frobenius partition to be \((c(\lambda_1)c(\mu_1)), \ldots, c(\lambda_s)c(\mu_s))\).

**Example 1.23.** The following is an example of 9-coloured Frobenius partition with colour sequence \((a_2b_2, a_0b_0, a_1b_0, a_2b_1)\) and weight 18:

\[
\left(\begin{array}{cccc}
3_{a_1} & 2_{a_0} & 0_{a_1} & 0_{a_2} \\
4_{b_2} & 4_{b_0} & 1_{b_0} & 0_{b_1}
\end{array}\right).
\]

Following the same reasoning as for classical Frobenius partitions, the generating function for the number \(F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})\) of \(n^2\)-coloured Frobenius partitions of \(m\) where for \(i \in \{0,\ldots, n-1\}\), the symbol \(a_i\) (resp. \(b_i\)) appears \(u_i\) (resp. \(v_i\)) times, is

\[
F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^{m u_0} \cdots u_{n-1} b_1 v_1 \cdots v_{n-1} = [x^0] \prod_{i=0}^{n-1} (-x a_i q) (-x^{-1} b_i q) \infty.
\]

Similar coloured Frobenius partitions had already been introduced by Andrews in [And84a], but he didn’t keep track of the colours in the generating function there.

1.2.4. **Generalisations of Capparelli and Primc’s identities.** The \(n^2\)-coloured Frobenius partitions are very natural objects to study our generalisations of Primc and Capparelli’s identities. Indeed their generating function (1.17) is exactly the generating function for the coloured partitions in \(\mathcal{P}_n\).

**Theorem 1.24** (Connection between \(\mathcal{P}_n\) and \(\mathcal{F}_n\)). Let \(n\) be a positive integer.

Let \(P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})\) be the number of \(n^2\)-coloured partitions of \(m\) in colours \(\{a_i b_k : 0 \leq i, k \leq n-1\}\), satisfying the difference conditions \(\Delta\) (see (1.8)), where for \(i \in \{0,\ldots, n-1\}\), the symbol \(a_i\) (resp. \(b_i\)) appears \(u_i\) (resp. \(v_i\)) times in their bound colours.

Let \(F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})\) be the number of \(n^2\)-coloured Frobenius partitions of \(m\) where for \(i \in \{0,\ldots, n-1\}\), the symbol \(a_i\) (resp. \(b_i\)) appears \(u_i\) (resp. \(v_i\)) times in their bound colours.
Then
\[
\sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1} = \sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1}.
\]

**Remark.** We actually prove a refinement of Theorem 1.25 according to the notion of reduced colour sequence defined in Section 2. This is given in Theorem 3.9. We do not state it in this introduction to avoid technicalities.

Moreover, when we set for all \(i\), \(b_i = a_i^{-1}\), then all free colours vanish and we have the following elegant expression for our generating functions as the constant term in an infinite product.

**Theorem 1.25** (Generalisation of Princ’s identity). Let \(n\) be a positive integer. We have
\[
\sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1} = \sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1}.
\]

From this theorem, it is easy to deduce a corollary, corresponding to the principal specialisation, which generalises both of Princ’s original identities. By performing the dilations \(q \to q^a\), and for all \(i \in \{0, \ldots, n-1\}\), \(a_i \to q^{-i}\), the generating function above becomes \([x^0](-xq; q)_\infty(-x^{-1}; q)_\infty\), which is also equal to \(1/(q; q)_\infty\).

**Corollary 1.26** (Principal specialisation). Let \(n\) be a positive integer. We have
\[
\sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1} = \frac{1}{(q; q)_\infty}.
\]

Moreover, by using Jacobi’s triple product repeatedly, we are able to give an expression of the generating function for coloured Frobenius partitions as a sum of infinite products, which gives yet another expression for the generating function for \(P_n\).

**Theorem 1.27**. Let \(P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n)\) be the number of \(n^2\)-coloured partitions of \(m\) in colours \(\{a_i, b_k : 0 \leq i, k \leq n - 1\}\), satisfying the difference conditions \(\Delta\) (see (1.8)), where for \(i \in \{0, \ldots, n-1\}\), the symbol \(a_i\) (resp. \(b_i\)) appears \(u_i\) (resp. \(v_i\)) times in the colour sequence. Then
\[
\sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_n \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_n) q^m a_{u_0} \cdots a_{u_{n-1}} b_{v_0} \cdots b_{v_n-1} = \frac{1}{(q; q)_\infty} \sum_{s_1, \ldots, s_n \in \mathbb{Z}} a_{-s} \prod_{i=1}^{n-1} a_i^{s_i-s_{i+1}} q^{s_i(s_i-s_{i+1})} \left(\prod_{i=1}^{n-1} a_i^{r_i-r_{i+1}} q^{r_i(r_i-r_{i+1})}\right) \sum_{0 \leq r_i \leq j-1} \prod_{i=1}^{n-1} a_i^{r_i-r_{i+1}} q^{r_i(r_i-r_{i+1})} \times \left(\prod_{i=0}^{i-1} a_i^{-1} q^{i(i+1)/2} \right) + \prod_{i=0}^{i+1} a_i^{-1} q^{i(i+1)/2} \right) \times \left(\prod_{i=0}^{i-1} a_i^{-1} q^{i(i+1)/2} \right) \right) \infty.
\]
This expression will allow us to deduce an expression for the character of level 1 irreducible highest weight modules of $A_{n-1}^{(1)}$ as a series with positive coefficients in our second paper [DK19].

Finally, through our bijections from Theorem 1.22, Theorem 1.25 also gives us two generalisations of Capparelli’s identities in terms of coloured Frobenius partitions.

**Theorem 1.28** (Two generalisations of Capparelli’s identity). Let $n$ be a positive integer.

Let $C_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ be the number of partitions of $m$ in $C_n$ (see (1.10)), where for $i \in \{0, \ldots, n-1\}$, the symbol $a_i$ (resp. $a_i^{-1}$) appears $u_i$ (resp. $v_i$) times in the colours.

Let $C'_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ be the number of partitions of $m$ in $C'_n$ (see (1.12)), where for $i \in \{0, \ldots, n-1\}$, the symbol $a_i$ (resp. $a_i^{-1}$) appears $u_i$ (resp. $v_i$) times in the colours.

Let $F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ of $n^2$-coloured Frobenius partitions of $m$ where for $i \in \{0, \ldots, n-1\}$, the symbol $a_i$ (resp. $a_i^{-1}$) appears $u_i$ (resp. $v_i$) times in the colours.

Then

$$
\sum_{m, u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1} \geq 0} C_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{-u_0} \ldots a_{n-1}^{-u_{n-1}} =
\sum_{m, u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1} \geq 0} C'_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{-u_0} \ldots a_{n-1}^{-u_{n-1}}
= (q; q)_\infty \times \sum_{m, u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1} \geq 0} F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{-u_0} \ldots a_{n-1}^{-u_{n-1}}
= (q; q)_\infty [x^0] \prod_{i=0}^{n-1} (-x a_i q; q)_\infty (-x^{-1} a_i^{-1}; q)_\infty
.$$  

**Remark.** When $n = 1$ or 2, the sets $C_n$ and $C'_n$ are the same. So when $n = 2$, this simply gives Capparelli’s identity. However, when $n \geq 3$, the sets $C_m$ and $C'_n$ are different, giving two different generalisations of Capparelli’s identity.

Again, performing the dilations $q \rightarrow q^n$, and for all $i \in \{0, \ldots, n-1\}$, $a_i \rightarrow q^{-i}$, gives us a very simple corollary corresponding to the principal specialisation.

**Corollary 1.29** (Principal specialisation). Let $n$ be a positive integer. We have

$$
\sum_{m, u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1} \geq 0} C_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{-u_0} \ldots a_{n-1}^{-u_{n-1}} =
\sum_{m, u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1} \geq 0} C'_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{-u_0} \ldots a_{n-1}^{-u_{n-1}}
= \frac{(q^n; q^n)_\infty}{(q; q)_\infty}.
$$

The remainder of this paper is organised as follows. In Section 2, we define the notion of kernel and reduced colour sequence, which will be key in our proof of Theorem 1.25, and compute the weight of the minimal partition with a given kernel. In Section 3, we study the combinatorics of coloured Frobenius, and prove Theorems 1.25 and 1.27. In section 4, we give the bijections between $P_n$ and $CC_n$, and between $P_n$ and $CC'_n$. Finally, in Section 5, we give the proof of a key Proposition of Section 2, which we postpone to the end of the paper as it is quite technical and is not necessary to the understanding of the rest of this paper.

### 2. Reduced colour sequences and minimal partitions

2.1. **Definition.** The original method of weighted words of Alladi and Gordon [AG93, AAG95] relies on the idea that any partition with $m$ parts and satisfying difference conditions can be obtained from the minimal partition satisfying difference conditions and adding a partition into with at most $m$ parts to it. For example, all partitions into $m$ parts satisfying the difference conditions at least 2 from the Rogers-Ramanujan identities can be obtained by starting with the minimal partition $(2m - 1) + (2m - 3) + \cdots + 3 + 1$, and adding some partition into at most $m$ parts to it.
Here, to compute the generating function for coloured partitions with difference conditions of $P_n$, we also use minimal partitions. But while Alladi, Andrews, and Gordon computed minimal partitions with a certain number of parts, here we compute minimal partitions with a certain kernel. Let us start by defining this terminology.

**Definition 2.1.** Let $c_1, \ldots, c_s$ be a sequence of colours taken from $\{a_i b_k : i, k \in \mathbb{N}\}$. The *minimal partition* associated to $c_1, \ldots, c_s$ according to the difference conditions $\Delta$ is the coloured partition $\lambda_1 + \cdots + \lambda_s$ with minimal weight such that for all $i \in \{1, \ldots, s\}$, $c(\lambda_i) = c_i$. We denote this partition by $\text{min}_\Delta(c_1, \ldots, c_s)$.

**Proposition 2.2.** The weight of $\text{min}_\Delta(c_1, \ldots, c_s)$ is equal to

$$|\text{min}_\Delta(c_1, \ldots, c_s)| = \sum_{k=1}^s k \Delta(c_k, c_{k+1}).$$

Here, we used again the convention that $c_{s+1} = a_\infty b_\infty$ and $\Delta(c_i, a_\infty b_\infty) = 1$ for every colour $c_i$.

**Proof:** Let $c_1, \ldots, c_s$ be a sequence of colours and $\text{min}_\Delta(c_1, \ldots, c_s)$ the corresponding minimal partition. By definition, the smallest part $\lambda_s$ of the minimal partition is equal to 1, which is also equal to $\Delta(c_s, c_{s+1})$. For all $i \in \{1, \ldots, s-1\}$ we have $\lambda_i = \lambda_{i+1} + \Delta(c_i, c_{i+1})$. Thus by induction,

$$\lambda_i = \sum_{k=i}^s \Delta(c_k, c_{k+1}).$$

Summing over all $i \in \{1, \ldots, s\}$, we get

$$|\text{min}_\Delta(c_1, \ldots, c_s)| = \sum_{i=1}^s \sum_{k=i}^s \Delta(c_k, c_{k+1})$$

$$= \sum_{k=1}^s \sum_{i=1}^k \Delta(c_k, c_{k+1})$$

$$= \sum_{k=1}^s k \Delta(c_k, c_{k+1}).$$

$\square$

**Example 2.3.** Considering the difference conditions $\Delta$ from matrix $P_3$ in (1.6), the minimal partition with colour sequence $a_1 b_0, a_0 b_0, a_2 b_2, a_1 b_1, a_1 b_1, a_0 b_1, a_1 b_2, a_0 b_2$ is

$$\text{min}_\Delta(a_1 b_0, a_0 b_0, a_2 b_2, a_1 b_1, a_1 b_1, a_0 b_1, a_1 b_2, a_0 b_2) = 9a_1b_0 + 8a_0b_0 + 7a_2b_2 + 6a_1b_1 + 6a_1b_1 + 4a_0b_1 + 3a_1b_2 + 1a_0b_2.$$  

It has weight 60.

Given a sequence $c_1, \ldots, c_s$ of colours taken from $\{a_i b_k : i, k \in \mathbb{N}\}$, we define the following operations:

- if there is some $i$ such that $c_i = a_k b_k$ and $c_{i+1} = a_\ell b_\ell$, then remove $c_{i+1}$ from the colour sequence,

- if there is some $i$ such that $c_i = a_k b_k$ and $c_{i+1} = a_\ell b_\ell$, then remove $c_i$ from the colour sequence.

Apply the operations above as long as it is possible. The sequence obtained in the end is called the reduction of $c_1, \ldots, c_s$, denoted by $\text{red}(c_1, \ldots, c_s)$. A colour sequence that is equal to its reduction is called a *reduced colour sequence*.

**Remark.** The reduction operation only removes free colours.

**Remark.** The order in which removals are done does not have any influence on the final result.

**Remark.** For each pair of free colours $(a_k b_k, a_\ell b_\ell)$ with $k \neq \ell$, there is exactly one bound colour $a_k b_\ell$ such that $a_k b_k$ can be removed to its left and $a_\ell b_\ell$ can be removed to its right.

**Remark.** For each bound colour $a_k b_\ell$ ($k \neq \ell$), there is exactly one free colour $a_k b_k$ that can be removed to its left, and exactly one free colour $a_\ell b_\ell$ that can be removed to its right.
Example 2.4. The reduction of 
\[ a_1b_1, a_1b_2, a_2b_2, a_3b_3, a_3b_1, a_1b_3, a_3b_3, a_3b_2, a_1b_1 \]
is 
\[ a_1b_2, a_3b_1, a_1b_3, a_3b_2, a_1b_1. \]

Let \( \lambda = \lambda_1 + \cdots + \lambda_s \) be a partition such that \( c(\lambda_1) = c_1, \ldots, c(\lambda_s) = c_s \). The kernel of \( \lambda \), denoted by \( \ker(\lambda) \), is the reduced colour sequence \( \text{red}(c_1, \ldots, c_s) \).

2.2. Combinatorial description of reduced colour sequences. We want to study the partitions of \( P_n \) having a given kernel. To do so, we need to understand combinatorially the set of colour sequences having a certain reduction.

Proposition 2.5. Let \( S \) be a reduced colour sequence. Any colour sequence \( C \) such that \( \text{red}(C) = S \) can be obtained by performing a certain number of insertions of the following types in \( S \):

1. if there is a free colour \( a_kb_k \) in \( S \), insert the same colour \( a_kb_k \) arbitrarily many times to its right,
2. if there is a bound colour \( a_kb_k \) in \( S \), insert the free colour \( a_kb_k \) arbitrarily many times to its left,
3. if there is a bound colour \( a_kb_k \) in \( S \), insert the free colour \( a_kb_k \) arbitrarily many times to its right.

The proof follows immediately from the definition of reduced colour sequences in the previous section.

Example 2.6. Let 
\[ S = a_1b_2, a_3b_1, a_2b_2, a_4b_3, a_3b_2. \]
The sequence 
\[ C = a_1b_1, a_1b_2, a_2b_2, a_3b_3, a_3b_1, a_1b_3, a_3b_3, a_3b_2, a_1b_1, a_2b_2, a_4b_3, a_3b_2 \]
is obtained from \( S \) by inserting \( a_1b_1 \) twice to the left of \( a_1b_2 \) (insertion (2)), \( a_2b_2 \) once to the right of \( a_1b_2 \) (insertion (3)), \( a_3b_3 \) three times to the left of \( a_3b_1 \) (insertion (2)), and \( a_2b_2 \) once to the right of \( a_2b_2 \) (insertion (1)).

Remark. The way one obtains \( C \) from \( S \) via the insertions above is not unique (even up to the order in which we perform the insertions). Indeed, it could be that in \( S = c_1, \ldots, c_s \), the colour that can be inserted to the right of some \( c_j \) is the same as the one that can be inserted to the left of \( c_{j+1} \).

For example, \( a_1b_2, a_2b_2, a_2b_3 \) can be obtained from \( a_1b_2, a_2b_3 \) either by inserting \( a_2b_2 \) to the right of \( a_1b_2 \) (insertion (3)) or to the left of \( a_2b_3 \) (insertion (2)).

To understand reduced colour sequences and insertions combinatorially, and make sure that we count our partitions in an unique way, we need some definitions.

Definition/Proposition 2.7. A primary pair is a pair \((c, c')\) of bound colours such that in the insertion rules of Proposition 2.5, the free colour that can be inserted to the right of \( c \) is the same as the one that can be inserted to the left of \( c' \).

These pairs are exactly those of the form \((a_kb_k, a_kb_k)\), where \( i \neq k \) and \( k \neq \ell \).

We will be interested in maximal sequences of primary pairs in \( S \).

Definition 2.8. Let \( S = c_1, \ldots, c_s \) be a reduced colour sequence. The maximal primary subsequences of \( S \) are subsequences \( c_i, c_{i+1}, \ldots, c_j \) of \( S \) such that

- for all \( k \in \{i, \ldots, j - 1\} \), \((c_k, c_{k+1})\) is a primary pair,
- \((c_{i-1}, c_i)\) and \((c_j, c_{j+1})\) are not primary pairs.

We denote by \( t(S) \) the number of maximal primary subsequences of \( S \), and by \( S_1, \ldots, S_{t(S)} \) these maximal primary subsequences.

Example 2.9. Let 
\[ S = a_1b_2, a_2b_3, a_2b_2, a_1b_4, a_3b_2, a_2b_1, a_3b_3, a_2b_2. \]
Here \( t(S) = 3 \) and the maximal primary subsequences of \( S \) are, from left to right,
\[ S_1 := a_1b_2, a_2b_3, \]
\[ S_2 := a_1b_4, \]
\[ S_3 := a_3b_2, a_2b_1. \]
Let us now define secondary pairs of colours, inside which two different colours can be inserted.

**Definition/Proposition 2.10.** A *secondary pair* is a pair \((c, c')\) of colours satisfying one of the following assertions:

1. The colours \(c\) and \(c'\) are both bound, and the free colour that can be inserted to the right of \(c\) is different from the one that can be inserted to the left of \(c'\). These pairs are of the form \((a_i b_j, a_k b_k)\), where \(i \neq j\), \(j \neq k\), and \(k \neq \ell\).

2. The colour \(c\) is free, \(c'\) is bound, and the colour that can be inserted to the left of \(c'\) is different from \(c\). These pairs are of the form \((a_i b_1, a_k b_1)\), where \(i \neq k\), and \(k \neq \ell\).

3. The colour \(c\) is bound, \(c'\) is free, and the colour which can be inserted to the right of \(c\) is different from \(c'\). These pairs are of the form \((a_i b_k, a_i b_k)\), where \(i \neq k\), and \(k \neq \ell\).

**Remark.** In the above, the colours \(c\) or \(c'\) can be equal to \(a_\infty b_\infty\) (when they are free). This allows us to avoid treating separately the case of insertions at one of the ends of the colour sequence \(C = c_1, \ldots, c_s\). Indeed, by our convention, inserting \(a_i b_i\) to the left of \(c_1 = a_\ell b_\ell\) is the same as inserting \(a_i b_i\) inside the pair \((c_0, c_1) = (a_\infty b_\infty, a_\ell b_\ell)\). This is included in Case (2). Similarly, inserting \(a_k b_k\) to the right of \(c_s = a_k b_k\) is the same as inserting \(a_k b_k\) inside the pair \((c_s, c_{s+1}) = (a_\ell b_\ell, a_\infty b_\infty)\), which is included in Case (3).

With the definitions and propositions above, we can now determine uniquely the places where insertions can occur in a reduced colour sequence.

Let \(S = c_1, \ldots, c_s\) be a reduced colour sequence of length \(s\). Then \(S\) can be written uniquely in the form

\[ S = T_1 S_1 T_2 S_2 \ldots T_\ell S_\ell T_{\ell+1}, \]

where \(S_1, \ldots, S_\ell\) are the maximal primary subsequences of \(S\), and \(T_1, \ldots, T_{\ell+1}\) are (possibly empty) sequences of consecutively distinct free colours.

For all \(u \in \{1, \ldots, \ell\}\), let \(i_{2u-1}\) (resp. \(i_{2u}\)) be the index of the first (resp. last) colour of \(S_u\), i.e.

\[ S_u = c_{i_{2u-1}}, \ldots, c_{i_{2u}}. \]

We have \(i_{2u-1} \leq i_{2u}\), with equality when \(S_u\) is a singleton. By the definition of maximal primary subsequences, for all \(u\), the pairs \((c_{i_{2u-1}}, c_{i_{2u}})\) and \((c_{i_{2u}}, c_{i_{2u}+1})\) are secondary pairs.

We can now state the following.

**Proposition 2.11.** Using the notation above, the insertions of free colours in \(S\) can occur exactly in the following \(s + t\) places (possibly multiple times in the same place):

- to the right of \(c_i\), for all \(i \in \{1, \ldots, s\}\),
- to the left of \(c_{i_{2u}+1}\), for all \(u \in \{1, \ldots, \ell\}\).

Let \(f_1, \ldots, f_{s+t}\) be the \(s + t\) free colours that can be inserted in \(S\) (in order).

Let \(n_1, \ldots, n_{s+t}\) be non-negative integers. We denote by \(S(n_1, \ldots, n_{s+t})\) the colour sequence obtained from \(S\) by inserting \(n_i\) times the colour \(b_i\) in \(S\), for all \(i\).

Using this notation, we finally have unicity of the insertions.

**Proposition 2.12.** For each colour sequence \(C\) such that \(\text{red}(C) = S\), there exist a unique \((s + t)\)-tuple of non-negative integers \((n_1, \ldots, n_{s+t})\) such that \(C = S(n_1, \ldots, n_{s+t})\).

**Example 2.13.** In Example 2.6, we have \(s = 5\), \(t = 3\),

\[
S_1 = a_1 b_2, \quad S_2 = a_3 b_1, \quad S_3 = a_4 b_3, a_3 b_2 \\
T_1 = \emptyset, \quad T_2 = \emptyset, \quad T_3 = a_2 b_2, \quad T_4 = \emptyset,
\]

and

\[
C = S(2, 1, 3, 0, 1, 0, 0, 0).
\]
2.3. **Influence of the insertions on the minimal partition.** We now study how insertions inside a colour sequence affect the minimal differences between the parts of the corresponding minimal partition. Let us start with a general lemma about the minimal differences $\Delta$.

**Lemma 2.14.** For all $k, \ell \in \mathbb{N}$ with $k \neq \ell$, we have
\[
\Delta(a_k b_k, a_k b_\ell) = \chi(k < \ell), \quad (2.1) \\
\Delta(a_k b_\ell, a_k b_\ell) = \chi(k > \ell), \quad (2.2) \\
\Delta(a_k b_k, a_k b_\ell) + \Delta(a_k b_\ell, a_k b_\ell) = 1. \quad (2.3)
\]

**Proof:** We only give the details for (2.1). Remembering that
\[
\lambda_d \text{ differences. The minimal partition after insertion will be}
\]
on the other hand, by the definition of $\Delta$, and using that
By (2.1) and (2.2), we have
\[
\text{Equation (2.2) is proved in the same way, and (2.3) is obtained by adding (2.1) and (2.2) together.} \quad \square
\]

If $S$ is a reduced colour sequence, we want to see how the insertion of some free colour in $S$ affects the minimal partition, or equivalently the minimal differences between successive parts.

Let us start with an observation. Because for all $k$, $\Delta(a_k b_k, a_k b_k) = 0$, inserting a free colour $a_k b_k$ once or multiple times inside a given pair has exactly the same effect on the rest of the minimal partition. Therefore we only need to study the case where we insert a single free colour inside a primary or secondary pair.

First, let us see what happens to the minimal differences if we insert a free colour inside a primary pair.

**Proposition 2.15.** Let $(a_i b_k, a_k b_\ell)$, with $i \neq k$ and $k \neq \ell$, be a primary pair. We have
\[
\Delta(a_i b_k, a_k b_k) + \Delta(a_k b_k, a_k b_\ell) = \Delta(a_i b_k, a_k b_\ell).
\]

**Proof:** By (2.1) and (2.2), we have
\[
\Delta(a_i b_k, a_k b_k) + \Delta(a_k b_k, a_k b_\ell) = \chi(i > k) + \chi(k < \ell).
\]
On the other hand, by the definition of $\Delta$, and using that $i \neq k$ and $k \neq \ell$, we have
\[
\Delta(a_i b_k, a_k b_\ell) = \chi(i > k) - \chi(i = k) + \chi(k < \ell) - \chi(k = \ell)
\]
This is the same expression as before. \quad \square

The above proposition shows that inserting a free colour inside a primary pair doesn’t disrupt the rest of the minimal partition.

**Corollary 2.16.** Let $C = c_1, \ldots, c_s$ be a colour sequence, and let $\min \Delta(C) = \lambda_1 + \cdots + \lambda_s$ be the corresponding minimal partition. Inserting a free colour $c'$ inside a primary pair $(c_i, c_{i+1})$ doesn’t disrupt the minimal differences. The minimal partition after insertion will be $\lambda_1 + \cdots + \lambda_i + \lambda' + \lambda_{i+1} + \cdots + \lambda_s$, with $\lambda' = \lambda_{i+1} + \Delta(c', c_{i+1})$.

We now turn to insertions inside secondary pairs. In certain cases, it will disrupt the minimal differences. We first study the case where we insert a free colour to the left of $c'$ in a secondary pair $(c, c')$.

**Proposition 2.17.** (Left insertion) Let $(a_i b_j, a_k b_\ell)$, with $j \neq k$ and $k \neq \ell$, be a secondary pair where $a_k b_\ell$ is a bound colour (Cases (1) and (2) in Definition 2.10). We have
\[
\Delta(a_i b_j, a_k b_k) + \Delta(a_k b_k, a_k b_\ell) - \Delta(a_i b_j, a_k b_\ell) = 0 \text{ or } 1.
\]

**Proof:** Let $D$ denote the difference above. By definition of $\Delta$ and the fact that $j \neq k$ and $k \neq \ell$, we have
\[
\Delta(a_i b_j, a_k b_\ell) = \chi(i > k) + \chi(j \leq \ell).
\]
On the other hand, we have
\[
\Delta(a_i b_j, a_k b_k) = \chi(i \geq k) + \chi(j < k),
\]
and
\[
\Delta(a_k b_k, a_k b_\ell) = \chi(k < \ell).
\]
Thus the difference is equal to
\[ D = \chi(j < k) + \chi(k < \ell) - \chi(j \geq k). \]
This is always equal to 0 or 1. Indeed, when the first two terms are 1, then we have \( j < k < \ell \) and the third term is 1 too. When the last term is 1, then at least one of the first two is 1 too. If it wasn’t the case, we would have \( j \geq k \geq \ell \) and \( j \leq \ell \), i.e. \( j = k = \ell \), which is impossible because \( j \neq k \).

**Definition 2.18.** When the difference in the Proposition 2.17 is 0 (resp. 1), we call \((a_i b_j, a_k b_\ell)\) a type 0 (resp. type 1) left pair, and the corresponding insertion a type 0 (resp. type 1) left insertion.

**Remark.** The type of the left pair \((a_i b_j, a_k b_\ell)\) in Proposition 2.17 doesn’t depend on \( i \). In particular \((a_i b_j, a_k b_\ell)\) and \((a_i b_j, a_k b_\ell)\) have the same type.

Similarly, we study the case where we insert a free colour to the right of \( c \) in a secondary pair \((c, c')\). This essentially works in the same way as left insertions.

**Proposition 2.19** (Right insertion). Let \((a_i b_j, a_k b_\ell)\), with \( i \neq j \) and \( j \neq k \), be a secondary pair where \( a_i b_j \) is a bound colour (Cases (1) and (3) in Definition 2.10). We have
\[ \Delta(a_i b_j, a_k b_\ell) + \Delta(a_j b_j, a_k b_\ell) - \Delta(a_i b_j, a_k b_\ell) = 0 \text{ or } 1. \]

**Proof:** Following the same reasoning as in the proof of Proposition 2.17, we show that the difference above is equal to
\[ \chi(i > j) + \chi(j > k) - \chi(i \geq k), \]
which again is always equal to 0 or 1. \( \square \)

As before, we define type 0 and type 1.

**Definition 2.20.** When the difference in the previous proposition is 0 (resp. 1), we call \((a_i b_j, a_k b_\ell)\) a type 0 (resp. type 1) right pair, and the corresponding insertion a type 0 (resp. type 1) right insertion.

**Remark.** The type of the right pair \((a_i b_j, a_k b_\ell)\) in Proposition 2.19 doesn’t depend on \( \ell \). In particular \((a_i b_j, a_k b_\ell)\) and \((a_i b_j, a_k b_\ell)\) have the same type.

From Propositions 2.17 and 2.19, we now understand the effect that an insertion inside a secondary pair has on the minimal partition, depending on the type of this insertion.

**Corollary 2.21** (Type 0 insertion). Let \( C = c_1, \ldots, c_s \) be a colour sequence, and let \( \min_\Delta(C) = \lambda_1 + \cdots + \lambda_s \) be the corresponding minimal partition. For any \( i \in \{0, \ldots, s\} \), the type 0 insertion of a free colour \( c' \) inside a secondary pair \((c_i, c_{i+1})\) doesn’t disrupt the minimal differences. The minimal partition after insertion will be \( \lambda_1 + \cdots + \lambda_i + \lambda' + \lambda_{i+1} + \cdots + \lambda_s \), with \( \lambda' = \lambda_{i+1} + \Delta(c', c_{i+1}) \).

**Example 2.22.** The minimal partition with colour sequence
\[ C = a_2 b_2, a_1 b_0, a_0 b_2, a_1 b_0, a_2 b_1 \]
is
\[ \min_\Delta(C) = 5a_2 b_2 + 4a_1 b_0 + 2a_0 b_2 + 2a_1 b_0 + 1a_2 b_1. \]
We insert \( a_1 b_1 \) inside \((a_0 b_2, a_1 b_0)\). The minimal partition with colour sequence
\[ C' = a_2 b_2, a_1 b_0, a_0 b_2, a_1 b_1, a_1 b_0, a_2 b_1 \]
is
\[ \min_\Delta(C') = 5a_2 b_2 + 4a_1 b_0 + 2a_0 b_2 + 2a_1 b_1 + 2a_1 b_0 + 1a_2 b_1. \]
The part \( 2a_1 b_1 \) was inserted, but all the other parts stay the same.

**Corollary 2.23** (Type 1 insertion). Let \( C = c_1, \ldots, c_s \) be a colour sequence, and let \( \min_\Delta(C) = \lambda_1 + \cdots + \lambda_s \) be the corresponding minimal partition. For any \( i \in \{0, \ldots, s\} \), the type 1 insertion of a free colour \( c' \) inside a secondary pair \((c_i, c_{i+1})\) adds 1 to the minimal difference between \( c_i \) and \( c_{i+1} \). This forces us to add 1 to each part to the left of the newly inserted part in the minimal partition, which become \((\lambda_1 + 1) + \cdots + (\lambda_i + 1) + \lambda' + \lambda_{i+1} + \cdots + \lambda_s \), with \( \lambda' = \lambda_{i+1} + \Delta(c', c_{i+1}) \).
Example 2.24. In the colour sequence $C$ of the previous example, we insert $a_2b_2$ inside $(a_0b_2, a_1b_0)$. The minimal partition with colour sequence

$$C'' = a_2b_2, a_1b_0, a_0b_2, a_2b_2, a_1b_0, a_2b_1$$

is

$$\min_\Delta(C'') = 6a_2b_2 + 5a_1b_0 + 3a_0b_2 + 3a_2b_2 + 2a_1b_0 + 1a_2b_1.$$ 

All the parts to the left of the newly inserted part are increased by one compared to $\min_\Delta(C)$.

So far we have only studied the case of a single insertion (either left or right) inside a secondary pair. We still need to understand what happens to the minimal differences if, inside a secondary pair $(a_ib_j, a_kb_k)$, we insert both $a_jb_j$ to the right of $a_ib_j$ and $a_kb_k$ to the left of $a_kb_k$.

Proposition 2.25 (Left and right insertion). Let $(a_ib_j, a_kb_k)$, with $j \neq k$, be a secondary pair. We have

$$\Delta(a_ib_j, a_kb_k) + \Delta(a_kb_j, a_kb_k) - \Delta(a_kb_j, a_kb_k) + \Delta(a_kb_k, a_kb_k)$$

is

0 if both the right and left insertions inside $(a_kb_j, a_kb_k)$ are of type 0,

1 if exactly one of the insertions inside $(a_kb_j, a_kb_k)$ is of type 1,

2 if both the right and left insertions inside $(a_kb_j, a_kb_k)$ are of type 1.

Proof: Let $D$ be the difference above. We have

$$D = \Delta(a_kb_j, a_kb_k) + \Delta(a_kb_k, a_kb_k) - \Delta(a_kb_k, a_kb_k) + \Delta(a_kb_k, a_kb_k).$$

The first line is equal to the right type of $(a_kb_j, a_kb_k)$, which by the remark after Proposition 2.19, is the same as the right type of $(a_kb_j, a_kb_k)$. The second line is simply the left type of $(a_kb_j, a_kb_k)$. Thus performing both a left and right insertion inside a secondary pair is the same as performing the two insertions separately.

We conclude this section by summarising the influence of all the possible insertions on the minimal partition.

Proposition 2.26 (Summary of the different types of insertion). Let $C = c_1, \ldots, c_s$ be a colour sequence, and let $\min_\Delta(C) = \lambda_1 + \cdots + \lambda_s$ be the corresponding minimal partition. When we insert a free colour $\ell'$ inside a pair $(c_i, c_{i+1})$, the minimal partition transforms as follows:

- if $c_i$ is a free colour and $\ell' = c_i$, the minimal partition becomes $\lambda_1 + \cdots + \lambda_i + \lambda_{i+1} + \cdots + \lambda_s$ (i.e. the part $\lambda_i$ repeats, and the rest of the partition remains unchanged);
- if $(c_i, c_{i+1})$ is a primary pair, the minimal partition becomes $\lambda_1 + \cdots + \lambda_i + \lambda_i' + \lambda_{i+1} + \cdots + \lambda_s$, with $\lambda_i' = \lambda_i + \Delta(\ell', c_{i+1})$;
- if $(c_i, c_{i+1})$ is a secondary pair and the insertion of $\ell'$ is of type 0, the minimal partition becomes $\lambda_1 + \cdots + \lambda_i + \lambda_i' + \lambda_{i+1} + \cdots + \lambda_s$, with $\lambda_i' = \lambda_i + \Delta(\ell', c_{i+1})$;
- if $(c_i, c_{i+1})$ is a secondary pair and the insertion of $\ell'$ is of type 1, the minimal partition becomes $(\lambda_1 + 1) + \cdots + (\lambda_i + 1) + \lambda_i' + \lambda_{i+1} + \cdots + \lambda_s$, with $\lambda_i' = \lambda_i + \Delta(\ell', c_{i+1})$ (i.e. we add 1 to all the parts to the left of the newly inserted part $\lambda_i'$).

We call the first two types of insertions above neutral insertions.

2.4. Generating function for partitions with a given kernel. Our goal is to count partitions of $\mathcal{P}_n$ with a given kernel. The results from the previous section will help us do so.

Let $S = c_1, \ldots, c_s$ be a reduced colour sequence of length $s$, having $t$ maximal primary subsequences. Let $f_1, \ldots, f_s + t$ be the free colours that can be inserted in $S$. In the following, we denote by $\mathcal{N}$ (resp. $\mathcal{T}_0$, $\mathcal{T}_1$) the set of indices $i$ such that the insertion of $f_i$ is neutral (resp. of type 0, of type 1). We have $\mathcal{N} \cup \mathcal{T}_0 \cup \mathcal{T}_1 = \{1, \ldots, s + t\}$.

Moreover, the secondary pairs in $S$ are exactly $(c_{i_2u-1}, c_{i_2u-1})$ and $(c_{i_2u}, c_{i_2u+1})$, for $u \in \{1, \ldots, t\}$, where $S_u = c_{i_2u-1}, \ldots, c_{i_2u}$. So we can write

$$\mathcal{T}_0 = \bigcup_{u=1}^{t} \mathcal{T}_0^u, \quad \mathcal{T}_1 = \bigcup_{u=1}^{t} \mathcal{T}_1^u,$$
where $T_0^n$ (resp. $T_1^n$) is the set of indices $j$ such that $f_j$ can be inserted inside $(c_{2u-1}, c_{2u})$ or $(c_{2u}, c_{2u+1})$ and is of type 0 (resp. 1). For all $u \in \{1, \ldots, t\}$, we have $|T_0^n| = 2 - |T_1^n|$. We want to study the minimal partition of the colour sequence $S(n_1, \ldots, n_{s+t})$. Denote by $S_1^n$ (resp. $S_1$) the indices $j$ of $T_1^n$ (resp. $T_1$) such that $n_j > 0$. We start with the following lemma.

**Lemma 2.27.** For all $j \in \{1, \ldots, s + t\}$, if $n_j > 0$, i.e. the colour $f_j$ is actually inserted, the corresponding part $\lambda(f_j)$ in the minimal partition of $S(n_1, \ldots, n_{s+t})$ is equal to

$$\lambda(f_j) = \# (\{j, \ldots, s + t\} \cap (N \cup T_0 \cup S_1)). \tag{2.4}$$

**Proof:** We proceed via backward induction on $j$.

- If $j = s + t$, $\lambda(f_{s+t})$ is the last part of the minimal partition and therefore has size 1. Equation (2.4) is correct, as $s + t \in N \cup T_0 \cup S_1$.
- Now assume that (2.4) holds for $f_{j+1}$, and prove it for $f_j$. Let $k$ and $\ell$ be such that $f_j = ak_bk$ and $f_{j+1} = a_kb\ell$. We always have $k \neq \ell$.

  (1) For now, let us assume that $n_{j+1} > 0$, i.e. that $f_{j+1}$ was actually inserted in the colour sequence.

  - If $j \in N$ or $j$ is a left secondary insertion, then the subsequence between $f_j$ and $f_{j+1}$ in $S(n_1, \ldots, n_{s+t})$ is $f_j, ak_bk, f_{j+1}$ or $f_j, a\ell_b\ell, f_{j+1}$. In the first case, we have
    $$\lambda(f_j) = \Delta(ak_bk, ak_bk) + \Delta(ak_bk, a\ell_b\ell) + \lambda(f_{j+1})$$
    $$= 1 + \lambda(f_{j+1}),$$
    where the second equality follows from Lemma 2.14.

    In the second case, we have also
    $$\lambda(f_j) = \Delta(ak_bk, a\ell_b\ell) + \Delta(a\ell_b\ell, a\ell_b\ell) + \lambda(f_{j+1})$$
    $$= 1 + \lambda(f_{j+1}),$$
    By the induction hypothesis, we have
    $$\lambda(f_j) = 1 + \# (\{j + 1, \ldots, s + t\} \cap (N \cup T_0 \cup S_1))$$
    $$= \# (\{j, \ldots, s + t\} \cap (N \cup T_0 \cup S_1)),$$
    because $j \in N \cup T_0 \cup S_1$.

  - If $j$ is a right secondary insertion, then $f_j$ appears directly before $f_{j+1}$ in $S(n_1, \ldots, n_{s+t})$.

    Thus we have
    $$\lambda(f_j) = \Delta(f_j, f_{j+1}) + \lambda(f_{j+1})$$
    $$= 1 + \lambda(f_{j+1}),$$
    and we can deduce (2.4) in the exact same way as before.

(2) Now we treat the case where $f_{j+1}$ was not inserted in the colour sequence. By Proposition 2.26, if $j + 1 \in N \cup T_0$, it does not change anything to the other parts in the minimal partition, so $\lambda(f_j)$ stays the same as in case (1).

If $j + 1 \in T_1$ and $b_{j+1}$ was not inserted, then by Proposition 2.26, the part $\lambda(f_j)$ decreases by one compared to the previous case. But in this case, $\# (\{j, \ldots, s + t\} \cap (N \cup T_0 \cup S_1))$ also decreases by one compared to case (1), so Equation (2.4) is still correct.

We can now give a formula for the weight of the minimal partition with colour sequence $S(n_1, \ldots, n_{s+t})$.

**Proposition 2.28.** With the notation above, the size of the minimal partition with colour sequence $S(n_1, \ldots, n_{s+t})$ is

$$|\min_\Delta(S(n_1, \ldots, n_{s+t})))| = |\min_\Delta(S)|$$

$$+ \sum_{j \in S_1} (P(j) + n_j \times \# (\{j, \ldots, s + t\} \cap (N \cup T_0 \cup S_1)))$$

$$+ \sum_{j \in N \cup T_0} n_j \times \# (\{j, \ldots, s + t\} \cap (N \cup T_0 \cup S_1)), \tag{2.5}$$

where $P(j)$ is the number of colours of $S$ that are to the left of $f_j$. 

\[18\]
Proof: We start with the minimal partition \( \text{min}_{\Delta}(S) \) with colour sequence \( S \). It has weight \( \lfloor \text{min}_{\Delta}(S) \rfloor \).

Then we insert the parts corresponding to colours of type 1. Let \( j \in S_1 \). By Proposition 2.26, inserting \( f_j \) adds 1 to all the parts of \( \text{min}_{\Delta}(S) \) which are to the left of \( \lambda(f_j) \). So this adds \( P(j) \) to the total weight. Moreover, by Lemma 2.27, the part \( \lambda(f_j) \) is of size \( \# \left( \{ j, \ldots, s+t \} \cap (\mathcal{N} \cup \mathcal{T}_0 \cup S_1) \right) \), and we insert it \( n_j \) times. Summing over all \( j \in S_1 \) gives the first sum.

Finally, the insertion of parts corresponding to colours \( f_j \) with \( j \in \mathcal{N} \cup \mathcal{T}_0 \) yields the last sum. \( \square \)

Starting from Proposition 2.28, we will show a key proposition, which will be very useful to establish the connection with coloured Frobenius partitions.

Recall that the \( q \)-binomial coefficient is defined as follows:

\[
\binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_{k}(q; q)_{n-k}},
\]

and we assume that \( \binom{n}{k}_q = 0 \) if \( k < 0 \) or \( k > n \).

**Proposition 2.29.** Let \( n \) be a positive integer and \( m \) a non-negative integer. Let \( S = c_1, \ldots, c_s \) be a reduced colour sequence of length \( s \), having \( t \) maximal primary subsequences. The generating function for minimal partitions in \( \mathcal{P}_n \) with kernel \( S \), having \( s + m \) parts, is the following:

\[
\sum_{\text{colour sequence of length } s+m \text{ such that red}(C)=S} q^{\text{val}_{\text{min}_{\Delta}}(C)} = q^{\lfloor \text{min}_{\Delta}(S) \rfloor + m} \sum_{u=0}^{t} q^{u(s-t)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ \frac{s + m - 1}{m - u} \right]_q, \tag{2.6}
\]

where \( g_{0,0} = 1 \), and for \( u \leq t \),

\[
g_{u,v}(q; x_1, \ldots, x_v) = \sum_{\epsilon_1, \ldots, \epsilon_v \in \{0,1\}} q^{uv+(\epsilon)} \prod_{k=1}^{v} q^{(x_k-1)\sum_{i=1}^{v} \epsilon_i}.
\]

By observing that all partitions of \( \mathcal{P}_n \) with a given colour sequence \( C \) of length \( s + m \) can be obtained in a unique way by adding a partition with at most \( s + m \) parts to the minimal partition \( \text{min}_{\Delta}(C) \), Proposition 2.29 is actually equivalent to the following generating function for all partitions of \( \mathcal{P}_n \) with a given kernel.

**Proposition 2.30.** Let \( n \) be a positive integer and \( m \) a non-negative integer. Let \( S = c_1, \ldots, c_s \) be a reduced colour sequence of length \( s \), having \( t \) maximal primary subsequences. The generating function for partitions in \( \mathcal{P}_n \) with kernel \( S \), having \( s + m \) parts, is the following:

\[
\sum_{\ell(\lambda)=s+m, \ker(\lambda)=S} q^{\text{val}(\lambda)} = q^{\lfloor \text{min}_{\Delta}(S) \rfloor + m} \sum_{u=0}^{t} q^{u(s-t)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ \frac{s + m - 1}{m - u} \right]_q. \tag{2.7}
\]

The proof of Proposition 2.29 from Proposition 2.28, quite technical, is postponed to Section 5. Its reading is not necessary to understand the connection between the generalised Primc partitions of \( \mathcal{P}_n \) and the \( n^2 \)-coloured Frobenius partitions, which we will study in the next section, nor the bijection with the generalisation of Capparelli’s identity, which we give in Section 4.

3. Coloured Frobenius partitions

In this section, we compute the generating function for \( n^2 \)-coloured Frobenius partitions with a given kernel and show that it is the same as the generating function (2.7) for generalised Primc partitions with the same kernel.

3.1. The difference conditions corresponding to minimal \( n^2 \)-coloured Frobenius partitions. We start by showing that minimal \( n^2 \)-coloured Frobenius partitions are in bijection with minimal coloured partitions satisfying some new difference conditions \( \Delta' \).

Let \( \left( \lambda_1, \lambda_2, \cdots, \lambda_s \right) \) be an \( n^2 \)-coloured Frobenius partition. Recall from the introduction that \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s \) is a partition into \( s \) distinct non-negative parts, each coloured with some \( a_i, i \in \{0, \ldots, n-1\}, \)
with the order (1.15). Similarly, $\mu = \mu_1 + \mu_2 + \cdots + \mu_s$ is a partition into $s$ distinct non-negative parts, each coloured with some $b_i, i \in \{0, \ldots, n-1\}$, with the order (1.16). The colour sequence of this $n^2$-coloured Frobenius partition is $(c(\lambda_1)c(\mu_1), \ldots, c(\lambda_s)c(\mu_s))$, and its kernel can be defined in the same way as for coloured partitions.

Given a colour sequence $c_1, \ldots, c_s$ taken from $\{a_i b_k : i, k \in \{0, \ldots, n-1\}\}$, the minimal $n^2$-coloured Frobenius partition associated to $c_1, \ldots, c_s$, is the $n^2$-coloured Frobenius partition $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix}$ with minimal weight such that for all $i \in \{1, \ldots, s\}$, $c(\lambda_i)c(\mu_i) = c_i$. We denote it by $\min^F(c_1, \ldots, c_s)$.

**Proposition 3.1.** Let $c_1, \ldots, c_s$ be a colour sequence taken from $\{a_i b_k : i, k \in \{0, \ldots, m-1\}\}$. There is a weight-preserving bijection between the minimal $n^2$-coloured Frobenius partition $\min^F(c_1, \ldots, c_s)$ and the minimal coloured partition $\min^C(c_1, \ldots, c_s)$, where for all $i, k, i', k' \in \mathbb{N}$,

$$\Delta'(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') + \chi(k \leq k').$$

**Proof:** Start with $\min^F(c_1, \ldots, c_s) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix}$, and transform it into the coloured partition $\nu = \nu_1 + \cdots + \nu_s$, where for all $j \in \{1, \ldots, s\}$,

$$\nu_j = \lambda_j + \mu_j,$$

$$c(\nu_j) = c(\lambda_j)c(\mu_j).$$

Clearly $\min^F(c_1, \ldots, c_s)$ and $\nu$ have the same weight and colour sequence.

Moreover, by definition of the order (1.15), and using the minimality of $\min^F(c_1, \ldots, c_s)$, the difference between $\lambda_j$ of colour $a_i$ and $\lambda_{j+1}$ of colour $a_{i'}$ is exactly $\chi(i \geq i')$, for all $j \in \{1, \ldots, s\}$. Similarly, the difference between $\mu_j$ of colour $a_k$ and $\mu_{j+1}$ of colour $a_{k'}$ is exactly $\chi(k \leq k')$.

Thus for all $j \in \{1, \ldots, s\}$, the difference between $\nu_j$ and $\mu_{j+1}$ is exactly $\chi(i \geq i') + \chi(k \leq k')$ and $\nu = \min^C(c_1, \ldots, c_s)$.

By unicity of the minimal partition (resp. Frobenius partition), this is indeed a bijection. □

We denote by $\mathcal{P}_n'$ the set of $n^2$-coloured partitions satisfying the minimal difference conditions $\Delta'$. 

**Remark.** When we don’t have the minimality condition, the $n^2$-coloured Frobenius partitions with colour sequence $c_1, \ldots, c_s$ are not in bijection with coloured partitions with colour sequence $c_1, \ldots, c_s$ and minimal differences $\Delta'$. For example, take the case of one colour $a_1 b_1$. The $n^2$-coloured Frobenius partitions with colour sequence $a_1 b_1$ are generated by $q/(1-q)^2$, as we can choose any value for both $\lambda_1$ and $\mu_1$. On the other hand, coloured partitions with colour sequence $a_1 b_1$ and difference $\Delta'$ are generated by $q/(1-q)$, as we can only choose the value of one part $\nu_1$.

However, for our purpose in this paper, we only need the generating function for minimal partitions. Moreover, we will be able to relate $\Delta'$ with the difference conditions $\Delta$ of Primc’s identity, which will allow us to reuse a lot of work done in Section 2.

Let us start with the following property, which follows from the definition of $\Delta$ (1.10) and $\Delta'$ (1.12).

**Property 3.2.** The minimal differences $\Delta(c, c')$ and $\Delta'(c, c')$ are equal, except in the following cases:

1. $c = c' = a_i b_i$, in which case $\Delta(a_i b_i, a_i b_i) = 0$ and $\Delta'(a_i b_i, a_i b_i) = 2$.
2. $c = a_i b_i$ and $c' = a_i b_i$, in which case $\Delta'(a_i b_i, a_i b_i) = \Delta(a_i b_i, a_i b_i) + 1$.
3. $c = a_i b_i$ and $c' = a_i b_i$, in which case $\Delta'(a_i b_i, a_i b_i) = \Delta(a_i b_i, a_i b_i) + 1$.

These particular cases correspond to the insertions of type (1), (2), and (3), respectively, in Proposition 2.5.

Using the fact that reduced colour sequences do not contain any pair $(c, c')$ of the types mentioned above, we have the following corollary.

**Corollary 3.3.** Let $S$ be a reduced colour sequence. Then

$$\min_\Delta(S) = \min_\Delta'(S).$$

But when $C$ is a coloured sequence which is not reduced, we do not have $\min_\Delta(C) = \min_\Delta'(C)$ in general. So to compute, we define one last difference condition

$$\Delta'' := 2 - \Delta',$$

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which shares many properties with $\Delta$.

**Proposition 3.4.** The difference conditions $\Delta''$ satisfy the following properties on free colours.

(1) Difference between two free colours: For all $i, k$, $\Delta''(a_ib_k, a_kb_k) = \chi(i \neq k) = \Delta(a_ib_i, a_kb_k)$.

(2) Insertion inside a primary pair: Let $(a_ib_k, a_kb_k)$, with $i \neq k$ and $k \neq \ell$, be a primary pair. We have

$$\Delta''(a_ib_k, a_kb_k) + \Delta''(a_kb_k, a_kb_k) = \Delta''(a_kb_k, a_kb_k).$$

(3) Left insertion inside a secondary pair: Let $(a_ib_j, a_kb_k)$, with $j \neq k$ and $k \neq \ell$, be a secondary pair. We have

$$\Delta''(a_ib_j, a_kb_k) + \Delta''(a_kb_k, a_kb_k) - \Delta''(a_kb_k, a_kb_k) = 0 \text{ or } 1.$$ Moreover such an insertion is of $\Delta'$-type 0 (resp. 1) if and only if it is of $\Delta$-type 1 (resp. 0).

(4) Right insertion inside a secondary pair: Let $(a_ib_j, a_kb_k)$, with $i \neq j$ and $j \neq k$, be a secondary pair. We have

$$\Delta''(a_ib_j, a_kb_k) + \Delta''(a_kb_k, a_kb_k) - \Delta''(a_kb_k, a_kb_k) = 0 \text{ or } 1.$$ Moreover such an insertion is of $\Delta'$-type 0 (resp. 1) if and only if it is of $\Delta$-type 1 (resp. 0).

**Proof:** Property (1) follows clearly from the definition of $\Delta'$. Let us now prove (2). We have:

$$\Delta''(a_ib_k, a_kb_k) + \Delta''(a_kb_k, a_kb_k) = 4 - \Delta'(a_ib_k, a_kb_k) - \Delta'(a_kb_k, a_kb_k) \text{ by definition of } \Delta''$$

$$= 2 - \Delta(a_ib_k, a_kb_k) - \Delta(a_kb_k, a_kb_k) \text{ by Property 3.2}$$

$$= 2 - \Delta(a_kb_k, a_kb_k) \text{ by Property 2.15}$$

$$= 2 - \Delta'(a_kb_k, a_kb_k) \text{ by Property 3.2}$$

$$= \Delta''(a_kb_k, a_kb_k) \text{ by definition of } \Delta''.$$ Let us finally turn to (3). Property (4) is proved in a similar way. We have

$$\Delta''(a_ib_j, a_kb_k) + \Delta''(a_kb_k, a_kb_k) - \Delta''(a_kb_k, a_kb_k) = 0 \text{ or } 1,$$

But by Proposition 2.17,

$$\Delta(a_ib_j, a_kb_k) + \Delta(a_kb_k, a_kb_k) - \Delta(a_kb_k, a_kb_k) = 0 \text{ or } 1,$$

and the value 0 or 1 is the $\Delta$-type of the insertion. This completes the proof of (3).

Proposition 3.4 shows that $\Delta''$ behaves exactly like $\Delta$ with respect to the insertion of free colours, except that the types of all insertions inside secondary pairs are reversed. In other words, using the notation at the beginning of Section 2.4, given a reduced colour sequence $S = c_1, \ldots, c_{n}$ and $f_1, \ldots, f_{n-\ell}$ the free colours that can be inserted in $S$, $N$ (resp. $T_0$, $T_1$) is exactly the set of indices $i$ such that the insertion of $f_i$ is neutral (resp. of type 1, of type 0) for the order $\Delta''$.

### 3.2. The generating function for $n^2$-coloured Frobenius partitions with a given kernel

Now that we understand the orders $\Delta'$ and $\Delta''$, we will use them to compute the generating function for $n^2$-coloured Frobenius partitions with a given kernel.

Before doing this, we need a technical lemma about the function $g_{u,v}$ defined in Proposition 2.30, which will appear again in this section.

**Lemma 3.5.** Let $g_{u,v}$ be the function defined in Proposition 2.30. We have

$$g_{u,v}(q^{-1}; 2 - x_1, \ldots, 2 - x_v) = q^{-u(2v+u-1)}g_{u,v}(q; x_1, \ldots, x_v).$$

**Proof:** When $u = v = 0$, this is trivially true. Otherwise, we have by definition:

$$g_{u,v}(q^{-1}; 2 - x_1, \ldots, 2 - x_v) = \sum_{\epsilon_1, \ldots, \epsilon_v \in \{0, 1\}; \epsilon_1 + \cdots + \epsilon_v = u} q^{-(u+\sum_{\epsilon_1}^v \epsilon_1)} \prod_{k=1}^v q^{-(2-x_k-1) \sum_{\epsilon_1}^{k-1} \epsilon_1}.$$
Proposition 3.6. Let $n$ be a positive integer and $m$ a non-negative integer. Let $S = c_1, \ldots, c_k$ be a colour sequence of length $s$, having $t$ maximal primary subsequences. Using the notation of Section 2.4, the generating function for minimal coloured partitions with order $\Delta'$ and a given kernel.

\[
\sum_{\text{Coarse sequence of length } s+m \text{ such that } \text{red}(C)=S} q^{\min_{\Delta'}(C)} = q^{\min_{\Delta'}(S) + m(s+m+1)} \sum_{u=0}^{t} q^{-u(t+m)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m-u} q^{-u}. \tag{3.2}
\]

Proof: Let $C = c_1, \ldots, c_{s+m}$ be a colour sequence whose reduction is $S$. The weight of the corresponding minimal partition in $P'_n$ is

\[
\text{min}_{\Delta'}(C) = \sum_{i=1}^{s+m} i\Delta'(c_i, c_{i+1}) = (s+m)\left(s + m + 1\right) - \text{min}_{\Delta'}(C), \tag{3.3}
\]

where the second equality follows from the definition of $\Delta'$.

On the other hand, by Corollary 3.3 and (3.3), we have

\[
\text{min}_{\Delta'}(S) = \text{min}_{\Delta'}(S) = s(m) - \text{min}_{\Delta'}(S). \tag{3.4}
\]

Given that, by Proposition 3.4, $\Delta$ and $\Delta'$ have exactly the same insertion properties up to exchanging the type 0 and 1 insertions, Proposition 2.29 immediately gives us that

\[
\sum_{\text{Coarse sequence of length } s+m \text{ such that } \text{red}(C)=S} q^{\min_{\Delta'}(C)} = q^{\min_{\Delta'}(S) + m} \sum_{u=0}^{t} q^{u(s-t)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m-u} q^{-u}. \tag{3.5}
\]

Combining this with (3.3), we get that the generating function for minimal partitions in $P'_n$ is

\[
G := \sum_{\text{Coarse sequence of length } s+m \text{ such that } \text{red}(C)=S} q^{\text{min}_{\Delta'}(C)} = q^{(s+m)(s+m+1) - \text{min}_{\Delta'}(S) - m} \sum_{u=0}^{t} q^{-u(s-t)} g_{u,t}(q^{-1}; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m-u} q^{-u}. \tag{3.6}
\]

By Lemma 3.5 and the fact that for all $k \in \{1, \ldots, t\}$, $|T_0^k| = 2 - |T_0^k|$, the above becomes

\[
G = q^{(s+m)(s+m+1) - \text{min}_{\Delta'}(S) - m} \sum_{u=0}^{t} q^{-u(s+t+u-1)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m-u} q^{-u}. \tag{3.7}
\]

Now using the fact that

\[
\left[ s + m - 1 \right]_{m-u} q^{-u} = q^{-(s+u-1)(m-u)} \left[ s + m - 1 \right]_{m-u} q^{-u}, \tag{3.8}
\]

we obtain

\[
G = q^{(s+m)(s+m+1) - \text{min}_{\Delta'}(S) - m} \sum_{u=0}^{t} q^{-u(t+m)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m-u} q^{-u}, \tag{3.9}
\]

where we used (3.4) in the last equality. This completes the proof. \qed
By Proposition 3.1, the generating function in (3.2) is also the generating function for minimal $n^2$-coloured Frobenius partitions with kernel $S$. Finally, using the fact that any $n^2$-coloured Frobenius partitions with colour sequence $C$ of length $s + m$ can be obtained in a unique way by adding a partition into at most $s + m$ parts to $\lambda$ and another partition into at most $s + m$ parts to $\mu$ in the minimal $n^2$-coloured Frobenius partition $\min F(C) = (\lambda_1 \lambda_2 \cdots \lambda_{s+m})$ we obtain the following key expression for the generating function of $n^2$-coloured Frobenius partitions with a given kernel $S$.

**Proposition 3.7.** Let $n$ be a positive integer and $m$ a non-negative integer. Let $S = c_1, \ldots, c_n$ be a reduced colour sequence of length $s$, having $t$ maximal primary subsequences. Using the notation of Section 2.4, the generating function for $n^2$-coloured Frobenius partitions with kernel $S$, having length $s + m$, is the following:

$$
\sum_{\ell(F) = s + m, \ker(F) = S} q^{\left[ F \right]} = \frac{q^{\min_u(S) + m(s + m + 1)}}{(q; q)_{s + m}} \sum_{u=0}^{t} q^{-u(t + m)} g_{u, t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ s + m - 1 \right]_{m - u}. \tag{3.5}
$$

3.3. **Equality of generating functions for $\mathcal{F}_n$ and $\mathcal{P}_n$.** Proposition 2.30 gives the generating function for coloured partitions of $\mathcal{P}_n$ with kernel $S$, and Proposition 3.7 gives the generating function for coloured Frobenius partitions of $\mathcal{F}_n$ with the same kernel $S$. In this section, we show that these two generating functions are actually equal, which will complete the proof of our generalisation of Primc’s identity (Theorem 1.25).

But before doing so, we need a lemma about $q$-binomial coefficients.

**Lemma 3.8.** Let $s$ be a positive integer and $m, u$ two non-negative integer. Then

$$
\frac{1}{(q; q)_{s + m}} = \sum_{m' \geq 0} \frac{q^{(m' - u)(s + m')}}{(q; q)_{s + m'}} \left[ m - u \right]_{m' - u}. \tag{3.5}
$$

**Proof:** Let us consider a partition into parts at most $s + m$, generated by $\frac{1}{(q; q)_{s + m}}$.

![Figure 2. Decomposition of the Ferrers board.](image)

Draw its Ferrers diagram on the plane as shown in Figure 2, and draw the line of equation $x - y = u + s$. This line intersects the boundary of the Ferrers board in a point with coordinates $(s + m', m' - u)$ for some integer $m' \in \{u, \ldots, m\}$. (we take the convention that the $x$-axis always belongs to the boundary of the Ferrers board). It defines three zones in the Ferrers diagram:

- a rectangle of size $(m' - u) \times (s + m')$ on the bottom-left of the intersection, generated by $q^{(m' - u)(s + m')}$,
- a partition into parts at most $s + m'$ on top on the rectangle, generated by $\frac{1}{(q; q)_{s + m'}}$.  

• a partition with at most \( m' - u \) parts, each of size at most \( m - m' \), generated by \( \binom{m-u}{m'-u} \).

Summing over all possible values of \( m' \) gives the desired result.

We are now ready to prove the following theorem, which implies Theorem 1.25.

**Theorem 3.9.** Let \( n \) be a positive integer and \( m \) a non-negative integer. Let \( S = c_1, \ldots, c_s \) be a reduced colour sequence of length \( s \), having \( t \) maximal primary subsequences. Then

\[
\sum_{\lambda \in \mathcal{P}_n: \ker(\lambda) = S} q^{\omega(\lambda)} = \sum_{F \in \mathcal{F}_n: \ker(F) = S} q^{\omega(F)}. 
\]

**Proof:** By Proposition 2.30,

\[
\sum_{\lambda \in \mathcal{P}_n: \ker(\lambda) = S} q^{\omega(\lambda)} = \sum_{m \geq 0} \frac{q^{\min_{\Delta}(S)+m}}{(q; q)_{s+m}} \sum_{u=0}^{t} q^{u(s-t)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ \frac{s+m-1}{m-u} \right]_q 
\]

and by Proposition 3.7,

\[
\sum_{F \in \mathcal{F}_n: \ker(F) = S} q^{\omega(F)} = \sum_{m \geq 0} \frac{q^{\min_{\Delta}(S)+m(s+m+1)}}{(q; q)_{s+m+1}^2} \sum_{u=0}^{t} q^{u(s-t)} g_{u,t}(q; |T_0^1|, \ldots, |T_0^t|) \left[ \frac{s+m-1}{m-u} \right]_q 
\]

Thus, to prove the theorem, it is sufficient to show that for \( u \in \{0, \ldots, t\},

\[
\sum_{m \geq 0} \frac{q^m}{(q; q)_{s+m+1}} \left[ \frac{s+m-1}{m-u} \right]_q = \sum_{m \geq 0} \frac{q^{(m-u)(s+m)+m+m'(t+m+1)}}{(q; q)_{s+m+1}^2} \left[ \frac{s+m-1}{m-u} \right]_q. 
\]

By Lemma 3.8,

\[
\frac{1}{(q; q)_{s+m}} \left[ \frac{s+m-1}{m-u} \right]_q = \sum_{m' \geq 0} \frac{q^{(m'-u)(s+m')}}{(q; q)_{s+m'}} \left[ \frac{s+m-1}{m'-u} \right]_q 
\]

Thus

\[
\sum_{m \geq 0} \frac{q^m}{(q; q)_{s+m}} \left[ \frac{s+m-1}{m-u} \right]_q = \sum_{m \geq 0} \sum_{m' \geq 0} \frac{q^{(m'-u)(s+m')}+m'}{(q; q)_{s+m'}} \left[ \frac{s+m-1}{m'-u} \right]_q \left[ \frac{s+m-1}{m'+u} \right]_q 
\]

The last thing to show is that

\[
\sum_{m \geq 0} q^{m'-m} \left[ \frac{s+m-1}{s+m'-1} \right] = \frac{1}{(q; q)_{s+m'}},
\]

which is true by separating the partitions into at most \( s + m' \) parts counted by \( \frac{1}{(q; q)_{s+m'}} \) according to the length \( m - m' \) of their largest part.

Thus (3.7) is true and the theorem is proved. \( \square \)
3.4. Proof of Theorem 1.27. In the last section, we proved our main theorem (Theorem 1.25) relating the generating function for generalised Primc partitions and the one of coloured Frobenius partitions. In this section, we study the particular case where we set $b_i = a_i^{-1}$ for all $i \in \{0, \ldots, n\}$. All the free colours vanish, and the generating function can now be written as a sum of infinite products, as stated in Theorem 1.27.

Let $n$ be a positive integer. By Theorem 1.25 in which we set $b_i = a_i^{-1}$ for all $i$, we have

$$P_n := \sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1} \geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{u_0 - v_0} \cdots a_{n-1}^{u_{n-1} - v_{n-1}}$$

$$= [x^n] \prod_{i=0}^{n-1} (-xa_i q; q)_\infty (-x^{-1} a_i^{-1}; q)_\infty.$$

Using the Jacobi triple product (1.14) in each term of this product, we obtain

$$P_n = \frac{1}{(q; q)_\infty^n} [x^n] \prod_{i=0}^{n-1} \left( \sum_{m_i \in \mathbb{Z}} x^{m_i} a_i^{m_i} q^{m_i(m_i+1)/2} \right)$$

$$= \frac{1}{(q; q)_{\infty}^n} \prod_{m_0, \ldots, m_{n-1} \in \mathbb{Z}} \frac{1}{m_0 + \cdots + m_{n-1} = 0} \left( \prod_{i=0}^{n-1} a_i^{m_i} \right) q^{\sum_{i=0}^{n-1} m_i(m_i+1)/2}.$$

Now replacing $m_0$ by $-m_1 - \cdots - m_{n-1}$ and using that

$$\frac{m_0(m_0 + 1)}{2} = \sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-1} m_i + \sum_{1 \leq i < j \leq n-1} m_i m_j,$$

we get

$$P_n = \frac{1}{(q; q)^n_{\infty}} \sum_{m_1, \ldots, m_{n-1} \in \mathbb{Z}} \left( \prod_{i=1}^{n-1} \left( a_i a_0^{-1} \right)^{m_i} \right) q^{\sum_{i=1}^{n-1} m_i^2 + \sum_{1 \leq i < j \leq n-1} m_i m_j}. \quad (3.8)$$

We want to apply the Jacobi triple product again inside the sum, in order to obtain a sum of infinite products. To do so, we perform some changes of variables. We first need the following lemma.

**Lemma 3.10.** Let

$$M(n) := \sum_{i=1}^{n-1} m_i^2 + \sum_{1 \leq i < j \leq n-1} m_i m_j.$$

Let $s_n = 0$ and for all $i \in \{1, \ldots, n-1\}$,

$$s_i := \sum_{j=i}^{n-1} m_j.$$

Then we have

$$M(n) = \sum_{i=1}^{n-1} s_i(s_i - s_{i+1}) = \sum_{i=1}^{n-1} \frac{(i + 1)s_i - is_{i+1})^2}{2i(i+1)}.$$

**Proof:** The first equality follows directly from the definition of the $s_i$’s. Let us now prove the second equality. We have

$$\sum_{i=1}^{n-1} \frac{(i + 1)s_i - is_{i+1})^2}{2i(i+1)} = \sum_{i=1}^{n-1} \frac{i + 1}{2i} s_i^2 - s_is_{i+1} + \frac{i}{2(i+1)} s_{i+1}^2$$

$$= -\sum_{i=1}^{n-1} s_is_{i+1} + s_i^2 + \sum_{i=2}^{n-1} \left( \frac{i + 1}{2i} s_i^2 + \frac{i - 1}{2i} s_i^2 \right)$$

$$= \sum_{i=1}^{n-1} s_i(s_i - s_{i+1}),$$

where the second equality followed from the change of variable $i \to i - 1$ in the last sum. \qed
By Lemma 3.10 and (3.8), we obtain
\[
P_n = \frac{1}{(q; q)_{\infty}^n} \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}} \left( \prod_{i=1}^{n-1} (a_i a_0^{-1})^{s_i - s_{i+1}} \right) q^{\sum_{i=1}^{n-1} s_i (s_i - s_{i+1})}
\]
\[
= \frac{1}{(q; q)_{\infty}^n} \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}} a_0^{-s_0} \prod_{i=1}^{n-1} a_i^{s_i - s_{i+1}} q^{s_i (s_i - s_{i+1})}.
\]

This is (1.18). Let us do perform a few more changes of variables to obtain (1.19).

For all \(i \in \{1, \ldots, n-1\}\), let us write \(s_i = i \times d_i + r_i\), with \(r_i \in \{0, \ldots, i-1\}\). This is the euclidian division by \(i\), so this expression is unique, and for \(r_1, \ldots, r_{n-1}\) fixed, there is a bijection between \(\{(s_1, \ldots, s_{n-1}) \in \mathbb{Z}^{n-1} : s_i \equiv r_i \mod i\}\) and \(\{(d_1, \ldots, d_{n-1}) \in \mathbb{Z}^{n-1}\}\). Moreover our choice \(s_n = 0\) corresponds to \(d_n = r_n = 0\).

We obtain
\[
M(n) = \sum_{i=1}^{n-1} \left( \frac{i(i+1)}{2} (d_i - d_{i+1})^2 + \frac{(i+1)(r_i - i r_{i+1})^2}{2(i+1)} + (d_i - d_{i+1})((i+1)r_i - i r_{i+1}) \right).
\]

By a last change of variable \(p_i = d_i - d_{i+1}\), equivalent to \(d_i = \sum_{j=i}^{n-1} p_j\), \(\{(d_1, \ldots, d_{n-1}) \in \mathbb{Z}^{n-1}\}\) is in bijection with \(\{(p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}\}\). This yields
\[
M(n) = \sum_{i=1}^{n-1} \left( \frac{i(i+1)}{2} p_i^2 + \frac{(i+1)r_i - i r_{i+1})^2}{2(i+1)} + p_i((i+1)r_i - i r_{i+1}) \right)
\]
\[
= \sum_{i=1}^{n-1} r_i (r_i - r_{i+1}) + \sum_{i=1}^{n-1} \left( \frac{i(i+1)}{2} p_i^2 + p_i((i+1)r_i - i r_{i+1}) \right)
\]

Backtracking all these changes of variables, we have for all \(i \in \{1, \ldots, n-1\}\),
\[
m_i = s_i - s_{i+1}
\]
\[
= i d_i + r_i - (i+1)d_{i+1} - r_{i+1}
\]
\[
= i \sum_{j=i}^{n-1} p_j + r_i - (i+1) \sum_{j=i+1}^{n-1} p_j - r_{i+1}
\]
\[
= i p_i - \sum_{j=i+1}^{n-1} p_j + r_i - r_{i+1}.
\]

Thus, by the above and Lemma 3.10, the generating function in (3.8) becomes
\[
P_n = \frac{1}{(q; q)_{\infty}^n} \sum_{0 \leq r_i \leq j-1} \sum_{p_1, \ldots, p_{n-1} \in \mathbb{Z}} \left( \prod_{i=1}^{n-1} (a_i a_0^{-1})^{p_i - \sum_{j=i+1}^{n-1} p_j + r_i - r_{i+1}} \right)
\]
\[
\times q^{\sum_{i=1}^{n-1} r_i (r_i - r_{i+1}) + \sum_{i=1}^{n-1} \left( \frac{i(i+1)}{2} p_i^2 + p_i((i+1)r_i - i r_{i+1}) \right)}.
\]

It can be shown by induction on \(n\) that
\[
\prod_{i=1}^{n-1} (a_i a_0^{-1})^{p_i - \sum_{j=i+1}^{n-1} p_j} = \prod_{i=1}^{n-1} \left( \prod_{\ell=0}^{i-1} a_{\ell} a_{\ell}^{-1} \right)^{p_i}
\]

Therefore reorganising (3.9) leads to
\[
P_n = \frac{1}{(q; q)_{\infty}^n} \sum_{0 \leq r_i \leq j-1} \left( \prod_{i=1}^{n-1} (a_i a_0^{-1})^{r_i - r_{i+1}} q^{r_i (r_i - r_{i+1})} \right)
\]
\[
\times \sum_{p_1, \ldots, p_{n-1} \in \mathbb{Z}} \left( \prod_{\ell=0}^{i-1} a_{\ell} a_{\ell}^{-1} \right)^{p_i} q^{i(i+1)r_i - i r_{i+1}}
\]

\[26\]
the key in proving our two new generalisations of Capparelli’s identity (Theorem 1.28).

We first give a variant of the bijection

\[
= \frac{1}{(q; q)_{\infty}^n} \sum_{r_1, \ldots, r_{n-1} \geq 0, 0 \leq r_j \leq -1} \left( \prod_{i=1}^{n-1} a_i^{r_i-r_{i+1}} q^{r_i(r_i-r_{i+1})} \right) \\
\times \prod_{i=1}^{n-1} \sum_{p_1, \ldots, p_{n-1} \in \mathbb{Z}} \left( \left( \prod_{\ell=0}^{i-1} a_{\ell}^{1-a_{\ell}} \right) q^{-(i+1)(i+1)+(i+1)r_i-r_{i+1}} \right) \prod_{i=1}^{n} q^{j(i+1)p_{j}(p_{j+1})}
\]

where in the last equality, we used Jacobi’s triple product identity in each of the sums in the \(p_i\)’s. Theorem 1.27 is proved.

Remark. Andrews [And84a] gave the particular cases \(n = 1, 2, 3\) of this formula, but without keeping track of the colours. Our result is more general, as it is both valid for all \(n\) and keeps track of the colours.

4. Bijections between Generalised Primc partitions and Generalised Capparelli partition pairs

Now that we have established the connection between the generalised Primc partitions of \(P_n\) and the \(n^2\)-coloured Frobenius partitions, this section is dedicated to the proof of Theorem 1.22, which connects generalised Primc partitions with two different generalisations of Capparelli partitions. This connection is the key in proving our two new generalisations of Capparelli’s identity (Theorem 1.28).

The proofs in this section are generalisations of the first author’s bijection between \(P_2\) and \(C_2\) in [Dou18b]. However, the partitions in \(P_m\), \(C_m\) and \(CC_m\) have a more intricate combinatorial description, so that it is better to reformulate and simplify the bijection between \(P_2\) and \(C_2\) before generalising it.

4.1. Reformulation of Dousse’s bijection between \(P_2\) and \(C_2\). We first give a variant of the bijection of [Dou18b]. The one-to-one correspondence is the same, but the intermediate steps are different.

Let \((\lambda, \mu) \in C_2\) be a partition pair of total weight \(n\), where \(\lambda \in C_2\) and \(\mu\) is an unrestricted partition coloured \(b\). The idea from [Dou18b] is to insert the parts of \(\mu\) inside \(\lambda\) and modify the colour of certain parts in order to obtain a partition in \(P_2\), all in a bijective way. Here we keep the same idea but perform the insertions in a different order, making the resulting partitions easier to describe at each step.

To make the comparison with [Dou18b] clear, we illustrate our variant of the bijection on the same example

\[
\begin{align*}
\lambda &= 8d + 8a + 6c + 5e + 3d + 1a, \\
\mu &= 8b + 8d + 7b + 5b + 3b + 2b + 2b + 1b + 1b,
\end{align*}
\]

First of all, recall that \(\lambda \in C_2\) satisfies the difference conditions from

\[
\begin{bmatrix}
a & c & d \\
a & 2 & 2 & 2 \\
c & 1 & 1 & 2 \\
d & 0 & 1 & 2
\end{bmatrix}
\]

Note also that the column and row \(b\) in matrix \(P_2\) from (1.4) mean that if there is a part \(k_b\) in the partition, then it can repeat but the number \(k\) cannot appear in any other colour.

**Step 1**: For all \(j\), if there are some parts of size \(j\) in \(\mu\) but none in \(\lambda\), then move these parts from \(\mu\) to \(\lambda\). Call \(\lambda_1\) and \(\mu_1\) the resulting partitions.
In our example, we obtain

\[
\lambda_1 = 8d + 8a + 7b + 6c + 5c + 3d + 2b + 2b + 1a, \\
\mu_1 = 8b + 8b + 5b + 3b + 1b + 1b.
\]

The pair \((\lambda_1, \mu_1)\) is such that \(\lambda_1\) satisfies the difference conditions in the matrix

\[
C_2^{\lambda} = \begin{pmatrix}
 a & b & c & d \\
 2 & 1 & 2 & 2 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 1 & 2 \\
 0 & 1 & 1 & 2
\end{pmatrix}
\] (4.1)

and \(\mu_1\) is a partition coloured \(b\) containing only parts of sizes that also appear in \(\lambda_1\) but in a colour different from \(b\). Indeed, in \(\lambda_1\), there can now be some parts coloured \(b\) which can repeat and are distinct from all the other parts, and the minimal differences between parts coloured \(a, c, d\) is the same as before.

This process is reversible, as one can simply move the \(b\)-parts of \(\lambda_1\) back to \(\mu_1\).

**Step 2:** For all \(j\), if there are some parts \(j_b\) in \(\mu_1\), and \(j_c\) appears in \(\lambda_1\) (by (4.1), it cannot repeat nor appear in another colour), then transform those \(j_b\)'s into \(j_c\)'s and move them from \(\mu_1\) to \(\lambda_1\). Call \(\lambda_2\) and \(\mu_2\) the resulting partitions.

In our example, we obtain

\[
\lambda_2 = 8d + 8a + 7b + 6c + 5c + 5c + 3d + 2b + 2b + 1a, \\
\mu_2 = 8b + 8b + 3b + 1b + 1b.
\]

Now the parts coloured \(c\) can repeat, and the rest of the partition was not affected at all. Thus the pair \((\lambda_2, \mu_2)\) is such that \(\lambda_2\) satisfies the difference conditions in the matrix

\[
C_2^{\lambda} = \begin{pmatrix}
 a & b & c & d \\
 2 & 1 & 2 & 2 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 2 \\
 0 & 1 & 1 & 2
\end{pmatrix}
\] (4.2)

and \(\mu_2\) is a partition coloured \(b\) containing only parts of sizes that also appear in \(\lambda_2\) but in colour \(a\) or \(d\).

This process is also reversible. If in \(\lambda_2\), there is a \(c\)-coloured part \(j_c\) that repeats, then transform all but one of the \(j_c\)'s into \(j_b\)'s and move them to \(\mu_2\).

**Step 3:** For all \(j\), if there are some parts \(j_b\) in \(\mu_2\), then \(j\) appears in \(\lambda_2\) in colour \(a\) or \(d\), but not \(c\). Transform those \(j_b\)'s into \(j_c\)'s and insert them inside \(\lambda_2\), with the colour order \(a < c < d\). Call \(\lambda_3\) the resulting partition.

In our example, we obtain

\[
\lambda_3 = 8d + 8c + 8a + 7b + 6c + 5c + 5c + 3d + 3c + 2b + 2b + 1c + 1c + 1a.
\]

Now the minimal difference between parts of colour \(c\) and \(a\) (resp. \(d\) and \(c\)) is 0, and the rest of the partition was not affected at all. Thus the partition \(\lambda_3\) satisfies exactly the difference conditions of Primc’s matrix \(P_2\) in (1.4).

This final step is also reversible. If in \(\lambda_3\), there are some parts \(j_c\) such that \(j_a\) or \(j_d\) also appears, then transform those \(j_c\)'s into \(j_b\)'s, remove them from \(\lambda_3\), and put them in a separate partition \(\mu_2\).

We obtain the same final partition as in [Dou18b], only the intermediate steps are different.

All the steps in this bijection preserve the weight, the number of parts, the size of the parts, and the number of \(a\)-parts and \(d\)-parts. Noting that \(CC_2 = CC'_2\), Theorem 1.22 is proved in the case \(n = 2\).

In the remainder of this section, we generalise this bijection for all \(m\). For \(m \geq 3\), \(CC_m\) and \(CC'_m\) are actually distinct, so there will be two different bijections.
4.2. Preliminary observations. Before we define our two bijections which will prove the two generalisations of Capparelli’s identity, we start with a few observations which help us understand better the combinatorial structure of the difference conditions $\Delta$.

Let us start with a remark about the colour $a_0b_0$, which plays a particular role in our reasoning, as it does not appear in the generalisations of Capparelli’s identity.

**Remark.** We have $\Delta(a_0b_0, a_0b_0) = 0$, and for all $c \neq a_0b_0$, 

$$
\Delta(c, a_0b_0) = \Delta(a_0b_0, c) = 1.
$$

This means that the colour $a_0b_0$ can repeat, but that if there is an integer $k$ of colour $a_0b_0$, then $k$ cannot appear in any other colour. This is the only restriction involving $a_0b_0$.

Our bijection will rely on the insertion of parts with free colours inside sequences of parts of the same size, so we need to understand the combinatorics of these sequences. The first step towards this is understanding pairs of colours $(c, c')$ such that $\Delta(c, c') = 0$.

**Proposition 4.1.** A pair of colours $(c, c')$ satisfies $\Delta(c, c') = 0$ if and only if it satisfies one of the following four conditions:

1. $c = c'$ and $c$ is a free colour,
2. $c = a_ib_i$ is a free colour, $c' = a_kb_k$ is a bound colour (i.e. $k \neq \ell$), and $\ell < i \leq k$,
3. $c = a_ib_i$ is a bound colour (i.e. $i \neq j$), $c' = a_kb_k$ is a free colour, and $i < k \leq j$,
4. $c = a_ib_i$ and $c' = a_kb_k$ are both bound colours (i.e. $i \neq j$ and $k \neq \ell$), and $i < k$ and $j > \ell$.

**Proof:** (1) This follows easily from Properties 1.11 and 1.12.

(2) By the definition (1.8) of $\Delta$, we have 

$$
\Delta(a_ib_i, a_kb_k) = \chi(i \geq k) - \chi(i = k) + \chi(i \leq \ell).
$$

If $i = k$, then $\Delta(a_ib_i, a_kb_k) = 0$ if and only if $i > \ell$. If $i \neq k$, then $\Delta(a_ib_i, a_kb_k) = 0$ if and only if $\ell < i < k$. Both cases can be summed up as $\ell < i \leq k$.

(3) Again by the definition of $\Delta$, we have 

$$
\Delta(a_ib_j, a_kb_k) = \chi(i \geq k) + \chi(j \leq k) - \chi(j = k).
$$

If $j = k$, then $\Delta(a_ib_j, a_kb_k) = 0$ if and only if $i < k$. If $j \neq k$, then $\Delta(a_ib_j, a_kb_k) = 0$ if and only if $i < k < j$. Both cases can be summed up as $i < k \leq j$.

(4) Finally, 

$$
\Delta(a_ib_j, a_kb_k) = \chi(i \geq k) + \chi(j \leq \ell).
$$

This is zero if and only if $i < k$ and $j > \ell$. \hfill \square

Proposition 4.1 allows us to understand exactly the shape of the colour sequences of subpartitions where all the parts have the same value.

**Proposition 4.2.** Let $C = c_1 \cdots c_s$ be a sequence of colours such that for all $i \in \{1, \ldots, s-1\}$, $\Delta(c_i, c_{i+1}) = 0$. Then, writing for all $i$, $c_i = a_i b_i$, the sequence $C$ satisfies one of the following:

**Case 1:** There is exactly one free colour $c_i$ in $C$ (which may repeat an arbitrary number of times). In this case, the inequalities between then $k_i$’s and $\ell_i$’s can be summarised as follows, where the numbers below indicate which case of Proposition 4.1 each pair of inequalities correspond to.

<table>
<thead>
<tr>
<th>index(a)</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$\cdots$</th>
<th>$c_{i-1}$</th>
<th>$c_i$</th>
<th>$\cdots$</th>
<th>$c_{i+j-1}$</th>
<th>$c_{i+j}$</th>
<th>$\cdots$</th>
<th>$c_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_i$</td>
<td>$k_2$</td>
<td>$\cdots$</td>
<td>$k_{i-1}$</td>
<td>$k_i$</td>
<td>$\cdots$</td>
<td>$k_{i+j}$</td>
<td>$\cdots$</td>
<td>$k_s$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4.3)

<table>
<thead>
<tr>
<th>index(b)</th>
<th>$\ell_1$</th>
<th>$\ell_2$</th>
<th>$\cdots$</th>
<th>$\ell_{i-1}$</th>
<th>$k_i$</th>
<th>$\cdots$</th>
<th>$k_i$</th>
<th>$\ell_{i+j}$</th>
<th>$\cdots$</th>
<th>$\ell_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4$</td>
<td>$4$</td>
<td>$\cdots$</td>
<td>$4$</td>
<td>$3$</td>
<td>$1$</td>
<td>$\cdots$</td>
<td>$1$</td>
<td>$2$</td>
<td>$\cdots$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

There are three possible sub-cases:

**Case 1a:** the free colour is on the left end ($i = 1$).

**Case 1b:** the free colour is on the right end ($i + j - 1 = s$).

**Case 1c:** there bound colours on both sides of the free colour ($i \neq 1, s + 1 - j$).
Case 2: There is no free colour in $C$. In this case, the inequalities between then $k_i$’s and $\ell_i$’s can be summarised as follows, where all the inequalities come from Case (4) of Proposition 4.1.

$$\begin{array}{ccccccc}
\text{index}(a) & k_1 & k_2 & \cdots & k_i & k_{i+1} & \cdots & c_s \\
\text{index}(b) & \ell_1 & > & \ell_2 & > & \cdots & > & \ell_i & > & \ell_{i+1} & > & \cdots & > & \ell_s
\end{array} \quad (4.4)$$

There are three possible sub-cases:
Case 2a: for all $i \in \{1, \ldots, s\}, k_i > \ell_i$.
Case 2b: for all $i \in \{1, \ldots, s\}, k_i < \ell_i$.
Case 2c: there is exactly one $i \in \{1, \ldots, s\}$ such that $k_i < \ell_i$ and $k_{i+1} > \ell_{i+1}$.

Proof: The fact that there is at most one free colour in $C$ follows from the triangular inequality. Assume there are two different free colours $c_i$ and $c_{i+j}$ in $C$, then by the triangular inequality, we have $1 = \Delta(c_i, c_j) \leq \Delta(c_i, c_i + 1) + \cdots + \Delta(c_{j-1}, c_j)$, contradicting the fact that each term in this sum is 0.

The inequalities presented in the tables above follow from a straightforward application of Proposition 4.1.

We conclude this section by characterising, using Proposition 4.2, the insertions of free colours that can be performed in the colour sequences in Case 2.

Proposition 4.3. Let $C = c_1 \cdots c_s$ be a sequence of bound colours such that for all $i \in \{1, \ldots, s-1\}$, $\Delta(c_i, c_{i+1}) = 0$. Then, writing for all $i$, $c_i = a_k b_k$, the insertions of free colours we can perform in $C$ are exactly the following.

- If $C$ is in Case 2a, then we can insert the free colour $a_k b_k$ to the left of $c_1$, where
  \[ \ell_1 < k \leq k_1. \]
  The sequence we obtain is in Case 1a.

- If $C$ is in Case 2b, then we can insert the free colour $a_k b_k$ to the right of $c_s$, where
  \[ k_s < k \leq \ell_s. \]
  The sequence we obtain is in Case 1b.

- If $C$ is in Case 2c, where $k_i \ell_i$ and $k_{i+1} > \ell_{i+1}$, then we can insert the free colour $a_k b_k$ between $c_i$ and $c_{i+1}$, where
  \[ k_i < k \leq k_{i+1} \quad \text{and} \quad \ell_{i+1} < k \leq \ell_i. \]
  The sequence we obtain is in Case 1c.

Remark. In Case 2c, we have
\[ 1 + \max(k_i, \ell_{i+1}) \leq k \leq \min(\ell_i, k_{i+1}). \]

Forbidding equality either on the right or on the left in this formula leads to our two generalisations of Capparelli’s identity and bijections.

4.3. Bijection between $\mathcal{P}_n$ and $\mathcal{C}C_n$. Now that we understood the colour sequences corresponding to parts of the same size in $\mathcal{P}_n$ and where free colours can be inserted in them to keep a difference 0 between the parts, we can present our bijection between $\mathcal{P}_n$ and $\mathcal{C}C_n$. The idea is similar to the bijection $\mathcal{P}_2$ and $\mathcal{C}C_2$ in Section 4.1 as we will insert parts coloured $a_k b_k$ inside a partition of $\mathcal{C}C_n$, but we will now need the observations of Section 4.2 to see how these insertions affect the partition.

Recall that coloured partitions in $\mathcal{C}_n$ are defined by the minimal difference conditions $\delta$ stated in (1.10). By definition of $\delta$, parts in free colours are not allowed to repeat. Moreover, the fact that for all $\ell < k$,
\[ \delta(a_k b_k, a_k b_k) = 1 \text{ and } \delta(a_k b_k, a_k b_k) = 1 \]
implies that sequences $C = c_1 \cdots c_s$ colours such that for all $i \in$
\{1, \ldots, s - 1\}, \Delta(c_i, c_{i+1}) = 0 are either in Case 2 from Proposition 4.2 or in Case 1’, which is the same as Case 1 except that now \(\ell_i - 1 > k_i, k_j < k_{i+1}\), and the free colour \(c_i\) cannot repeat:

\[
\begin{array}{cccccccc}
index(a) & c_1 & c_2 & \cdots & c_{i-1} & c_i & c_{i+1} & \cdots & c_s \\
\hline
\ell_1 < \ell_2 < \cdots < \ell_{i-1} < k_i < k_{i+1} < \cdots < \ell_s \\
\wedge \\
\ell_i - 1 & > & 1 & \cdots & > & k_i & > & \ell_{i+j} & > & \cdots & > & \ell_s
\end{array}
\tag{4.5}
\]

Let us now describe our bijection. Let \((\lambda, \mu) \in C_C n\) be a partition pair of total weight \(m\), where \(\lambda \in C_n\) and \(\mu\) is an unrestricted partition coloured \(a_0b_0\). The idea is again to insert the parts of \(\mu\) inside \(\lambda\) and modify the colour of certain parts in order to obtain a partition in \(P_n\), in a bijective way. We illustrate this bijection on an example in the case \(n = 3\):

\[
\lambda = 4a_0b_1 + 4a_2b_0 + 2a_0b_2 + 2a_1b_1 + 1a_2b_0, \\
\mu = 5a_0b_0 + 5a_0b_0 + 4a_0b_0 + 4a_0b_0 + 2a_0b_0 + 2a_0b_0 + 1a_0b_0.
\]

**Step 1:**

For all \(j\), if there are some parts of size \(j\) in \(\mu\) but none in \(\lambda\), then move these parts directly from \(\mu\) to \(\lambda\). Call \(\lambda_1\) and \(\mu_1\) the resulting partitions.

\[
\lambda_1 = 5a_0b_0 + 5a_0b_0 + 4a_0b_1 + 4a_2b_0 + 3a_0b_0 + 2a_0b_2 + 2a_1b_1 + 1a_2b_0, \\
\mu_1 = 4a_0b_0 + 4a_0b_0 + 2a_0b_0 + 2a_0b_0 + 1a_0b_0.
\]

The pair \((\lambda_1, \mu_1)\) is such that \(\lambda_1\) satisfies the difference conditions

\[
\begin{align*}
\delta_1(a_0b_0, a_0b_0) &= 0, \\
\delta_1(a_0b_0, a_kb) &= 1 \text{ for all } \ell, k, \\
\delta_1(a_kb, a_0b_0) &= 1 \text{ for all } \ell, k, \\
\delta_1(a_kb, a_kb') &= \delta(a_kb, a_kb') \text{ in all the other cases},
\end{align*}
\tag{4.6}
\]

and \(\mu_1\) is a partition coloured \(a_0b_0\) containing only parts of sizes that also appear in \(\lambda_1\) but in a colour different from \(a_0b_0\). Indeed, in \(\lambda_1\), there can now be some parts coloured \(a_0b_0\) which can repeat and are distinct from all the other parts, and the minimal differences between parts with other colours is the same as before.

In the case \(n = 3\), the minimal differences \(\delta_1\) can be summarised in the following matrix, where we underlined the difference with Primc’s matrix \(P_3\) (1.6).

\[
C^1_3 = \begin{pmatrix}
a_{2b0} & a_{2b1} & a_{1b0} & a_{2b2} & a_{1b1} & a_{0b1} & a_{1b2} & a_{0b2} \\
a_{2b1} & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
a_{1b0} & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\
a_{1b1} & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\
a_{0b0} & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
a_{1b2} & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\
a_{1b1} & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\
a_{0b2} & 0 & 0 & 0 & 1 & 1 & 2 & 1
\end{pmatrix}
\tag{4.7}
\]

This first step is reversible, as one can simply move the parts coloured \(a_0b_0\) of \(\lambda_1\) back to \(\mu_1\).

**Step 2:** For all \(j\), if the part \(j\) appears in \(\lambda_1\) in a free colour \(a_kb\) \((k \neq 0)\), then by definition of the difference conditions \(\delta\), it cannot repeat. In that case, if there are also some parts \(a_kb\) in \(\mu_1\), then change their colour to \(a_kb\) and move them to \(\lambda_1\). Call \(\lambda_2\) and \(\mu_2\) the resulting partitions.

In our example, we obtain

\[
\lambda_2 = 5a_0b_0 + 5a_0b_0 + 4a_0b_1 + 4a_2b_0 + 3a_0b_0 + 2a_0b_2 + 2a_1b_1 + 2a_1b_1 + 2a_1b_1 + 1a_2b_0.
\]
\[ \mu_2 = 4a_0b_0 + 4a_0b_0 + 1a_0b_0. \]

Now parts coloured with free colours can repeat in \( \lambda_2 \), and the rest of the partition was not affected at all. Indeed, this step creates sequences as in (4.5), where the free colour \( a_i \) can now repeat. Thus the pair \((\lambda_2, \mu_2)\) is such that \( \lambda_2 \) satisfies the difference conditions
\[
\begin{align*}
\delta_2(a_k b_k, a_k b_k) &= 0 \text{ for all } k, \\
\delta_2(a_i b_k, a_i b) &= \delta_1(a_i b_k, a_i b) \text{ in all the other cases,}
\end{align*}
\]
and \( \mu_2 \) is a partition coloured \( a_0b_0 \) containing only parts of sizes that also appear in \( \lambda_2 \) but in a bound colour.

In the case \( n = 3 \), the matrix representing the minimal differences \( \delta_2 \) become the following, where the differences with (1.6) are still underlined:
\[
C_3^2 = \begin{pmatrix}
a_2 b_0 & a_2 b_1 & a_1 b_0 & a_0 b_0 & a_0 b_1 & a_1 b_1 & a_0 b_1 & a_1 b_2 & a_0 b_2 \\
a_2 b_0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
a_2 b_1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 \\
a_1 b_0 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\
a_0 b_0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
a_2 b_2 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 \\
a_1 b_1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 2 \\
a_0 b_1 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\
a_1 b_2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
a_0 b_2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
\end{pmatrix}
\]
\[ (4.9) \]

This step is also reversible. If in \( \lambda_2 \), there is a free-coloured part \( j_{a_k b_k} \) that repeats, then transform all but one of the \( j_{a_k b_k} \)'s into \( j_{a_0 b_0} \)'s and move them to \( \mu_2 \).

**Step 3:** For all \( j \), if there are some parts \( j_{a_0 b_0} \) in \( \mu_2 \), then \( j \) appears in \( \lambda_3 \) in a certain number of bound colours, but not in any free colour. These colours form a sequence of the shape (4.4) (Case 2) in Proposition 4.2. By Proposition 4.3, there is only one position \( p_j \) where a free colour can be inserted in this sequence in a way that all the corresponding parts keep the same size.

Transform all these \( j_{a_0 b_0} \) into \( j_{a_k b_k} \), where \( k = \min(\ell_{p_j - 1}, k_{p_j}) \), and insert them in \( \lambda_2 \) in the only position possible. Here we take the convention that \( \ell_{p_j - 1} = \infty \) (resp. \( k_{p_j} = \infty \)) if there is no \( c_{p_j - 1} \) (resp. \( c(p_j) \)) in the colour sequences. This happens in Case 2a (resp. Case 2b) of Proposition 4.2. This insertion process creates sequences of the type (4.3) where \( \ell_{i - 1} = k_i \) or \( k_i = k_{i+j} \). Call \( \lambda_3 \) the resulting partition.

In our example, we obtain
\[
\lambda_3 = 5a_0b_0 + 5a_0b_0 + 4a_0b_1 + 4a_1b_1 + 4a_1b_1 + 4a_1b_1 + 3a_0b_0 + 3a_0b_0 + 2a_0b_0 + 2a_1b_1 + 2a_1b_1 + 2a_1b_1 + 2a_1b_1 + 2a_2b_2 + 2a_2b_2.
\]

Indeed, in the sequence \( a_2b_0 \) (Case 2a), the only place a free colour can be inserted is to the left of \( a_2b_0 \), and our rule sets this free colour to be \( a_2b_2 \). In the sequence \( a_0b_1, a_2b_0 \) (Case 2c), the only place a free colour can be inserted is between \( a_0b_1 \) and \( a_2b_0 \), and this colour should be \( a_1b_1 \).

Now the partition \( \lambda_3 \) satisfies the difference conditions
\[
\begin{align*}
\delta_3(a_k b_k, a_k b_k) &= 0 \text{ for all } \ell < k, \\
\delta_3(a_k b_k, a_k b_k) &= 0 \text{ for all } \ell < k, \\
\delta_3(a_i b_k, a_i b_k) &= \delta_2(a_i b_k, a_i b) \text{ in all the other cases.}
\end{align*}
\]
\[ (4.8) \]

By (4.8), (4.6), and (1.10), we see that \( \delta_3 = \Delta \), so the partition \( \lambda_3 \) belongs to \( \mathcal{P}_n \).

This final step is also reversible. If in \( \lambda_3 \), there is a \( j \) such that the sequence of colours of parts of size \( j \) are of the type (4.3) where \( \ell_{i - 1} = k_i \) or \( k_i = k_{i+j} \), then take all the parts with free colour \( a_k b_k \), change their colour to \( a_0b_0 \), remove them from \( \lambda_3 \), and put them in a separate partition \( \mu_2 \).

All the steps in this bijection are simply colour modification on free colours, so this bijection preserves the weight, the number of parts, the size of the parts, and the number of appearances of each bound colour.
4.4. **Bijection between** $\mathcal{P}_m$ **and** $\mathcal{C}_m'$. The idea behind the definition (1.10) of $\delta$ and the previous bijection was to forbid some sequences of the shape (4.3) by:

- forbidding repetition of free colours,
- forbidding that $\ell_{i-1} = k_i$ or $k_i = \ell_{i+j}$, i.e. modifying (4.3) in the following way:

\[
\begin{align*}
& k_{i-1} < k_i = \cdots = k_i \leq k_{i+j} \quad \wedge \quad k_{i-1} < k_i < k_{i+j} \\
& \ell_{i-1} \geq k_i = \cdots = k_i > \ell_{i+j} \quad \vee \quad \ell_{i-1} > k_i > \ell_{i+j}
\end{align*}
\]

The idea behind the definition (1.12) of $\delta'$ and our second bijection, which we describe in this section, is to forbid some other sequences of the shape (4.3) by:

- again forbidding repetition of free colours,
- forbidding that $k_{i-1} + 1 = k_i$ or $k_i = \ell_{i+j} + 1$, i.e. modifying (4.3) in the following way:

\[
\begin{align*}
& k_{i-1} < k_i = \cdots = k_i \leq k_{i+j} \quad \wedge \quad k_{i-1} + 1 < k_i \leq k_{i+j} \\
& \ell_{i-1} \geq k_i = \cdots = k_i > \ell_{i+j}) \quad \vee \quad \ell_{i-1} \geq k_i > 1+ \ell_{i+j}
\end{align*}
\]

Let us now describe our second bijection. Let $(\lambda', \mu') \in \mathcal{C}_n'$ be a partition pair of total weight $m$, where $\lambda' \in \mathcal{C}_n'$ and $\mu'$ is an unrestricted partition coloured $a_0b_0$. The idea is again to insert the parts of $\mu'$ inside $\lambda'$ and modify the colour of certain parts in order to obtain a partition in $\mathcal{P}_n$, in a bijective way. We illustrate this bijection on an example in the case $n = 3$:

\[
\begin{align*}
\lambda' &= 4a_0b_1 + 4a_2b_0 + 2a_0b_2 + 2a_2b_1 + 1a_0b_1, \\
\mu' &= 5a_0b_0 + 5a_0b_0 + 4a_0b_0 + 4a_0b_0 + 3a_0b_0 + 2a_0b_0 + 2a_0b_0 + 1a_0b_0.
\end{align*}
\]

**Step 1:** This step is the same as in the previous bijection. For all $j$, if there are some parts of size $j$ in $\mu'$ but none in $\lambda'$, then move these parts directly from $\mu'$ to $\lambda'$. Call $\lambda_1'$ and $\mu_1'$ the resulting partitions.

\[
\begin{align*}
\lambda_1' &= 5a_0b_0 + 5a_0b_0 + 4a_0b_0 + 4a_0b_0 + 3a_0b_0 + 2a_0b_0 + 2a_0b_0 + 1a_0b_0, \\
\mu_1' &= 4a_0b_0 + 4a_0b_0 + 2a_0b_0 + 2a_0b_0 + 1a_0b_0.
\end{align*}
\]

The pair $(\lambda_1', \mu_1')$ is such that $\lambda_1'$ satisfies the difference conditions

\[
\begin{align*}
\delta_1'(a_0b_0, a_0b_0) &= 0, \\
\delta_1'(a_0b_0, a_kb) &= 1 \text{ for all } \ell, k, \\
\delta_1'(a_kb, a_0b_0) &= 1 \text{ for all } \ell, k, \\
\delta_1'(a_kb, a_kb') &= \delta'(a_kb, a_kb') \text{ in all the other cases,}
\end{align*}
\]

and $\mu_1'$ is a partition coloured $a_0b_0$ containing only parts of sizes that also appear in $\lambda_1'$ but in a colour different from $a_0b_0$.

In the case $n = 3$, the minimal differences $\delta_1'$ can be summarised in the following matrix, where we underlined the difference with Prime’s matrix $P_3$ (1.6).

\[
D_3^1 = \begin{pmatrix}
2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2
\end{pmatrix}
\]

This first step is reversible, as one can simply move the parts coloured $a_0b_0$ of $\lambda_1'$ back to $\mu_1'$. 


**Step 2:** Again, the second step is similar to our first bijection. For all \( j \), if the part \( j \) appears in \( \lambda'_1 \) in a free colour \( a_k b_k \) \((k \neq 0)\), then by definition of the difference conditions \( \delta'_1 \), it cannot repeat. In that case, if there are also some parts \( j_{a_i b_i} \) in \( \mu'_1 \), then change their colour to \( a_k b_k \) and move them to \( \lambda'_1 \). Call \( \lambda'_2 \) and \( \mu'_2 \) the resulting partitions.

\[
\lambda'_2 = 5_{a_0 b_0} + 5_{a_0 b_0} + 4_{a_0 b_0} + 4_{a_2 b_2} + 4_{a_2 b_2} + 2_{a_2 b_2} + 2_{a_2 b_2} + 2_{a_2 b_2} + 1_{a_0 b_1},
\]
\[
\mu'_2 = 4_{a_0 b_0} + 5_{a_0 b_0} + 1_{a_0 b_1}.
\]

Now parts coloured with free colours can repeat in \( \lambda'_2 \), and the rest of the partition was not affected at all. Thus the pair \( (\lambda'_2, \mu'_2) \) is such that \( \lambda'_2 \) satisfies the difference conditions

\[
\delta'_2(a_k b_k, a_k b_k) = 0 \text{ for all } k,
\]
\[
\delta'_2(a_i b_i, a_i b_i) = \delta'_1(a_i b_k, a_i b_k) \text{ in all the other cases},
\]

and \( \mu'_2 \) is a partition coloured \( a_0 b_0 \) containing only parts of sizes that also appear in \( \lambda'_2 \) but in a bound colour.

In the case \( n = 3 \), the matrix representing the minimal differences \( \delta_3 \) become the following:

\[
D_3^2 = \begin{pmatrix}
  a_2 b_0 & a_2 b_1 & a_1 b_0 & a_0 b_0 & a_2 b_2 & a_1 b_1 & a_0 b_1 & a_1 b_2 & a_0 b_2 \\
 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 \\
 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\
\end{pmatrix}
\]

This step is again reversible. If in \( \lambda'_2 \), there is a free-coloured part \( j_{a_i b_k} \) that repeats, then transform all but one of the \( j_{a_i b_k} \)'s into \( j_{a_0 b_0} \)'s and move them to \( \mu'_2 \).

**Step 3:** This last step is different from the one of the first bijection, and corresponds to our new modification of (4.3).

For all \( j \), if there are some parts \( j_{a_i b_i} \) in \( \mu'_2 \), then \( j \) appears in \( \lambda'_2 \) in a certain number of bound colours, but not in any free colour. These colours form a sequence of the shape (4.4) (Case 2) in Proposition 4.2. By Proposition 4.3, there is only one position \( p_j \) where a free colour can be inserted in this sequence in a way that all the corresponding parts keep the same size.

Transform all these \( j_{a_0 b_0} \) into \( j_{a_k b_k} \), where \( k = 1 + \max(k_{p_j-1}, \ell_{p_j}) \), and insert them in \( \lambda'_2 \) in the only position possible. Here we take the convention that \( k_{p_j-1} = 0 \) (resp. \( \ell_{p_j} = 0 \)) if there is no \( c_{p_j-1} \) (resp. \( c_{p_j} \)) in the colour sequences. This happens in Case 2a (resp. Case 2b) of Proposition 4.2. This insertion process creates sequences of the type (4.3) where \( k_i = 1 = k_i \) or \( \ell_i = \ell_{i+1} = 1 \). Call \( \lambda'_3 \) the resulting partition.

In our example, we obtain

\[
\lambda'_3 = 5_{a_0 b_0} + 5_{a_0 b_0} + 4_{a_0 b_1} + 4_{a_1 b_1} + 4_{a_2 b_2} + 4_{a_2 b_2} + 2_{a_2 b_2} + 2_{a_2 b_2} + 2_{a_2 b_2} + 2_{a_2 b_2} + 1_{a_0 b_1} + 1_{a_1 b_1}.
\]

Indeed, in the sequence \( a_0 b_1 \) (Case 2b), the only place a free colour can be inserted is to the right of \( a_0 b_1 \), and our rule sets this free colour to be \( a_1 b_1 \). In the sequence \( a_0 b_1, a_2 b_0 \) (Case 2c), the only place a free colour can be inserted is between \( a_0 b_1 \) and \( a_2 b_0 \), and this colour should be \( a_1 b_1 \).

Now the partition \( \lambda'_3 \) satisfies the difference conditions

\[
\delta'_3 (a_k b_k, a_k b_k-1) = 0 \text{ for all } \ell \geq k,
\]
\[
\delta'_3 (a_{k-1} b_k, a_k b_k) = 0 \text{ for all } \ell \geq k,
\]
\[
\delta'_3 (a_i b_k, a_i b_{k'} = \delta'_3 (a_i b_k, a_i b_{k'}) \text{ in all the other cases},
\]

By (4.12), (4.10), and (1.12), we see that \( \delta'_3 = \Delta \), so the partition \( \lambda'_3 \) belongs to \( \mathcal{P}'_n \).
We also rewrite

\[ \lambda_j, \ \text{there is a } j \text{ such that the sequence of colours of parts of size } j \]

are of the type (4.3) where \( k_{i-1} + 1 = k_i \) or \( \ell_i = \ell_{i+j} + 1 \), then take all the parts with free colour \( a_{k_i}b_{k_i} \), change their colour to \( a_{q_{k_i}}b_{q_{k_i}} \), remove them from \( \lambda_j \), and put them in a separate partition \( \mu_j' \).

Just like our first bijection, this one preserves the weight, the number of parts, the size of the parts, and the number of appearances of each bound colour.

5. PROOF OF PROPOSITION 2.29

In this last section, we give a proof of Proposition 2.29. Let \( S = c_1, \ldots, c_s \) be a reduced colour sequence of length \( s \), having \( t \) maximal primary subsequences. We use the same notation as in Section 2.4. In addition, we define for all \( u \in \{1, \ldots, t\} \), \( j_{2u-1} \) (resp. \( j_{2u} \)) to be the index of the free colour which can be inserted to the left (resp. right) of \( S_u \).

Thus we have \( T_u^0 = \{j_{2u-1}, j_{2u}\} \cap T_0 \) and \( T_u^1 = \{j_{2u-1}, j_{2u}\} \cap T_1 \).

For brevity, we denote from now on the set of all integers between \( i \) and \( j \) by \( \llbracket i; j \rrbracket \).

Our starting point is the equality

\[
G_{S,m}(q) := \sum_{C \text{colour sequence of length } s+m \text{ such that } \text{red}(C) = S} q^{|\min_{\Delta}(C)|} = \sum_{n_1+\ldots+n_{s+t} = m} q^{|\min_{\Delta}(S(n_1,\ldots,n_{s+t}))|}, \tag{5.1}
\]

which simply follows from the definition of reduced colour sequences.

Proposition 2.28 gives us an expression for \( |\min_{\Delta}(S(n_1,\ldots,n_{s+t}))| \), which we will use to derive Proposition 2.29. Let us start with a lemma which evaluates a sum appearing in the formula for \( |\min_{\Delta}(S(n_1,\ldots,n_{s+t}))| \).

**Lemma 5.1.** Let

\[ \Sigma_1 := \sum_{j \in S_1} (\#(\llbracket j; s+t \rrbracket \cap (N \cup T_0 \cup S_1)), \]

where \( P(j) \) is the number of colours of \( S \) that are to the left of \( f_j \). We have

\[ \Sigma_1 = \sum_{u=1}^{t} \left( |N| + u + 1 + \sum_{v=u}^{t} (|T_v^0| + |S_v^0|) \right) |S_1^1| + \sum_{j \in S_1^1} \#(\llbracket j' < j : j' \in \mathcal{S}_1^1 \rrbracket), \]

where \( \mathcal{S}_1^1 := T_1^1 \setminus \mathcal{S}_1^1 \) is the set of indices \( j \) of \( T_1^1 \) such that the free colour \( f_j \) is not inserted.

**Proof:** First, writing \( S_1 = \bigcup_{u=1}^{t} S_1^u \), we have

\[ \Sigma_1 = \sum_{u=1}^{t} \sum_{j \in S_1^u} (P(j) + \#(\llbracket j; s+t \rrbracket \cap (N \cup T_0 \cup S_1))). \]

Now, noticing that for \( j \in S_1^u \), \( P(j) = j - u \), we can write

\[ \Sigma_1 = \sum_{u=1}^{t} \sum_{j \in S_1^u} (j_{2u-1} - u + j - j_{2u-1} + \#(\llbracket j; s+t \rrbracket \cap (N \cup T_0 \cup S_1))). \tag{5.2} \]

We first note that

\[
\begin{align*}
    j_{2u-1} - u &= 1 - u + j_{2u-1} - 1 \\
    &= 1 - u + \#(\llbracket 1; j_{2u-1} - 1 \rrbracket \cap N) + \#(\llbracket 1; j_{2u-1} - 1 \rrbracket \cap (T_0 \cup T_1)) \\
    &= 1 - u + \#(\llbracket 1; j_{2u-1} - 1 \rrbracket \cap N) + 2u - 2 \\
    &= \#(\llbracket 1; j_{2u-1} - 1 \rrbracket \cap N) + u - 1.
\end{align*}
\]

We also rewrite \( j - j_{2u-1} \) as

\[
\begin{align*}
    j - j_{2u-1} &= \#(\llbracket j_{2u-1}; j - 1 \rrbracket \cap T_0^u) + \#(\llbracket j_{2u-1}; j - 1 \rrbracket \cap S_1^u) + \#(\llbracket j_{2u-1}; j - 1 \rrbracket \cap \mathcal{S}_1^1) + \#(\llbracket j_{2u-1}; j - 1 \rrbracket \cap N) \\
    \text{Finally, we have} \#(\llbracket j; s+t \rrbracket \cap (N \cup T_0 \cup S_1)) &= \#(\llbracket j; s+t \rrbracket \cap N) + \#(\llbracket j; j_{2u} \rrbracket \cap (T_0^u \cup S_1^u)) + \#(\llbracket j_{2u} + 1; s+t \rrbracket \cap (T_0 \cup S_1)) \\
    &= \#(\llbracket j; s+t \rrbracket \cap N) + \#(\llbracket j; j_{2u} \rrbracket \cap (T_0^u \cup S_1^u)) + \sum_{v=u+1}^{t} (|T_v^0| + |S_v^0|).
    
\end{align*}
\]
Combining the three observations above, (5.2) becomes
\[
\Sigma_1 = \sum_{u=1}^{t} \sum_{j \in S^*_1} \left( |\mathcal{N}| + u - 1 + \sum_{v=0}^{t} (|\mathcal{T}_v^1| + |S_v^1|) + \#(\{j_{2u-1}; j - 1\} \cap \mathcal{S}_v^1) \right).
\]
Noticing that \(|\mathcal{N}| + u - 1 + \sum_{v=0}^{t} (|\mathcal{T}_v^1| + |S_v^1|)\) does not depend on \(j\), and that \(\#(\{j_{2u-1}; j - 1\} \cap \mathcal{S}_v^1)\) yields the desired formula.

We can now give a formula for the generating function for minimal partitions \(\min_{\bar{\Delta}}(S(n_1, \ldots, n_{s+t})\) for a fixed set \(S_1\). The desired generating function \(G_{S_1,m}(q)\) of (5.1) will then be obtained by summing over all possible sets \(S_1\).

**Lemma 5.2.** Let \(S_1\) be fixed. Define
\[
H_{S_1}(q) := \sum_{n_1, \ldots, n_{s+t} : n_1 + \cdots + n_{s+t} = m, \{j \in T_1; n_j > 0\} = S_1} q^{\min_{\bar{\Delta}}(S(n_1, \ldots, n_{s+t}))}.
\]
We have
\[
H_{S_1}(q) = q^{\min_{\bar{\Delta}}(S)} + \Sigma_1 + m - |S_1| \left( m - 1 + |\mathcal{N}| + |T_0| \right) \left( m - |S_1| \right).
\]

**Proof:** By Proposition 2.28 and Lemma 5.1, we have
\[
H_{S_1}(q) = \sum_{n_1, \ldots, n_{s+t} : n_1 + \cdots + n_{s+t} = m, \{j \in T_1; n_j > 0\} = S_1} q^{\min_{\bar{\Delta}}(S)} + \Sigma_1 + \sum_{j \in S_1} \#(\{j; s + t\} \cap \mathcal{N} \cup T_0, 1) + \sum_{j \in \mathcal{N} \cup T_0} n_j \#(\{j; s + t\} \cap \mathcal{N} \cup T_0, 1).\]

Thus by the changes of variables
\[
n_j' = \begin{cases} n_j & \text{if } j \in \mathcal{N} \cup T_0, \\ n_j - 1 & \text{if } j \in S_1. \end{cases}
\]
and noticing that \(\min_{\bar{\Delta}}(S)\) and \(\Sigma_1\) do not depend on the \(n_j\)’s, we obtain
\[
H_{S_1}(q) = q^{\min_{\bar{\Delta}}(S)} + \Sigma_1 + \sum_{\left( n_j' \right) \in \mathcal{N} \cup T_0, 1} \sum_{\sum_{j \in S_1} n_j' = m - |S_1|} q^{\sum_{j \in \mathcal{N} \cup T_0, 1} n_j' \#(\{j; s + t\} \cap \mathcal{N} \cup T_0, 1)}.
\]

Moreover, we can interpret the sum above as the generating function for partitions into exactly \(m - |S_1|\) parts, each part being at most \(|\mathcal{N}| + |T_0|\). Indeed, for all \(j \in \mathcal{N} \cup T_0 \cup S_1\), \(n_j\) can be interpreted as the number of parts of size \(\#(\{j; s + t\} \cap \mathcal{N} \cup T_0, 1)\) (see Figure 3 below).

\[\text{Figure 3. Decomposition of the Ferrers board.}\]
The generating function for such partitions is given by $q^{m-|S_1|} \binom{m-1+|N|+|T_0|}{m-|S_1|}_q$, which yields the desired formula (5.3) for $H_{S,S_1}(q)$.

Before we compute $G_{S,m}(q)$, we still need one more lemma about $q$-binomial coefficients.

Lemma 5.3. Let $a$ and $b$ be non-negative integers. We have

$$\sum_{A \subseteq \llbracket 1 : a + b \rrbracket, |A| = a} q^{\sum_{j \in A} \#(j' < j : j' \in \llbracket 1 : a + b \rrbracket \setminus A)} = \binom{a + b}{a}_q.$$

Proof: Partitions whose Ferrers diagram fits inside a $a \times b$ box, generated by $\binom{a+b}{a}_q$, are in bijection with walks on the plane going from $(0,0)$ to $(b,a)$, having $b$ right steps and $a$ up steps. The partition can be seen on top of the path, as shown in Figure 4.

![Figure 4. A partition as a path.](image)

If $A \subseteq \llbracket 1 : a + b \rrbracket, |A| = a$ is the set of up steps, then for each position $j \in A$, the part of the partition corresponding to this up step has its size equal to the number of right steps that have been done before, i.e. $\#(j' < j : j' \in \llbracket 1 : a + b \rrbracket \setminus A)$.

We are now ready to sum $H_{S,S_1}(q)$ over all possible sets $S_1$ to obtain a formula for $G_{S,m}(q)$.

Proposition 5.4. Let $S$ be a reduced colour sequence, and $m$ a non-negative integer. We have

$$G_{S,m}(q) = \sum_{k_1, \ldots, k_t} q^{\min \Delta(S)} + \sum_{u=1}^t k_u(|N|+u-1+\sum_{v=u}^t (|T_0^v|+k_v)) q^{m-\sum_{u=1}^t k_u} \binom{m-1+|N|+|T_0|}{m-\sum_{u=1}^t k_u}_q \prod_{u=1}^t \binom{|T_0^u|}{k_u}_q.$$

Proof: By Lemma 5.2, we have:

$$G_{S,m}(q) = \sum_{k_1, \ldots, k_t} \sum_{k_u \in |T_0^u|} \sum_{S_1 : k_1, \ldots, k_t, \sum_{u=1}^t k_u \leq |T_0^u|, \forall u, S_1^u \subseteq T_0^u \text{ and } |S_1^u| = k_u} H_{S,S_1}(q) = \sum_{k_1, \ldots, k_t} \sum_{k_u \in |T_0^u|} \sum_{S_1 : k_1, \ldots, k_t, \sum_{u=1}^t k_u \leq |T_0^u|, \forall u, S_1^u \subseteq T_0^u \text{ and } |S_1^u| = k_u} q^{\min \Delta(S)} + \sum_{u=1}^t k_u(|N|+u-1+\sum_{v=u}^t (|T_0^v|+k_v)) q^{m-\sum_{u=1}^t k_u} \binom{m-1+|N|+|T_0|}{m-\sum_{u=1}^t k_u}_q.$$

By Lemma 5.1, this becomes

$$G_{S,m}(q) = \sum_{k_1, \ldots, k_t} q^{\min \Delta(S)} + \sum_{u=1}^t k_u(|N|+u-1+\sum_{v=u}^t (|T_0^v|+k_v)) q^{m-\sum_{u=1}^t k_u} \binom{m-1+|N|+|T_0|}{m-\sum_{u=1}^t k_u}_q.$$
Exchanging the final sum and product, and then using Lemma 5.3 for each \( u \in \{1, \ldots, t\} \) with \( a = k_u \) and \( b = |T_u^u| - k_u \) gives the desired formula. \( \square \)

What remains to do is show that the expression for \( G_{S,m}(q) \) in Proposition 5.4 is actually the same as (2.6).

First, let us give yet another lemma about \( q \)-binomial coefficients.

**Lemma 5.5.** Let \( m, \ell_1, \ldots, \ell_t \) be non-negative integers. We have

\[
q^m \left[ m + \ell_1 + \cdots + \ell_t - 1 \right]_q = q^m \sum_{0 = x_0 \leq x_1 \leq \cdots \leq x_t = m} \prod_{r=1}^{t} q^{\ell_{r}x_{r-1}} \left[ \frac{x_r - x_{r-1} + \ell_r - 1}{x_r - x_{r-1}} \right]_q.
\]

In the above, we use the convention that \([-1]_0 = 1\), corresponding to the case where a certain \( \ell_r \) is equal to 0.

**Proof:** The left-hand side is the generating function for partitions fitting inside a \( m \times (\ell_1 + \cdots + \ell_t) \) box, such that the largest part is equal to \( m \). Take the Ferrers board of such a partition, and draw it in the plane as shown on Figure 5 (where the partition is above the path).

![Figure 5. Decomposition of the Ferrers board.](image)

For all \( i \in \{1, \ldots, t\} \), let \( x_i \) be the size of the \( \sum_{k=i+1}^{t} \ell_k + 1 \)-th part (with \( x_i = 0 \) if there are less than \( \ell_1 + \cdots + \ell_t - y_i + 1 \) parts).

For all \( i \in \{1, \ldots, t\} \), let \( y_i := \sum_{k=1}^{i} \ell_k \). For fixed \( 0 \leq x_1 \leq \cdots \leq x_t = m \), these partitions are generated by

\[
\prod_{r=1}^{t} q^{\ell_{r}x_{r-1}} \times q^{x_r - x_{r-1}} \left[ \frac{x_r - x_{r-1} + \ell_r - 1}{x_r - x_{r-1}} \right]_q,
\]

where \( q^{\ell_{r}x_{r-1}} \) generates the rectangle between the \( y \)-axis, the lines \( y = y_r \) and \( y = y_{r-1} \), and the line \( x = x_{r-1} \), and the second term generates partitions fitting inside a \( (x_r - x_{r-1}) \times \ell_r \) box, such that the largest part is equal to \( x_r - x_{r-1} \).

The above is equal to

\[
q^m \prod_{r=1}^{t} q^{\ell_{r}x_{r-1}} \left[ \frac{x_r - x_{r-1} + \ell_r - 1}{x_r - x_{r-1}} \right]_q,
\]

and summing over all possible values for \( x_1, \ldots, x_{t-1} \) gives the desired result. \( \square \)
We use the lemma above to rewrite a part of the expression in Proposition 5.4.

**Lemma 5.6.** We have:

\[
q^{m - \Sigma_{u=1}^t k_u \left[ m - 1 + |\mathcal{N}| + |\mathcal{T}_0| \right]} \times \sum_{0 = m_0 \leq m_1 \leq \cdots \leq m_t = m} \left( \prod_{u=1}^t q^{(k_u |\mathcal{T}_0^u|)m_{u-1}} \left[ m_u - m_{u-1} + |\mathcal{T}_0^u| - 1 \right] m_u - m_{u-1} - k_u \right) q^{\mathcal{N}m_t} \left[ m - m_t + |\mathcal{N}| - 1 \right] q^{-m - \Sigma_{u=1}^t k_u (1 + |\mathcal{N}| + \Sigma_{e=u+1}^1 (k_u + |\mathcal{T}_0^u|))}
\]

**Proof:** Let us start by applying Lemma 5.5 with \( t = t + 1 \), \( m = m - \Sigma_{u=1}^t k_u \), \( \ell_u = k_u + |\mathcal{T}_0^u| \) for all \( u \in \{1, \ldots, t\} \), and \( \ell_{t+1} = |\mathcal{N}| \). We have

\[
X := q^{m - \Sigma_{u=1}^t k_u \left[ m + |\mathcal{T}_0| + |\mathcal{N}| - 1 \right]} = q^{m - \Sigma_{u=1}^t k_u} \sum_{0 = x_0 \leq x_1 \leq \cdots \leq x_{t+1} = m - \Sigma_{u=1}^t k_u} \left( \prod_{u=1}^t q^{(k_u + |\mathcal{T}_0^u|)x_{u-1}} \left[ x_u - x_{u-1} + k_u + |\mathcal{T}_0^u| - 1 \right] x_u - x_{u-1} \right) q^{\mathcal{N}|x_t|} \left[ m - \Sigma_{u=1}^t k_u - x_t + |\mathcal{N}| - 1 \right] q.
\]

By the changes of variables \( x_u = m_u - \Sigma_{u=1}^u k_v \), we obtain

\[
X = q^{m - \Sigma_{u=1}^t k_u} \sum_{0 = m_0 \leq m_1 \leq \cdots \leq m_t = m} \left( \prod_{u=1}^t q^{(k_u + |\mathcal{T}_0^u|)(m_{u-1} - \Sigma_{v=1}^{u-1} k_v)} \left[ m_u - m_{u-1} + |\mathcal{T}_0^u| - 1 \right] m_u - m_{u-1} - k_u \right) q^{\mathcal{N}(m_t - \Sigma_{u=1}^t k_u)} \left[ m - m_t + |\mathcal{N}| - 1 \right] q = q^{m - \Sigma_{u=1}^t k_u (1 + |\mathcal{N}|) - \Sigma_{u=1}^t (k_u + |\mathcal{T}_0^u|) \Sigma_{v=1}^{u-1} k_v} \times \sum_{0 = m_0 \leq m_1 \leq \cdots \leq m_t = m} \left( \prod_{u=1}^t q^{(k_u + |\mathcal{T}_0^u|)m_{u-1}} \left[ m_u - m_{u-1} + |\mathcal{T}_0^u| - 1 \right] m_u - m_{u-1} - k_u \right) q^{\mathcal{N}m_t} \left[ m - m_t + |\mathcal{N}| - 1 \right] q.
\]

We deduce the final formula by using that

\[
\sum_{u=1}^t (k_u + |\mathcal{T}_0^u|) \sum_{v=1}^{u-1} k_v = \sum_{v=1}^t k_v \sum_{u=v+1}^t (k_u + |\mathcal{T}_0^u|).
\]

\[
G_{S,m}(q) = q^{\min_{\Delta}(S) + m} \sum_{k_1, \ldots, k_t \leq |\mathcal{T}_0^t|} \left( \prod_{u=1}^t q^{k_u (u+1 + k_u + |\mathcal{T}_0^u|)} \left[ |\mathcal{T}_0^u| \right] q^{-k_u} \right) \times \sum_{0 = m_0 \leq m_1 \leq \cdots \leq m_t = m} \left( \prod_{u=1}^t q^{(k_u + |\mathcal{T}_0^u|)m_{u-1}} \left[ m_u - m_{u-1} + |\mathcal{T}_0^u| - 1 \right] m_u - m_{u-1} - k_u \right) q^{\mathcal{N}m_t} \left[ m - m_t + |\mathcal{N}| - 1 \right] q.
\]

\[
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\]
Exchanging the summations, we obtain:

\[
G_{S,m}(q) = q^{\min \Delta(S)+m} \sum_{0=m_0 \leq m_1 \leq \ldots \leq m_t \leq m} \left( \prod_{u=1}^{t} q^{k_u(u-2+k_u+|T_u^u|)+(k_u+|T_u^u|)|T_{u-1}^u|} \right) \times \left[ \sum_{k_u \leq |T_u^u|} q^{k_u} \left[ \begin{array}{c} m_u - m_{u-1} + |T_u^u| - 1 \\ m_u - m_{u-1} - k_u \end{array} \right] \right] q^{\sum_{0 \leq m_t \leq m} \left[ \begin{array}{c} m - m_t + |N| - 1 \\ m - m_t \end{array} \right]}.
\]

We need one last lemma to complete our proof of Proposition 2.29.

**Lemma 5.7.** We have

\[
\sum_{0=m_0 \leq m_1 \leq \ldots \leq m_t} \prod_{u=1}^{t} q^{k_u(u-2+k_u+|T_u^u|)+(k_u+|T_u^u|)|T_{u-1}^u|} \left[ \begin{array}{c} m_u - m_{u-1} + |T_u^u| - 1 \\ m_u - m_{u-1} - k_u \end{array} \right] q^{\sum_{0 \leq m_t \leq m} \left[ \begin{array}{c} m - m_t + |N| - 1 \\ m - m_t \end{array} \right]}
= \sum_{v=0}^{t} g_{v,t}(q; |T_0^0|, \ldots, |T_0^t|) \left[ \begin{array}{c} m_t + t - 1 \\ m_t - v \end{array} \right],
\]

where \( g_{v,t} \) was defined in Proposition 2.29.

Indeed, once Lemma 5.7 is proved, we can write

\[
G_{S,m}(q) = q^{\min \Delta(S)+m} \sum_{v=0}^{t} g_{v,t}(q; |T_0^0|, \ldots, |T_0^t|) \sum_{0 \leq m_t \leq m} q^{\sum_{0 \leq m_t \leq m} \left[ \begin{array}{c} m - m_t + |N| - 1 \\ m - m_t \end{array} \right]}
= q^{\min \Delta(S)+m} \sum_{v=0}^{t} q^{v \sum_{0 \leq m_t \leq m-v} \left[ \begin{array}{c} m - m_t + v + |N| - 1 \\ m - m_t - v \end{array} \right]}
\]

where the second equality follows from the change of variable \( m_t' = m_t - v \). Using Lemma 5.5 with \( t = 2 \)

\[
G_{S,m}(q) = q^{\min \Delta(S)+m} \sum_{v=0}^{t} q^{v \sum_{0 \leq m_t \leq m-v} \left[ \begin{array}{c} m + v - |N| - 1 \\ m - v \end{array} \right]}
\]

Observing that \( |N| = s - t \) concludes the proof of Proposition 2.29.

We conclude this section by the proof of Lemma 5.7.

**Proof of Lemma 5.7:** Let us define \( G_0(q; m) = \chi(m=0) \), and for \( v \geq 1 \)

\[
G_v(q; x_1, \ldots, x_v; m) := \sum_{0=m_0 \leq m_1 \leq \ldots \leq m_v \leq m} \prod_{u=1}^{v} q^{k_u(u-2+k_u+x_u)+(k_u+x_u)|T_{u-1}^u|} \left[ \begin{array}{c} 2 - x_u \\ k_u \end{array} \right] \left[ \begin{array}{c} m_u - m_{u-1} + x_u - 1 \\ m_u - m_{u-1} - k_u \end{array} \right]
\]

So that the function in Lemma 5.7 is \( G_v(q; |T_0^0|, \ldots, |T_0^t|; m_t) \).

We show by induction on \( v \) that

\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{u=0}^{v} g_{v,u}(q; x_1, \ldots, x_v) \left[ \begin{array}{c} m + v - 1 \\ m - u \end{array} \right].
\]

Recall from [And84b, p. 37, (3.3.10)] that

\[
\left[ \begin{array}{c} a + b \\ c \end{array} \right]_q = \sum_{a' \geq 0} \left[ \begin{array}{c} a \\ a' \end{array} \right]_q \left[ \begin{array}{c} b \\ c - a' \end{array} \right]_q q^{a'(b-c+a')}.
\]
By (5.7) with \( a = 2 - x_1, b = m + x_1 - 1, \) and \( c = m, \) we have
\[
G_1(q; x_1; m) = \begin{bmatrix} m+1 \\ m \end{bmatrix}_q
= \begin{bmatrix} m \\ m \end{bmatrix}_q + q \begin{bmatrix} m \\ m-1 \end{bmatrix}_q
= g_{0,1}(q; x_1) \begin{bmatrix} m \\ m \end{bmatrix}_q + g_{1,1}(q; x_1) \begin{bmatrix} m \\ m-1 \end{bmatrix}_q.
\]
So (5.6) is true for \( v = 1. \)

Now assume that it is true for \( v - 1 \geq 1 \) and prove it for \( v. \) We have
\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{0=m_0 \leq m_1 \leq \ldots \leq m_v=m} \prod_{u=1}^{2-x_v} \left( \sum_{k_u=0}^{2-x_u} q^{k_u(u-2+k_u+x_u)+(k_u+x_u)m_{u-1}} \begin{bmatrix} 2-x_u \\ k_u \end{bmatrix}_q \begin{bmatrix} m_u - m_{u-1} + x_u - 1 \\ m_u - m_{u-1} - k_u \end{bmatrix}_q \right)
\]
\[
= \sum_{m_{v-1}=0}^{m} \prod_{u=1}^{v-1} \left( \sum_{k_u=0}^{2-x_u} q^{k_u(u-2+k_u+x_u)+(k_u+x_u)m_{u-1}} \begin{bmatrix} 2-x_u \\ k_u \end{bmatrix}_q \begin{bmatrix} m_u - m_{u-1} + x_u - 1 \\ m_u - m_{u-1} - k_u \end{bmatrix}_q \right)
\]
\[
= \sum_{m_{v-1}=0}^{m} G_{v-1}(q; x_1, \ldots, x_{v-1}; m_{v-1}) \sum_{k_{v-1}=0}^{2-x_{v-1}} q^{k_{v}(v-2+k_{v}+x_{v})+(k_{v}+x_{v})m_{v-1}} \begin{bmatrix} 2-x_{v} \\ k_{v} \end{bmatrix}_q \begin{bmatrix} m_{v-1} + v - 1 \\ m_{v-1} - k_{v} \end{bmatrix}_q,
\]
where we used the induction hypothesis in the last equality.

Rearranging the order of summation leads to
\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{u=0}^{v-1} q^{ux_u} g_{u,v-1}(q; x_1, \ldots, x_{v-1}) \sum_{k_{v}=0}^{2-x_{v}} q^{k_{v}(v-2+u+k_{v}+x_{v})} \begin{bmatrix} 2-x_{v} \\ k_{v} \end{bmatrix}_q
\]
\[
\times \sum_{m_{v+1}=0}^{m} q^{(k_{v}+x_{v})(m_{v+1}-u)} \begin{bmatrix} m_{v-1} + v - 2 \\ m_{v-1} - u \end{bmatrix}_q \begin{bmatrix} m - m_{v-1} + x_{v} - 1 \\ m - m_{v-1} - k_{v} \end{bmatrix}_q.
\]

Using Lemma 5.5 with \( t = 2, m = m - u - k_{v}, \ell_1 = v - 1 + u, \) and \( \ell_2 = k_{v} + x_{v}, \) and the change of variable \( x_1 = m_{v-1} - u, \) this yields:
\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{u=0}^{v-1} q^{ux_u} g_{u,v-1}(q; x_1, \ldots, x_{v-1}) \sum_{k_{v}=0}^{2-x_{v}} q^{k_{v}(v-2+u+k_{v}+x_{v})} \begin{bmatrix} 2-x_{v} \\ k_{v} \end{bmatrix}_q
\]
\[
\times \begin{bmatrix} m + v + x_{v} - 2 \\ m - u - k_{v} \end{bmatrix}_q.
\]

Using (5.7) again with \( a = 2 - x_v, b = m + v + x_v - 2, c = m - u, \) and \( a' = k_v, \) we obtain
\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{u=0}^{v-1} q^{ux_u} g_{u,v-1}(q; x_1, \ldots, x_{v-1}) \begin{bmatrix} m + v \\ m - u \end{bmatrix}_q.
\]

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By the \( q \)-analogue of Pascal’s triangle, this becomes
\[
G_v(q; x_1, \ldots, x_v; m) = \sum_{u=0}^{v-1} q^{uw} g_{u,v}(q; x_1, \ldots, x_{v-1}) \left[ \frac{m + v - 1}{m - u} \right]_q
\]
\[
+ \sum_{u=0}^{v-1} q^{uw+u} g_{u,v-1}(q; x_1, \ldots, x_{v-1}) \left[ \frac{m + v - 1}{m - u - 1} \right]_q
\]
\[
= \sum_{u=0}^{v-1} \left( q^{uw} g_{u,v}(q; x_1, \ldots, x_{v-1}) + q^{uw+u} g_{u,v-1}(q; x_1, \ldots, x_{v-1}) \right) \left[ \frac{m + v - 1}{m - u} \right]_q
\]  
(5.8)

Recall that
\[
g_{u,v}(q; x_1, \ldots, x_v) = \sum_{\epsilon_1, \ldots, \epsilon_v \in \{0,1\} : \epsilon_1 + \cdots + \epsilon_v = u} q^{u\epsilon + \epsilon(v)} \prod_{k=1}^{v} q^{(x_k-1) \sum_{i=1}^{k-1} \epsilon_i + x_k \epsilon_k} = 1.
\]

So, separating the case where \( \epsilon_v = 0 \) from the case where \( \epsilon_v = 1 \), we have
\[
g_{u,v}(q; x_1, \ldots, x_v) = \sum_{\epsilon_1, \ldots, \epsilon_{v-1} \in \{0,1\} : \epsilon_1 + \cdots + \epsilon_{v-1} = u} q^{u\epsilon + \epsilon(v)} \prod_{k=1}^{v-1} q^{(x_k-1) \sum_{i=1}^{k-1} \epsilon_i + x_k \epsilon_k} (x_v-1)^u
\]
\[
+ \sum_{\epsilon_1, \ldots, \epsilon_{v-1} \in \{0,1\} : \epsilon_1 + \cdots + \epsilon_{v-1} = u-1} q^{u\epsilon + \epsilon(v)} \prod_{k=1}^{v-1} q^{(x_k-1) \sum_{i=1}^{k-1} \epsilon_i + x_k \epsilon_k} (x_v-1)^{u-1}.
\]

After simplification, this is exactly (5.8). The lemma is proved.

\( \square \)

Acknowledgements

The research of the first author was supported by the project IMPULSION of IdexLyon. Part of this research was conducted while the second author was visiting Lyon, funded by the same project.

References


