Rigidity of Schubert closures and invariance of Kazhdan-Lusztig polynomials

Fokko du Cloux

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Institut Girard Desargues
UMR 5028 CNRS
Université Lyon-I
69622 Villeurbanne Cedex FRANCE
ducloux@desargues.univ-lyon1.fr

Abstract. Let $[e, w]$, $[e, w']$ be intervals from the origin in two Coxeter groups W, W'. We prove that any poset isomorphism $\varphi : [e, w] \to [e, w']$ preserves Kazhdan-Lusztig polynomials, in the sense that $P_{\varphi(x),\varphi(y)} = P_{x,y}$ for any $x \leq y$ in $[e, w]$, in the case where one of the two groups W, W' has the property that the Coxeter graph of each of its irreducible constituents is either a tree, or affine of type $\tilde{\mathrm{A}}_n$; in particular, the result holds for all finite or affine Coxeter groups. We obtain this result as a consequence of an explicit description of all the poset isomorphisms from $[e, w]$ to $[e, w']$.

1 Introduction

1.1. Let (W, S) be a Coxeter system, where we assume that the cardinality $|S| = n$ of S is finite; usually we omit S from the notation and simply speak of the Coxeter group W. We will denote $\leq$ the Bruhat ordering on W (see [10] for its definition and basic properties). In their celebrated paper [11], Kazhdan and Lusztig have defined polynomials $P_{x,y} \in \mathbf{Z}[q]$ for each $x \leq y$ in W, which are the components of a remarkable basis of the Hecke algebra of W. When W is the Weyl group of a finite or Kač-Moody Lie algebra $\mathfrak{g}$, these polynomials hold the key to the representation theories of $\mathfrak{g}$ and associated reductive groups, and also describe the singularities of the Schubert varieties for $\mathfrak{g}$. For general W, they might point to yet-to-be-discovered representation theories and/or geometries—in fact, a conjectural representation-theoretic interpretation of the polynomials $P_{x,y}$ has been proposed by Dyer [8].

In view of the fundamental role played by Kazhdan-Lusztig polynomials in representation theory, it is natural to wonder about their true nature. In particular, it has been conjectured by Dyer [7] that in fact $P_{x,y}$ depends only on the isomorphism type of the poset $[x, y]$, where $[x, y]$ is the interval between x and y in the Bruhat ordering (see 2.1 for basic poset terminology). This turns out to be a rather delicate problem; the only substantial known result is a positive answer by Brenti [2], building on the work of Dyer, in the case where $[x, y]$ is *adihedral, i.e.*, where $[x, y]$ contains no subinterval of length three isomorphic to the Bruhat ordering on the Coxeter group of type A_2 (see also [5] for a slightly different approach.) In fact, Brenti proves that in the adihedral case, the polynomial $P_{x,y}$ is the Stanley g-polynomial of the dual poset $[x, y]^\circ$, as defined in [13], section 3.14.

Note that if the answer to Dyer's question were affirmative, then for any other Coxeter group W', and for any two pairs of elements $v \leq w$ in W, $v' \leq w'$ in W', any poset isomorphism $\varphi : [v, w] \to [v', w']$ would *preserve Kazhdan-Lusztig polynomials*, in the sense that for any $x \leq y$ in $[v, w]$, we would have $P_{\varphi(x),\varphi(y)} = P_{x,y}$. In this form, we shall give an affirmative answer for intervals from the identity in W, W' (to be called *Schubert closures* in the sequel), for a class of groups which covers the main cases of interest :

1.2 Theorem. — *Let W, W' be two Coxeter groups, and consider elements $w \in W$, $w' \in W'$ such that the Schubert closures $[e, w]$, $[e, w']$ are isomorphic as posets for the Bruhat orderings on W, W' respectively. Then any poset isomorphism $\varphi : [e, w] \to [e, w']$ preserves Kazhdan-Lusztig polynomials, in the following cases :*

(a) *the Coxeter graph of one of the two groups W, W' is a tree;*

(b) *one of the two groups W, W' is affine of type $\tilde{\mathrm{A}}_n$;*

(c) *each irreducible component of one of the two groups W, W' is of one of the types* (a), (b) *above.*

In particular, the result holds in all cases where one of the two groups W, W' is finite or affine.

(as we learned from the referee, this result was also obtained by Brenti [3], in the case where W, W' are both of type A_n, as a corollary of a purely combinatorial construction of the $P_{x,y}$.)

1.3. In contrast with Brenti's results in the adihedral case and in type A_n, we are not able to give a definition of $P_{x,y}$ in poset-theoretic terms (indeed, we do not know if a poset isomorphism $\varphi : [v,w] \to [v',w']$ preserves Kazhdan-Lusztig polynomials if only *one* of v, v' is the identity.) Our approach relies on a sufficiently explicit description of all poset isomorphisms between $[e,w]$ and $[e,w']$.

This turns out to be an interesting subject in itself. Assume first that $W = W'$, and, for simplicity, that W is irreducible; assume also that φ induces the identity on S (it is easy to reduce to this case). For the case of the full group, it has been proved by van den Hombergh [9] (and rediscovered by Waterhouse [14]) that if the rank of W is at least 3, the group $\mathrm{Aut}_S(W)$ of automorphisms of the Bruhat order inducing the identity on S has a single non-trivial element, *viz.* the inversion; in contrast, when W is dihedral, say finite of cardinality $2m$, $\mathrm{Aut}_S(W)$ is isomorphic to $(\mathbf{Z}/2)^{m-2}$. In effect, Waterhouse proves that if B denotes the set of dihedral elements in W (see Definition 2.7), then any order automorphism φ of W is entirely determined by its restriction to B; he then shows that if the rank is > 2 and $\varphi \in \mathrm{Aut}_S(W)$, the only possibilities for the restriction are identity and inversion. Let $\mathrm{Aut}_S(B)$ be the group of poset automorphisms of B inducing the identity on S; then again (Lemma 4.3) $\mathrm{Aut}_S(B)$ is an elementary abelian 2-group. Our approach is to start with *any* $\varphi \in \mathrm{Aut}_S(B)$. It is easy to see that there exists a unique maximal decreasing subset $\mathrm{dom}(\varphi) \subset W$ such that φ extends to an isomorphism, again denoted by φ, from $\mathrm{dom}(\varphi)$ onto a decreasing subset of W (and in fact, $\mathrm{dom}(\varphi)$ is φ-stable.) What we show is that if W satisfies the hypotheses of Theorem 1.2, φ preserves Kazhdan-Lusztig polynomials on $\mathrm{dom}(\varphi)$. Assume that $w \in W$ is such that $B \subset [e,w]$ (in other words, that w is full, in the terminology of Definition 2.7). Then any isomorphism $\varphi : [e,w] \to [e,w']$ inducing the identity on S restricts to an element of $\mathrm{Aut}_S(B)$, and by definition $[e,w] \subset \mathrm{dom}(\varphi)$; so Theorem 1.2 will be proved in this case.

In fact it will turn out that if φ is not the identity or the inversion, $\mathrm{dom}(\varphi)$ will always be a rather small subset of W (for instance, when W is affine, $\mathrm{dom}(\varphi)$ is finite). This means that for sufficiently general full elements of W, the only Schubert closures isomorphic to $[e,w]$ are $[e,w]$ and $[e,w^{-1}]$, and the only non-identity isomorphism fixing S betweeen them is given by inversion; in other words, the rigidity result of van den Hombergh and Waterhouse extends to embeddings of sufficiently general Schubert closures. On the other hand, for the exceptional elements, there can be many embeddings and automorphisms. The typical example is given in section 4.8 : If we have a decomposition of S into two disjoint non-empty subsets

S' and S'', and denote W', W'' the subgroups of W generated by S' and S'' respectively, we can define a poset automorphism of the decreasing subset $W'W''W' \subset W$ which induces the inversion on W'' and the identity on W'; we show that this automorphism preserves Kazhdan-Lusztig polynomials. Under the hypotheses of Theorem 1.2, it turns out that for any $\varphi \in \mathrm{Aut}_S(B)$, the action of φ on $\mathrm{dom}(\varphi)$ can ultimately be described in terms of this example; this is how the Theorem is proved.

In the general case where $W \neq W'$ in the statement of 1.2, we show in Theorem 3.5 that to each $w \in W$ we can associate another Coxeter group $W(w)$ over the set $\mathrm{supp}(w) \subset S$, and a canonical isomorphism of $[e, w]$ onto a Schubert closure in $W(w)$, which preserves Kazhdan-Lusztig polynomials. The Coxeter system $(W(w), \mathrm{supp}(w))$ depends only on the poset $[e, w]$; so if $[e, w] \subset W$ and $[e, w'] \subset W'$ are isomorphic, there will be an isomorphism of Coxeter systems from $(W(w), \mathrm{supp}(w))$ onto $(W(w'), \mathrm{supp}(w'))$, which carries the Schubert closure in $W(w')$ corresponding to $[e, w']$ to a Schubert closure of $W(w)$ isomorphic to $[e, w]$. Moreover, the image of w under the canonical isomorphism is always a full element of $W(w)$; so we are reduced to the situation handled above.

Of course, we expect Theorem 1.2 to be true for arbitrary W, W'. However, the explicit analysis of $\mathrm{dom}(\varphi)$ for $\varphi \in \mathrm{Aut}_S(B)$ seems to get more complicated when the graph is no longer a tree; this is already apparent in our treatment of case $\tilde{\mathrm{A}}_n$ in section 7. Even the general case where the graph is a cycle has defeated us so far, and seems to call for more complicated constructions than the ones used in this paper.

2 Preliminaries and basic definitions

2.1. We refer to [13], section 3.1, for all poset terminology not explained here. Let P be a graded poset with smallest element $\mathbf{0}$; we denote l the length function on P. For each $x \leq y$ in P, we set $[x, y] := \{x \in P \mid x \leq z \leq y\}$; $[x, y]$ is called the *interval* between x and y. We say that P is *locally finite*, if all intervals $[x, y] \subset P$ are finite. A *decreasing* (resp. *increasing*) subset of P is a subset Q such that for each element x in Q, all smaller (resp. larger) elements are also in Q.

Assume henceforth that P is locally finite. For each $x < y$ in P, the *atoms* (resp. *coatoms*) of $[x, y]$ are the elements z in $[x, y]$ of relative length (resp. colength) one, *i.e.*, such that $l(z) - l(x) = 1$ (resp. $l(y) - l(z) = 1$). If $x > \mathbf{0} \in P$, the coatoms of $[\mathbf{0}, x]$ will simply be called the coatoms of x, and denoted $\mathrm{coat}(x)$; the set of atoms of $[\mathbf{0}, x]$ will be called the *support* of

x, and denoted supp(x). More generally, for any decreasing subset $Q \subset P$, we will denote supp(Q) the set of elements of length one in Q.

2.2. Let (W, S) be as in the introduction. We refer to [10] section 5.9 for a definition of the Bruhat ordering on W. For this ordering, W is locally finite, graded and with smallest element e, the identity in W. The length function on W associated with the Bruhat ordering is the same one as the usual length function on W as a Coxeter group. From the properties of the Bruhat ordering it follows that for each $x \in W$, and each reduced expression $x = s_1 \dots s_p$ of x, the support of x as defined in 2.1 is equal to the set $\{s_1, \dots, s_p\}$ (of course there will usually be many repetitions of each element of supp(x) among the s_j.) We shall sometimes denote $L(x)$ (resp. $R(x)$) the set of $s \in S$ such that $sx < x$ (resp. $xs < x$); we say that $L(x)$ ($R(x)$) is the *left* (right) *descent set* of x.

Recall that the *Coxeter matrix* of W is the symmetric square matrix $M = (m_{s,t})_{s,t \in S}$, with entries in $\{1, 2, \dots, \infty\}$, where $m_{s,t}$ is the order in W of the element st. The *Coxeter graph* of W is the graph $\Gamma = \Gamma_S$ with vertex set S, and edges the sets $\{s, t\}$ such that $m_{s,t} > 2$ (corresponding to pairs of generators which do *not* commute in W); the edge $\{s, t\}$ carries the label $m_{s,t}$, usually omitted when $m_{s,t} = 3$. The group W is said to be *irreducible* if Γ is connected.

For each subset $J \subset S$, we denote W_J the subgroup of S generated by J, which is again a Coxeter group, with Coxeter matrix the restriction of M to $J \times J$. It follows that W_J is also the set of $x \in W$ such that supp(x) $\subset J$.

2.3. We recall the fundamental property of the Bruhat ordering (see for instance the Proposition in section 5.9 of [10]) :

Proposition. — *Let $x \leq y$ in W and $s \in S$ be such that $xs > x$ and $ys < y$. Then the interval $[x, y]$ is stable under right multiplication by s.*

(Of course the analogous property holds for left multiplications.)

Corollary. — *Let $y \in W$ and $s \in S$ be such that $ys < y$. Then $[e, y]$ is the (non-necessarily disjoint) union of $[e, ys]$ and $[e, ys]s$.*

2.4. We will say that a finite graded poset P is *dihedral*, if it is isomorphic to an interval in a dihedral Coxeter group. This is equivalent to saying that P has one element in top and bottom lengths, two in each intermediate length, and that each element in each length is comparable to each element in the previous length. Also, by abuse of language, we will say that $x \in W$ is dihedral, if the interval $[e, x]$ is dihedral. Clearly, the set of dihedral elements in W is decreasing; in particular, the identity element e is dihedral.

A key role in our investigations will be played by the following theorem, due to Matthew Dyer [6] (see also [4] for the main application which we shall need here) :

Theorem. — (Dyer) *Let $x < y$ be two elements in W with $l(y) - l(x) > 1$. Then the following are equivalent :*

(i) *$[x, y]$ has two atoms;*

(ii) *$[x, y]$ has two coatoms;*

(iii) *$[x, y]$ is dihedral.*

Corollary. — ([4] Corollary 3.7) *Let $x \in W$ be non-dihedral, and let $s \in S$. Then if $zs < z$ for all $z \in \operatorname{coat}(x)$ except at most one, we have $xs < x$.*

2.5. The following proposition had already been remarked by van den Hombergh ([9] lemma 7) :

Proposition. — (van den Hombergh) *Let w be any element in W, and let J be an arbitrary subset of S. Denote $[e, w]_J = [e, w] \cap W_J$. Then $[e, w]_J$ has a largest element, which we shall denote $w|_J$ (and hence $[e, w]_J = [e, w|_J]$).*

Proof. — We argue by induction on $l(w)$. If $w = e$, $w|_J = e$ and there is nothing to prove. Otherwise, let $s \in S$ be such that $ws < w$. Let $v' = ws|_J$, and set $v = v's$ if $s \in J$ and $v's > v'$, $v = v'$ otherwise. We claim that v has the required property. By definition, $\operatorname{supp}(v) \subset J$, and $v \in [e, w]$ by 2.3. Now, let $x \in [e, w]_J$. If $x \leq ws$, we have $x \leq v'$ and we are done. Otherwise, it follows from 2.3 that $xs < x$; in particular, x will have a reduced expression ending in s, and from the remarks in 2.2, this implies that $s \in J$. But then we have $xs \leq ws$, hence $xs \leq v' \leq v$, and since $vs < v$ in all cases when $s \in J$, again from 2.3 we have $x = xs.s \leq v$.

Remark. — It follows from the proof of the Proposition that $w|_J$ may be constructed from any reduced expression $w = s_1 \ldots s_p$ of w as follows : let $j_1 < \ldots < j_q$ be the set of indices j such that $s_j \in J$, and define by induction a sequence $(v_k)_{0 \leq k \leq q}$ in W_J as follows : $v_0 = e$, and for $k > 0$, $v_k = v_{k-1}s_{j_k}$ if $v_{k-1}s_{j_k} > v_{k-1}$, $v_k = v_{k-1}$ otherwise. Then $w|_J = v_q$.

2.6 Lemma. — *Let $w \in W$, and let J be any subset of S; let $v = w|_J$ (see 2.5). Then for each $s \in J$ such that $ws < w$, we have $vs < v$.*

Proof. — This is clear : if $ws < w$, the interval $[e, w]$ is stable under right multiplication by s, whence $vs \leq w$; but $\mathrm{supp}(vs) \subset J$, so $vs \leq v$, and of course $vs \neq v$.

2.7 Definition. — For each non-empty decreasing subset Q in W, define the *bud* $B = B(Q)$ of Q to be the set of dihedral elements in Q. For each $s \neq t$ in $\mathrm{supp}(Q)$, define the *leaf* $L_{\{s,t\}}$ of B to be the set of all elements in B (or equivalently, in Q) with support contained in $\{s, t\}$. In particular, $B(W)$ is the union over all $s \neq t$ of the dihedral subgroups $W_{\{s,t\}}$.

When $w \in W$, we shall simply denote $B(w)$ instead of $B([e, w])$, and say that $B(w)$ is the bud of w. If $l(w) > 1$, $B(w)$ is the union over all pairs $\{s, t\} \subset \mathrm{supp}(w)$, $s \neq t$, of the intervals $[e, v_{s,t}]$, where $v_{s,t} = w|_{\{s,t\}}$. We say that an element $w \in W$ is *full*, if $B(w) = B(W)$.

2.8. The following result was also proved in [14] :

Proposition. — (Waterhouse) *Let x and y be two non-dihedral elements in W. Then if* $\mathrm{coat}(x) = \mathrm{coat}(y)$, $x = y$.

Proof. — Clearly, if two elements x and y in W share the same coatom set, they must have equal length p. If $p \leq 2$, x and y are dihedral, so there is nothing to prove. Let $p \geq 3$, and let $s \in S$ be such that $xs < x$. Then since $zs \in [e, x]$ for all $z \in \mathrm{coat}(x)$, we have $zs < z$ for all $z \in \mathrm{coat}(x)$, $z \neq xs$. But since $\mathrm{coat}(x) = \mathrm{coat}(y)$, we see that Corollary 2.4 applies to y, so that $ys < y$; and then ys is the only element $z \in \mathrm{coat}(y)$ such that $zs > z$, which is also xs. So $ys = xs$, and $y = x$.

Let P, P' be two locally finite graded posets with smallest element $\mathbf{0}$. We say that a poset homomorphism $\varphi : P \to P'$ is an *embedding*, if φ is an isomorphism from P onto a decreasing subset of P'. It is equivalent to say that φ is an injection $P \to P'$ such that $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathrm{coat}(x)) = \mathrm{coat}(\varphi(x))$ for all $x > \mathbf{0}$ in P.

Corollary. — *Let φ, φ' be two embeddings from a non-empty decreasing subset $Q \subset W$ into a Coxeter group W'. Then φ and φ' are equal if and only if they agree on the bud B of Q.*

Proof. — Assume that φ and φ' agree on B, and let us prove by induction on $l(x)$ that $\varphi(x) = \varphi'(x)$ for all $x \in Q$. Note that as a consequence of the Theorem in 2.4, an element in a Coxeter group is dihedral if and only if it has at least three coatoms. Since B is the set of dihedral elements in Q, we may assume that x is non-dihedral, and in particular that $l(x) > 2$. Then $\varphi(z) = \varphi'(z)$ for all $z \in \mathrm{coat}(x)$ from the inductive hypothesis; hence $\varphi(x)$

and $\varphi'(x)$ are two non-dihedral elements in W' which have the same coatom set, so $\varphi(x) = \varphi'(x)$.

2.9 Proposition. — *Let W, W' be two Coxeter groups, let B be a non-empty decreasing subset of $B(W)$ (Definition 2.7), and let $\varphi : B \to W'$ be an embedding (see 2.8). Then there is a unique largest decreasing subset $Q \subset W$ such that $B = B(Q)$ and φ extends to an embedding $Q \to W'$; moreover the extension of φ to Q is unique.*

We denote $Q = \mathrm{dom}(\varphi)$, and say that $\mathrm{dom}(\varphi)$ is the *domain* of φ in W; the unique extension of φ to $\mathrm{dom}(\varphi)$ will also be denoted φ.

Proof. — Let $\mathcal{P}$ be the set of all the non-empty decreasing subsets $P \subset W$ such that $B(P) = B$, and φ extends to P. From the Corollary in 2.8, the extension of φ to each $P \in \mathcal{P}$ is unique. From this it follows immediately that the set $\mathcal{P}$ is stable under unions; hence it has a unique largest element (*viz.* the union of all its elements), which is the required decreasing subset Q.

Remark. — In fact, the elements of $\mathrm{dom}(\varphi)$ may be enumerated in order of increasing length through the following algorithm. Run through the elements of W in order of increasing length. Then $x \in \mathrm{dom}(\varphi)$ if *(a)* $x \in B$ or *(b)* $x \notin B(W)$, $\mathrm{coat}(x) \subset \mathrm{dom}(\varphi)$ and there exists $y \in W'$, necessarily unique, such that $\varphi(\mathrm{coat}(x)) = \mathrm{coat}(y)$; in case *(b)*, set $\varphi(x) = y$. In order to find out if a given $w \in W$ belongs to $\mathrm{dom}(\varphi)$, apply the same algorithm, but restrict it to $[e, w]$; if we find $x \in [e, w]$ which is not in $\mathrm{dom}(\varphi)$, we may stop and conclude that $w \notin \mathrm{dom}(\varphi)$. Otherwise $\varphi(w)$ is found at the end of the algorithm.

3 Canonical realization of a Schubert closure

3.1. We say that a non-empty decreasing subset $B \subset B(W)$ (Definition 2.7) is *group-like*, if for each $s \neq t$ in $\mathrm{supp}(B)$, the leaf $L_{\{s,t\}} \subset B$ has at most one maximal element (as opposed to two). From the Proposition in 2.5 it follows that for each $w \in W$, the bud $B(w)$ of $[e, w]$ is group-like. To each group-like $B \subset B(W)$ we associate a Coxeter matrix $M^B = (m^B_{s,t})$ over $S^B = \mathrm{supp}(B) \subset S$, by setting $m^B_{s,t}$ equal to the length of the maximal element in $L_{\{s,t\}}$, or $m^B_{s,t} = \infty$ if there is no such maximal element. We denote W^B the Coxeter group defined by the set S^B and the Coxeter matrix M^B.

In this section 3, we will have to deal with several Coxeter matrices defined over subsets of the same set S. For such a matrix $M = (m_{s,t})_{s,t\in J}$, we shall denote $J = \mathrm{supp}(M)$; we will say that $M \leq M'$ if $\mathrm{supp}(M) \subset \mathrm{supp}(M')$, and $m_{s,t} \leq m'_{s,t}$ for all $s, t \in \mathrm{supp}(M)$.

3.2. Fix a non-empty decreasing subset $Q \subset W$. For any $s \in S$, we denote $D_s = D_s(Q)$ the set of $x \in Q$ such that $xs \in Q$; we say that D_s is the *domain* of the (right) action of s on Q. Clearly $x \in D_s$ if and only if there exists $y \geq x \in Q$ such that $ys < y$; in particular, D_s is a (possibly empty) decreasing subset of Q. Note that $xs > x$ for any $x \in Q$, $x \notin D_s$.

We say that an embedding (see 2.8) $\varphi : Q \to W'$ from Q to another Coxeter group W' with generating set S', $\mathrm{supp}(Q) \subset S' \subset S$, is *compatible with the action of* $s \in \mathrm{supp}(Q)$, if $\varphi(xs) = \varphi(x)s$ for all $x \in D_s$. If for a given $J \subset \mathrm{supp}(Q)$ this holds for every $s \in J$, we say that φ is compatible with the action of J.

3.3 Proposition. — *Let $B \subset B(W)$ be any non-empty decreasing subset. Set $S^B = \mathrm{supp}(B)$, and $Q^B = \{x \in W \mid B(x) \subset B\}$. Let W' be a Coxeter group with generating set S' such that $S^B \subset S' \subset S$, and let $\varphi : B \to W'$ be an embedding compatible with the action of S^B. Then* $\mathrm{dom}(\varphi) = Q^B$, *and φ is compatible with the action of S^B on* $\mathrm{dom}(\varphi)$.

Proof. — Since by definition $B(\mathrm{dom}(\varphi)) = B$, the inclusion $\mathrm{dom}(\varphi) \subset Q^B$ is clear. We shall show by induction on $l(x)$ that each $x \in Q^B$ belongs to $\mathrm{dom}(\varphi)$, and that φ is compatible with the action of S^B on $[e, x]$. Clearly we may assume that x is non-dihedral.

From the Remark in 2.9, in order to prove that $x \in \mathrm{dom}(\varphi)$, it suffices to show that $\varphi(\mathrm{coat}(x))$ is the coatom set of some $y \in W'$. Since $x \neq e$, we may choose $s \in S^B$ such that $xs < x$. From the inductive hypothesis, we have $[e, x[\subset \mathrm{dom}(\varphi)$, and in particular $xs \in \mathrm{dom}(\varphi)$. We will show that $y = \varphi(xs)s$ satisfies $\varphi(\mathrm{coat}(x)) = \mathrm{coat}(y)$. Since $zs < z$ for each $z \in \mathrm{coat}(x)$, $z \neq xs$, we see that $D_s([e, x[) = [e, x[\setminus\{xs\}$. But since φ is compatible with the action of s on $[e, x[$, it follows that $P = \varphi([e, x[) \setminus \varphi(xs)$ is stable under right multiplication by s, and since $\mathrm{coat}(\varphi(xs)) = \varphi(\mathrm{coat}(xs)) \subset P$, $\varphi(xs)s < \varphi(xs)$ would imply $\varphi(xs) \in P$, contradiction. Hence $y = \varphi(xs)s > \varphi(xs)$. Now it was shown in [5], section 2.9, that the coatoms of x in W are xs, and the zs, where z ranges through the set of coatoms of xs such that $zs > z$. A similar description holds for the coatoms of y in W'. Since φ is compatible with the action of s on $[e, x[$, it is then clear that φ takes $\mathrm{coat}(x)$ to $\mathrm{coat}(y)$, proving our contention. Note that from the uniqueness of y, this already shows that $\varphi(xs)s$ is independent of the choice of s.

It remains to show that φ is compatible with the action of each $t \in S^B$, $t \neq s$ on $[e,x]$. If $xt > x$, this is clear, for then $D_t([e,x]) \subset [e,x[$, so we may directly apply the inductive hypothesis. So assume $xt < x$. The only thing that we have to prove is that $\varphi(x) = \varphi(xt)t$; but this is precisely the fact that the element y constructed above is independent of the choice of s.

3.4 Remark. — In fact, it is clear from the condition $\varphi(x) = \varphi(xs)s$ if $xs < x$, that for any non-empty decreasing subset $Q \subset W$ with $\mathrm{supp}(Q) = S^B$, there is at most one embedding $\varphi : Q \to W'$ compatible with the action of S^B. What the Proposition does is reduce the question of existence to the dihedral case.

But this is easy to settle. Let $s \neq t \in S^B$, and let $L = L_{\{s,t\}}$ be the corresponding leaf in B. Extend the definition of M^B in 3.1 to arbitrary B by defining $m^B_{s,t}$ to be the length of the maximal elements of L if there are such elements, ∞ otherwise. It is clear that φ has to induce the identity on S^B. In particular, φ restricts to an embedding from L to the corresponding leaf $L' = W'_{\{s,t\}} \subset W'$. An obvious necessary condition for the existence of φ is that $m^B_{s,t} \leq m'_{s,t}$, where $M' = (m'_{s,t})$ is the Coxeter matrix of W', if L is group-like, and $m^B_{s,t} < m'_{s,t}$ otherwise.

Let us examine under which circumstances this necessary condition is also sufficient. We may assume $B \neq \{e\}$. Let B' be the set of $u \in B$ such that the right descent set $R(u)$ (see 2.2) contains at most one element. Equivalently, B' is the set of $u \in B$ such that u is not maximal in $B(W)$. Then for each $u \neq e$ in B', there is a single $s \in S^B$ such that $us < u$, so that the element $\varphi(us)s \in W'$ is well-defined; elementary considerations about dihedral groups will show that this defines an embedding $\varphi : B' \to W'$ which is compatible with the action of S^B on B'. If $u \in B$ is maximal in $B(W)$, we have $R(u) = \{s,t\}$ with $s \neq t$, and u is the longest element in $W_{\{s,t\}}$. Then if $m^B_{s,t} = m_{s,t} < m'_{s,t}$, the elements $\varphi(us)s$ and $\varphi(ut)t$ in W' are distinct, and φ cannot be extended to $B' \cup \{u\}$. If on the other hand $m_{s,t} = m'_{s,t}$, we may set $\varphi(u) = \varphi(us)s = \varphi(ut)t$. So the necessary and sufficient condition is that $M^B \leq M'$, with $m^B_{s,t} < m'_{s,t}$ if $L_{\{s,t\}}(B)$ has two maximal elements, and $m^B_{s,t} = m'_{s,t}$ if $m^B_{s,t} = m_{s,t}$.

3.5 Theorem. — *Let B be any group-like non-empty subset of $B(W)$. Define S^B, M^B as in 3.1, and Q^B as in 3.3. Let W' be a Coxeter group over a generating set S' with $S^B \subset S' \subset S$, such that its Coxeter matrix M' satisfies $M^B \leq M' \leq M$, where M is the Coxeter matrix of W. Then there is a unique embedding $\varphi : Q^B \to W'$, compatible with the action of S^B. Moreover, φ preserves reduced expressions, in the sense that if $x \in Q^B$,* *and if $x = s_1 \dots s_p$ is a reduced expression for x in W, then $\varphi(x) = s_1 \dots s_p$ is a reduced expression for $\varphi(x)$ in W'.*

Proof. — From the fact that B is group-like, and the inequalities $M^B \leq M' \leq M$, we see that the necessary and sufficient condition from Remark 3.4 for the existence of an embedding $\varphi : B \to W'$ which is compatible with the action of S^B is fulfilled. Then Proposition 3.3 ensures that φ extends to Q^B, and that the extension is still compatible with the action of S^B. We have already remarked that uniqueness is obvious in any case.

It remains to show that φ preserves reduced expressions. We shall argue by induction on $l(x)$. There is nothing to prove if $x = e$, so we may assume $l(x) > 0$. Let $x = s_1 \dots s_p$ be any reduced expression, and set $s = s_p$, $z = xs$. Then from the inductive hypothesis, $s_1 \dots s_{p-1}$ is a reduced expression for $\varphi(z)$, and since $\varphi(x) = \varphi(z)s$, and $l(\varphi(x)) = l(x)$, $s_1 \dots s_p$ is a reduced expression for $\varphi(x)$.

3.6 Definition. — Let $w \in W$, let $B = B(w)$ be the bud of $[e, w]$, and let W^B be the Coxeter group corresponding to the matrix M^B defined in 3.1. Let φ be as in Theorem 3.5, with $W' = W^B$. Then $[e, \varphi(w)] \subset W^B$ is called the *canonical realization* of $[e, w]$. Note that we have $W^B = W$ if and only if w is full (cf. 2.7)

3.7 Remark. — It is important to realize that $w' = \varphi(w)$ may possess reduced expressions in W' which do not come from reduced expressions of w in W. In other words, there may be additional "flexibility" in the group W'. The simplest case is the one where W and W' are dihedral, say finite with orders $2m$, $2m'$, $m' < m$, and $w \in W$ is of length m'. Then w has a single reduced expression in W but two in W'.

Another simple example is the case of Coxeter elements. If w is a Coxeter element in W, we have $m^B_{s,t} = 2$ for all $s \neq t$ in S. If we set $W' = W^B$, then $W' \simeq (\mathbf{Z}/2)^n$ is commutative, hence w' has $n!$ reduced expressions in W', whereas w might have just a single one in W.

The underlying reason is that there might be elements in W with the same support and the same matrix M^B, but with distinct (albeit isomorphic) buds. Any reduced expression for w' in W' will correspond to a reduced word in W, but not necessarily to an expression of our original w.

3.8 Proposition. — *Let $Q \subset W$ be any non-empty decreasing subset, and set $S^Q = \operatorname{supp}(Q)$. Let W' be a Coxeter group with generating set S' such that $S^Q \subset S' \subset S$, and let $\varphi : Q \to W'$ be an embedding which is compatible with the action of S^Q. Then φ preserves Kazhdan-Lusztig polynomials on Q.*

Proof. — We prove that $P_{\varphi(x),\varphi(y)} = P_{x,y}$ for all $x \leq y$ in Q by induction on $l(y)$, and for fixed y, by induction on $l(y) - l(x)$. If $x = y$ there is nothing to prove. Assume $x < y$, and choose $s \in S^Q$ such that $ys < y$. Then we have $\varphi(zs) = \varphi(z)s$ for all $z \in [e, y]$. From the recursion formulæ in [11], (2.2.c) and (2.3.g), we have three cases :

(a) $xs > x$. Then $P_{x,y} = P_{xs,y}$.

(b) $xs < x$, $x \not\leq ys$. Then $P_{x,y} = P_{xs,ys}$.

(c) $xs < x \leq ys$. Then we have the recursion formula :

$$P_{x,y} = P_{xs,ys} + qP_{x,ys} - \sum_{\substack{x \leq z < ys \\ zs < z}} q^{\frac{1}{2}(l(y)-l(z))} \mu(z, ys) P_{x,z}$$

where $\mu(x, y)$ is the coefficient of degree $\frac{1}{2}(l(y) - l(x) - 1)$ of $P_{x,y}$ if $l(y) - l(x)$ is odd, zero otherwise.

(analogous formulæ hold for the left action of s.)

It is clear that in all three cases, the analogous case holds for the computation of $P_{\varphi(x),\varphi(y)}$, and that the formulæ are transformed by φ into the corresponding formulæ in W'.

3.9. The results in section 3 enable us to reduce the proof of Theorem 1.2 to the case where $W = W'$. Indeed, in the notation of 1.2, φ restricts to a bijection from $\mathrm{supp}(w)$ onto $\mathrm{supp}(w')$, which transforms the Coxeter matrix $M^{B(w)}$ into $M^{B(w')}$. Hence the Coxeter systems $(W^{B(w)}, \mathrm{supp}(w))$ and $(W^{B(w')}, \mathrm{supp}(w'))$ are canonically isomorphic. Composing with the canonical realizations, we see that $[e, w]$ and $[e, w']$ are isomorphic to Schubert closures in the same Coxeter group $W^{B(w)}$, via isomorphisms which preserve Kazhdan-Lusztig polynomials.

Also, it is enough to consider the case were the two elements are full. In particular, we may assume that all the coefficients in the Coxeter matrix of W are finite.

4 Embeddings in the canonical group

4.1. From now on, we denote B the bud of our Coxeter group W (see 2.7). The problem we wish to consider is the classification of poset isomorphisms $\varphi : [e, w] \to [e, w']$, where w and w' are full elements of W. If we fix w, this amounts to the consideration of the set of embeddings $[e, w] \to W$, which we shall denote $\mathrm{Hom}([e, w], W)$. As we have remarked in 3.9, we may and shall from here on assume that all the entries in the Coxeter matrix M of W are finite.

The crucial remark is the following : since for each full element w we have $B(w) = B$, each $\varphi \in \mathrm{Hom}([e,w],W)$ induces an automorphism of the poset B; and from the Corollary in 2.8, the map $\mathrm{Hom}([e,w],W) \to \mathrm{Aut}(B)$ thus defined is *injective.* We reverse our viewpoint somewhat, and fix $\varphi \in \mathrm{Aut}(B)$. Then the set of $w \in W$ for which there exists an isomorphism $[e,w] \to [e,w']$ inducing φ on B is exactly the set of full elements in the domain $\mathrm{dom}(\varphi)$ as defined in 2.9. The main question which will occupy us in the sequel, and which is in fact equivalent to the statement of Theorem 1.2, is the following : *is it true that in all cases φ preserves Kazhdan-Lusztig polynomials on* $\mathrm{dom}(\varphi)$ *?*

4.2. First we wish to make some remarks about the structure of the group $\mathrm{Aut}(B)$. Consider the Coxeter graph Γ of W as an edge-labeled graph. We shall denote E the set of edges in Γ. Then it is clear that the datum of the poset B is equivalent to the datum of the graph Γ. Hence, each $\varphi \in \mathrm{Aut}(B)$ induces an automorphism of Γ. Conversely, from the Coxeter presentation it is clear that each automorphism of Γ extends uniquely to an automorphism of the Coxeter system (W,S). The extension induces an automorphism of the Bruhat ordering, and in particular an automorphism of B. So the group $\mathrm{Aut}(B)$ decomposes as a semidirect product $\mathrm{Aut}(W,S) \ltimes \mathrm{Aut}_S(B)$, where $\mathrm{Aut}(W,S) \simeq \mathrm{Aut}(\Gamma)$ is the automorphism group of the Coxeter system (W,S), and $\mathrm{Aut}_S(B)$ is the group of automorphisms of B inducing the identity on S.

Clearly each element of $\mathrm{Aut}(W,S)$ induces an embedding $[e,w] \to W$ which preserves Kazhdan-Lusztig polynomials. So our main object of study will be the set $\mathrm{Hom}_S([e,w],W)$ of elements in $\mathrm{Hom}([e,w],W)$ inducing the identity on S, identified as above with a subset of $\mathrm{Aut}_S(B)$.

4.3 Lemma. — *The group* $\mathrm{Aut}_S(B)$ *is an elementary abelian* 2*-group.*

Proof. — The set of elements of length ≥ 2 in B is the disjoint union of the sets $L_{\{s,t\}} \setminus \{e,s,t\}$, where $\{s,t\}$ runs through the two-element subsets of S. From the description of the Bruhat order on dihedral groups, it is clear that to define an automorphism of B fixing S we may arbitrarily permute the two elements in each $L_{s,t}$ of length p, $2 \leq p < m_{s,t}$. Each such automorphism is obviously of order two, hence $\mathrm{Aut}_S(B)$ is an elementary abelian 2-group. Its order is 2^r, where $r = \sum_{\{s,t\}\in E}(m_{s,t} - 2)$.

4.4. Let us next reduce to the case where W is irreducible. Indeed, if $S = \coprod_{j=1}^m S_j$ is the decomposition of S into connected components for the graph Γ, we have the corresponding decomposition $W = \prod_{j=1}^m W_j$. Set $B_j = B(W_j)$. Clearly each $\varphi \in \mathrm{Aut}_S(B)$ stabilizes the various B_j, and since for each $x = x_1 \dots x_m$ in W with $x_j \in W_j$ we have $[e,x] = [e,x_1] \times \dots \times [e,x_m]$, we see easily that $x \in \mathrm{dom}(\varphi)$ if and only if each x_j is in $\mathrm{dom}(\varphi_j)$, where φ_j is the restriction of φ to B_j. Hence $\mathrm{dom}(\varphi) = \prod_{j=1}^m \mathrm{dom}(\varphi_j)$.

It is also true that if $x, y \in W$, $x = x_1 \dots x_m$, $y = y_1 \dots y_m$, with $x_j, y_j \in W_j$, then $P_{x,y} = \prod_{j=1}^m P_{x_j,y_j}$ (this is easily checked using the recursion formulæ recalled in the proof of Proposition 3.8). So our problem readily reduces to the case where W is irreducible, which we shall assume from now on. We may also assume, to avoid trivialities, that the rank of W is at least 3.

4.5. One very special element of $\mathrm{Aut}_S(B)$ is inversion, $x \to x^{-1}$; we shall denote this special element by I. Clearly, $\mathrm{dom}(I) = W$; from [9], and also [14], it follows in fact that I is the *only* non-identity element of $\mathrm{Aut}_S(B)$ with this property. As is well-known, I preserves Kazhdan-Lusztig polynomials (again this is easily checked by induction on the lengths of x and y, using the left and right versions of the recursion formulæ for the $P_{x,y}$.)

Lemma. — *Let* $\varphi, \varphi' \in \mathrm{Aut}_S(B)$, *and assume that* $\mathrm{dom}(\varphi) \subset \mathrm{dom}(\varphi')$. *Then* φ' *stabilizes* $\mathrm{dom}(\varphi)$, *and* φ *and* φ' *commute on* $\mathrm{dom}(\varphi)$. *In particular,* $\mathrm{dom}(\varphi)$ *is* φ*-stable.*

Proof. — The fact that $\mathrm{dom}(\varphi)$ is φ-stable, and that φ defines an involution on its domain, follows easily from the Remark in 2.9. Let $x \in \mathrm{dom}(\varphi)$. We wish to prove by induction on $l(x)$ that $\varphi'(x) \in \mathrm{dom}(\varphi)$, and $\varphi\varphi'(x) = \varphi'\varphi(x)$ (the right-hand side is well-defined since $\varphi(x) \in \mathrm{dom}(\varphi)$). We may assume x non-dihedral, and in particular $l(x) > 0$. From the inductive hypothesis, we have $\mathrm{coat}(\varphi'(x)) = \varphi'(\mathrm{coat}(x)) \subset \mathrm{dom}(\varphi)$, and $\varphi\varphi'(\mathrm{coat}(x)) = \varphi'\varphi(\mathrm{coat}(x))$ is the coatom set of $\varphi'\varphi(x)$. Again from the Remark in 2.9, this means that $\varphi'(x) \in \mathrm{dom}(\varphi)$ and $\varphi\varphi'(x) = \varphi'\varphi(x)$.

Corollary. — *Each* $\mathrm{dom}(\varphi)$ *is stable under inversion, and* $\varphi(x^{-1}) = \varphi(x)^{-1}$.

4.6. Our approach will rely on the decomposition of an arbitrary $\varphi \in \mathrm{Aut}_S(B)$ as a product of simpler ones, which afford a direct description. In this context, the following lemma will be useful :

Lemma. — *Let* $\varphi_1, \varphi_2 \in \mathrm{Aut}_S(B)$, $\varphi = \varphi_1\varphi_2$. *Let* $Q = \mathrm{dom}(\varphi_1) \cap \mathrm{dom}(\varphi_2)$, $Q_1 = \varphi_1^{-1}(Q)$, $Q_2 = \varphi_2^{-1}(Q)$. *Then* $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\varphi_1) = Q_1$, $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\varphi_2) = Q_2$, *and* $\varphi = \varphi_2\varphi_1$ *on* Q_1, $\varphi = \varphi_1\varphi_2$ *on* Q_2.

Proof. — Clearly, $\varphi_2\varphi_1$ is defined on Q_1, and coincides with φ on B, so $Q_1 \subset \mathrm{dom}(\varphi)$, and $\varphi = \varphi_2\varphi_1$ on Q_1. We show by induction on $l(x)$ that for any $x \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\varphi_1)$, we have $x \in Q_1$. We may assume x non-dihedral, and in particular $l(x) > 0$. Then from the inductive hypothesis, $\mathrm{coat}(x) \subset Q_1$, and

$$\varphi_2(\mathrm{coat}(\varphi_1(x))) = \varphi_2\varphi_1(\mathrm{coat}(x)) = \varphi(\mathrm{coat}(x)) = \mathrm{coat}(\varphi(x))$$

so $\varphi_1(x) \in \mathrm{dom}(\varphi_2)$ from the Remark in 2.9, and since $\varphi_1(x) \in \mathrm{dom}(\varphi_1)$ in any case, $x \in Q_1$. The case of $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\varphi_2)$ is similar.

4.7. In preparation for 4.8, we recall some results on double cosets in W. For each pair J, J' of subsets of S, we say that $x \in W$ is (J, J')-*reduced* if $sx > x$ for all $s \in J$, and $xt > x$ for all $t \in J'$. We denote $W^{(J,J')}$ the set of all (J, J')-reduced elements in W. We denote $W^J = W^{(J,\emptyset)}$. The following proposition may be found in [9] (see also [1], chapitre 4, §1, exercice 3):

Proposition. — (van den Hombergh) *Let J, J' be arbitrary subsets of S.*

(a) *Each double coset $W_J x W_{J'}$ in W contains a unique element y of minimal length, which is also the unique element in $W_J x W_{J'}$ which is (J, J')-reduced.*

(b) *Let y be as in* (a). *Then the stabilizer of $yW_{J'}$ for the action of W_J on the left is W_K, where K is the set of $s \in J$ such that $y^{-1}sy \in J'$. Similarly, the stabilizer of $W_J y$ for the action of $W_{J'}$ on the right is W_L, where L is the set of $s \in J'$ such that $ysy^{-1} \in J$, and $s \to ysy^{-1}$ is a bijection from L to K preserving the Coxeter diagram.*

(c) *Let the notation be as in* (b). *Then each $x \in W_J yW_{J'}$ has a unique decomposition $x = x'yx''$, with $x' \in W_J$ and $x'' \in W^L_{J'}$. In this decomposition, $l(x) = l(x') + l(y) + l(x'')$.*

4.8. Let us now consider an important class of examples. Fix a partition $S = S' \coprod S''$ of S into two disjoint non-empty sets, and let $W' = W_{S'}$, $W'' = W_{S''}$. Each $y \in W''$ is the distinguished double coset representative in $W'yW'$ defined in the Proposition in 4.7.

Proposition. — *Let B'' be the bud of W'', and $\varphi_0 \in \mathrm{Aut}_{S''}(B'')$. Extend φ_0 to an automorphism φ of B by setting $\varphi(x) = x$ if $x \notin B''$, and set $Q = W'\mathrm{dom}(\varphi_0)W'$. Then Q is a decreasing subset of W contained in $\mathrm{dom}(\varphi)$.*

Proof. — The fact that Q is a decreasing subset follows from the decomposition in *(c)* of the Proposition in 4.7. Fix $y \in \mathrm{dom}(\varphi_0)$, and let L be as in *(b)* of the said Proposition. We remark that in our situation, the set L is simply the set of $s \in S'$ such that $ys = sy$. Indeed, if $ys = ty$ with $t \in S'$, we see that $\mathrm{supp}(ys) \cap S'$ is equal to $\{s\}$ from the first decomposition, and to $\{t\}$ from the second, so we must have $s = t$. Another remark is that $s \in S'$ commutes with $y \in W''$ if and only if s commutes with all $t \in \mathrm{supp}(y)$. Indeed, if $x = ys = sy$, taking a reduced expression $y = s_1 \dots s_p$ we see that $s_1 \dots s_p s$ and $s s_1 \dots s_p$ are two reduced expressions for x; in the notation of 2.5, for each $t \in \mathrm{supp}(y)$ we get $x_{\{s,t\}} = ts$ from the first one, and $x_{\{s,t\}} = st$ from the second one. Hence $ts = st$. The converse is clear.

For each $x = x'yx'' \in Q$, with $x' \in W'$, $x'' \in W'^L$, we set $\varphi(x) = x'\varphi_0(y)x''$. To see that φ is indeed a poset automorphism of Q, it is enough to show that for each $x > e$ in Q we have $\varphi(\mathrm{coat}(x)) = \mathrm{coat}(\varphi(x))$. Writing $x = x'yx''$ as above, we see that the coatoms of x are of three types : *(a)* elements of the form $z'yx''$, $z' \in \mathrm{coat}(x')$; *(b)* elements of the form $x'ux''$, $u \in \mathrm{coat}(y)$, for which $x'ux''$ remains reduced (in the sense that $l(x'ux'') = l(x') + l(u) + l(x''))$; *(c)* elements of the form $x'yz''$, $z'' \in \mathrm{coat}(x'')$, for which $x'yz''$ remains reduced. From our opening remarks, we see that the reduction in case *(b)* may fail only when there is a (necessarily unique) $t \in \mathrm{supp}(y)$ such that $t \notin \mathrm{supp}(u)$. Then there may be a larger commuting set $S' \supset L' \supset L$ corresponding to u, and we must decompose x'' in the form $x'' = x''_1 x''_2$, $x''_1 \in W^L_{L'}$, $x''_2 \in W'^{L'}$, and write $x'ux'' = x'x''_1 u x''_2$; this will be reduced if and only if $l(x'x''_1) = l(x') + l(x''_1)$. Similarly in case *(c)*, we must write $z'' = z''_1 z''_2$, $z''_1 \in W_L$, $z''_2 \in W'^L$, and the condition will be $l(x'z''_1) = l(x') + l(z''_1)$. Clearly, coatoms of type *(a)* are taken by φ to the corresponding coatoms of $\varphi(x)$. For the coatoms of type *(b)*, we remark that φ_0 preserves coatom sets and supports; so from the above analysis we see that $x'ux''$ is reduced if and only if $x'\varphi_0(u)x''$ is reduced. The equality $\varphi(x'ux'') = x'\varphi_0(u)x''$ is clear if $\mathrm{supp}(u) = \mathrm{supp}(y)$, and in the case where $\mathrm{supp}(u)$ decreases, we have :

$$\varphi(x'ux'') = \varphi(x'x''_1 u x''_2) = x'x''_1\varphi_0(u)x''_2 = x'\varphi_0(u)x''_1x''_2 = x'\varphi_0(u)x''$$

so φ induces a bijection on this type of coatoms as well. Finally, for coatoms of type *(c)*, a similar argument shows that again $x'\varphi_0(y)z''$ is a coatom of $\varphi(x)$, and $\varphi(x'yz'') = x'\varphi_0(y)z''$.

4.9 Proposition. — *Let the notation be as in proposition 4.8. Then if φ_0 preserves Kazhdan-Lusztig polynomials on $\mathrm{dom}(\varphi_0)$, φ preserves Kazhdan-Lusztig polynomials on Q.*

Proof. — In fact, it is clear that φ is compatible with the actions of S' on the left and on the right. Let us prove that for each $x \leq y \in Q$ we have

$P_{\varphi(x),\varphi(y)} = P_{x,y}$, reasoning as usual by induction on $l(y) + (l(y) - l(x))$. If there is an $s \in S'$ such that $ys < y$ or $sy < y$, we can argue as in 3.8, using the fact that φ commutes with the appropriate action of s, and conclude from the inductive hypothesis. Otherwise, $y \in \mathrm{dom}(\varphi_0)$, hence $x \in \mathrm{dom}(\varphi_0)$, and there is nothing to prove.

5 The second layer : tree case

5.1. In this section, we will consider the case where the Coxeter graph Γ is a *tree*. We set $B = B(W)$, and maintain our assumption that the entries in the Coxeter matrix of W are finite. We shall concentrate on the action of $\varphi \in \mathrm{Aut}_S(B)$ on the elements of length two in B (this is what we call the second layer of the automorphism.) Since leaves of length two in B, corresponding to pairs of commuting generators, contain just one element in length two, this must be fixed by φ, and φ induces the identity on such leaves. The other leaves correspond precisely to the edges in the Coxeter graph; so the action of φ on elements of length 2 may be described by the choice of a sign $\sigma(\gamma)$ for each element γ in the edge set E of Γ : if $\gamma = \{s,t\}$, we set $\sigma(\gamma) = +$ if $\varphi(st) = st$, $\sigma(\gamma) = -$ if $\varphi(st) = ts$. Henceforth we shall use the edge set E to index the leaves of B of length greater than two.

For each choice of σ, let φ_σ be the automorphism of B whose restriction to L_γ, $\gamma \in E$, is identity if $\sigma(\gamma) = +$, inversion if $\sigma(\gamma) = -$. Then we shall show that if $\varphi \in \mathrm{Aut}_S(B)$ is arbitrary, and if σ is the labelling of E defined by φ, we have $\mathrm{dom}(\varphi) \subset \mathrm{dom}(\varphi_\sigma)$, so that composing with φ_σ we can reduce to the case where φ induces the identity on elements of length two.

5.2 Lemma. — *For each labelling σ on E, we have $\mathrm{dom}(\varphi_\sigma) = \mathrm{dom}(\varphi_{-\sigma})$, and $\varphi_{-\sigma} = I \circ \varphi_\sigma$, where as usual I denotes inversion on W.*

Proof. — Clearly $I \circ \varphi_\sigma$ is defined on $\mathrm{dom}(\varphi_\sigma)$ and coincides with $\varphi_{-\sigma}$ on B. Hence $\mathrm{dom}(\varphi_\sigma) \subset \mathrm{dom}(\varphi_{-\sigma})$, and $\varphi_{-\sigma} = I \circ \varphi_\sigma$ on $\mathrm{dom}(\varphi_\sigma)$. But interchanging the roles of σ and $-\sigma$ we get the reverse inclusion, whence our result.

5.3. A crucial role in our approach is played by some remarkable elements in length four, which had already appeared both in [9] and [14].

Lemma. — *For each connected three-element subset $J \subset S$ there exists a unique $r_J \in W_J$ such that $l(r_J) = 4$ and $B(r_J)$ defines a connected Coxeter graph on J (necessarily of type A_3). If $J = \{a,b,c\}$, with $ab \neq ba$, $bc \neq cb$,*

$ac = ca$, the left and right descent sets of r_J are equal to $\{b\}$, so that $r_J = bacb = bcab$ are the only reduced expressions for r_J.

Proof. — A connected three-element subset of a tree is a string, so it contains a unique middle generator b, and two extremal elements a and c, which commute with each other but do not commute with b. If r_J induces on J a connected Coxeter graph with middle generator b, we must have $\mathrm{supp}(r_J) = J$, and $[e, r_J]$ must contain at least one of aba, bab and one of bcb, cbc. But since r_J is of length four, in any reduced expression for r_J only one generator can appear twice. If the repeated generator is extremal in J, say equal to a, $[e, r_J]$ can contain only one of bc, cb, contradicting the fact that it contains one of bcb, cbc. So the repeated generator is b, and for the same reason as above, both a and c have to lie in between the two occurrences of b. This shows that necessarily $r_J = bacb = bcab$, and these are the two only possible reduced expressions for r_J.

5.4. The following Proposition shows how the elements r_J impose rigidity conditions on the intervals containing them :

Proposition. — *Let $\varphi \in \mathrm{Aut}_S(B)$, let σ be the labelling on E induced by φ, and let $J = \{a, b, c\} \subset S$ be a connected three-element set, with middle generator b. Let r_J be as in 5.3, and assume that $r_J \in \mathrm{dom}(\varphi)$. Then $\sigma(\{a, b\}) = \sigma(\{b, c\})$.*

Proof. — From the uniqueness property of r_J, it is clear that if $r_J \in \mathrm{dom}(\varphi)$, $\varphi(r_J) = r_J$. The interval $[e, r_J]$ contains four elements in length three : two Coxeter elements bac and acb in W_J, and two dihedral elements bab and bcb. For reasons of support, say, both dihedral elements are fixed by φ, whereas the two Coxeter elements are either fixed or interchanged. Now if $x = bac$, $x|_{\{a,b\}} = ba$, $x|_{\{b,c\}} = bc$; if $x = acb$, $x|_{\{a,b\}} = ab$, $x|_{\{b,c\}} = cb$. So the pairs ab, ba and bc, cb are either both fixed or both interchanged by φ.

5.5. For any $\gamma = \{s, t\} \in E$, we set $\mathrm{supp}(\gamma) = \{s, t\}$, considered as a two-element subset of S. For any subset E' of E, we set $\mathrm{supp}(E') = \bigcup_{\gamma \in E'} \mathrm{supp}(\gamma)$. We say that E' is *connected*, if the graph $\Gamma_{E'}$ defined on $\mathrm{supp}(E')$ by the edges in E' is connected. For general E', we define the connected components of E' to be the edge sets of the connected components of $\Gamma_{E'}$; clearly these components define a partition of E'.

For any labelling σ on E, let $E_\pm := \{\gamma \in E \mid \sigma(\gamma) = \pm\}$. We shall determine $\mathrm{dom}(\varphi_\sigma)$ in the special case where $E_\pm$ are both connected. We shall give a description in terms of *obstructions* : if F is any subset of W, the set Q_F of all x in W such that $[e, x] \cap F = \emptyset$ is clearly a decreasing subset;

we say that F is a set of obstructions defining the decreasing subset Q_F. Any decreasing subset Q is defined by a unique smallest set of obstructions : take for F the set of minimal elements in the complement of Q.

Let us assume that $E_\pm$ are both non-empty. Then a moment's thought will show that our connectedness assumption is equivalent to the assertion that $\mathrm{supp}(E_+)$ and $\mathrm{supp}(E_-)$ intersect in a single element $s \in S$, which is not extremal in Γ. Let us denote $\mathrm{star}(s)$ the set of elements $t \in S$ such that $\{s,t\} \in E$ (in other words, $\mathrm{star}(s)$ is the set of $t \in S$ which do not commute with s.) Then we have a partition $\mathrm{star}(s) = \mathrm{star}_+(s) \coprod \mathrm{star}_-(s)$ by intersection with $\mathrm{supp}(E_+)$ and $\mathrm{supp}(E_-)$, and in fact $\mathrm{supp}(E_\pm) \setminus \{s\}$ is the union of the connected components of the elements of $\mathrm{star}_\pm(s)$ in $S \setminus \{s\}$.

5.6 Proposition. — *In the above situation, set* $W_+ = W_{\mathrm{supp}(E_+)}$, $W'_+ = W_{\mathrm{supp}(E_+)\setminus\{s\}}$, *and define* W_-, W'_- *analogously. Then*

$$\mathrm{dom}(\varphi_\sigma) = W'_+W_-W'_+ \cup W'_-W_+W'_-$$

and $\mathrm{dom}(\varphi_\sigma)$ *is defined by the obstructions* r_J, *where* J *runs through the subsets* $\{a,s,c\}$, $a \in \mathrm{star}_+(s)$, $c \in \mathrm{star}_-(s)$. *Also,* φ_σ *preserves Kazhdan-Lusztig polynomials on* $\mathrm{dom}(\varphi_\sigma)$.

Proof. — Set $W' = W_{S\setminus\{s\}}$. Then it is clear that W' is equal to the direct product $W'_+ \times W'_-$. Set $Q' = W'_-W_+W'_- \cup W'_+W_-W'_+$, and let Q be the decreasing subset defined by the obstructions r_J. Let $x \in Q$, and let us show that $x \in Q'$. If $x \in W'W_{\{s\}}W'$, which is equivalent to the property that x has a reduced expression with at most one occurrence of s, we have $x \in W'_+W_-W_+ \cap W'_-W_+W'_-$.

Otherwise, let y be the unique element of minimal length in $W'xW'$ (see 4.7). Then $l(y) \geq 3$, and any reduced expression of y starts and ends with s. Set $S_+ = \mathrm{supp}(E_+)$, $S_- = \mathrm{supp}(E_-)$, $S'_+ = S_+ \setminus \{s\}$, $S'_- = S_- \setminus \{s\}$. Choose a reduced expression $y = ss_2 \dots s_{p-1}s$, and let us show that if $s_2 \in S'_+$, then $\mathrm{supp}(y) \subset S_+$. Indeed, from the fact that y is (S',S')-reduced, it follows that $s_2 \in \mathrm{star}_+(s)$. But any $t \in S'_-$, $t \notin \mathrm{star}_-(s)$ commutes with S_+. So if there is an occurrence of an element in S'_- among the s_j, the first such occurrence lies in $\mathrm{star}_-(s)$; but then $[e,y]$ would contain one of the r_J, contradiction. Similarly, $\mathrm{supp}(y) \subset S_-$ if $s_2 \in S'_-$, and we conclude that $y \in W_+$ or $y \in W_-$. In the first case, we write $x \in W'_-W'_+W_+W'_+W'_- = W'_-W_+W'_-$, and we are done; in the second case, $x \in W'_+W_-W'_+$.

So we have the inclusion $Q \subset Q'$. The inclusion $\mathrm{dom}(\varphi_\sigma) \subset Q$ is the Proposition in 5.4. So to get equality throughout, it suffices to prove $Q' \subset \mathrm{dom}(\varphi_\sigma)$. But from the Proposition in 4.8, with φ_0 equal to inversion on W_-, we see that $W'_+W_-W'_+ \subset \mathrm{dom}(\varphi_\sigma)$. Symmetrically, we have $W'_-W_+W'_- \subset \mathrm{dom}(\varphi_{-\sigma})$, and the conclusion follows from 5.2. The fact that φ_σ preserves Kazhdan-Lusztig polynomials on $\mathrm{dom}(\varphi_\sigma)$ follows from Proposition 4.9 and the corresponding property for the inversion.

5.7 Remarks. — *(a)* In the notation of 5.6, the intersection of $W'_+W_-W'_+$ and $W'_-W_+W'_-$ is exactly $W'W_{\{s\}}W'$. Indeed, we have seen that $W'W_{\{s\}}W'$ is contained in the intersection; conversely, if $x \in W'_+W_-W'_+ \cap W'_-W_+W'_-$, and if y is the unique element of minimal length in the double coset $W'xW'$, we see that $y \in W_+ \cap W_-$, and in particular $\mathrm{supp}(y)$ is contained in $\mathrm{supp}(E_+) \cap \mathrm{supp}(E_-) = \{s\}$.

(b) Consider the case where W is of type A_n. Let a be one of the two extremal elements in S. Then it is well-known that if $W' = W_{S\setminus\{a\}}$, we have $W = W'W_{\{a\}}W'$; in fact one can even show that this is the only irreducible Coxeter group affording such a decomposition. In particular, in the situation of the Proposition we will have $W_+ = W'_+W_{\{s\}}W'_+$, and similarly for W_-; it follows that in this case we simply have $\mathrm{dom}(\varphi_\sigma) = W'W_{\{s\}}W'$, and there is a unique involution $w_\sigma \in W$ such that $\mathrm{dom}(\varphi_\sigma) = [e, w_\sigma]$.

5.8 Theorem. — *Let σ be any labelling of E. Let $\mathcal{J}$ be the set of connected three-element subsets in S such that the two edges in Γ_J have unequal labels. Then $\mathrm{dom}(\varphi_\sigma)$ is defined by the obstructions r_J, $J \in \mathcal{J}$, and φ_σ preserves Kazhdan-Lusztig polynomials on its domain.*

Proof. — We may of course assume that the labelling is not uniformly $+$ or uniformly $-$. Let Q be the decreasing subset in W defined by the obstructions r_J. Again, from the Proposition in 5.4 we see that $\mathrm{dom}(\varphi_\sigma) \subset Q$.

We say that a non-extremal $s \in S$ is a *point of discontinuity* for the labelling σ, if the edges from s to the various $t \in \mathrm{star}(s)$ do not all carry the same label. For each point of discontinuity s, we denote $T_+(s)$ the union of $\{s\}$ and the connected components in $S\setminus\{s\}$ of the elements of $\mathrm{star}_+(s)$, and define $T_-(s)$ analogously. Then $T_+(s) \cup T_-(s) = S$, and $T_+(s) \cap T_-(s) = \{s\}$. We say that $T_+(s)$ and $T_-(s)$ are the two subtrees of S defined by s. We claim that we can always find s such that the labelling on one of the subtrees defined by s is constant. Indeed, let s be any point of discontinuity. If one of the labellings induced on $T_+(s)$, $T_-(s)$ is constant, we are done. Otherwise, set $S_1 = T_+(s)$, and choose a point of discontinuity $s_1 \in S_1$. Since s is not a point of discontinuity in S_1, $s_1 \neq s$. Let S_2 be the subtree of S_1 defined by s_1, not containing s. Then S_2 is also one of the subtrees of S defined by s_1. If the labelling on S_2 is constant, we are done. Otherwise, we continue the process. Since the sequence (S_k) is strictly decreasing, the process must stop.

Now we prove the theorem by induction on the number of points of discontinuity of the labelling. If the labelling has just one point of discontinuity, we are in the situation of Proposition 5.6, so there is nothing to prove. Otherwise, choose a point of discontinuity s such that the labelling on one of the two subtrees $T_+(s)$, $T_-(s)$ is constant. Composing with I if necessary (see 5.2) we may assume that the labelling is constant on $T_-(s)$. Now let σ_1 be the labelling which is uniformly $-$ on the edges of $T_-(s)$, and uniformly $+$ on the edges of $T_+(s)$, and let $\varphi_1 = \varphi_{\sigma_1}$; also, let σ' be the labelling $\sigma\sigma_1$, and let $\varphi' = \varphi_{\sigma'}$. Then from the description of $\mathrm{dom}(\varphi_1)$ in Proposition 5.6, and from the inductive hypothesis, we see that $Q = \mathrm{dom}(\varphi_1) \cap \mathrm{dom}(\varphi')$. For each $J \in \mathcal{J}$, the middle generator of J is a point of discontinuity of σ, so it is either equal to s_1, or it is a point of discontinuity for σ'. Write $\mathcal{J} = \mathcal{J}_1 \coprod \mathcal{J}'$ accordingly. Now we note that $r_J \in \mathrm{dom}(\varphi')$ for every $J \in \mathcal{J}_1$, and that Q is in fact the decreasing subset of $\mathrm{dom}(\varphi')$ defined by the r_J, $J \in \mathcal{J}_1$. Since these elements are fixed by φ', we see that Q is φ'-stable subset of $\mathrm{dom}(\varphi')$. An analogous argument shows that it is also a φ_1-stable subset of $\mathrm{dom}(\varphi_1)$. Hence the composition $\varphi'\varphi_1 = \varphi_1\varphi'$ makes sense on Q, and coincides with φ_σ on B. This shows that $Q \subset \mathrm{dom}(\varphi_\sigma)$.

The fact that φ preserves Kazhdan-Lusztig polynomials now follows immediately from the corresponding property for φ' (from the inductive hypothesis), and for φ_1.

Corollary. — *Let φ be an arbitrary element of $\mathrm{Aut}_S(B)$, and let σ be the labelling of E defined by φ. Then $\mathrm{dom}(\varphi) \subset \mathrm{dom}(\varphi_\sigma)$. The r_J, $J \in \mathcal{J}$, will be called the* rigidity obstructions *for $\mathrm{dom}(\varphi)$.*

Proof. — This is clear from Theorem 5.8 and the Proposition in 5.4 : if $x \in \mathrm{dom}(\varphi)$, then $[e, x]$ cannot contain any of the r_J.

5.9 Example. — Although it is too simple to serve as a guide for the general situation, let us consider the picture in type A_n in more detail. We have already seen in Remark 5.7 *(b)* that in the case where E_+ and E_- are connected, $\mathrm{dom}(\varphi_\sigma)$ is an interval of the form $[e, w_\sigma]$, where w_σ is an involution in W. We show that this is in fact the case for *any* σ. We may assume that σ is non-constant. Choose an enumeration $S = \{s_1, \ldots, s_n\}$ such that two consecutive generators are connected by an edge in Γ; then s_1 and s_n are the two extremal generators. Let T be the set of points of discontinuity of the labelling, and set $T' = T \cup \{s_1, s_n\}$. Define on T' the graph structure $\Gamma_{T'}$ whose edges are the pairs $\{t_1, t_2\}$ such that the geodesic $[t_1, t_2]$ connecting t_1 and t_2 in Γ does not pass through any other $t \in T$; this is of course again a string. Choose any $t_0 \in T$, and enumerate $T' = \{t_0, t_1, \ldots, t_q\}$ in such a way that the elements of the two components of $\Gamma_{T'}$ separated by t_0 are enumerated outwardly from t_0. For $1 \leq k \leq q$, let $u_k \in T$ be the first intersection of $]t_k, t_0]$ with T', and set $S_k = [t_k, u_k[$, $W_k = W_{S_k}$; set $S_0 = \{t_0\}$, $W_0 = W_{\{t_0\}}$. Then $S = \coprod_{k=0}^{q} S_k$, and we claim that $\mathrm{dom}(\varphi_\sigma) = W_q \ldots W_1 W_0 W_1 \ldots W_q$, which is easily shown to be of the form $[e, w_\sigma]$. Indeed, for each $t \in T$ there is a unique rigidity obstruction r_t; if $x \in \mathrm{dom}(\varphi)$, the obstruction r_{u_q} already ensures that $x = x_1 y x_2 \in W_q W_{S \setminus S_q} W_q$, and $[e, y]$ cannot contain any of the rigidity obstructions corresponding to the other points in T. The conclusion follows by induction on the number of points of discontinuity.

6 Higher layers

6.1. Going back to the general case of an arbitrary Coxeter group W with Coxeter matrix M, and $B = B(W)$, for each $k \geq 1$ we denote $\mathrm{Aut}_{S,k}(B)$ the subgroup of all $\varphi \in \mathrm{Aut}_S(B)$ which induce the identity on all elements of length $\leq k$ in B (so that $\mathrm{Aut}_{S,1}(B) = \mathrm{Aut}_S(B)$.) Our goal in this section 6 is to prove the following theorem :

Theorem. —*For any $k \geq 2$, any $\varphi \in \mathrm{Aut}_{S,k}(B)$ preserves Kazhdan-Lusztig polynomials on* $\mathrm{dom}(\varphi)$.

The proof of the Theorem will occupy the remainder of this section.

6.2. Clearly, the statement of the Theorem in 6.1 is equivalent to the fact that for each $w \in \mathrm{dom}(\varphi)$, the poset isomorphism $[e, w] \to [e, \varphi(w)]$ induced by φ preserves Kazhdan-Lusztig polynomials. From the results in section 3, this statement reduces easily to the case where w is full (Definition 2.7). Indeed, φ induces an isomorphism from $B(w)$ to $B(\varphi(w))$, so that as in 3.9, w and $\varphi(w)$ may be identified to full elements in the same Coxeter group W', with generating set $S' \subset S$. The two embeddings $[e, w] \to W'$, $[e, \varphi(w)] \to W'$ coincide on $[e, w] \cap [e, \varphi(w)]$, and in particular on all elements of length $\leq k$ in $[e, w]$. So the automorphism φ' of $B(W')$ arising from φ under the above identifications belongs to $\mathrm{Aut}_{S',k}(B(W'))$.

Another easy reduction is to the case where W is irreducible; this goes exactly as in 4.4. So we shall henceforth assume that W is irreducible with finite-valued Coxeter matrix M and Coxeter graph Γ. We fix a full element $w \in \mathrm{dom}(\varphi)$, and we shall prove that φ preserves Kazhdan-Lusztig polynomials on $[e, w]$.

6.3. For each $\gamma \in E$, denote L_γ the corresponding leaf in B, as in 5.1, and v_γ the maximal element in L_γ; similarly, write $m_{s,t} = m_\gamma$ if $\gamma = \{s,t\}$. For any $\gamma \in E$, let $k_\gamma = k_\gamma(\varphi)$ be the smallest integer j such that φ interchanges the two elements of length j in L_γ, or m_γ if there is no such j. Write $|\varphi| = \sum_{\gamma \in E}(m_\gamma - k_\gamma)$. We shall argue by induction on $|\varphi|$.

If $|\varphi| = 0$, then φ is the identity, and there is nothing to prove. So we may assume that $|\varphi| > 0$, and choose $\gamma \in E$ such that k_γ is minimal with the property that $k_\gamma < m_\gamma$. In particular, $k = k_\gamma - 1$ is the largest value of k such that $\varphi \in \mathrm{Aut}_{S,k}(B)$.

6.4. In 5.3, we have defined an element r_J of length four for each connected three-element subset J of Γ_S, in the case where Γ_S is a tree. It is clear that the definition goes through in general, provided only that the induced graph Γ_J is a string. The only other possibility is that Γ_J is a three-cycle. Then there will be six elements r of length four in W_J such that the Coxeter graph defined by $B(r)$ is connected : if $J = \{a,b,c\}$, they are $abca$, $acba$, and the four elements deduced from them by circular permutation (the proof is the same as for the Proposition in 5.3.) Since the Coxeter graph defined by $B(r)$ is of type A_3, each of these elements will have a well-defined middle generator (a in the case of $abca$, $acba$). This defines a partition of the six elements into three pairs of mutually inverse ones.

6.5 Lemma. — *Let $\gamma = \{a,b\}$ be the edge chosen in 6.3, and assume that $k_\gamma = 3$. Let $c \in S$ be such that $J = \{a,b,c\}$ is connected. Then $[e,w]$ does not contain any of the elements r associated to J in 6.4, with middle generator a or b.*

Proof. — Assume to the contrary. Then r has exactly one coatom z with support in $\{a,b\}$, *viz.* aba if the middle generator is a, bab otherwise. This is the same for r and r^{-1}. But clearly $B(r)$ and $B(\varphi(r))$ define the same Coxeter matrix over J, hence $\varphi(r)$ is again one of the six special elements, with the same middle generator as r. Therefore $\varphi(r) = r$ or $\varphi(r) = r^{-1}$, and in both cases the coatom z is fixed, contrary to the hypothesis $k_\gamma = 3$.

6.6 Lemma. — *Let x, x' in W be non-dihedral, such that there exists a subset Z of W, and $v \neq v' \in W$, not belonging to Z, with* $\mathrm{coat}(x) = Z \cup \{v\}$, $\mathrm{coat}(x') = Z \cup \{v'\}$ *(we shall say that the coatom sets of x and x' differ by a single element.) Then* $\mathrm{supp}(x) = \mathrm{supp}(x')$ *is a three-element set $J = \{a,b,c\}$, v and v' are dihedral with same support, say $\{a,b\}$, and we have either $x = vc$, $x' = v'c$ or $x = cv$, $x' = cv'$.*

Proof. — *(a)* Clearly the two elements x, x' have the same length, which is at least three. Let us first prove that v and v' are dihedral with same support. From 2.8, this will follow if we prove that $\mathrm{coat}(v) = \mathrm{coat}(v')$. Let $u \in \mathrm{coat}(v)$. Then there are exactly two elements strictly between u and x : one of them is v, and the other one, which we denote z, belongs to Z. But then $u < z < x'$, and again there are two elements between u and x'. One of them is z, and since z is the only element in Z lying over u, the other one must be v'. Hence $u < v'$, and we have proved that $\mathrm{coat}(v) \subset \mathrm{coat}(v')$. The other inclusion is similar.

Set $\mathrm{supp}(v) = \mathrm{supp}(v') = \{a, b\}$. Since v is obtained from a reduced expression for x by erasing a single generator, and x is non-dihedral, we see that $\mathrm{supp}(x)$ is a three-element set $J = \{a, b, c\}$; and similarly, $\mathrm{supp}(x') = \{a, b, c'\}$. Now for any $z \in Z$, we have $c' \in \mathrm{supp}(z) \subset \{a, b, c\}$, so $c = c'$.

(b) Let $s \in J$ be such that $xs < x$. If $xs = v$, we have $s = c$, and x has the required form. But then $\mathrm{coat}(x) = \{v, uc, u'c\}$ is a three-element set, where u, u' are the coatoms of v, and $zc < z$ for all $z \in Z$. From the Corollary in 2.4, this implies that $x'c < x'$; hence $x' = v'c$, and we are done.

Assume now that $xs = z \in Z$. Then we have $vs < v$, hence $s \in \{a, b\}$, and since v and v' are of the same length in a dihedral group, this implies $v's > v'$. But then $x's > x'$, because we already have $zs > s$, so there are two coatoms of x' moving up under s. Set $y = x's$. Then $z < x' < y$, and the interval $[z, y]$ is stable under right multiplication by s, so $x = zs < y$.

Recall from the proof of Proposition 3.3 that in the above situation the coatoms of y are x', and the $z's$, where z' runs through the set of coatoms of x' such that $z's > z'$. But since all elements of Z other than z are taken down by s, there are exactly two such coatoms z', *viz.* v' and z. Hence $\mathrm{coat}(y) = \{x', v's, x\}$ is a three-element set. Clearly, y has a reduced expression with exactly one occurrence of the generator c. A moment's thought will show that if on each side of c there is an occurrence of a generator with which it does not commute, then y must have at least four coatoms. Hence the only possibility is that y is one of $v'sc$, $cv's$. Since the only two coatoms of $v's$ are v and v', the three coatoms of y are vc, $v'c$ and $v's$ in the first case, cv, cv' and $v's$ in the second. This proves the Lemma.

6.7 Lemma. — *Let the notation be as in Lemma 6.5, with $k_\gamma \geq 3$ arbitrary. Let $x \in W_J \cap [e, w]$ be non-dihedral of length $k_\gamma + 1$, with a dihedral coatom $z \in L_\gamma$. Then x is one of zc, cz.*

Proof. — We recall that $k_\gamma < m_\gamma$, so that the two elements of length k_γ in L_γ have unique reduced expressions; therefore any reduced expression for x is obtained by inserting one occurrence of c in the unique reduced expression for z. We distinguish two cases :

(a) $k_\gamma = 3$. Then if x is not one of the required forms, it is one of $abca$, $acba$, $bacb$, $bcab$. But this is impossible from Lemma 6.5.

(b) $k_\gamma > 3$. Then z is the only dihedral coatom of x. An easy induction will show that all elements of length $< k_\gamma$ in W lie in $\mathrm{dom}(\varphi)$ and are fixed by φ. From the Proposition in 2.8 it follows that all coatoms of x other than z are fixed by φ. Hence $\mathrm{coat}(x)$ and $\mathrm{coat}(\varphi(x))$ differ by a single element. But then the conclusion follows from Lemma 6.6

6.8 Proposition. — *Let w, γ be as in 6.2, 6.3. Set* $\mathrm{supp}(\gamma) = \{a, b\}$, *and write* $W_\gamma = W_{\{a,b\}}$, $W' = W_{S\setminus\{a,b\}}$. *Then* $w \in W'W_\gamma W'$.

Proof. — Let y be the unique shortest representative in the double coset $W'wW'$ (see 4.7); then, since w is full, the Proposition is equivalent to the equality $y = v_\gamma$.

We shall assume to the contrary, and reach a contradiction. In any case we have $v_\gamma \leq y$. Fix a reduced expression $y = s_1 \dots s_p$. Then if we look at the sequence of occurrences of a, b in the expression, we see that we must have at least $m_\gamma - 1$ alternations (switches from a to b or vice-versa). And of course, s_1 and s_p belong to $\{a, b\}$. Interchanging a and b if necessary, we may assume that $s_1 = a$. We wish to prove that we can extract from $s_1 \dots s_p$ a word of length $m_\gamma + 1$, which is a reduced expression of v_γ with the insertion of a single $c \notin \{a, b\}$, and which is still (S', S')-reduced, where $S' = S \setminus \{a, b\}$. Although this is elementary, it has to be done with some care, since it does *not* work, for instance, if $m_\gamma = 3$.

We are looking for an element of the form $z = u'cu''$, with $u', u'' \in W_\gamma$, $l(u') + l(u'') = m_\gamma$, $u'u'' = v_\gamma$, and which is moreover (S', S')-reduced. The only way in which the (S', S')-reduction could fail, is if $cz < z$ or $zc < z$. But since v_γ is a coatom of z, and $cv_\gamma > v_\gamma$ in any case, $cz < z$ can only happen if $z = cv_\gamma$, hence $u'c = cu'$. Similarly $zc < z$ happens if and only if $cu'' = u''c$.

Let $s_j = c$ be the first occurrence in the reduced expression of a generator not in $\{a, b\}$, and suppose first that $j = 2$. Then c does not commute with a. Extract a subword of length $m_\gamma + 1$ from $s_1 \dots s_p$ by erasing all generators after s_2 until the next occurrence of b, then all generators until the next a, and so on. This is possible because there are at least $m_\gamma - 1$ alternations of a, b. The resulting word is $acbab\dots$, satisfying our requirement. If $s_{p-1} \notin \{a, b\}$, we can argue similarly, starting from the end of the expression.

So we may assume that s_1, s_2, s_{p-1}, s_p all belong to $\{a, b\}$, and in particular that $j > 2$ (note that $m_\gamma \geq 4$, so that p is at least equal to five, since we have assumed that $v_\gamma < y$.) Start with $s_1 \dots s_{j-1}c = aba \dots c$, and proceed as above, erasing generators after s_j until the next occurrence of a generator in $\{a,b\}$ not equal to s_{j-1}, and so on. We end up with $z = u'cu''$ as above, with $u' = s_1 \dots s_{j-1}$. Since $j > 2$, $u'c = cu'$ is certainly impossible, and if $l(u'') > 1$, $cu'' = u''c$ is impossible as well, and we are done; this will happen if $j < m_\gamma - 1$. If the last $m_\gamma - 1$ generators in $s_1 \dots s_p$ do not all lie in $\{a,b\}$, we similarly get a word of the required form starting from the end of the expression. So we are left with the case where our word starts and ends with at least $m_\gamma - 1 \geq 3$ generators in $\{a,b\}$. Then we start with $s_1 \dots s_{m_\gamma - 2}c$ instead, and proceed; our assumption about the end of the expression ensures that there are at least two alternations of a,b after c, so that the construction indeed reaches a word of length $m_\gamma + 1$.

Let $z = u'cu'' \leq y$ be the element constructed above. Transferring the leftmost generator in u'' to the other side of c if necessary, we may assume that u'' starts with a generator not commuting with c. Using the remarks in the proof of the Proposition in 4.8, it is then easy to see that z is written in $(\{a,b\},\{a,b\})$-canonical form as defined in *(c)* of the Proposition in 4.7. Now we get an element $x \leq z$ of length $k_\gamma + 1$, by erasing generators in u'' from the right while there are at least two left, then generators in u' from the left. We get $x = u'_1cu''_1$, still in $(\{a,b\},\{a,b\})$-canonical form, with $l(u'_1u''_1) = l(u'_1) + l(u''_1) = k_\gamma$. Now $v = u'_1u''_1$ is a dihedral coatom of x, so from Lemma 6.7 we conclude that $x = vc$ or $x = cv$. The first case is already impossible since then vc would be the $(\{a,b\},\{a,b\})$-canonical form of x; the second case is possible only if $l(u'_1) = 1$ and u'_1 commutes with c. But it is easy to see that $l(u'_1) = 1$ happens only if $u'_1 = u'$, so $u'_1c = cu'_1$ is impossible as well. This contradiction concludes the proof.

6.9 Remark. — As a corollary of the Proposition, we get severe restrictions on the groups W for which there exist non-trivial $\varphi \in \mathrm{Aut}_{S,k}(B)$ such that $\mathrm{dom}(\varphi)$ contains full elements. Indeed, from the fact that $w \in W'W_\gamma W'$, it follows that for any edge γ' which is adjacent to γ, we have $m_{\gamma'} = 3$. (Proof : interchanging a and b if necessary, we may assume that $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\gamma') = \{b\}$, $\mathrm{supp}(\gamma') = \{b,c\}$. But if we choose a reduced expression of w where all occurrences of a,b are contiguous, then all occurrences of c will be either to the left or to the right of all occurrences of b, hence $l(w|_{\{b,c\}}) \leq 3$, and $l(w|_{\{b,c\}}) = m_{\gamma'}$ because w is full.) So the edges γ such that φ can induce a non-identity automorphism on L_γ must be isolated, in an obvious sense, in the set of edges such that $m_\gamma > 3$.

6.10. Now it is easy to conclude the proof of the Theorem in 6.1. Let w, γ be as in 6.2, 6.3. Then in the notation of 6.8, $[e,w] \subset W'W_\gamma W'$. But if φ_0 is the automorphism of W_γ leaving all elements fixed except the two elements in length k_γ which are interchanged, we see from the Proposition in 4.8 that if φ_γ is the extension of φ_0 leaving all elements outside L_γ fixed, then $W'W_\gamma W' \subset \mathrm{dom}(\varphi_\gamma)$, and φ_γ preserves Kazhdan-Lusztig polynomials on $W'W_\gamma W'$ from 4.9. Let $w' = \varphi_\gamma(w)$, and let $\varphi' = \varphi\varphi_\gamma \in \mathrm{Aut}_{S,k}(B)$. Then it follows from 4.6 that $[e, w'] \subset \mathrm{dom}(\varphi')$, and $\varphi = \varphi'\varphi_\gamma$ on $[e, w]$. But clearly $|\varphi'| < |\varphi|$, so we may apply the inductive hypothesis to φ' and get the desired conclusion.

It is also easy to conclude the proof of Theorem 1.2 in case *(a)*. We have seen in 4.1 that it is enough to prove that any $\varphi \in \mathrm{Aut}(B)$ preserves Kazhdan-Lusztig polynomials on its domain. This has been reduced to the case where W is irreducible with finite-valued Coxeter matrix, and $\varphi \in \mathrm{Aut}_S(B)$. Now with the notation of the Corollary in 5.8, we have $\mathrm{dom}(\varphi) \subset \mathrm{dom}(\varphi_\sigma)$; moreover from the Theorem in 5.8, φ_σ preserves Kazhdan-Lusztig polynomials. Using the Lemma in 4.5, we may replace φ by $\varphi' = \varphi\varphi_\sigma \in \mathrm{Aut}_{S,2}(B)$, and conclude from the Theorem in 6.1.

7 Type $\tilde{\mathrm{A}}_n$

7.1. In order to deal with case *(b)* in Theorem 1.2, we first need some facts about Coxeter elements. Let W be an arbitrary Coxeter group. Recall that a *Coxeter element* in W is an element x which is the product of the generators $s \in S$ in some chosen order. This can also be expressed by saying that x is an element of length n, such that $\mathrm{supp}(x) = S$. Let us denote C the set of Coxeter elements in W.

To any $x \in C$ we may associate an orientation τ of the edges of the Coxeter graph Γ_S as follows. For each edge $\gamma = \{a, b\}$ of Γ, the element $u = x|_{\{a,b\}}$ defined in 2.5 is of length two, since a and b occur each exactly once in any reduced expression of x. If $u = ab$, we orient γ from a to b, and the other way around if $u = ba$. It is known (see for instance Shi [12] that this procedure yields a bijection between C and the set of orientations of Γ *without oriented cycles.* The inverse construction, going from an orientation ω to a Coxeter element $x = s_1 \ldots s_n$, is at follows : choose for s_1 any source for ω, set $S' = S \setminus \{s_1\}$, let Γ' be the graph on S' obtained by erasing all the edges in Γ containing s_1, let ω' be the orientation on Γ' induced by ω, and let $s_2 \ldots s_n$ be the Coxeter element in $W_{S'}$ associated to ω'. The element x thus obtained is independent of the choice of s_1.

If Γ is a tree, then of course there can never be any oriented cycles, and all orientations occur; hence the number of Coxeter elements is $2^{|E|}$, where $|E| = |S| - 1$ is the number of edges in Γ. It turns out that in this case,

$C \subset \mathrm{dom}(\varphi)$ for any $\varphi \in \mathrm{Aut}_S(B)$. Indeed, if $\varphi \in \mathrm{Aut}_{S,k}(B)$, $k \geq 2$, it is easy to see by induction on the length that any $x \in W$ such that $[e, x]$ does not contain any dihedral element of length $k+1$ is in $\mathrm{dom}(\varphi)$, and in fact fixed by φ; in particular, this holds for any Coxeter element. On the other hand, for any labelling σ of E, the description of $\mathrm{dom}(\varphi_\sigma)$ given in Theorem 5.8, in terms of obstructions r_J, again shows that C is contained in $\mathrm{dom}(\varphi_\sigma)$. But we have seen in 6.10 that any φ decomposes on its domain as a composition $\varphi = \varphi'\varphi_\sigma$ for some labelling σ and $\varphi' \in \mathrm{Aut}_{S,2}(B)$, so our claim follows.

When Γ has cycles, this will no longer be true. Indeed, let $\varphi \in \mathrm{Aut}_S(B)$, and let σ be the labelling of E defined by φ, as defined in 5.1. Then the action of φ on $x \in C$ is easy to describe. For any edge $\gamma \in E$, φ fixes the elements of length two in L_γ if $\sigma(\gamma) = +$, and interchanges them otherwise. If we define the action of a labelling on an orientation in the obvious way, using the fact that $\varphi(x|_J) = \varphi(x)|_J$ for any $J = \{a, b\} \subset S$ it follows that if ω is the orientation of Γ defined by x, $\varphi(x)$ is the Coxeter element corresponding to the orientation $\sigma\omega$ if this orientation has no oriented cycles, and is undefined otherwise.

7.2 Definition. — Let W be any Coxeter group, B its bud, and $\varphi \in \mathrm{Aut}_S(B)$. We define the *Coxeter obstructions* for φ to be the set of Coxeter elements x such that $x \notin \mathrm{dom}(\varphi)$.

7.3. In the sequel, we shall assume that W is an affine Coxeter group of type $\tilde{\mathrm{A}}_{n-1}$, $n \geq 3$ (the shift in the index ensures that n is still the rank of our Coxeter group). Since $m_{s,t} \leq 3$ for all $s, t \in S$, the elements of $\mathrm{Aut}_S(B)$ are entirely defined by their corresponding edge-labellings; $|\mathrm{Aut}_S(B)| = 2^n$. Our goal is to prove that in this case, $\mathrm{dom}(\varphi)$ can always be described in terms of the obstructions r_J appearing in Theorem 5.8, and the Coxeter obstructions defined above.

7.4. Let us first deal with the case where $n = 3$ (in fact, it is not hard to describe $\mathrm{dom}(\varphi)$ for any $\varphi \in \mathrm{Aut}_S(B)$ when W is an arbitrary Coxeter group of rank three, and deduce that φ preserves Kazhdan-Lusztig polynomials on $\mathrm{dom}(\varphi)$.) Set $S = \{a, b, c\}$. Assume that σ is not uniformly $+$ or $-$; replacing σ by $-\sigma$ if necessary, we may assume that there is a single edge labelled $-$, and renaming the vertices if necessary, that the distinguished edge is $\{a, b\}$. The Coxeter graph Γ possesses exactly two cyclic orientations; this shows that there are six Coxeter elements in W, and that for any non-constant σ, there will be exactly two Coxeter obstructions, inverse to each other. In our case, these are acb and bca. We show that these obstructions suffice to determine $\mathrm{dom}(\varphi)$. Indeed, if $[e, x]$ does not contain acb nor bca, then in any reduced expression for x there can be no occurrence of c between the two extremal occurrences of a, b, unless one of a, b does not occur at all, and we have $x = aca = cac$ or $x = bcb = cbc$. But this shows that in all cases, $x \in W'W''W'$, where $W' = W_{\{c\}}$, $W'' = W_{\{a,b\}}$. Since the reverse inclusion $W'W''W' \subset \mathrm{dom}(\varphi)$ is clear from the Proposition in 4.8, this proves our claim. Note that in this case the rigidity obstructions $abca$, $acba$, $bacb$, $bcab$ are already excluded from the Coxeter obstructions.

7.5. Assume from now on that $n \geq 4$. Then any connected three-element subset of S is a string, so we may define the element r_J as in the Lemma in 5.3. In fact, since Γ has no nodes, J is entirely determined by its middle generator; so for each $a \in S$ we shall write r_a for the rigidity obstruction with middle generator a.

For each labelling σ on E, we define the subsets $E_\pm$ of E as in 5.5. We define the *components* of the labelling to be the connected components of E_+ and E_-, as defined in 5.5; so the components define a partition of E. Assume that σ is non-constant; then the number of components is always even. Again it is true that there are exactly two mutually inverse Coxeter obstructions for φ_σ; they correspond to $\sigma\omega_+$ and $\sigma\omega_-$, where ω_+ is the counterclockwise cyclic orientation of Γ, ω_- the clockwise one; denote the corresponding Coxeter elements by u_+, u_-. Note that $\sigma\omega_+$ and $\sigma\omega_-$ are "constant" on each component of the labelling, and switch directions when we pass from one component to the next.

We first treat the case where there are just two components (*i.e.*, E_+ and E_- are connected.) Let $\mathrm{supp}(E_-)$ be defined as in 5.5. The only case where we can have $\mathrm{supp}(E_-) = S$ is the case where E_+ is a single edge; replacing σ by $-\sigma$ if necessary (which amounts to composing with inversion), we may assume that this is not the case. We write $S_+ = \mathrm{supp}(E_+)$, $S_- = \mathrm{supp}(E_-)$, $S'_+ = S \setminus S_-$, $S'_- = S \setminus S_+$, and define $W_+ = W_{S_+}$, $W_- = W_{S_-}$, $W'_+ = W_{S'_+}$, $W'_- = W_{S'_-}$.

7.6 Proposition. — *In the above situation, let a and b be the two extremal points in S_-; let r_a, r_b be the corresponding rigidity obstructions. Then $\mathrm{dom}(\varphi_\sigma)$ is the decreasing subset of W defined by r_a, r_b and the two Coxeter obstructions u_+, u_-. If $S_- = \{a, b\}$ is a two-element set, we have $\mathrm{dom}(\varphi_\sigma) = W'_+W_-W'_+$; otherwise, we have $\mathrm{dom}(\varphi_\sigma) = W'_+W_-W'_+ \cup W'_-W_+W'_-$.*

Proof. — We may assume that $a < b$ counterclockwise. Denote I, J the connected three-element subsets of S with midpoints a, b respectively. It is clear that $\mathrm{dom}(\varphi)$ is contained in the decreasing subset Q of W defined by the four obstructions r_a, r_b, u_+ and u_-. Now let $x \in Q$ be arbitrary.

Assume first that $S_- = \{a, b\}$. We wish to show that $x \in W'_+ W_- W'_+$. If $\mathrm{supp}(x)$ does not contain both a and b, this is clear : indeed, if for example $x \in W_{S\setminus\{a\}}$, which is of type A_{n-1}, we have seen in 5.7 *(b)* that $W_{S\setminus\{a\}} = W'_+ W_{\{b\}} W'_+ \subset W'_+ W_- W'_+$. So we assume henceforth that $\{a, b\} \subset \mathrm{supp}(x)$. Start with any reduced expression for x, and assume that a occurs more than once; look at the subexpression contained between the two extremal occurrences of a. Since $[e, x]$ does not contain $abca$, where c is the third element of I, in the above subexpression there is either no occurrence of b or no occurrence of c. So the corresponding element lies in either $W_{S\setminus\{b\}}$ or $W_{S\setminus\{c\}}$; both these groups are finite of type A_{n-1}, and a is an extremal generator in the corresponding Coxeter graph. But as we have recalled above, any element in such a group has a reduced expression in which a occurs at most once; so changing the reduced expression for x we may assume that a occurs exactly once, and write $x = x'ax''$, with $x', x'' \in W_{S\setminus\{a\}}$. Again we may choose expressions for x' and x'' in which b occurs at most once. Now we have two cases :

(a) b occurs on both sides of a. Then between the two occurrences of b, there is already a, so the third element, call it d, in J does not occur. But all elements of S other than d and a commute with b; so we may assume that the occurrences of b are next to a, and we have $x = z'babz'' \in W'_+ W_- W'_+$.

(b) b occurs on one side of a only, say on the left. Let y be the unique shortest representative in $W'_+ x W'_+$ defined in 4.7. Then $y \leq x$ and $\{a, b\} \subset \mathrm{supp}(y)$, so we have a reduced expression for y with exactly one occurrence of a and one of b, to the left of a. But since $sy > y$ for all $s \notin \{a, b\}$, any reduced expression of y must start with b, and similarly, finish with a; write $y = bs_2 \ldots s_{p-1}a$, and let $y' = bs_2 \ldots s_{p-1} = ya \in W_{S\setminus\{a\}}$. Then y' is an element in $W_{S\setminus\{a\}}$ with left descent set $L(y') = \{b\}$. If we enumerate $S \setminus \{a\}$ clockwise in the form $c = t_1, t_2, \ldots, t_{n-1} = b$, it is well-known that the only such elements are of the form $t_{n-1}t_{n-2} \ldots t_k$, for some $1 \leq k \leq n-1$. If $k = n-1$, $y = ba \in W_-$, and we are done. But all t_j other than b and c commute with a, so if $1 < k < n-1$ we can write

$$y = bt_{n-2} \ldots t_k a = bat_{n-2} \ldots t_k \in W_- W'_+$$

contradicting the minimality of y. Thus the only case left is the one where $k = 1$. But then we see that $y = bt_{n-2} \ldots t_1 a$ is exactly the Coxeter obstruction corresponding to the orientation $\sigma\omega_+$, so this case is excluded. The case where b occurs on the right of a is similar, with the appearance of the Coxeter obstruction $\sigma\omega_-$.

So we have the inclusions $\mathrm{dom}(\varphi) \subset Q \subset W'_+W_-W'_+$, and since $W'_+W_-W'_+ \subset \mathrm{dom}(\varphi)$ is clear from the Proposition in 4.8, our result is proved in this case.

Now assume that $|S_-| > 2$. We wish to prove that x belongs to $W'_+W_-W'_+ \cup W'_-W_+W'_-$. If, say, a is not in $\mathrm{supp}(x)$, x belongs to the subgroup $W_{S\setminus\{a\}}$ which is of type A_{n-1}, and since the groups $W'_\pm$ are included in $W_{S\setminus\{a\}}$, and are the corresponding subgroups for the induced labeling on $S \setminus \{a\}$, the conclusion follows from Proposition 5.6. So we may assume that both a and b are in $\mathrm{supp}(x)$. Then as before we see from the obstruction r_a that x can be written in the form $x'ax''$, with $x', x'' \in W_{S\setminus\{a\}}$. If b belongs to both $\mathrm{supp}(x')$ and $\mathrm{supp}(x'')$, then from the obstruction r_b, x' and x'' lie in the double coset $W'bW'$, where $W' = W'_+ \times W'_-$. Write $J = \{b_-, b, b_+\}$, with $b_- \in \mathrm{supp}(E_-)$, $b_+ \in \mathrm{supp}(E_+)$, and $L_+ = S'_+ \setminus \{b_+\}$, $L_- = S'_- \setminus \{b_-\}$. Then $b_\pm$ are the only elements in $S' = S'_+ \coprod S'_-$ which do not commute with b. So from the description of this type of double coset in the proof of the Proposition in 4.8, we see that x' has a unique decomposition of the form $x' = x'_1bz'_+z'_-$, where $x'_1 \in W'$, $z'_+ \in {W'}_+^{L_+}$, $z'_- \in {W'}_-^{L_-}$, and similarly we may write $x'' = z''_+z''_-bx''_1$, $z''_+ \in {W'}_+^{(\emptyset,L_+)}$, $z''_- \in {W'}_-^{(\emptyset,L_-)}$. But from the obstruction r_b it follows that between the two occurrences of b we cannot have both b_+ and b_-. It follows that either $z'_+ = z''_+ = e$, or $z'_- = z''_- = e$. Hence we have either $x \in W'W_-W' = W'_+W_-W'_+$ or $x \in W'W_+W' = W'_-W_+W'_-$.

Finally, if b occurs on one side of a only, say on the left, we consider as before the shortest length representative y in $W'xW'$. We can write $y = by_+y_-a$, with $y_+ \in W'_+$, $y_- \in W'_-$. But by_+ has to belong to $W_+^{S'_+}$; hence if we enumerate $\mathrm{supp}(E_+) \setminus \{a, b\}$ counterclockwise in the form $t'_1, \ldots, t'_p$ we must have $by_+ = bt'_1 \ldots t'_j$ for some $0 \leq j \leq p$, and $y_+ = t'_1 \ldots t'_p$. Since y_+ and y_- commute, we must also have $y_+a \in W_+^{(\emptyset,S'_+)}$, which is possible only if $j = 0$ or $j = p$. Similarly, if we enumerate $\mathrm{supp}(E_-)$ in the form $t''_1, \ldots, t''_q$, we must have $y_- = e$ or $y_- = t''_q \ldots t''_1$. The only case where we cannot conclude that $y \in W'_+W_-W'_+$ or $y \in W'_-W_+W'_-$ is when $y = bt'_1 \ldots t_pt''_q \ldots t''_1a$, which is precisely the Coxeter obstruction $\sigma\omega_+$, hence is excluded.

So $\mathrm{dom}(\varphi_\sigma) = Q = W'_+W_-W'_+ \cup W'_-W_+W'_-$, and the conclusion follows as before from the Proposition in 4.8.

7.7. Let σ be a non-constant labelling on S; denote T the set of boundary points of the various components of σ (remark that for each component C, $\mathrm{supp}(C)$ is a string, hence has exactly two boundary points). Each $a \in T$ bounds exactly two components C, C'; we say that a *separates* C and C'.

From the remarks in 7.5, we see that T is also the set of elements in S which are sources or sinks for $\sigma\omega_+$ and $\sigma\omega_-$; and when we run through them in counterclockwise order, say, they are alternately sources and sinks for each of $\sigma\omega_+$, $\sigma\omega_-$ (sources for $\sigma\omega_+$ being sinks for $\sigma\omega_-$ and conversely.) For each component C, let σ_C be the labelling equal to $-$ on C, and $+$ elsewhere; let φ_C be the corresponding element in $\mathrm{Aut}_S(B)$.

Lemma. —*Let $x \in W$ be such that for each component C of σ, $[e, x]$ contains exactly one Coxeter obstruction u_C for φ_C, and let ω_C be the corresponding orientation. Assume that for each boundary point $a \in T$, separating two components C and C', the orientations on the two edges containing a are the same in ω_C and $\omega_{C'}$ (this implies that either a is a source in both ω_C and $\omega_{C'}$, or a is a sink in both.) Then $[e, x]$ contains one of the two Coxeter obstructions for φ_σ.*

Proof. — We say that $a \in T$ is a source, if it is a source in the two orientations ω_C, $\omega_{C'}$ corresponding to the two components separated by a, a sink otherwise. Clearly, if we go through the elements of T in clockwise order, they are alternately sources and sinks (recall that the cardinality of T is even.) So there is exactly one of the two orientations $\sigma\omega_+$, $\sigma\omega_-$ for which the nature of the points of T is as defined above; call this orientation ω, and let u be the corresponding Coxeter element. We shall prove that $u \in [e, x]$. If σ has two components C and C', ω is either ω_C or $\omega_{C'} = -\omega_C$, so there is nothing to prove. Assume from now on that there are at least four components.

Let T' (resp. T'') be the set of elements $a \in T$ which are sources (resp. sinks) for ω. Let $S' = S \setminus T''$. For each $a \in T'$, separating two components C and C', we set $S_a = \mathrm{supp}(C) \cup \mathrm{supp}(C')$, and $S'_a = S_a \setminus \{b, b'\}$, where $b \in \mathrm{supp}(C)$ and $b' \in \mathrm{supp}(C')$ are the boundary points other than a. The orientations on S_a induced by ω, ω_C and $\omega_{C'}$ are the same, with a as the only source, b and b' as sinks.

Choose a reduced expression $s_1 \dots s_p$ for x. For each $a \in T'$, we can extract from this expression a reduced expression for the Coxeter element u_a of W_{S_a} corresponding to the above orientation; removing the generators b and b', we obtain a Coxeter element u'_a for $W_{S'_a}$. Since the subexpressions for the various u'_a have disjoint supports, they correspond to disjoints subsets of indices in $\{1, \dots, p\}$. Consider the union of these subexpressions, with the elements arranged in the order of their appearance in $s_1 \dots s_p$. To obtain a reduced expression for u, it remains to add the elements of T''. Fix $b \in T''$, separating two components C and C', corresponding to endpoints $a, a' \in T'$. Let c be the last generator in the expression already constructed, belonging to $\operatorname{supp}(C) \cup \operatorname{supp}(C')$; say $c \in \operatorname{supp}(C)$. From the fact that u'_a was extracted from u_a, and the fact that b can occur in any reduced expression for u_a only after all other elements of $\operatorname{supp}(C)$ have occurred (because b is the only sink in $\operatorname{supp}(C)$), we see that there is an occurrence of b after c. For each $b \in T''$, we take the first such occurrence. We insert these elements in the expression already constructed, again in the order of appearance in $s_1 \dots s_p$, and we are done.

7.8 Theorem. — *Let σ be an arbitrary (non-constant) labelling of E, and let T be the set of extremal points of the components of σ. Let $Q \subset W$ be the decreasing subset defined by the rigidity obstructions r_a, $a \in T$, and the two Coxeter obstructions, corresponding to the orientations $\sigma\omega_+$ and $\sigma\omega_-$. Then $\operatorname{dom}(\varphi_\sigma) = Q$, and φ_σ preserves Kazhdan-Lusztig polynomials on its domain.*

Proof. — We shall argue by induction on the number of connected components of E_-. If E_- is connected, the result follows from Propositions 7.6 and 4.9.

Assume E_- is not connected. For each component C of σ, define φ_C as in 7.7. From Proposition 7.6, $\operatorname{dom}(\varphi_C)$ is defined by the two rigidity obstructions corresponding to the extremal points of $\operatorname{supp}(C)$, and the two Coxeter obstructions u_C, u'_C corresponding to $\sigma_C\omega_+$, $\sigma_C\omega_-$.

We already know that $\operatorname{dom}(\varphi) \subset Q$. Now let x be an arbitrary element in Q. Then we claim that x is contained in $\operatorname{dom}(\varphi_C)$ for some component C. Assume to the contrary. Since $[e, x]$ does not contain any of the r_J, this is equivalent to saying that for each component C, $[e, x]$ contains one of the Coxeter obstructions u_C, u'_C. Now let C, C' be two consecutive components of the labelling, so that $\operatorname{supp}(C) \cap \operatorname{supp}(C')$ is a single point $\{a\}$ (here we use the fact that there are at least four components.) Let v, v' be the two Coxeter obstructions in $[e, x]$, corresponding to C, C' respectively, and let ω, ω' be the corresponding orientations of E. Then a is either a source or a sink for both ω and ω'. We claim that in fact the nature of a is the same in the two orientations. Indeed, assume for instance that a were a source in ω, and a sink in ω'. Let b be the other extremity in C'. Because x satisfies the rigidity obstruction corresponding to a, it has a reduced expression with a single occurrence of a, as was shown in the proof of the Proposition in 7.5. But since ω has a as its only source, in any reduced expression for v, a is the first term, hence there has to be an occurrence of b after a. Similarly, a is the only sink in ω', so in any reduced expression for v', there has to be an occurrence of b before a. Since from our reduced expression of x we can extract reduced expressions for both v and v' there has to be an occurrence of

b on each side of a. But as we have seen in the proof of the Proposition in 7.5, this condition and the rigidity obstructions corresponding to a and b suffice to ensure that x is in $\mathrm{dom}(\varphi_{C'})$, hence $[e, x]$ cannot contain v', contradiction. So $[e, x]$ contains exactly one Coxeter obstruction for each component C, and if we orient each component accordingly, these orientations alternate, hence define on E the orientation of one of the Coxeter obstructions for φ_σ. But then from the Lemma in 7.7, $[e, x]$ contains that Coxeter obstruction, contrary to our assumption $x \in Q$.

So x is contained in $\mathrm{dom}(\varphi_C)$ for some component C. Let $\sigma' = \sigma\sigma_C$, and let $\varphi' = \varphi_{\sigma'}$; let E'_- be the set of edges labelled $-$ in σ'. Then E'_- has one less connected component than E_- : either $C \subset E_-$, and it will disappear, or $C \subset E_+$ separates two components in E_-, which will get glued together. Clearly φ_C takes the Coxeter obstructions for φ_σ to those for φ'; also, φ_C fixes the rigidity obstructions for φ other than those corresponding to the extremities of $\mathrm{supp}(C)$ (for which φ_C is not defined). Hence $\varphi_C(x)$ belongs to the decreasing subset Q' defined by the obstructions for σ', which is equal to $\mathrm{dom}(\varphi')$ from the inductive hypothesis. So $\varphi = \varphi'\varphi_C$ is defined on $[e, x]$, and $x \in \mathrm{dom}(\varphi)$.

The fact that φ preserves Kazhdan-Lusztig polynomials on $\mathrm{dom}(\varphi)$ is clear : we know that φ_C and φ' preserve them on their respective domains, hence φ preserves them on each $[e, x]$.

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