ASYMPTOTICS FOR SKEW STANDARD YOUNG TABLEAUX VIA BOUNDS FOR CHARACTERS

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Abstract. We are interested in the asymptotics of the number of standard Young tableaux $f^{\lambda/\mu}$ of a given skew shape λ/μ. We restrict ourselves to the case where both diagrams are balanced, but investigate all growth regimes of $|\mu|$ compared to $|\lambda|$, from $|\mu|$ fixed to $|\mu|$ of order $|\lambda|$. When $|\mu| = o(|\lambda|^{1/3})$, we get an asymptotic expansion to any order. When $|\mu| = o(|\lambda|^{1/2})$, we get a sharp upper bound. For bigger $|\mu|$, we prove a weaker bound and give a conjecture on what we believe to be the correct order of magnitude.

Our results are obtained by expressing $f^{\lambda/\mu}$ in terms of irreducible character values of the symmetric group and then applying known upper bounds on characters.

1. Introduction and statement of results

Background. Standard Young tableaux of a given shape λ are standard combinatorial objects coming from the representation theory of symmetric groups and the theory of symmetric functions; we refer the reader to [1] for a recent survey on the topic. The number f^λ of such tableaux is given by the well-known hook-length formula of Frame, Robinson and Thrall [5]. This exact product formula is also suited for asymptotic analysis: for example, if λ has fewer than $L\sqrt{|\lambda|}$ rows and columns (for some constant L), then we have

$$\log f^\lambda = \tfrac{1}{2}|\lambda|\log|\lambda| + O(|\lambda|). \tag{1.1}$$

This formula (with a precise version of the $O(|\lambda|)$) is a key ingredient to find the limit shape of random Young diagrams distributed according to the Plancherel measure; see [11, Chapter 1] for an introduction to this wide subject.

A natural generalization of the problem is to consider skew shapes λ/μ, that is the diagram obtained by removing a smaller diagram μ from the top-left corner of a bigger diagram λ. The number of standard Young tableaux of shape λ/μ is usually denoted $f^{\lambda/\mu}$ and is an object of interest in algebraic combinatorics. In general, there is no product formula for $f^{\lambda/\mu}$, but several papers have been devoted to finding special shapes for which product formulas hold. The most recent result in this direction is a formula for a six-parameter family of skew shapes given by Morales, Pak and Panova [9], which generalizes previous results of Kim and Oh [6] and DeWitt [2]. Another interesting problem is the asymptotic analysis of $f^{\lambda/\mu}$, which we shall discuss in this paper.

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Even in the simplest case of a (non-skew) shape λ, the asymptotics of f^λ largely depend on how the shape λ tends to infinity: does it have rows and/or columns of linear size (this is sometimes referred to as the Thoma-Vershik-Kerov regime) or, at the opposite, are the numbers of rows and columns of the same order as the square-root of the size? Since the numbers $f^{\lambda/\mu}$ depend on two partitions, there are even more possible asymptotic regimes. In this article, to simplify the discussion, we focus on the case where **both diagrams λ and μ are balanced**; i.e. we assume throughout the paper that there exists L such that λ (resp. μ) has less than $L\sqrt{|\lambda|}$ (resp. $L\sqrt{|\mu|}$) rows and columns. L should be considered fixed and all constants, including the ones in the O, o symbols, might depend on L. However, the methods developed here can probably be used in more generality.

A pioneering work in the asymptotic study of $f^{\lambda/\mu}$ is due to Stanley [13]: he considered the case where μ is fixed and proved that for balanced diagrams λ (in fact, we only need the number of rows and columns to be sublinear here), we have

$$A_{\lambda/\mu} := |\mu|! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu} \underset{|\lambda|\to\infty}{\sim} 1. \tag{1.2}$$

On the other side of the spectrum, Morales, Pak and Panova [8] investigated various situations where the size k of μ grows linearly with the size n of λ. In all cases with balanced diagrams, they obtained

$$\log f^{\lambda/\mu} = \tfrac{1}{2}|\lambda/\mu| \log |\lambda/\mu| + O(|\lambda/\mu|), \tag{1.3}$$

and gave a precise estimate for the $O(|\lambda/\mu|)$ term. In this framework, again, it seems that $A_{\lambda/\mu} = |\mu|! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu}$ is a meaningful normalization since $\log(A_{\lambda/\mu}) = O(|\lambda|)$ while all factors in the definition of $A_{\lambda/\mu}$ are significantly bigger.

The goal of this paper is to investigate the behaviour of $A_{\lambda/\mu}$ in intermediate regimes, that is when $1 \ll |\mu| \ll |\lambda|$. We get various results, depending on the growth of $k := |\mu|$ compared to $n := |\lambda|$.

Results. When $k = o\left(n^{1/3}\right)$, we get an asymptotic expansion of $A_{\lambda/\mu}$ to any order. This extends previous results of Stanley for fixed k (see the discussion after Theorem 3.2 in [13]). The terms in this expansion involve characters of the symmetric group, so we first need to introduce some terminology. For a permutation σ, we denote by $|\sigma|$ its *absolute length*, i.e the number of transpositions (not necessarily adjacent) needed to factorize σ. Also, $\chi^\lambda(\sigma)$ is the character of the irreducible symmetric group representation associated to λ evaluated on σ. (If λ has size n and σ is a permutation in the symmetric group S_k with $k < n$, we implicitly use the injection $S_k \subset S_n$ consisting in fixing integers $j > k$.)

Theorem 1. *Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced, with $k = o\left(n^{1/3}\right)$. Then for any natural integer r (not depending on k and n), we have as n tends to infinity,*

$$A_{\lambda/\mu} = \sum_{\substack{\sigma \in S_k,\\ |\sigma| \le r}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + O\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right).$$

In particular for $r = 0$, this gives the following character-free asymptotic estimate.

Corollary 2. *Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced, with $k = o\left(n^{1/3}\right)$. Then we have*

$$A_{\lambda/\mu} = 1 + O\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right).$$

In other words,

$$f^{\lambda/\mu} = \frac{f^\lambda f^\mu}{k!}\left(1 + O\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)\right).$$

When k is at most of order $n^{1/2}$, we can prove the following upper bound.

Theorem 3. *There exist constants C_1 and C_2 such that the following holds. Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced, with $k < C_1 n^{1/2}$. Then we have*

$$A_{\lambda/\mu} \leq e^{C_2 k^{\frac{3}{2}} n^{-\frac{1}{2}}}.$$

We give an explicit formula for C_1 in Section 2.3. The bound of Theorem 3 is in some sense tight; in Section 4, we give families of skew shapes for which $\log A_{\lambda/\mu} = \Theta\left(|\mu|^{\frac{3}{2}}|\lambda|^{-\frac{1}{2}}\right)$ (here $f = \Theta(g)$ means that there exists a constant C such that $f = Cg + o(g)$). These skew shapes have been chosen so that $f^{\lambda/\mu}$ (and hence $A_{\lambda/\mu}$) admits a product formula. Even if the formula is explicit, the derivation of its asymptotics is cumbersome and was done on a computer.

We next investigate the case where $k \geq C_1 n^{1/2}$. In this case, we can only prove the following upper bound.

Theorem 4. *Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced, with $k \geq C_1 n^{1/2}$. Then we have*

$$A_{\lambda/\mu} \leq e^{k\left(\log \frac{k^2}{n} + O(1)\right)}.$$

Note that when k is of order $n^{1/2}$, the upper bound in Theorem 3 is $e^{\Theta(n^{1/4})}$, while the one in Theorem 3 is $e^{\Theta(n^{1/2})}$. We believe that this does not reflect the real behaviour of $A_{\lambda/\mu}$, but that this is rather an artifact of our method. In fact, we conjecture that the bound given in Theorem 3 holds without hypothesis on k:

Conjecture 5. *Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced. Then we have*

$$e^{-Ck^{\frac{3}{2}} n^{-\frac{1}{2}}} \leq A_{\lambda/\mu} \leq e^{Ck^{\frac{3}{2}} n^{-\frac{1}{2}}},$$

for some positive constant C.

Note that this conjecture, unlike our previous theorems, also includes a lower bound. The conjecture is supported by numerical evidence obtained as above: we computed the asymptotic expansion of $\log A_{\lambda/\mu}$, in various cases with product formulas. These computations are also presented in Section 4.

We believe that the techniques developed by Morales, Pak and Panova can be used to prove the conjecture when k is of order n. We do not know however how to attack it in the regime $n^{1/2} \ll k \ll n$, or how to prove the lower bound for $k \ll n^{1/2}$. The approach of this paper is not suited for lower bounds. In Section 3, we explain why it is not possible to obtain a sharper upper bound when $n^{1/2} \ll k \ll n$ with our method and the bounds on characters available at the time.

Method. We finish this introduction by a short discussion on the method used to prove our results. As Stanley, we start with an expression of $f^{\lambda/\mu}$ (or equivalently $A_{\lambda/\mu}$) as a sum involving irreducible characters of the symmetric group (see Lemma 6 below). We then control the sum thanks to an upper bound on these characters due to the second author and Śniady [4]. As shown in Section 3, the other bounds for characters known to date would not allow us to improve the bound in Theorem 4. In comparison, the method of Morales, Pak and Panova [8] is completely different, being based on a (non-multiplicative) hook-length formula for skew diagrams.

2. Proofs of the asymptotic estimates

Let us consider two Young diagrams $\lambda \vdash n$ and $\mu \vdash k$, and denote by $r(\lambda)$ (resp. $c(\lambda)$) the number of rows (resp. colums) of λ (idem for μ). We assume that λ and μ are balanced, i.e.

$$r(\lambda), c(\lambda) \leq L\sqrt{n}, \quad r(\mu), c(\mu) \leq L\sqrt{k},$$

for some positive constant L.

2.1. Preliminaries.

We start with a lemma expressing $f^{\lambda/\mu}$ (or equivalently $A_{\lambda,\mu}$) as a sum of irreducible character values of the symmetric group. This formula already appears in a slightly different form in Stanley [13, Theorem 3.1], but here we give a more direct proof.

Lemma 6 (Stanley)**.** *Let μ be a partition of k, and let $n \geq k$. Then for all partitions λ of n, we have*

$$A_{\lambda/\mu} = \sum_{\sigma \in S_k} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}.$$

Proof: Iterating the branching rule for representations of the symmetric group (see, e.g., [12, Section 2.8]), we get that, if $\sigma \in S_k$,

$$\chi^\lambda(\sigma) = \sum_{\mu \vdash k} f^{\lambda/\mu} \chi^\mu(\sigma),$$

where the sum runs over partitions of k. Since $(\sigma \mapsto \chi^\mu(\sigma))_{\mu \vdash k}$ forms an orthogonal basis of central functions on S_k [12, Theorem 1.9.3], the coefficient $f^{\lambda/\mu}$ of $\chi^\mu(\sigma)$ in the expansion of $\chi^\lambda(\sigma)$ can be obtained by a scalar product computation:

$$f^{\lambda/\mu} = \langle \chi^\lambda(\sigma), \chi^\mu(\sigma) \rangle = \frac{1}{k!} \sum_{\sigma \in S_k} \chi^\lambda(\sigma)\chi^\mu(\sigma).$$

Dividing by $f^\lambda f^\mu$ gives the result. □

Our results are also based on asymptotic bounds for characters due to the second author and Śniady [4].

Theorem 7 (Féray-Śniady)**.** *There exists a constant $a > 1$ such that for every partition $\nu \vdash m$ and every permutation $\sigma \in S_m$,*

$$\left| \frac{\chi^\nu(\sigma)}{f^\nu} \right| \leq \left[a \max\left(\frac{r(\nu)}{m}, \frac{c(\nu)}{m}, \frac{|\sigma|}{m} \right) \right]^{|\sigma|}.$$

For balanced Young diagrams, $\frac{r(\nu)}{m}$ and $\frac{c(\nu)}{m}$ are both of order $m^{-1/2}$. The upper bound thus depends on the absolute length $|\sigma|$ of σ.

Corollary 8. *Let $\nu \vdash m$ be a partition with*

$$r(\nu), c(\nu) \leq L\sqrt{m},$$

and σ be a permutation in S_m. Then the following holds.

- *When $|\sigma| \leq L\sqrt{m}$,*

$$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq \left(\frac{aL}{\sqrt{m}}\right)^{|\sigma|}. \tag{2.1}$$

- *When $|\sigma| > L\sqrt{m}$,*

$$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq \left(\frac{a|\sigma|}{m}\right)^{|\sigma|}. \tag{2.2}$$

2.2. **Proof of Theorem 1.** Let us now prove Theorem 1. We assume $k = o(n^{1/3})$. By Lemma 6, we have

$$A_{\lambda/\mu} = \sum_{\sigma \in S_k} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} = \sum_{i=0}^{k} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}$$

Let r be a fixed positive integer. We split this sum into three parts:

$$A_{\lambda/\mu} = \sum_{i=0}^{r} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + S_1 + S_2, \tag{2.3}$$

where

$$S_1 := \sum_{i=r+1}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}, \tag{2.4}$$

$$S_2 := \sum_{i=L\sqrt{k}+1}^{k} \sum_{\substack{\sigma \in S_k, \\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}. \tag{2.5}$$

We now wish to bound S_1 and S_2. To do so, we use a lemma from [4].

Lemma 9 (Lemma 14 from [4])**.** *For all $k, i \in \mathbb{N}$, we have*

$$\#\left\{\sigma \in S_k : |\sigma| = i\right\} \leq \frac{k^{2i}}{i!}.$$

Using Lemma 9 and Equation (2.1) for $\nu = \lambda$ and $\nu = \mu$, we obtain

$$|S_1| \leq \sum_{i=r+1}^{L\sqrt{k}} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}}\right)^i \left(\frac{aL}{\sqrt{k}}\right)^i.$$

We can now bound S_1 by the tail of an exponential series:

$$|S_1| \leq \sum_{i=r+1}^{+\infty} \frac{\left(a^2 L^2 k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^i}{i!}.$$

As $k = o(n^{1/3})$, we conclude that

$$|S_1| = O\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right). \tag{2.6}$$

Let us now turn to S_2. Since $|\sigma| = i \geq L\sqrt{k}$ in the summation index of S_2, we must now apply (2.2) to μ. On the other hand, since $k = o(n^{1/3})$, we still have $|\sigma| \leq L\sqrt{n}$ and we can still apply (2.1) to λ. Combining with Lemma 9, we have:

$$|S_2| \leq \sum_{i=L\sqrt{k}+1}^{k} \frac{k^{2i}}{i!} \left(\frac{aL}{\sqrt{n}}\right)^i \left(\frac{ai}{k}\right)^i.$$

Using $i! \geq \frac{i^i}{e^i}$, we obtain

$$|S_2| \leq \sum_{i=L\sqrt{k}+1}^{k} \left(a^2 Lekn^{-\frac{1}{2}}\right)^i.$$

This geometric series converges for $k < \left(a^2 Le\right)^{-1} n^{\frac{1}{2}}$, and thus $|S_2|$ is bounded by

$$|S_2| \leq \left(a^2 Lekn^{-\frac{1}{2}}\right)^{L\sqrt{k}+1} \frac{1}{1-\left(a^2 Lekn^{-\frac{1}{2}}\right)}. \tag{2.7}$$

When $k = o(n^{1/3})$ and $k \gg 1$, this is negligible compared to (2.6). For fixed k, S_2 can be bounded as S_1. Thus combining Equations (2.3), (2.6) and (2.7), we obtain

$$A_{\lambda/\mu} = \sum_{\substack{\sigma \in S_k,\\ |\sigma| \leq r}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + O\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right).$$

Theorem 1 is proved. □

Proof of Corollary 2: To prove Corollary 2, we apply Theorem 1 with $r = 0$. The only permutation σ such that $|\sigma| = 0$ is the identity, and in this case $\chi^\lambda(\mathrm{Id}) = f^\lambda$ and $\chi^\mu(\mathrm{Id}) = f^\mu$. Thus the theorem becomes

$$A_{\lambda/\mu} = 1 + O\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right),$$

as claimed. □

2.3. **Proof of Theorem 3.** Let us assume that $k < C_1 n^{1/2}$, with $C_1 := \left(a^2 Le\right)^{-1}$. We proceed similarly to Theorem 1. Let us write

$$A_{\lambda/\mu} = S_1' + S_2,$$

where S_2 is defined in (2.5) and

$$S_1' := \sum_{i=0}^{L\sqrt{k}} \sum_{\substack{\sigma \in S_k,\\ |\sigma| = i}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu}. \tag{2.8}$$

By the same argument as in the proof of Theorem 1, we can bound this by an exponential sum and obtain

$$|S_1'| \leq e^{a^2 L^2 k^{\frac{3}{2}} n^{-\frac{1}{2}}}. \tag{2.9}$$

Thanks to our choice for C_1, the sum S_2 is still a truncated convergent geometric series and (2.7) still holds. Combining this with (2.9) completes the proof. □

We note that the proof also gives some information on the value of C_2 in the theorem: any value $C_2 > a^2L^2$ works for n and k large enough. However the authors of [4] do not give estimates on the value of a in Theorem 7, so this is not useful in practice to compute C_2.

2.4. **Proof of Theorem 4.** We now prove the bound for large k, i.e. $k \geq (a^2Le)^{-1}\sqrt{n}$. Again, we cut $A_{\lambda/\mu}$ into several parts:

$$A_{\lambda/\mu} = S_1' + S_2' + S_3',$$

where S_1' was defined in (2.8),

$$S_2' := \sum_{i=L\sqrt{k}+1}^{L\sqrt{n}} \sum_{\substack{\sigma\in S_k,\\ |\sigma|=i}} \frac{\chi^\lambda(\sigma)}{f^\lambda}\frac{\chi^\mu(\sigma)}{f^\mu},$$

and

$$S_3' := \sum_{i=L\sqrt{n}+1}^{k} \sum_{\substack{\sigma\in S_k,\\ |\sigma|=i}} \frac{\chi^\lambda(\sigma)}{f^\lambda}\frac{\chi^\mu(\sigma)}{f^\mu}.$$

Note that it might happen that $k \leq L\sqrt{n}$. In this case, $S_3' = 0$ and all terms for $i > k$ in S_2' are 0 as well, but this will not affect the bounds below.

Consider first S_1'. Since it is a sum over permutations σ with $|\sigma| \leq L\sqrt{k} \leq L\sqrt{n}$, we can still apply (2.1) and Equation (2.9) still holds.

For S_2', we should apply (2.1) to λ and (2.2) to μ. Combining with Lemma 9, we have:

$$\begin{aligned}
|S_2'| &\leq \sum_{i=L\sqrt{k}+1}^{L\sqrt{n}} \frac{k^{2i}}{i!}\left(\frac{aL}{\sqrt{n}}\right)^i\left(\frac{ai}{k}\right)^i \\
&\leq \left(a^2Lekn^{-\frac{1}{2}}\right)^{L\sqrt{k}+1} \times \frac{\left(a^2Lekn^{-\frac{1}{2}}\right)^{L\sqrt{n}-L\sqrt{k}} - 1}{\left(a^2Lekn^{-\frac{1}{2}}\right) - 1}. \\
&\leq C_3\left(a^2Lekn^{-\frac{1}{2}}\right)^{L\sqrt{n}},
\end{aligned}$$

for some constant C_3. Note that this bound is different from (2.7), since we now have a *divergent* geometric series. Thus

$$|S_2'| \leq e^{\sqrt{n}\left(\log(kn^{-\frac{1}{2}})+O(1)\right)}. \tag{2.10}$$

Let us finally turn to S_3'. By Lemma 9 and Equation (2.2) applied to λ and μ, we have

$$|S_3'| \leq \sum_{i=L\sqrt{n}+1}^{k} \frac{k^{2i}}{i!}\left(\frac{ai}{n}\right)^i\left(\frac{ai}{k}\right)^i.$$

As before we bound the first factor i^i by $i!e^i$, and using $i \leq k$, we bound $\left(\frac{ai}{k}\right)^i$ by a^i. This gives

$$\begin{aligned}|S_3'| &\leq \sum_{i=L\sqrt{n}+1}^{k} \left(\frac{a^2k^2e}{n}\right)^i. \\ &\leq \left(a^2k^2en^{-1}\right)^{L\sqrt{n}+1} \frac{\left(a^2k^2en^{-1}\right)^{k-L\sqrt{n}}-1}{(a^2k^2en^{-1})-1} \\ &\leq C_4\left(a^2k^2en^{-1}\right)^k = e^{k\left(\log\frac{k^2}{n}+O(1)\right)},\end{aligned}$$

for some constant C_4.

From (2.9) and (2.10), we know that S_1' and S_2' are also bounded by $e^{k(\log\frac{k^2}{n}+O(1))}$ (for $k = \Theta(n^{1/2})$, the upper bound given in (2.10) is of this order; otherwise it is smaller). This completes the proof of Theorem 4. □

3. Limitations of the method

As mentioned in the introduction, we do not believe the bound in Theorem 4 to be sharp. Nevertheless we argue in this section that, with the known bounds on characters and the method used in this paper, we cannot improve Theorem 4. Such an improvement would require either new bounds on characters, or taking signs and cancellations appearing in Lemma 6 into account, or a completely different method. Improving upper bounds on characters or having a good understanding of their signs are in general believed to be hard problems, so we think that an improvement is more likely to come from a totally different method.

As far as we are aware of, the best bounds on characters available in the literature are the following.

- In [10], Roichman proved that there exist constants $b > 0$ and $q \in (0,1)$ such that for any partition ν and permutation σ of the same size m, we have

(3.1) $$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq \left[\max\left(q, \tfrac{r(\nu)}{m}, \tfrac{c(\lambda)}{m}\right)\right]^{b|\sigma|}.$$

  (In Roichman's paper, the exponent is in fact $b\,|\mathrm{supp}(\sigma)|$, where $|\mathrm{supp}(\sigma)|$ is the size of the support of σ, *i.e.* the number of its non-fixed points. This modification is however irrelevant since the value of b is not known and, for all σ, we have $|\sigma| \leq \mathrm{supp}(\sigma) \leq 2|\sigma|$.)
- In [7], Larsen and Shalev proved the following bound: for all $\varepsilon > 0$, there exists N such that, for all integers $m > N$, partitions ν and permutations σ both of size m, we have

(3.2) $$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq (f^\nu)^{B(\sigma)-1+\varepsilon},$$

  where $B(\sigma)$ is some parameter between 0 and 1 associated to σ. Its complete definition is technical and irrelevant here, but we note that $B(\sigma)$ is small (i.e. far from one) if and only if σ has few (i.e. a sublinear number of) fixed points.

- Lastly, we restate for the reader's convenience the bound of the second author and Śniady [4]: there exists a constant $a > 1$ such that for every partition $\nu \vdash m$ and every permutation $\sigma \in S_m$,

$$\left|\frac{\chi^\nu(\sigma)}{f^\nu}\right| \leq \left[a \max\left(\frac{r(\nu)}{m}, \frac{c(\nu)}{m}, \frac{|\sigma|}{m}\right)\right]^{|\sigma|}. \tag{3.3}$$

The last bound is better than the first one if the two following conditions hold simultaneously: the diagrams have no rows or columns of linear size (which we always assume here) and the length of the permutation is sublinear. On the opposite, the second bound is better for permutations with a sublinear number of fixed points, which implies that the length is linear in n. It seems that to get an optimal result we might need all these bounds, i.e. take the minimal one for each permutation σ (and diagram ν).

Call $U_{\mathrm{R}}(\sigma,\nu)$ (resp. $U_{\mathrm{LS}}(\sigma,\nu)$, $U_{\mathrm{F\acute{S}}}(\sigma,\nu)$) the right-hand side of (3.1) (resp. (3.2) and (3.3)) and set

$$U_{\mathrm{all}}(\sigma,\nu) = \min\big(U_{\mathrm{R}}(\sigma,\nu), U_{\mathrm{LS}}(\sigma,\nu), U_{\mathrm{F\acute{S}}}(\sigma,\nu)\big).$$

From Lemma 6, we know that

$$A_{\lambda/\mu} \leq B_{\lambda/\mu}, \text{ with } B_{\lambda/\mu} = \sum_{\sigma\in S_k} U_{\mathrm{all}}(\sigma,\lambda)U_{\mathrm{all}}(\sigma,\mu).$$

The main result of this section is a lower bound for $B_{\lambda/\mu}$, which matches the upper bound in Theorem 4.

Proposition 10. *Let $\lambda \vdash n$ and $\mu \vdash k$ balanced, and assume $k \geq 2Ln^{\frac{1}{2}}$. Then we have*

$$B_{\lambda/\mu} \geq e^{k\left(\log\frac{k^2}{n} + O(1)\right)}.$$

Proof: The idea of the proof consists in identifying in the sum defining $B_{\lambda/\mu}$ a large subfamily of permutations σ for which $U_{\mathrm{all}}(\sigma,\lambda)U_{\mathrm{all}}(\sigma,\mu)$ is large and reasonably simple, so that the sum on this smaller set is already big enough.

Fix a small number $\eta > 0$. We denote by $F(\sigma)$ the number of fixed points of σ. We will consider, for each $k \geq 1$, the set

$$\Omega_k^\eta = \big\{\sigma \in S_k : \; F(\sigma) = \eta\, k \text{ and } |\sigma| \geq (1-\eta)^2 k\big\}.$$

In the following, all constants are independent from k and n (so a fortiori of λ and μ), but might depend on η. Our first claim is that there exists a constant C_5 such that

$$|\Omega_k^\eta| \geq (C_5)^k\, k^{(1-\eta)k}. \tag{3.4}$$

First consider the superset $\tilde{\Omega}_k^\eta \supset \Omega_k^\eta$ obtained by removing the condition on the length of $|\sigma|$. Permutations in $\tilde{\Omega}_k^\eta$ are uniquely obtained by choosing the $\eta\, k$ fixed points and choosing a derangement (i.e. a fixed-point free permutation) of the remaining $(1-\eta)k$ points. Therefore

$$|\tilde{\Omega}_k^\eta| = \binom{k}{\eta k} D_{k(1-\eta)},$$

where D_K is the number of derangements of K elements. It is well known that $D_K \sim \frac{K!}{e}$, so that a simple computation using Stirling's approximation gives

$$|\tilde{\Omega}_k^\eta| \geq \gamma_1^k\, k^{(1-\eta)k},$$

for some constant γ_1 and k large enough.

It remains to see that Ω_k^η covers at least a proportion γ_2^k of $\tilde{\Omega}_k^\eta$ (for some constant γ_2). In fact we will see that this proportion tends to 1. Equivalently, we will prove that the proportion of derangements of K elements with length at least $(1-\eta)K$ tends to 1.

It is well-known since Feller [3, p. 815] that the number of cycles in a uniform random permutation of size K is asymptotically normal with mean and variance $\log(K)$. In particular the proportion of permutations of size K with less than ηK cycles - or equivalently with length at least $(1-\eta)K$ - tends to 1. On the other hand the proportion of derangements tends to $1/e$. So the proportion of derangements with length at least $(1-\eta)K$ among derangements should tend to 1 as well. This completes the proof of (3.4).

In the following we assume that σ is in Ω_k^η. We now give a lower bound for the summand corresponding to such permutations in the definition of $B_{\lambda/\mu}$. It easy to check that for k large enough, the parameter $B(\sigma)$ appearing in the Larsen-Shalev bound will be arbitrary close to 1 uniformly for σ in Ω_k^η (since σ has linearly many fixed points) so that the exponent $1-B(\sigma)+\varepsilon$ in the Larsen-Shalev bound becomes eventually positive. The Larsen-Shalev bound does not give any information in this case (the RHS is bigger than 1, while we know that the LHS is always smaller than 1) and is in particular bigger than Roichman's bound. We can therefore completely forget about this bound for both $U_{\mathrm{all}}(\sigma,\lambda)$ and $U_{\mathrm{all}}(\sigma,\mu)$.

Besides, since the diagrams λ and μ are balanced and since the length of σ in Ω_k^η is always at least $(1-2\varepsilon)k \geq L\sqrt{n}$ (assuming $\varepsilon < 1/4$), the maximum in the other two bounds will never be reached by $\frac{r(\lambda)}{\sqrt{n}}$ or $\frac{c(\lambda)}{\sqrt{n}}$. We therefore have

$$\begin{aligned} U_{\mathrm{all}}(\sigma,\lambda) &= \left[\min\left(q^b, \tfrac{a|\sigma|}{n}\right)\right]^{|\sigma|} \geq \left[\min\left(q^b, \tfrac{a(1-\varepsilon)^2 k}{n}\right)\right]^k \geq \left(C_6 \tfrac{k}{n}\right)^k,\\ U_{\mathrm{all}}(\sigma,\mu) &= \left[\min\left(q^b, \tfrac{a|\sigma|}{k}\right)\right]^{|\sigma|} \geq \left[\min\left(q^b, a(1-\varepsilon)^2\right)\right]^k = C_7^k, \end{aligned}$$

for some constants C_6 and C_7. Combining this with (3.4), we get

$$\sum_{\sigma\in\Omega_k^\eta} U_{\mathrm{all}}(\sigma,\lambda)U_{\mathrm{all}}(\sigma,\mu) \geq (C_5)^k\, k^{(1-\varepsilon)k} \left(C_6 \tfrac{k}{n}\right)^k C_7^k.$$

Call RHS the right-hand side of the previous display. We have

$$\log(\mathrm{RHS}) = (2k-\varepsilon)\log k - k\log n + k\log(C_5\, C_6\, C_7) = k\big(\log \tfrac{k^2}{n} + O(1)\big).$$

This completes the proof of the proposition. □

4. Numerical evidence

In this section, we give numerical evidence to support Conjecture 5 and the fact that the bound of Theorem 3 is sharp.

In [6], Kim and Oh proved an exact product formula for the number of standard Young tableaux of shape λ/μ represented in Figure 1. This formula was then independently rediscovered by Morales, Pak and Panova [9], under the following form (which is diffent from Kim-Oh's original formulation).

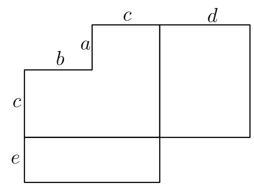

FIGURE 1. The skew shape λ/μ

Theorem 11 (Kim-Oh, Morales-Pak-Panova). *For all $a, b, c, d, e \in \mathbb{N}$, the number $f^{\lambda/\mu}$ of standard Young tableaux of shape λ/μ given in Figure 1 is equal to*

$$f^{\lambda/\mu} = \big((a+c+e)(b+c+d) - ab - ed\big)! \\ \times \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+d+e)}{\Phi(a+b)\Phi(d+e)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+d+e)},$$

where Φ is the superfactorial defined as

$$\Phi(n) := 1! \times 2! \times \cdots \times (n-1)!.$$

For such shapes, the numbers f^λ and f^μ can also easily be written as quotients of superfactorials by using the hook-length formula. Asymptotic expansions of logarithms of factorials (and thus of superfactorials) are known at any order, so it is possible in principle to compute asymptotic expansions of $\log A_{\lambda,\mu}$ at any order for such shapes. The computation is however very cumbersome in practice and best done with a computer by specializing a, b, c, d, e to some functions of a single parameter n.

We summarise some examples of the asymptotic behaviour of $\log A_{\lambda/\mu}$ in the tables below. Note that here $|\lambda|$ is $\Theta(n)$ and not exactly n as before. It makes the presentation simpler and hopefully doesn't create any confusion for the reader, since we are only interested in the order of magnitude of $A_{\lambda,\mu}$ and do not care about constants.

a	b	c	d	e	$\log A_{\lambda/\mu}$	$\lvert\mu\rvert$	$\lvert\mu\rvert^{\frac{3}{2}}\lvert\lambda\rvert^{-\frac{1}{2}}$
$n^{\frac{1}{17}}$	$2n^{\frac{1}{17}}$	$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$\frac{1}{4}n^{-\frac{11}{34}} + O\left(n^{-\frac{13}{17}}\right)$	$2n^{\frac{2}{17}}$	$\Theta\left(n^{-\frac{11}{34}}\right)$
$2n^{\frac{1}{13}}$	$n^{\frac{1}{13}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{3}}$	$-\frac{1}{2}n^{-\frac{7}{26}} + O\left(n^{-\frac{17}{39}}\right)$	$2n^{\frac{2}{13}}$	$\Theta\left(n^{-\frac{7}{26}}\right)$
$2n^{\frac{1}{10}}$	$n^{\frac{1}{10}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{3}}$	$n^{\frac{1}{2}}$	$\frac{1}{2}n^{-\frac{1}{5}} + O\left(n^{-\frac{11}{30}}\right)$	$2n^{\frac{1}{5}}$	$\Theta\left(n^{-\frac{1}{5}}\right)$
$n^{\frac{1}{6}}$	$2n^{\frac{1}{6}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{3}}$	$\frac{1}{2} + O\left(n^{-\frac{1}{6}}\right)$	$2n^{\frac{1}{3}}$	$\Theta(1)$
$n^{\frac{1}{6}}$	$n^{\frac{1}{6}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$\frac{2}{27}n^{-\frac{1}{3}} + O\left(n^{-\frac{2}{3}}\right)$	$n^{\frac{1}{3}}$	$\Theta(1)$
$n^{\frac{1}{5}}$	$3n^{\frac{1}{5}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{3}}$	$\frac{3}{2}n^{\frac{1}{10}} + O\left(n^{-\frac{1}{15}}\right)$	$3n^{\frac{2}{5}}$	$\Theta\left(n^{\frac{1}{10}}\right)$

TABLE 1. Asymptotics for $\log A_{\lambda/\mu}$ when $|\mu| = o(|\lambda|^{\frac{1}{2}})$

Table 1 gives evidence that the bound in Theorem 3 is sharp. Indeed, this bound is reached for skew diagrams with $|\mu| = \Theta(|\lambda|^\alpha)$, for various values of α between 0 and 1/2.

a	b	c	d	e	$\log A_{\lambda/\mu}$	$\lvert\mu\rvert$	$\lvert\mu\rvert^{\frac{3}{2}}\lvert\lambda\rvert^{-\frac{1}{2}}$
$n^{\frac{1}{4}}$	$3n^{\frac{1}{4}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{3}}$	$\frac{3}{2}n^{\frac{1}{4}} + O\left(n^{\frac{1}{12}}\right)$	$3n^{\frac{1}{2}}$	$\Theta\left(n^{\frac{1}{4}}\right)$
$2000n^{\frac{1}{4}}$	$400n^{\frac{1}{4}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$160000000n^{\frac{1}{4}} + O(1)$	$800000n^{\frac{1}{2}}$	$\Theta\left(n^{\frac{1}{4}}\right)$
$n^{\frac{1}{3}}$	$2n^{\frac{1}{3}}$	$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$\frac{1}{4}n^{\frac{1}{2}} + O\left(n^{\frac{1}{3}}\right)$	$2n^{\frac{2}{3}}$	$\Theta\left(n^{\frac{1}{2}}\right)$
$n^{\frac{2}{5}}$	$2n^{\frac{2}{5}}$	$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$\frac{1}{4}n^{\frac{7}{10}} + O\left(n^{\frac{3}{5}}\right)$	$2n^{\frac{4}{5}}$	$\Theta\left(n^{\frac{7}{10}}\right)$
$n^{\frac{2}{5}}$	$2n^{\frac{2}{5}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$-\frac{1}{4}n^{\frac{7}{10}} + O\left(n^{\frac{3}{5}}\right)$	$2n^{\frac{4}{5}}$	$\Theta\left(n^{\frac{7}{10}}\right)$
$n^{\frac{2}{5}}$	$n^{\frac{2}{5}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$\frac{2}{27}n^{\frac{2}{5}} + O\left(n^{\frac{2}{5}}\right)$	$n^{\frac{4}{5}}$	$\Theta\left(n^{\frac{7}{10}}\right)$
$n^{\frac{1}{2}}$	$2n^{\frac{1}{2}}$	$3n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$n^{\frac{1}{2}}$	$0.56746n + O(1)$	$2n$	$\Theta(n)$

Table 2. Asymptotics for $\log A_{\lambda/\mu}$ when $|\mu| \geq C|\lambda|^{\frac{1}{2}}$

In the cases represented in Table 2 (and in all other cases that we have investigated), we observe that $\log A_{\lambda/\mu}$ is at most of order $|\mu|^{\frac{3}{2}}|\lambda|^{-\frac{1}{2}}$. Here are some further comments:

- The first and second lines focus on the case where $|\mu| \sim C|\lambda|^{1/2}$ for some constant C. Comparing Theorem 3 and Theorem 4 could let us think that the order of magnitude of $\log A_{\lambda/\mu}$ depends on the constant C. The first and second lines of the table do not exhibit such a dependence.
- The fifth line of the table gives an example, where $\log A_{\lambda/\mu}$ behaves as $\Theta(|\mu|^{\frac{3}{2}}|\lambda|^{-\frac{1}{2}})$, with a negative constant. In this case $A_{\lambda/\mu}$ reaches the lower bound given in Conjecture 5. Intermediate situations, where $\log A_{\lambda/\mu}$ is negligible compared to $|\mu|^{\frac{3}{2}}|\lambda|^{-\frac{1}{2}}$, do also occur, see the second last line of the table.

All the results are evidence for Conjecture 5. However, as argued in the previous section, proving this conjecture would require new techniques or ideas.

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