An overview on phase field method to approximate evolving interfaces by geometric law

Elie Bretin

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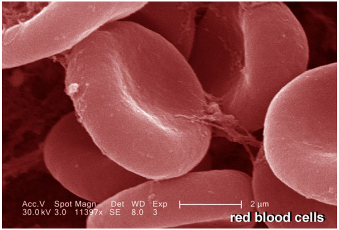
the slides are available at
http://fex.insa-lyon.fr/get?k=6WUxf7sb8mH7ZatQPCF

Examples of geometric energies

$$P(\Omega) = \int_{\partial\Omega} 1 d\mathcal{H}^{n-1}, \quad \mathcal{W}(\Omega) = \frac{1}{2}\int_{\partial\Omega} H^2 d\mathcal{H}^{n-1}, \quad P_\gamma(\Omega) = \int_{\partial\Omega} \gamma(\vec{n}) d\mathcal{H}^{n-}$$



Applications : biology, material sciences, image processing, shapes optimization ...

Mumford Shah functional [Mumford Shah 1989]

- Approximate an image $I(x)$ with piecewise smmooth function $u(x)$ by minimizing the functional

$$E(u,K) = \int_{\Omega} (u(x) - I(x))^2 dx + \alpha \int_{\Omega \backslash K} |\nabla u|^2 dx + \beta \text{Length}(K).$$

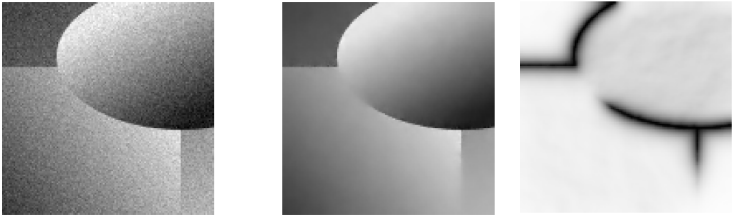
(a) Input image (b) Approximation

fig : Example of regularization obtained with Mumford Shah approach [Pock,Cremers, Bischof and Chambolle, 2009]

Mumford Shah approximation

- Ambrosio Tortorelli approximation [Ambrosio,Tortorelli,90]

$$E_\epsilon(u,\varphi) = \int_\Omega (u(x) - I(x))^2 dx + \alpha \int_\Omega \varphi |\nabla u|^2 dx$$
$$+\beta \int_\Omega \epsilon |\nabla \varphi|^2 + \frac{1}{\epsilon}(1-\varphi)^2 dx,$$



- Example of denoising obtained with Ambrosio Tortorelli approximation
  Left ; I, middle u and right : φ

Piecewise constant Mumford-Shah ($\alpha \to \infty$)

- An image segmentation model :
  Find a partition $\{\Omega_i\}_{i=1:N}$ and a color vector $c = (c_1, c_2, \cdots, c_N)$ as a minimizer of

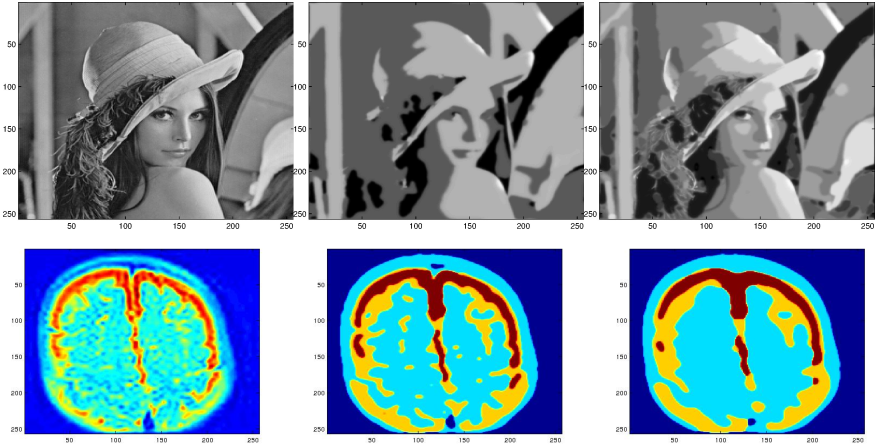
$$J(\Omega_1, \Omega_2, \cdots, \Omega_N, c) = \sum_{i=1}^{N} \left(\int_{\Omega_i} (I(x) - c_i)^2 dx + \beta P(\Omega_i) \right).$$

- An approximation :

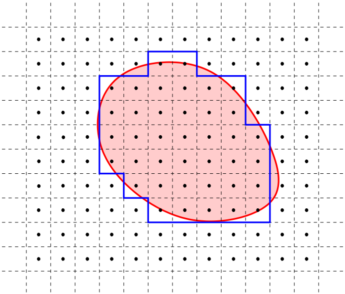
$$\begin{aligned} J_\epsilon(\mathbf{u}, c) &= \sum_{i=1}^{N} \left[\int u_i \, (I(x) - c_i)^2 dx \right] \\ &\quad + \beta \sum_{i=1}^{N} \left[\int \epsilon \frac{|\nabla u_i|^2}{2} + \frac{1}{2\epsilon} u_i^2 (1 - u_i)^2 dx \right] \end{aligned}$$

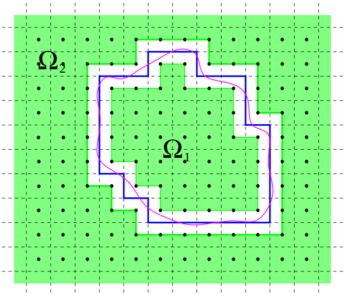
for all $\mathbf{u} = (u_1, u_2, \cdot, u_N)$ satisfying $\sum u_i = 1$.

Examples of image segmentation



Regularization of discrete contour by Willmore energy



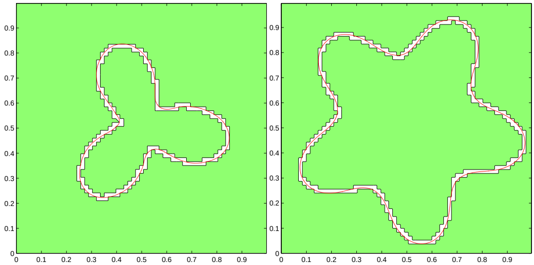


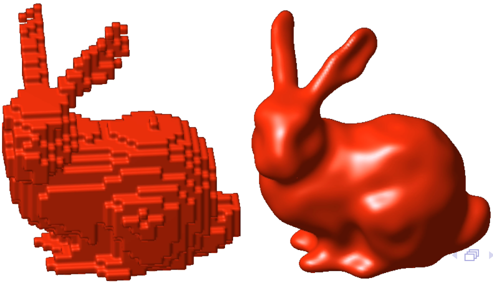
Find the set Ω^* such as

$$\Omega^* = \underset{\Omega_1 \subset \Omega \subset \Omega_2^c}{\arg\min}\, \mathcal{W}(\Omega), \quad \text{with } \mathcal{W}(\Omega) = \int_{\partial\Omega} H^2 d\mathcal{H}^{n-1}$$

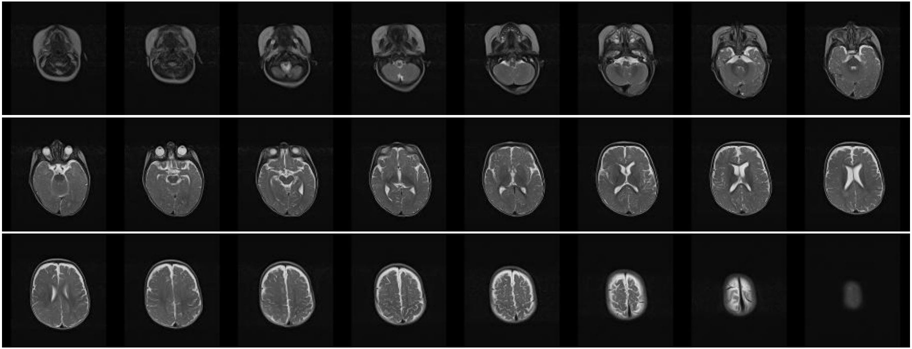
where Ω_1 and Ω_2 are two given set such as $\Omega_1 \subset \Omega_2^c$

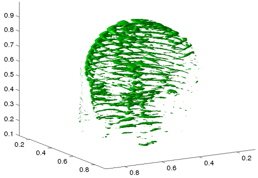
Numerical experiments

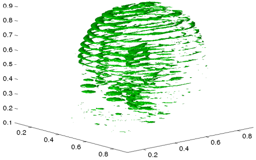


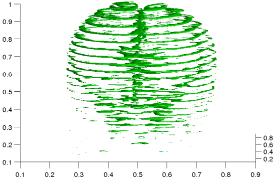


Motivation : Magnetic resonance Imaging







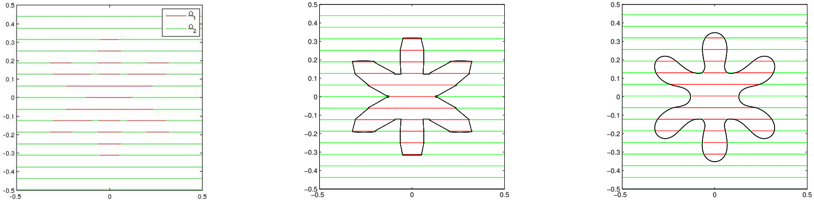


Surface reconstruction from orthogonal slice

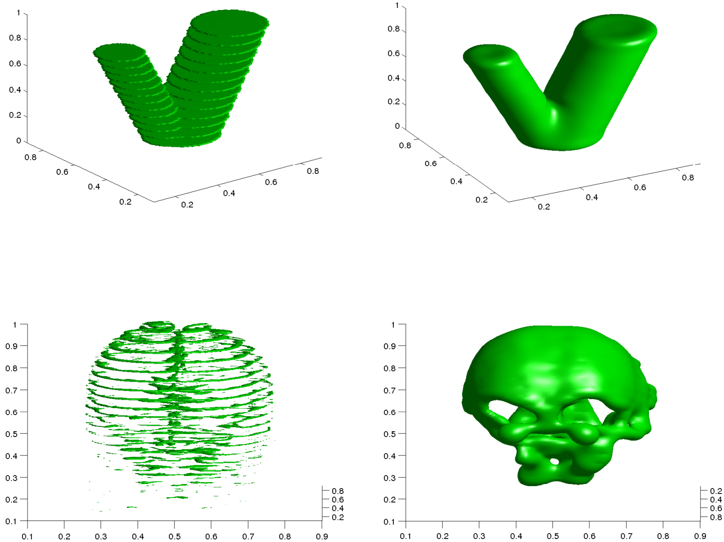
- Find the set Ω^* as a minimizer of

$$J_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} J(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2^c \\ +\infty & \text{otherwise} \end{cases},$$

where J is a surface geometric energy as the perimeter or the Willmore energy.



Example of reconstruction in dimension 3



1. Introduction
   - Examples of applications in image processing
   - Definitions of the curvature
   - Shape derivative of classical geometric energies
   - Numerical algorithms for mean curvature flow
2. Phase field approximation of mean curvature flow
3. Conserved and multiphase mean curvature flow
4. Approximation of Willmore energy and flow:

Curvature of a smooth curve in $\mathbb{R}^2$

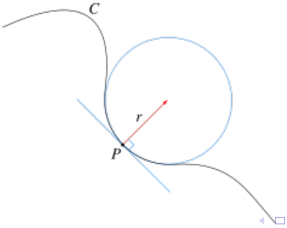
- A parametric representation of Γ

$$\Gamma = \left\{X(s) = (x(s), y(s)) \in R^2; s \in [0,1]\right\}.$$

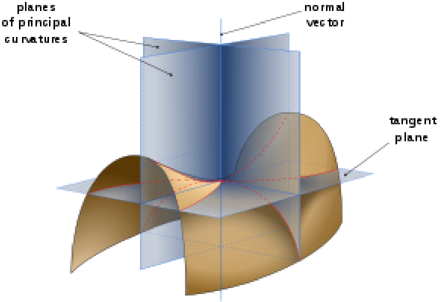
- Normal and curvature at $X(s)$

$$n(X(s)) = \frac{X_s^{\perp}}{|X_s|} = \frac{(y'(s), -x'(s,t))}{\sqrt{x'(s)^2 + y'(s)^2}}$$

$$\kappa(X(s)) = \frac{1}{|X_s|}\left(\frac{X_s}{|X_s|}\right)_s \cdot n(X(s)) = \frac{x'(s)y''(s) - y'(s)x''(s)}{(x'(s)^2 + y'(s)^2)^{3/2}}.$$



Curvature of a smooth surface in $\mathbb{R}^3$



- Principal curvature : κ_1, κ_2
- The mean curvature : $H = \kappa_1 + \kappa_2$
- The Gauss curvature : $G = \kappa_1 \kappa_2$

Using the second fundamental form :

Let $\Gamma \subset \mathbb{R}^d$ be a smooth manifold of co-dimension 1

- $T_x\Gamma$ is the tangent plan at $x \in \Gamma$
- Second fundamental form : $B_x : T_x \times T_x \to \mathbb{R}$ defined by

$$B_x(\xi, \eta) = \langle \xi, \partial_\eta n \rangle, \quad \forall (\xi, \eta) \in T_x \times T_x$$

  Note that B_x is bilinear and symmetric with eigenvalues $\kappa_1, \kappa_2 \cdots \kappa_{d-1}$.
- Mean and Gauss curvature

$$H = Trace(B_x) = \sum_{i=1}^{d-1} \kappa_i$$

$$G = det(B_x) = \prod_{i=1}^{d-1} \kappa_i$$

Using surface differential operators

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ and $X : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth scalar field and vectorfield

- Tangential gradient operator

$$\nabla^\Gamma \phi = (Id - n \otimes n)\nabla\phi) = \nabla\phi - \langle \nabla\phi, n\rangle n$$

- Tangential divergence operator

$$\mathrm{div}^\Gamma(X) = Trace\,((Id - n \otimes n)\nabla X) = \mathrm{div}(X) - \langle (n \cdot \nabla)X, n\rangle$$

- Remark that

$$\mathrm{div}^\Gamma(fX) = f\,\mathrm{div}^\Gamma(X) + X \cdot \nabla^\Gamma f$$

- Mean curvature

$$H = \mathrm{div}^\Gamma(n) = \mathrm{div}(n) \text{ as } |n|^2 = 1$$

Gauss Green and Stokes formula

- The divergence formula : if Y is a C^1 tangential vectorfield, then

$$\int_\Gamma \text{div}^\Gamma(Y)d\sigma(x) = 0$$

- if X is a C^1 vectorfield, then

$$\int_\Gamma \text{div}^\Gamma(X)d\sigma(x) = \int_\Gamma H\, X \cdot n d\sigma(x).$$

- Moreover,

$$\int_\Gamma f\,\text{div}^\Gamma(X)d\sigma = -\int_\Gamma \nabla^\Gamma(f)\cdot X d\sigma + \int_\Gamma H\, f\, X \cdot n d\sigma(x).$$

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Notion of shape derivative

[Henrot,Pierre 2004]

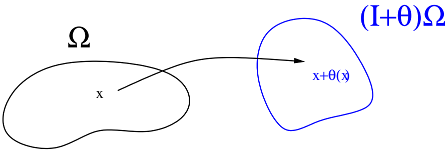
- First example of geometric energies

$$J_1(\Omega) = \int_{\Omega} f(x)dx \quad \text{and} \quad J_2(\Omega) = \int_{\partial\Omega} g(x)dx.$$

- Shape derivative in the direction θ

$$J'(\Omega)(\theta) = \lim_{\epsilon \to 0} \frac{J((Id + \epsilon\theta)\Omega) - J(\Omega)}{\epsilon},$$

where $\theta : \mathbb{R}^d \to \mathbb{R}^d$ is a vectorfield



Case of J_1

- Substitution in the integral :
  let τ be a diffeomorphism in $\mathbb{R}^d$, then

$$\int_{\tau(\Omega)} f(x)dx = \int_{\Omega} f(\tau(x))|det\nabla\tau|dx$$

- With $\tau = Id + \epsilon\theta$, we have

$$f(\tau(x)) = f(x + \epsilon\theta(x)) = f(x) + \epsilon\nabla f(x)\cdot\theta + o(\epsilon),$$

$$det\nabla\tau = 1 + \epsilon\,\mathrm{div}(\theta) + o(\theta),$$

  and then

$$J_1'(\Omega)(\theta) = \int_{\Omega} \nabla f\cdot\theta + f\,\mathrm{div}(\theta)dx = \int_{\partial\Omega} f(x)\theta\cdot n d\sigma(x).$$

Case of J_2

- Substitution in the integral :
  let τ be a C^1 diffeomorphism in $\mathbb{R}^d$, then

$$\int_{\partial(\tau(\Omega))} g(x)d\sigma(x) = \int_{\partial\Omega} g(\tau(x))\,|det\nabla\tau|\,\left|((\nabla\tau)^{-1})^T n\right| d\sigma(x)$$

- With $\tau = Id + \epsilon\theta$, we have

$$\left|((\nabla\tau)^{-1})^T n\right| = 1 - \epsilon\langle\nabla\theta n, n\rangle + o(\epsilon)$$

  and then

$$\begin{aligned} J_2'(\Omega)(\theta) &= \int_{\partial\Omega} g\,\mathrm{div}(\theta) - g\langle\nabla\theta n, n\rangle + \nabla g\cdot\theta d\sigma(x) \\ &= \int_{\partial\Omega} \partial_n g\,\theta\cdot n + \left(g\,\mathrm{div}^\Gamma(\theta) + \nabla^\Gamma g\cdot\theta\right) d\sigma(x) \\ &= \int_{\partial\Omega} \partial_n g\,\theta\cdot n + H\,g\,\theta\cdot n d\sigma(x) \end{aligned}$$

Application for the Volume and the Perimeter energy

- With $Vol(\Omega) = \int_\Omega 1dx$, then

$$Vol'(\Omega)(\theta) = \int_{\partial\Omega} 1\ \theta \cdot n d\sigma(x),$$

and it's L^2-gradient flow $\Rightarrow$ the normal velocity V_n satisfies

$$V_n = -1.$$

- With $P(\Omega) = \int_{\partial\Omega} 1dx$, then

$$P'(\Omega)(\theta) = \int_{\partial\Omega} H\ \theta \cdot n d\sigma(x),$$

and it's L^2-gradient flow $\Rightarrow$ the normal velocity V_n satisfies

$$V_n = -H.$$

General result

- Consider Energy on the form

$$J(\Omega) = \int_{\partial\Omega} F(x, n, H) d\sigma,$$

where $F = F(x, y, z)$ is assumed to be sufficiently smooth.

- Shape derivative [Dogan, Nochetto 2012]

$$J'(\Omega)(\theta) = \int_{\partial\Omega} \Big(\mathrm{div}^{\Gamma} [\nabla_y F]^{\Gamma} - \Delta_{\Gamma} [\partial_z F] + FH - \partial_z F |A|^2 + \nabla_x F \cdot n \Big) \theta \cdot n d\sigma$$

where

$$\mathrm{div}^{\Gamma} [\nabla_y F]^{\Gamma} = \mathrm{div}^{\Gamma} (\nabla_y F) - H \nabla_y F \cdot n$$

and

$$|A|^2 = \sum \kappa_i^2.$$

Application to Willmore energy

- Willmore energy :

$$\mathcal{W}(\Omega) = \frac{1}{2}\int_{\partial\Omega} H^2 d\sigma,$$

- Shape derivative

$$\mathcal{W}'(\Omega)(\theta) = \int_{\partial\Omega}\left(-\Delta_\Gamma[H] + \frac{1}{2}H^3 - H\sum|A|^2\right)\theta\cdot n d\sigma$$

and it's L^2-gradient flow $\Rightarrow$ the normal velocity V_n satisfies

$$V_n = \Delta_\Gamma[H] - \frac{1}{2}H^3 + H\sum|A|^2 = \Delta_\Gamma[H] + \frac{1}{2}H(H^2 - 4G)$$

Application to anisotropic perimeter

- Anisotropic perimeter :

$$P_\gamma(\Omega) = \int_{\partial\Omega} \gamma(n) d\sigma,$$

where γ is a smooth function, positively homogeneous of degree 1 :

- Some properties of γ

$$\gamma(\lambda y) = |\lambda| \gamma(y), \quad \text{and} \quad \nabla_y[\gamma] \cdot y = \gamma(y)$$

- Shape derivative

$$J'(\Omega)(\theta) = \int_{\partial\Omega} \left(\text{div}^\Gamma [\nabla_y \gamma]^\Gamma + \gamma H \right) \theta \cdot n d\sigma = \int_{\partial\Omega} H_\gamma \theta \cdot n d\sigma$$

where $H_\gamma = \text{div}^\Gamma(n_\gamma)$ and $n_\gamma = \nabla_y \gamma(n)$

Application to anisotropic perimeter

- Anisotropic curvature in dimension 2

$$H_\gamma = \operatorname{div}^\Gamma(\nabla_y \gamma(n)) = H \langle \nabla_y^2 \gamma(n) n^\perp, n^\perp \rangle$$

- In polar coordinate system :

$$\gamma(y) = \rho\phi(\theta) \text{ with } \rho = \sqrt{y_1^2 + y_2^2} \text{ and } \theta = \operatorname{atan}(y_2/y_1),$$

then, the anisotropic curvature reads

$$H_\gamma = H(\phi(\theta) + \phi''(\theta))$$

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Mean curvature flow

- mean curvature flow :

$$P(\Omega) = \int_{\partial\Omega} 1 \, d\mathcal{H}^{n-1}$$

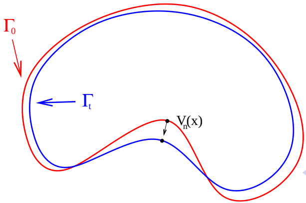
- Shape derivative

$$P'(\Omega)(\theta) = \int_{\partial\Omega} H \, \theta.n \, d\mathcal{H}^{n-1},$$

where n and H denote the normal and the mean curvature.

- L^2 gradient flow of $P \Rightarrow$ the normal velocity V_n satisfies

$$V_n = -H.$$

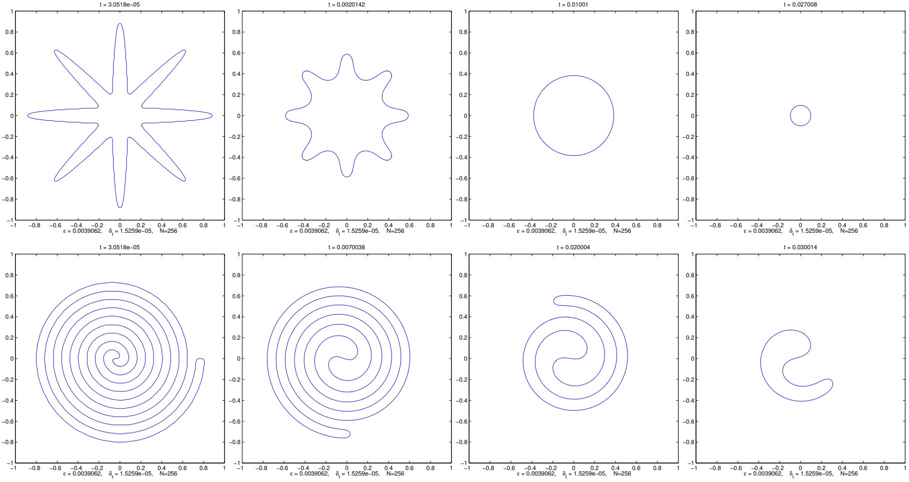


Some properties of mean curvature flow $t \to \Omega(t)$

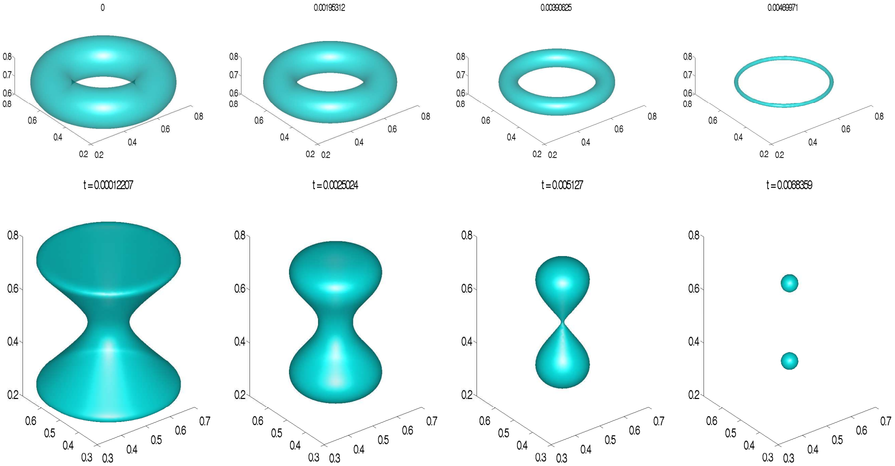
- Local existence for convex initial set. The set $\Omega(t)$ stay convex, converges to a point and becomes asymptotically spherical [Huisken 1984]
- In dimension 2, local existence for smooth closed curves. The set $\Omega(t)$ becomes convex in finite time, converges to a point and becomes asymptotically spherical [Gage and Hamilton 1986], [Grayson 1987]
- In dimension $n > 2$: singularities in finite time [Grayson 1989]
- Inclusion principle [Ecker 2002]:

$$\Omega_1(0) \subset \Omega_2(0) \text{ then } \Omega_1(t) \subset \Omega_2(t), \quad \forall t \in [0, T]$$

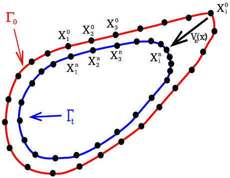
Example of mean curvature flow in dimension two



Example of mean curvature flow in dimension three



A Parametric approach ([Deckelnick,Dziuk,Elliott])



- A parametric representation

$$\Gamma = \{X(s) = (x(s), y(s))\}_{s\in[0,2\pi]}$$

- Normal vector $n(s)$ at $X(s)$:

$$n(X(s)) = \frac{X_s(s)^\perp}{|X_s(s)|} = \frac{(y'(s), -x'(s,t))}{\sqrt{x'(s)^2 + y'(s)^2}}$$

- Mean curvature at $X(s)$

$$\kappa(X(s)) = \frac{1}{|X_s(s)|}\left(\frac{X_s(s)}{|X_s(s)|}\right)_s \cdot n(X(s)) = \frac{x'(s)y''(s) - y'(s)x''(s)}{(x'(s)^2 + y'(s)^2)^{3/2}}$$

- Mean curvature flow

$$X_t(s) = \kappa(X(s))n(X(s)) \quad \text{or equivalently} \quad X_t = \frac{1}{|X_s|}\left(\frac{X_s}{|X_s|}\right)_s.$$

An explicit discretization :

- Discretization of $X(s,t)$:

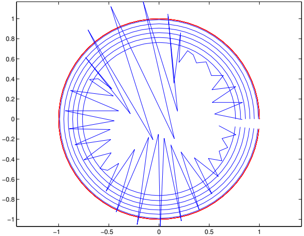
$$X(i\delta_s, n\delta_t) \simeq X_i^n = (x_i^n, y_i^n)$$

- Approximation of n and κ

$$\begin{cases} n(X_n^i) & = \dfrac{\left(y_{i+1}^n - y_{i-1}^n, -(x_{i+1}^n - x_{i-1}^n)\right)}{\left((x_{i+1}^n - x_{i-1}^n))^2 + (y_{i+1}^n - y_{i-1}^n)^2\right)^{1/2}} \\ \kappa(X_n^i) & = 4\dfrac{(y_{i+1}^n - 2y_i + y_{i-1}^n)(x_{i+1}^n - x_{i-1}^n) - (x_{i+1}^n - 2x_i + x_{i-1}^n)(y_{i+1}^n - y_{i-1}^n)}{\left((x_{i+1}^n - x_{i-1}^n))^2 + (y_{i+1}^n - y_{i-1}^n)^2\right)^{3/2}} \end{cases}$$

- An Euler explicit scheme

$$X_i^{n+1} = X_i^n + \delta_t\, \kappa(X_n^i)\, n(X_n^i).$$



A semi-implicit discretization,

- A weak formulation

$$X_t = \frac{1}{|X_s|}\left(\frac{X_s}{|X_s|}\right)_s \Rightarrow \int_0^{2\pi} X_t |X_s| \phi ds = \int_0^{2\pi} \frac{X_s}{|X_s|}\phi_s ds, \forall \phi \in H^1$$

- A finite element approach :

$$X(s,t) = \sum_{i=1}^{M} X_i(t)\phi_i(s)$$

- Spatial discretization

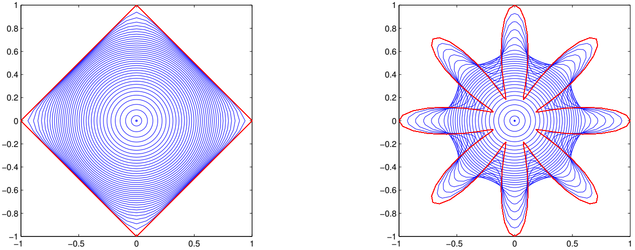
$$\frac{1}{2}\partial_t(X_i)(h_i + h_{i+1}) = \frac{X_{i+1} - X_i}{h_{i+1}} - \frac{X_i - X_{i-1}}{h_i}, \quad \text{with } h_i = |X_i - X_{i-1}|$$

- Semi-implicit time discretization

$$\frac{1}{2\delta_t}(X_i^{n+1} - X_i^n)(h_i^n + h_{i+1}^n) = \left(\frac{X_{i+1}^{n+1} - X_i^{n+1}}{h_{i+1}^n} - \frac{X_i^{n+1} - X_{i-1}^{n+1}}{h_i^n}\right).$$

A semi-implicit discretization,

- the scheme presents no problem of stability but



- Extension in greater dimension ? (see [Barett,Garcke and Nurnberg]
- Problem : how to deal with topology change

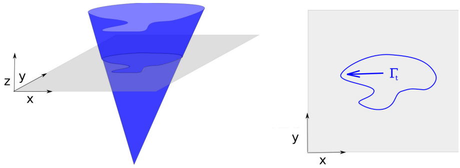
The level set method ([Osher,Sethian])

- An implicit representation of the interface

$$\Gamma = \{x \ ; \ \varphi(x,t) = 0\}$$

- Normal vector n and curvature κ :

$$n(\varphi) = \frac{\nabla\varphi}{|\nabla\varphi|} \quad \text{and} \quad \kappa(\varphi) = \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)$$



- Mean curvature flow

$$\partial_t\varphi = \kappa(\varphi)|\nabla\varphi| = \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)|\nabla\varphi|$$

The level set method

- A Hamilton-Jacobi equation

$$\partial_t \varphi = \mathrm{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right)|\nabla \varphi| = \Delta \varphi - \frac{\langle \nabla^2 \varphi \nabla \varphi, \nabla \varphi \rangle}{|\nabla \phi|^2}$$

- Weak solution in sense of viscosity [Evan,Spruck][Chen,Giga,Goto])
- Numerical approach : fast marching method (for transport equation) where the velocity $\kappa(\varphi)$ is estimated explicitly
- Stability problems as for the explicit parametric approach

The Allen Cahn equation as an approximate level set equation

- Idea : Choose a particular form of level set function

$$u(x,t) = q\left(\frac{d(x,t)}{\epsilon}\right),$$

where q is a profile satisfying $q''(s) = W'(q)$ and d is the signed distance function to a evolving set $\Omega(t)$.

- Remarks that

$$\nabla u = \frac{\nabla d}{\epsilon} q'\left(\frac{d}{\epsilon}\right) \quad \text{and} \quad \nabla^2 u = \frac{\nabla^2 d}{\epsilon} q'\left(\frac{d}{\epsilon}\right) + \frac{\nabla d \otimes \nabla d}{\epsilon^2} q''\left(\frac{d}{\epsilon}\right)$$

- Then the Hamilton-Jacobi equation reads now as

$$\partial_t u = \Delta u - \frac{\langle \nabla^2 u \nabla u, \nabla u \rangle}{|\nabla u|^2} = \Delta u - \frac{1}{\epsilon^2} q''\left(\frac{dist}{\epsilon}\right) = \Delta u - \frac{1}{\epsilon^2} W'(u)$$

The Allen Cahn equation as an approximate level set equation

- We obtain a reaction diffusion equation

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u),$$

- It's the L^2 gradient flow of Cahn Hillard energy

$$P_\epsilon(u) = \int \left(\epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} W(u) \right) dx,$$

- Numerical scheme :
  A splitting approach with implicit treatment of diffusion term
- Link with mean curvature flow ?

Principe of phase field method

- Approximation of energy

$$\begin{array}{ccc} J_\epsilon(u) & \longmapsto & J(\Omega) \\ \downarrow & & \downarrow \\ u_t = -\nabla J_\epsilon(u) & \dashrightarrow & V = -\nabla J(\Omega) \end{array}$$

- Perimeter

$$P(\Omega) = \int_{\partial\Omega} 1 d\sigma \quad \leftrightsquigarrow \quad P_\epsilon(u) = \int \epsilon|\nabla u|^2 + \frac{1}{\epsilon}W'(u)dx$$

- Willmore energy

$$\mathcal{W}(\Omega) = \frac{1}{2}\int_{\partial\Omega} H^2 d\sigma \quad \leftrightsquigarrow \quad \mathcal{W}_\epsilon(u) = \frac{1}{2\epsilon}\int_{\mathbb{R}^d} \left(\epsilon\Delta u - \frac{1}{\epsilon}W'(u)\right)^2 dx$$

Notion of Γ-convergence [Dal Maso 93]

Definition (Γ-convergence)

Let (X, d) be a metric space and let $G_\epsilon : X \to \overline{\mathbb{R}}$ be functions. We say that G_ϵ Γ-converges in X to $G : X \to \overline{\mathbb{R}}$ if

$$(1) \qquad \forall u_\epsilon \to u \text{ in } X, \qquad \liminf_{\epsilon\to 0} G_\epsilon(u_\epsilon) \geq G(u)$$

$$(2) \quad \forall u \quad \exists u_\epsilon \to u \text{ in } X \quad \text{such that} \quad \limsup_{\epsilon\to 0} G_\epsilon(u_\epsilon) \leq G(u)$$

Properties of the Γ-convergence

Property

Let (X, d) be a metric space and let $G_\epsilon : X \to \overline{\mathbb{R}}$ be functions which Γ-converge to $G : X \to \overline{\mathbb{R}}$. Then

- *G is lower semi-continuous on X*
- *if $F : X \to \overline{R}$ is continuous, then $G_\epsilon + F$ Γ-converges to $G + F$.*
- *If G_ϵ is equi-coercive, i.e.*

$$\forall t \in \mathbb{R}, \exists K_t \subset\subset X \text{ such as } \{G_\epsilon \leq t\} \subset K_t,$$

then G is coercive and reaches it's minimum on X. Moreover

$$\min_{u \in X}\{G(u)\} = \lim_{\epsilon \to 0} \inf_{u \in X} \{G_\epsilon(u)\}$$

Definition of a Generalized perimeter P

- The total variation $|Du|$ is defined $\forall u \in L^1(\mathbb{R}^n)$ by

$$|Du|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} u \,\mathrm{div}(\varphi) dx; \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \text{ and } ||\varphi||_\infty \leq 1 \right\}$$

- If it exits a smooth set Ω such as $u = \chi_\Omega$ then

$$P(\Omega) = \int_{\partial\Omega} 1 d\sigma(x) = |D\chi_\Omega|(\mathbb{R}^d)$$

- Generalized perimeter (Caccioppoli) : $\forall u \in L^1$,

$$P(u) := \begin{cases} |Du|(\mathbb{R}^n) & \text{if } u = \chi_\Omega \\ +\infty & \text{otherwise} \end{cases},$$

and P is lower semi-continuous on L^1.

Approximation with the Cahn Hilliard energy

Definition (Cahn Hilliard energy)

P_ϵ is defined $\forall u \in L^1(\mathbb{R}^n)$ by

$$P_\epsilon(u) = \begin{cases} \int_{\mathbb{R}^d} \left(\epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} W(u) \right) dx, & \text{if } u \in H^1(\mathbb{R}^n) \\ +\infty & \text{otherwise} \end{cases},$$

where W is positive, continuous and satisfies $W(s) = 0$ if and only if $t \in \{0, 1\}$. See for instance $W(s) = \frac{1}{2} s^2 (1 - s)^2$

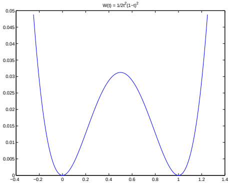


fig : $s \to W(s)$

Modica-Mortola Γ-convergence result

Theorem ([Modica-Mortola77])

$$\Gamma - \lim_{\epsilon \to 0} P_\epsilon = c_W P \;\; in\; L^1, \qquad with\; c_W = \int_0^1 \sqrt{2W(s)}ds,$$

which means that,

$$\begin{aligned} &(1) && \forall u_\epsilon \to u, && \liminf_{\epsilon \to 0} P_\epsilon(u_\epsilon) \geq c_W P(u) \\ &(2) \quad \forall u && \exists u_\epsilon \to u \quad such\ that && \limsup_{\epsilon \to 0} P_\epsilon(u_\epsilon) \leq c_W P(u) \end{aligned}$$

Lower bound inequality :

$$\forall u_\epsilon \to u, \qquad \liminf_{\epsilon\to 0} P_\epsilon(u_\epsilon) \geq c_W P(u)$$

- We can assume that $\liminf P_\epsilon(u_\epsilon) < \infty$ then

$$u_\epsilon \to u = \chi_\Omega$$

-
$$P_\epsilon(u_\epsilon) \geq \int |\nabla u_\epsilon| \sqrt{2W(u_\epsilon)} dx = \int |D\phi(u_\epsilon)| dx,$$

  with $\phi(s) = \int_0^s \sqrt{2W(s)} ds$

-
$$\lim_{inf} \int |D\phi(u_\epsilon)| \geq \int |D\phi(\chi_\Omega)| = \phi(1) P(\chi_\Omega) = c_W P(\chi_\Omega)$$

Upper bound inequality:

$$\forall u, \quad \exists u_\epsilon \to u \quad \text{such that} \quad \limsup_{\epsilon \to 0} P_\epsilon(u_\epsilon) \le c_W P(u)$$

- By density argument, we can assume that there is a smooth set Ω such as $u = \chi_\Omega$.
- The we can choose $u_\epsilon(x) = q\left(\frac{dist(x,\Omega)}{\epsilon}\right)$, where the profile function q depends only on W.

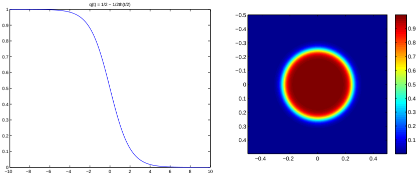


fig : $s \to q(s)$ and $u_\epsilon(x)$

Some properties of the profile function q

- The profile q is defined as

$$q = \arg\min_{\gamma \in H^1_{loc}(\mathbb{R})} \left\{ \int_{\mathbb{R}} \left(\frac{1}{2} |\gamma^{'}(s)|^2 + W(\gamma(s)) \right) ds \ ; \ \gamma(-\infty) = 1, \gamma(\infty) = 0 \right\}$$

  with $q(0) = 1/2$.

- Euler equation shows that q satisfies

$$q'' = W'(q).$$

- When W is C^2, then q satisfies $q' = -\sqrt{2W(q)}$ and then

$$c_W = \int_0^1 \sqrt{2W(s)} ds = \int_{\mathbb{R}} \frac{|q'(s)|^2}{2} + W(s) ds.$$

- Note that for $W(s) = \frac{1}{2} s^2 (1-s)^2$, $q(s) = \frac{1}{2} - \frac{1}{2} tanh\left(\frac{s}{2}\right)$.

Upper bound inequality:

Recall that $u = \chi_\Omega$ and consider the sequence

$$u_\epsilon(x) = q\left(\frac{dist(x,\Omega)}{\epsilon}\right) = q(d/\epsilon).$$

Then

$$\begin{aligned} P_\epsilon(u_\epsilon) &= \int_{\mathbb{R}^d}\left(\epsilon\frac{|\nabla u_\epsilon|^2}{2} + \frac{1}{\epsilon}W(u_\epsilon)\right)dx \\ &= \frac{1}{\epsilon}\int_{\mathbb{R}^d}\left(\frac{q'(d/\epsilon)}{2} + \frac{1}{\epsilon}W(q(d/\epsilon))\right)|\nabla d|dx \end{aligned}$$

Property (Co-area formula)

Let Ω be an open set of $\mathbb{R}^n$ and ϕ be a real-valued Lipschitz function on Ω. Then $\forall u \in L^1(\Omega)$,

$$\int_\Omega u(x)|\nabla\phi|dx = \int_{\mathbb{R}}\left(\int_{\phi^{-1}(s)} u(x)d\mathcal{H}^{n-1}\right)ds$$

Upper bound inequality:

$$
\begin{aligned}
P_\epsilon(u_\epsilon) &= \frac{1}{\epsilon}\int_{\mathbb{R}^d}\left(\frac{q'(d/\epsilon)}{2}+\frac{1}{\epsilon}W(q(d/\epsilon))\right)|\nabla d|dx \\
&= \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{d^{-1}(s)}\frac{q'(d/\epsilon)}{2}+\frac{1}{\epsilon}W(q(d/\epsilon))d\mathcal{H}^{\mathrm{n}-1}\right) \\
&= \frac{1}{\epsilon}\int_{\mathbb{R}} g(s)\left[\frac{|q'(s/\epsilon)|^2}{2}+W(q(s/\epsilon))\right]ds \\
&= \int_{-\infty}^{+\infty} g(\epsilon s)\left[\frac{|q'(s)|^2}{2}+W(q(s))\right]ds
\end{aligned}
$$

with $g(s)=|D\chi_{\{dist(x,\Omega)\le s\}}|$.
Then, by smoothness of Ω, $g(\epsilon s)\to g(0)=P(u)$ as $\epsilon\to 0$, and

$$
\lim_{\epsilon\to 0} P_\epsilon(u_\epsilon)\le c_W P(u).
$$

Allen Cahn equation

- Cahn Hilliard energy:

$$P_\epsilon(u) = \int_{\mathbb{R}^d} \left(\epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} W(u) \right) dx,$$

- L^2 gradient flow of $P_\epsilon \Longrightarrow$ Allen Cahn equation

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u)$$

- Existence, comparison principle
  [Ambrosio2000] [Almgren-Taylor-Wang93]

Existence of solution

Theorem

Consider the Allen Cahn equation in a box $Q = [0,1]$ with periodic boundary conditions and with a C^2 double well potential which satisfies

$$W'' \in L^\infty_{loc}(\mathbb{R}), \quad and \quad W(t) \geq \alpha t^2 + \beta, \text{ with } \alpha > 0, \beta \in \mathbb{R}$$

Then, for all $u_0 \in\in H^1(Q) \cap L^\infty(Q)$, there exists a function

$$u \in L^\infty(\mathbb{R}^+, H^1(Q)) \cap H^1_{loc}(\mathbb{R}^+, L^2(Q)),$$

with $u(x,0) = u_0$ such as for all $\phi \in C^\infty_c(\mathbb{R}^+ \times Q)$

$$\int_{Q\times R^+} u\phi_t dxdt = \int_{Q\times R^+} (\frac{1}{\epsilon^2} W'(u)\phi + \nabla u \cdot \nabla \phi) dxdt.$$

Outline of the proof

1 Build a linear piecewise approximating solution

$$u_h = u_h^{[t/h]}(x) + (t/h - [t/h])(u_h^{[t/h+1]}(x) - u_h^{[t/h](x)}),$$

where

$$u_h^{j+1} = \arg\min_{v \in H^1(Q)} \left\{ \frac{1}{\epsilon} P_\epsilon(v) + \frac{1}{2h} \int_Q (v - u_h^j)^2 dx \right\}$$

2 Prove that u_h is uniformly bounded in $L^\infty(\mathbb{R}^+, H^1(Q)) \cap H^1_{loc}(\mathbb{R}^+, L^2(Q))$ and extract a limit u (up to a subsequence) such as $\forall t \in [0, T]$.

$$u_h(t) \rightharpoonup u(t) \text{ in } H^1(Q)$$

3 Show that u is the solution of our problem : for all $\phi \in C_c^\infty(\mathbb{R}^+ \times Q)$

$$\int_{Q \times R^+} u \phi_t dxdt = \int_{Q \times R^+} (\frac{1}{\epsilon^2} W'(u)\phi + \nabla u \cdot \nabla \phi) dxdt.$$

Comparison principle and uniqueness

Theorem

Let $\epsilon > 0$ *and* $u, v \in L^\infty([0,T], H^1(\mathbb{R})) \cap H^1([0,T], L^2(\mathbb{R}))$ *such as*

$$\begin{cases} u_t - \Delta u + \frac{1}{\epsilon^2} W'(u) \geq v_t - \Delta v + \frac{1}{\epsilon^2} W'(v), & in\ \mathbb{R}^d \times [0,T] \\ u(x,0) \geq v(x,0), & in\ \mathbb{R}^d \end{cases}$$

then

$$u(x,t) \geq v(x,t) \quad in\ \mathbb{R}^d \times [0,T]$$

Lemma (Gronwall)

Let $\varphi : [0,T] \to \mathbb{R}$ *be continuous such as* $\forall t \in [0,T]$, $\varphi(t) \leq C + L \int_0^t \varphi(s) ds$, *then* $\forall t \in [0,T]$, $\varphi(t) \leq Ce^{Lt}$.

Consider the function $\psi(t) = (C + L \int_0^t \varphi(s))e^{-Lt}$ and show that $\psi'(t) \leq 0$.

Proof of comparison principle

- For all positive function $\varphi \in L^\infty([0,T], H^1(\mathbb{R})) \cap H^1([0,T], L^2(\mathbb{R}))$,

$$\int_{\mathbb{R}^d} (v_t - u_t)\varphi dx \leq \int_{\mathbb{R}^d} \nabla(u-v)\nabla\varphi dx + \int_{\mathbb{R}^d} \frac{1}{\epsilon^2}(W'(u) - W'(v))\varphi dx.$$

- Consider $\varphi = max(v-u, 0)$ and then

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi^2 dx \leq \frac{2}{\epsilon^2}\int_{\mathbb{R}^d} (W'(u) - W'(v))\varphi dx.$$

- Using a decomposition of W' on the form $W' = W_l' + W_i'$ where W_i is increasing and W_l is M-Lipschitz leads to

$$\frac{d}{dt}\phi(t) \leq \frac{2M}{\epsilon^2}\phi(t), \text{ with } \phi(t) = \int_{\mathbb{R}^d} \varphi(x,t) dx$$

- Apply the Gronwall lemma to conclude.

Formal asymptotic expansion

- Let u_ϵ be a solution of the Allen Cahn equation and introduce the set

$$\Omega_\epsilon = \left\{ x \in \mathbb{R}^d \; ; \; u_\epsilon(x,t) \geq \frac{1}{2} \right\}.$$

- In a small neighborhood of $\Gamma_\epsilon = \partial\Omega_\epsilon$, consider the stretched variable

$$z = \frac{d(x,t)}{\epsilon} = \frac{dist(x, \Omega_\epsilon)}{\epsilon}$$

Outer and inner expansion

- Outer expansions (far from the interface)

$$u_\epsilon(x,t) = u_0^\pm(t) + \epsilon u_1^\pm(t) + \epsilon^2 u_2^\pm(t) + O(\epsilon^3),$$

for $x \in \Omega_\epsilon$ (corresponding to u_i^-) and for $x \in Q \setminus \Omega_\epsilon$) (corresponding to u_i^+).

- Inner expansions (around the interface)

$$u_\epsilon(x,t) = U(z,x,t) = U_0(z,x,t) + \epsilon U_1(z,x,t) + \epsilon^2 U_2(z,x,t) + O(\epsilon^3),$$

with the assumption : $\nabla_x U \cdot \nabla d(x,t) = 0$.

- Matching condition

$$\lim_{z \to \pm\infty} U_i(z,x,t) = u_i^\pm(t).$$

Outer expansion and Matching condition

- Order ϵ^{-2} :

$$W'(u_0) = 0 \Rightarrow u_0^- = 1 \text{ and } u_0^+ = 0.$$

This implies that U_0 satisfies the following boundary conditions

$$U_0(0, x, t) = 0, \lim_{z \to -\infty} U_0(z, x, t) = 1 \text{ and } \lim_{z \to +\infty} U_0(z, x, t) = 0.$$

- Order ϵ^{-1} and 1 :

$$W''(u_0)u_1 = 0 \Rightarrow u_1^{\pm} = 0,$$

and

$$W''(u_0)u_2 = 0 \Rightarrow u_2^{\pm} = 0.$$

Then we obtain that

$$U_i(0, x, t) = 0 \text{ and } \lim_{z \to \pm\infty} U_i(z, x, t) = 0 \text{ for } i \in \{1, 2\}.$$

Formal asymptotic expansion

- Velocity of the front

$$V_\epsilon = -\partial_t d(x,t) = V_0 + \epsilon V_1 + O(\epsilon^2)$$

- About derivative of u

$$\begin{cases} \nabla u_\epsilon = \nabla_x U_\epsilon + \epsilon^{-1} m \partial_z U_\epsilon \quad \text{where } m = \nabla d(x,t) \\ \Delta u_\epsilon = \Delta_x U_\epsilon + \epsilon^{-1} \Delta_d \partial_z U_\epsilon + \epsilon^{-2} \partial_{zz}^2 U_\epsilon \quad \text{as} \quad \nabla_x U \cdot m = 0 \\ \partial_t u_\epsilon = \partial_t U_\epsilon - \epsilon^{-1} V_\epsilon \partial_z U_\epsilon \end{cases}$$

- Geometric properties of the signed distance

$$\Delta d(x,t) = \sum \frac{\kappa_i}{1 + \kappa_i d(x,t)} \Rightarrow \Delta d = H - \epsilon z |A|^2 + O(\epsilon^2),$$

where $|A|^2 = \sum \kappa_i^2$.

Formal asymptotic expansion

- Order ϵ^{-2}

$$\partial^2_{zz} U_0 - W'(U_0) = 0 \Rightarrow U_0(z, x, t) = q(z)$$

- Order ϵ^{-1}

$$\partial_z U_0(H + V_0) + \partial^2_{zz} U_1 - W''(U_0) U_1 = 0$$

Multiplying by $\partial_z U_0$ and integrating in z over $\mathbb{R}$ leads that $V_0 = -H$. Moreover, $U_1 = 0$ satisfies

$$\partial^2_{zz} U_1 - W''(q) U_1 = 0,$$

and is on the form $U_1(x, z, t) = c(x, t) q'(z)$. The boundary condition on the surface ($U_1(x, 0, t) = 0$) finally shows that

$$U_1 = 0.$$

Formal asymptotic expansion

- Order ϵ^0

$$-V_1\partial_z U_0 = |A^2| z\partial_z U_0 + \partial^2_{zz} U_2 - W''(U_0)U_2,$$

implies that

$$V_1 = 0 \text{ and } U_2(z,x,t) = -|A(x)|^2\eta_1(z),$$

where η_1 is the function defined as the solution of

$$\eta_1'' - W''(q)\eta_1 = sq', \quad \text{with} \quad \lim_{\pm\infty}\eta_1(s) = 0 \text{ and } \eta_1(0) = 0.$$

Formal asymptotic expansion

- In conclusion, the solution u_ϵ as expected on the form

$$u_\epsilon(x,t) = q\left(\frac{dist(x,\Omega_\epsilon)}{\epsilon}\right) - \epsilon^2 |A(x)|^2 \, \eta_1\left(\frac{dist(x,\Omega_\epsilon)}{\epsilon}\right) + O(\epsilon^3),$$

where the normal velocity V_ϵ satisfies

$$V^\epsilon = H + O(\epsilon^2).$$

Convergence of the Allen Cahn equation

- Let $\Omega(t)$ a regular motion by mean curvature, $t \in [0, T]$
- Allen Cahn equation solution u_ϵ with initial condition

$$u_\epsilon(x, 0) = q\left(\frac{dist(x, \Omega(0))}{\epsilon}\right)$$

- Convergence of $\Omega^\epsilon \to \Omega$:
  [Mottoni-Schatzmann89] [Chen92] [Bellettini-Paolini95]

$$\sup_{t \in [0,T]} dist\left(\partial\Omega^\epsilon(t), \partial\Omega(t)\right) = O\left(\epsilon^2 \log(\epsilon)^2\right)$$

Idea of the proof

1. Construct a sub and a super solution of the Allen Cahn equation on the form

$$v_\epsilon^\pm(x,t) = q^\epsilon\left(\frac{dist_\epsilon^\pm(x,t)}{\epsilon}\right) - \epsilon^2|A(x,t)|^2\eta_1^\epsilon\left(\frac{dist_\epsilon^\pm(x,t)}{\epsilon}\right) \pm c_2\epsilon^3 log(\epsilon)^3,$$

where

$$dist_\epsilon^\pm = dist(\Omega(t),x) \mp c_1\epsilon^2 log(\epsilon)^2.$$

2. From comparison principle, deduce that

$$v_\epsilon^-(x,t) \le u_\epsilon \le v_\epsilon^+(x,t),$$

and then that

$$dist(\partial\Omega^\epsilon,\partial\Omega) \le C\epsilon^2 log(\epsilon)^2.$$

Resolution of Allen Cahn equation

Resolution of the Allen Cahn equation in $Q = [0,1]^n$ with periodic boundary condition :

$$\begin{cases} u_t(x,t) = \Delta u(x,t) - \frac{1}{\epsilon^2} W'(u(x,t)), \text{ for } (x,t) \in Q \times [0,T], \\ u(x,0) = u_0 \in [0,1] \end{cases}$$

- An Euler explicit scheme : $u^n \simeq u(n\delta_t)$ where

$$u^{n+1} - u^n = \delta_t(\Delta u^n) - \frac{1}{\epsilon^2} W'(u^n),$$

  but stability issue.

Euler implicit

- An Euler implicit scheme :

$$u^{n+1} - u^n = \delta_t \left(\Delta u^{n+1} - \frac{1}{\epsilon^2} W'(u^{n+1}) \right),$$

or

$$u^{n+1} = \arg\min_u \left\{ \frac{1}{\epsilon} P_\epsilon(u) + \frac{1}{2\delta_t} \int (u - u^n)^2 dx \right\}.$$

- Resolution with a fixed point iteration

$$\phi(u) = (1 - \delta_t \Delta)^{-1} \left(u^n - \frac{\delta_t}{\epsilon^2} W'(u) \right),$$

which locally converges if $\delta_t \leq M^{-1}\epsilon^2$ where $M = \sup_{s \in [0,1]} \{|W''(s)|\}$

Lie and Strang splitting algorithm

- Let $y : [0, T] \to \mathbb{R}^n$ be solution of

$$\begin{cases} y_t(t) = (A + B)y(t), & \forall t \in [0, T] \\ y(0) = y_0 \end{cases},$$

  and satisfies $y(t) = e^{(A+B)t} y_0$
- Lie splitting

$$e^{(A+B)t} = e^{At} e^{Bt} + \frac{t^2}{2}[A, B] + O(t^3),$$

  where $[A, B] = AB - BA$.
- Symmetric Strange splitting

$$e^{(A+B)t} = e^{At/2} e^{Bt} e^{At/2} + \frac{t^3}{24}([A, [A, B]] - 2[B, [A, B]]) + O(t^4).$$

A splitting algorithm

- Allen-Cahn equation :

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) - \frac{1}{\epsilon^2} W'(u(x,t)) \\ u(x,0) = u_0 \in W^{1,\infty}(\mathbb{R}^d) \end{cases} \qquad \begin{cases} S(t) & : \text{total flow} \\ e^{t\Delta} & : \text{diffusion part} \\ Y(t) & : \text{reaction part} \end{cases}$$

- A Splitting Lie formula with $L(t) = Y(t)e^{t\Delta}$,

$$\|L(\delta_t)^n u_0 - S(n\delta_t)u_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{M}{\epsilon^2} \sqrt{\delta_t}\, \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}$$

Resolution of each operator

- Exact resolution of diffusion part in Fourier space : with

$$U_N^n(x) = \sum_{k_1,k_2,\cdots k_n=-N/2+1}^{N/2} c_k^n e^{2i\pi k\cdot x},$$

then

$$U_N^{n+1}(x) = e^{\Delta\delta_t} U_N(x,t) = \sum_{k_1,k_2,\cdots k_n=-N/2+1}^{N/2} e^{-4\pi^2|k|^2\delta_t} c_k e^{2i\pi k\cdot x}$$

- Treatment of the reaction part

$$\begin{cases} \text{By ODE integration} & \rightarrow \text{ unconditionally stable} \\ \text{Explicitly} & \rightarrow \text{ Stability under condition} \end{cases}$$

$$\delta_t \leq \left(\max_{s\in[0,1]} \left\{ W''(s) \right\} \right)^{-1} \epsilon^2$$

Matlab code

```
%%%%%%%%%%%%%%%% parameters %%%%%%%%%
N = 2^8;                % spatial resolution
epsilon = 2/N;
delta_t = 2*epsilon^2;  % time step
T = 0.05;

%%%%%%%%%%%%% initialisation u(x,0) : \Omega_0 is a circle radius of size 0.3
X1 = ones(N,1)*linspace(-1/2,1/2,N);
X2 = X1';
dist = sqrt(X1.^2 + X2.^2) - 0.3;
U1 = 1/2 - tanh(dist/(2*epsilon))/2;

%%%%%%%%%%%%%%%%% Heat kernel  %%%%%%%%%%%%%%%
K1 = [0:N/2,-N/2+1:-1]'*ones(1,N);
K2 = K1';
M = exp(-delta_t*4*pi^2*(K1.^2 + (K2).^2));

%%%%%%%%%%%%%%%  Allen Cahn solution computation %%%%%%%%%%
for n=1:T/delta_t,
W_prim = U1.*(U1-1).*(2*U1 -1);
U1 = ifft2(M.*fft2(U1 - delta_t/epsilon^2*(W_prim)));

%%%%%%%%%%%%% plot the solution all the 10 iterations
if (mod(n,1)==0)
   imagesc(U1);
   caxis([0,1])
   pause(0.01);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

end
```

Validation of this numerical method

- Initial set : a circle of radius R_0
- The motion by mean curvature remains a circle of radius $R(t) = \sqrt{R_0^2 - 2t}$,
- Extinction time : $t_{ext} = \frac{1}{2} R_0^2$

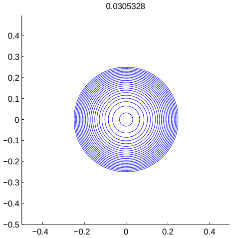


fig : $\partial\Omega^\epsilon(t)$ at different times t

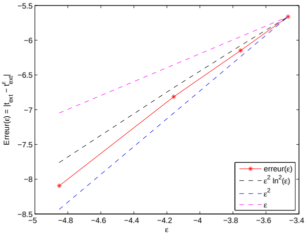
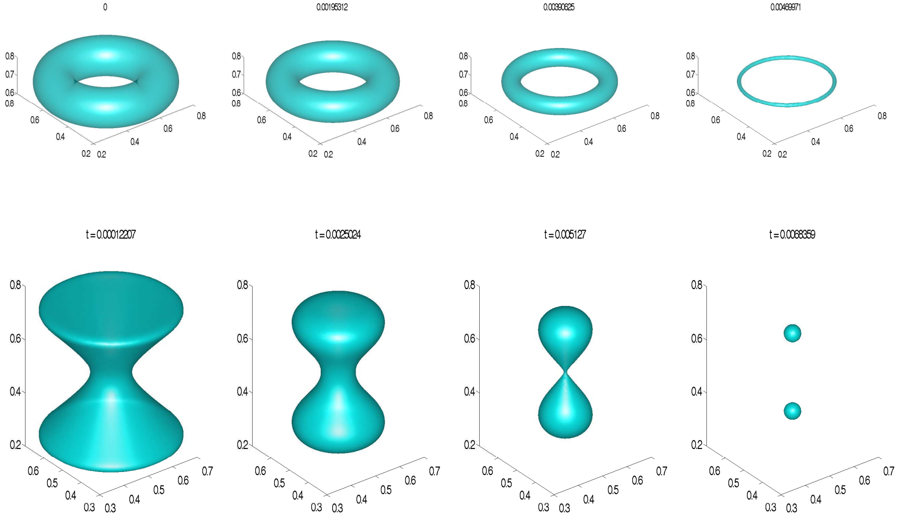


fig : Error $\epsilon \to |t_{ext} - t_{ext}^\epsilon|$

Some simulations



1. Introduction
2. Phase field approximation of mean curvature flow
3. Conserved and multiphase mean curvature flow
   - Conserved mean curvature flow
   - Inclusion-Exclusion boundary constraints
   - Multiphase mean curvature flow
4. Approximation of Willmore energy and flow:

Conserved mean curvature flow

- L^2-gradient flow of the perimeter under the constraint $Vol(\Omega) = C$:

$$V_n = -H + \lambda 1,$$

where λ is the Lagrange multiplier associated to the constraint.

- Then, the equality $\frac{d}{dt} Vol(\Omega(t)) = 0$ implies that

$$\int_{\partial\Omega} V_n 1 d\sigma = 0 \quad \Rightarrow \quad \lambda = \fint_{\partial\Omega} H d\sigma = \overline{H}.$$

- Conserved mean curvature law

$$V_n = -H + \overline{H}.$$

Some properties of conserved mean curvature flow

- Local existence of smooth solution in dimension 2 [Elliott and Garcke 1997]
- Local existence of smooth solution in arbitrary dimension + global solution for initial set sufficiently closed to the sphere [Esher and Simonett 1998]
- Global existence and uniqueness for convex initial set [Huysken 1987]
- Singularities in finite time and no inclusion principle !

An example of conserved mean curvature flow

- Initial set Ω_0 : union if two disjoint circles of radii R_0 and R_1 with $R_0 < R_1$
- Then, $\Omega(t)$ remains the union of two circles of radii $R_0(t)$ and $R_1(t)$, defined as the solutions of

$$\begin{cases} \frac{dR_0}{dt} = -\frac{1}{R_0} + \frac{2}{R_0+R_1} \\ \frac{dR_1}{dt} = -\frac{1}{R_1} + \frac{2}{R_0+R_1} \end{cases}$$

- Singularities in finite time

$$t_s = -\frac{R_0 R_1}{2} + \frac{R_0^2 + R_1^2}{4} ln\left(1 + \frac{2R_0 R_1}{(R_1 - R_0)^2}\right).$$

Phase field versus

- Approximation of the volume :

$$Vol(\Omega) = \int u dx \quad \text{if } u = \chi_\Omega.$$

- L^2-gradient flow of P_ϵ under the constraint $\int u dx = Const$.
- Conserved Allen Cahn equation

$$u_t = \Delta u - \frac{1}{\epsilon^2}(W'(u) + \epsilon\lambda)$$

where

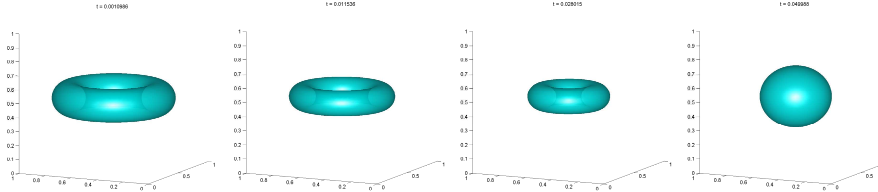
$$\lambda = \frac{1}{\epsilon} \fint W'(u) dx.$$

- Convergence to conserved mean curvature flow [Chen, Hilhorst and Logak 2009]

Numerical experiments in dimension 3

- Scheme : a Fourier-splitting approach with a explicit traitement of the reaction terms :

$$\begin{cases} u^{n+1/2} = e^{\Delta \delta_t} u^n \\ u^{n+1} = u^{n+1/2} + \frac{\delta_t}{\epsilon^2} \left(W'(u^{n+1/2}) - \fint W'(u^{n+1/2}) dx \right) \end{cases}$$



- Losses of volume observed but $\int u^n dx$ does not moved !

Asymptotic expansion

- Conserved Allen Cahn equation

$$u_t = \Delta u - \frac{1}{\epsilon^2}\left(W'(u) + \epsilon\lambda\right) \text{ with } \lambda = \frac{1}{\epsilon}\fint W'(u)dx.$$

- Inner expansions (around the interface)

$$u_\epsilon(x,t) = U(z,x,t) = U_0(z,x,t) + \epsilon U_1(z,x,t) + \epsilon^2 U_2(z,x,t) + O(\epsilon^3).$$

- Expansion of the Lagrange multiplier λ

$$\lambda(t) = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + O(\epsilon^3)$$

Formal asymptotic expansion

- Order ϵ^{-2}

$$\partial^2_{zz} U_0 - W'(U_0) = 0 \Rightarrow U_0(z, x, t) = q(z)$$

- Order ϵ^{-1}

$$\partial_z U_0(H + V_0) - \lambda_0 + \partial^2_{zz} U_1 - W''(U_0) U_1 = 0$$

Multiplying by $\partial_z U_0$ and integrating in z on $\mathbb{R}$ leads that

$$V_0 = -H + \lambda_0 / c_W.$$

Moreover, $U_1 = \frac{\lambda_0}{c_W} \eta(s)$ where η is solution of

$$\begin{cases} \eta''(s) - W''(q)\eta = q'(s) - c_W \\ \eta(0) = 0. \end{cases}$$

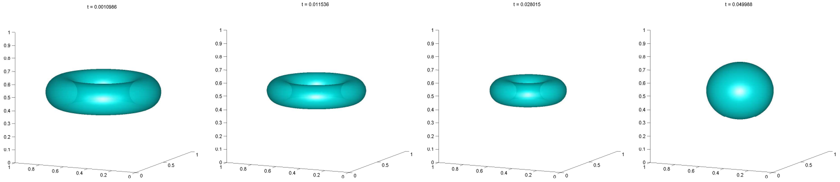
Formal asymptotic expansion

- The solution u_ϵ is then expected on the form

$$u_\epsilon(x,t) = q\left(\frac{dist(x,\Omega^\epsilon)}{\epsilon}\right) + \epsilon\frac{\lambda_0(t)}{c_W}\eta\left(\frac{dist(x,\Omega^\epsilon)}{\epsilon}\right) + O(\epsilon^2)$$

- Approximation of the volume :

$$Vol(\Omega^\epsilon(t)) = \int_\epsilon u_\epsilon(x,t)dx + O(\epsilon) \text{ only .}$$



- Losses of volume observed of order $O(\epsilon)$!

How to obtain a more efficient model ?

- Asymptotic expansion of u_ϵ :

$$\partial_z U_0(H + V_0) - \lambda_0 + \partial^2_{zz} U_1 - W''(U_0)U_1 = 0$$

- Remark that if

$$\partial_z U_0(H + V_0 - \lambda_0) + \partial^2_{zz} U_1 - W''(U_0)U_1 = 0,$$

then

$$V_0 = -H + \lambda_0 \text{ and } U_1 = 0$$

A modified conserved Allen Cahn equation

- A conserved Allen Cahn equation

$$u_t = \Delta u - \frac{1}{\epsilon^2}\left(W'(u) + \epsilon\lambda\sqrt{2W(u)}\right) \text{ with } \lambda = \frac{1}{\epsilon}\frac{\int W'(u)dx}{\int \sqrt{2W(u)}dx.}.$$

- Asymptotic expansion

$$u_\epsilon(x,t) = q\left(\frac{dist(x,\Omega^\epsilon)}{\epsilon}\right) + O(\epsilon^2)$$

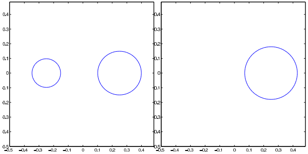
- Approximation of the volume :

$$Vol(\Omega^\epsilon(t)) = \int_\epsilon u_\epsilon(x,t)dx + O(\epsilon^2).$$

- Convergence to conserved mean curvature flow [Alfaro and Alifrangis 2014]

Numerical evidence of order of convergence

Case of two disjoint circles of radii R_0 and R_1 with $R_0 < R_1$



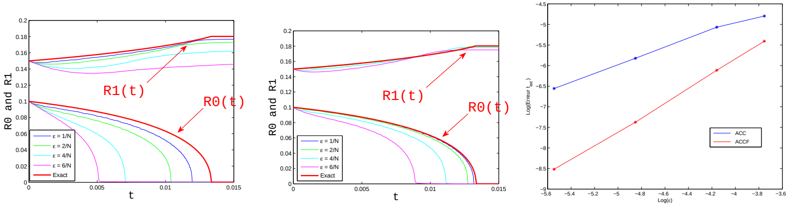
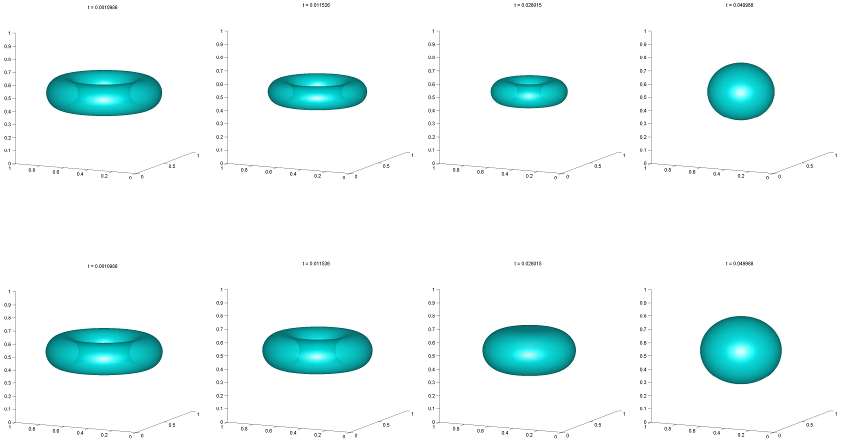


fig : Left : ACC, Center : ACCM, Right : $\epsilon \to |t_{ext} - t^{\epsilon}_{ext}|$

Comparison of the two models in dimension 3



1. Introduction
2. Phase field approximation of mean curvature flow
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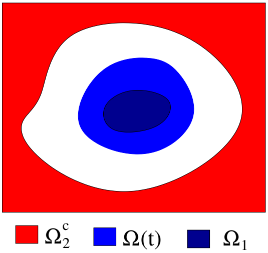
Inclusion-Exclusion boundary constraints

Minimization of

$$P_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} P(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2 \\ +\infty & \text{otherwise,} \end{cases}$$

for two given smooth sets Ω_1 and Ω_2 with $dist(\partial\Omega_1, \partial\Omega_1) > 0$.



Dirichlet boundary conditions :

- Dirichlet Cahn Hilliard energy approximation

$$\tilde{P}_{\epsilon,\Omega_1,\Omega_2}(u) = \begin{cases} \int_{\Omega_2 \setminus \Omega_1} \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) dx & \text{if } u \in X_{\Omega_1,\Omega_2} \\ +\infty & \text{otherwise.} \end{cases}$$

with

$$X_{\Omega_1,\Omega_2} = \left\{ u \in H^1(\Omega_2 \setminus \Omega_1) \; ; \; u_{|\partial\Omega_1} = 1 \quad , u_{|\partial\Omega_2} = 0 \right\},$$

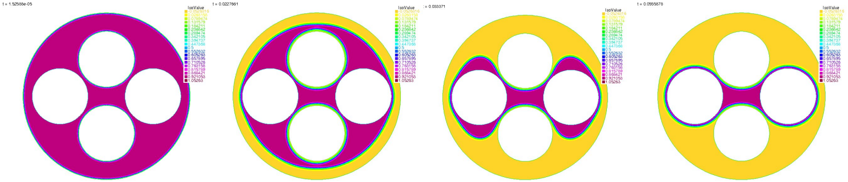
- Γ-convergence [Chambolle Bourdin] of $\tilde{P}_{\epsilon,\Omega_1,\Omega_2}$ *to* $c_W P_{\epsilon,\Omega_1,\Omega_2}$ in the $L^1(\mathbb{R}^d)$ topology :
- Order of convergence ? But $dist(\partial\Omega, \partial\Omega_1) > \epsilon$ and $dist(\partial\Omega, \partial\Omega_1) > \epsilon$!

Dirichlet boundary conditions :

- Allen Cahn equation with boundary Dirichlet conditions

$$\begin{cases} u_t = \triangle u - \frac{1}{\epsilon^2} W'(u), \quad \text{on} \quad \Omega_2 \setminus \Omega_1 \\ u_{|\partial\Omega_1} = 1, \quad u_{|\partial\Omega_2} = 0 \\ u(0,x) = u_0 \in X_{\Omega_1,\Omega_2}. \end{cases}$$

- Numerical scheme : Implicit Euler scheme in time and finite elements discretization in space.
- A numerical experiment with Freefem++



An other penalized Cahn Hilliard energy

- Cahn Hilliard Energy

$$P_{\epsilon,\Omega_1,\Omega_2}(u) = \begin{cases} P_\epsilon(u) & \text{if } u_{1,\epsilon} \leq u \leq 1 - u_{2,\epsilon} \\ +\infty & \text{otherwise,} \end{cases} ,$$

where $u_{1,\epsilon}$ and $u_{2,\epsilon}$ are defined by

$$u_{1,\epsilon} = q\left(\frac{dist(x,\Omega_1)}{\epsilon}\right) \quad \text{and } u_{2,\epsilon} = q\left(\frac{dist(x,\Omega_2)}{\epsilon}\right)$$

- Γ-convergence of $P_{\epsilon,\Omega_1,\Omega_2}$ to $c_W P_{\Omega_1,\Omega_2}$?
  Yes, slightly adaptation of Modica-Mortola proof !
- Order of convergence ? but $dist(\partial\Omega, \partial\Omega_1)$ and $dist(\partial\Omega, \partial\Omega_1)$ can be equal to zero !

Numerical scheme

- An Euler implicit scheme

$$u^{n+1} = \underset{u_{1,\epsilon} \leq v \leq 1 - u_{2,\epsilon}}{\arg\min} \left\{ P_\epsilon(v) + \frac{1}{2\delta_t} \int (v - u^n)^2 dx \right\}.$$

- The solution u^{n+1} can be obtained by a fixed point iteration

$$\phi(u) = Proj_{u_{1,\epsilon}, u_{2,\epsilon}} \left[(Id - \delta_t \Delta)^{-1} \left(u^n - \frac{\delta_t}{\epsilon^2} W'(u) \right) \right]$$

where the projector $P_{u_{1,\epsilon} \leq v \leq 1 - u_{2,\epsilon}}$ is defined by

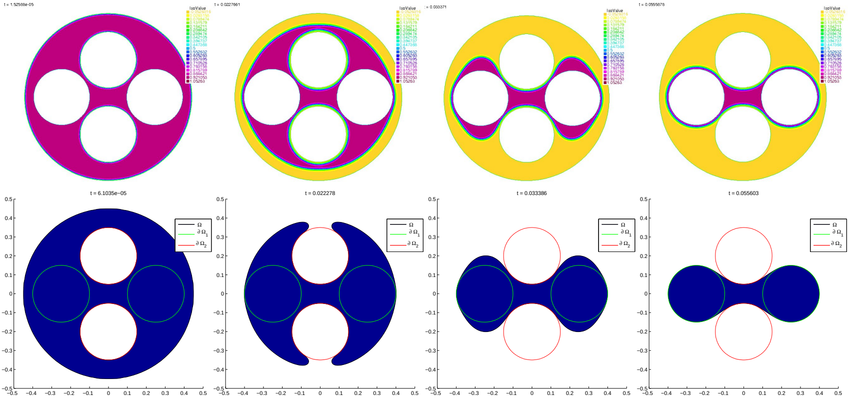
$$Proj_{u_1 \leq u \leq 1 - u_2}[v] = \min(\max(u_{1,\epsilon}, v), 1 - u_{2,\epsilon}),$$

and $(1 - \delta_t \Delta)^{-1}$ can be solved in Fourier space

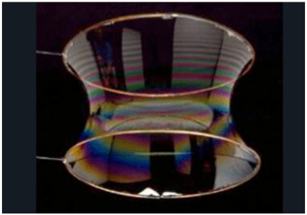
Matlab code

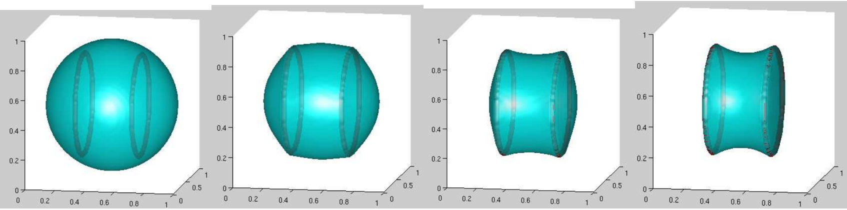
```
%%%%%%%%%%%%%% Parameters  %%%%%%%%%%%%%%%%
epsilon = 2/N;
T =1;
delta_t = 1/N^2;
%%%%%%%%%%%%%% Heat Kernel %%%%%%%%%%%%%%%%%%
 K1 = ones(N,1)*[0:N/2,-N/2+1:-1];
 M = 1./(1+4*pi^2*delta_t*(K1.^2 + K1'.^2));
%%%%%%%%%%%%%% Minimization scheme %%%%%%%%%%%%
for n=1:T/delta_t,
  U = U1_0;
  U1_0_fourier = fft2(U1_0);
  res = 1;

  %%%%%%%% fixed point iteration %%%%%%%%%%%%
  while res > 10^(-4),
  U_plus = ifft2( M.*(U1_0_fourier - delta_t/epsilon^2*fft2(U.*(U-1).*(2*U-1))));
  U_plus = max(min(1-U2,U_plus),U1);
  res = norm((U_plus-U));
  U = U_plus;
  end
  U1_0 = U;

end
```

Numerical experiments



Numerical experiment : example of minimal surface



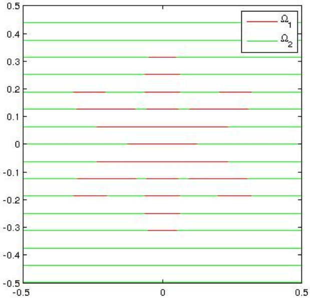


Case of thin constraints

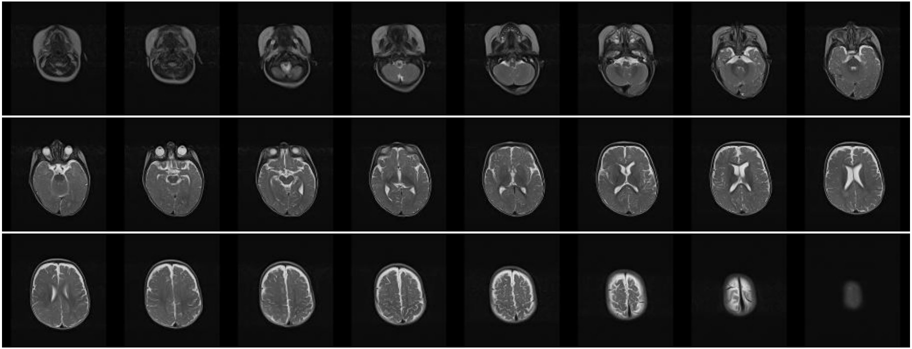
- Find the set Ω^* as a minimizer of

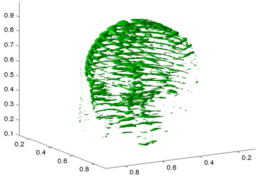
$$P_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} P(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2^c \\ +\infty & \text{otherwise} \end{cases},$$

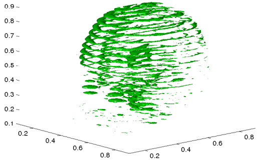
with $\mathring{\Omega}_1 = \mathring{\Omega}_2 = \emptyset$.

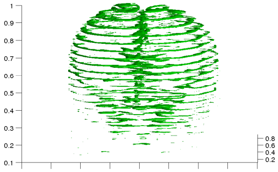


Application in Magnetic Resonance Imaging



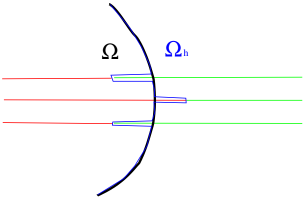






About semi-continuity of P_{Ω_1,Ω_2} in L^1-topology

- Note that P_{Ω_1,Ω_2} is not lower semi-continuous



$$P_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} P(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2^c \\ +\infty & \text{otherwise} \end{cases}$$

- Relaxation of the penalized perimeter

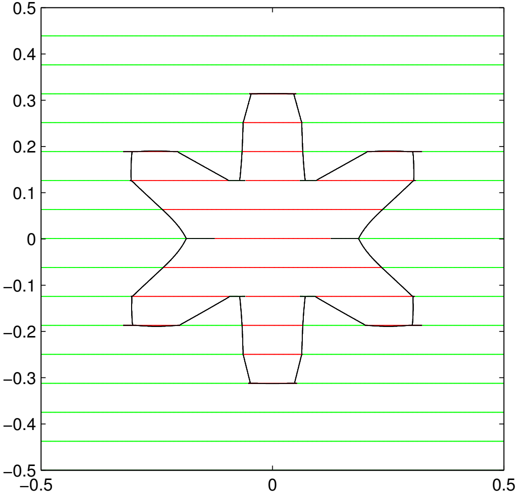
$$\overline{P_{\Omega_1,\Omega_2}}(\Omega) = \inf\{\liminf P_{\Omega_1,\Omega_2}(\Omega_h),\ \partial\Omega_h \in \mathrm{C}^2,\ \Omega_h \to \Omega \text{ in } \mathrm{L}^1(\Omega)\}.$$

- Identification of $\overline{P_{\Omega_1,\Omega_2}}$?

$$\overline{P_{\Omega_1,\Omega_2}}(\Omega) = P(\Omega) + 2\mathcal{H}^{n-1}(\Omega^0 \cap \Omega_1) + 2\mathcal{H}^{n-1}(\Omega^1 \cap \Omega_2)$$

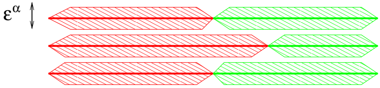
Numerical experiment

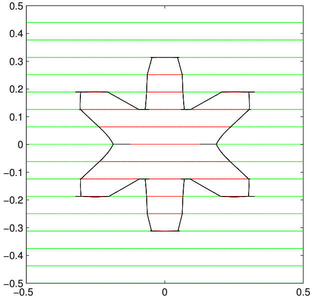
The constraint are not satisfied at the limit when $\epsilon \to 0$!

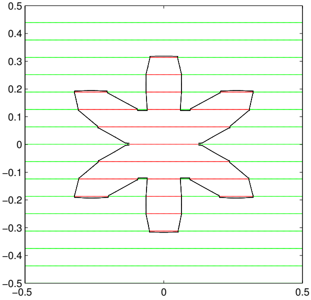


An idea : Use some thickened constraints

$P_{\epsilon,u^e_{1,\epsilon},u^e_{2,\epsilon}}$ where $u^e_{i,\epsilon} = q\left(\frac{dist(x,\Omega^\epsilon_i)}{\epsilon}\right)$.







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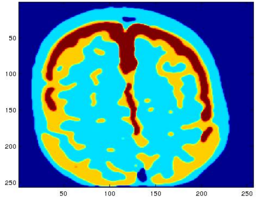
Multiphase perimeter

$$P(\Omega_1, \Omega_2, \dots, \Omega_N) = \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_j} 1 d\sigma(x),$$

where $\{\Omega\}_{i=1:N}$ formed a partition of Ω :

$$\Omega = \cup_{i=1}^{N} \Omega_i, \quad \text{and} \quad |\Omega_i \cap \Omega_j| = 0, \forall i \neq j.$$

Motivations : Image segmentation, optimal partition, bubble conjecture !



Multiphase Cahn Hilliard Energy

- Generalized Perimeter

$$P(\mathbf{u}) = \frac{1}{2}\sum_{i=1}^{N} |Du_i|(\Omega),$$

if it exists a partition $\{\Omega_i\}_{i=1:N}$ of Ω such as $\mathbf{u} = (1_{\Omega_1}, \cdots, 1_{\Omega_N})$.

- Multiphase Cahn Hilliard Energy,

$$P_\epsilon(\mathbf{u}) = \begin{cases} \frac{1}{2}\sum_i P_\epsilon(u_i), & \text{if } \mathbf{u} \in \Sigma, \\ +\infty \text{otherwise} \end{cases},$$

where $\Sigma = \left\{\mathbf{u} = (u_1, u_2, \cdots, u_N) \in \mathbb{R}^N \;;\; \sum_{i=1}^N u_i = 1\right\}$.

Property

P_ϵ Γ-converges to $c_W P$ for the L^1 topology.

Γ liminf inequality :

$$\forall \mathbf{u}_\epsilon \to \mathbf{u} \Longrightarrow \liminf_{\epsilon \to 0} P_\epsilon(\mathbf{u}_\epsilon) \geq c_W P(\mathbf{u})$$

- Modica Mortola applied to u_i shows that the existence of a set Ω_i such as $u_i = 1_{\Omega_i}$ and

$$\liminf_{\epsilon \to 0} \int \frac{\epsilon}{2} |\nabla u_{\epsilon,i}|^2 + \frac{1}{\epsilon} W(u_{\epsilon,i}) dx \geq c_W P(\Omega_i).$$

- Moreover, the constraint $\mathbf{u}_\epsilon \in \Sigma$ shows that $\{\Omega_i\}_{i=1:N}$is a partition of Ω and then

$$\liminf_{\epsilon \to 0} P_\epsilon(\mathbf{u}_\epsilon) \geq \frac{1}{2} \sum_{i=1}^{N} c_W P(\Omega_i) = c_W P(\mathbf{u}).$$

Γ limsup inequality :

$\forall \mathbf{u} = (\chi_{\Omega_1}, \chi_{\Omega_2}, \cdots \chi_{\Omega_N})$, where $\{\Omega_i\}_{i=1}^N$ is a partition of Ω,

it exists a sequence $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ such as $\limsup_{\epsilon\to 0} P_\epsilon(\mathbf{u}_\epsilon) \leq c_W P(\mathbf{u})$.

- We would like to take

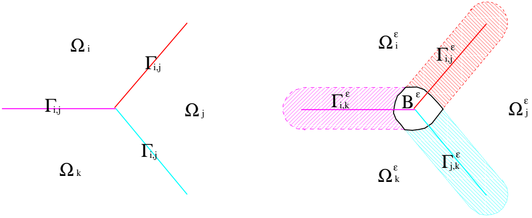
$$\mathbf{u}^\epsilon = \sum_{i=1}^N q\left(\frac{dist(x, \Omega_i)}{\epsilon}\right)\mathbf{e_i},$$

  but $\mathbf{u}_\epsilon \notin \Sigma$, !
- Restriction to polygonal partition [Baldo1990]
- Approximation q_ϵ of $q = (1 - \tanh(s))/2$ such as

$$\begin{cases} q_\epsilon(s) & = 0 \text{ if } s < -s_\epsilon \\ q_\epsilon(s) & = 1 \text{ if } s > s_\epsilon \\ q_\epsilon(s) & = q(s) \text{ if } |s| < s_\epsilon/2 \end{cases}, \text{ with } s_\epsilon = O(\epsilon)$$

Γ limsup inequality

- New partition of the domain



- Construction of $\mathbf{u}_\epsilon$:

$$\mathbf{u}_\epsilon(x) = \sum_{i=1}^{N} q_\epsilon\left(\frac{d_i}{\epsilon}\right)\mathbf{e_i} = \begin{cases} \mathbf{e_i} & \text{if } x \in \Omega_i^\epsilon, \\ q_\epsilon(d_i/\epsilon)\mathbf{e_i} + (1 - q_\epsilon(d_i/\epsilon))\mathbf{e_j} & \text{if } x \in \Gamma_{i,j}^\epsilon \\ \dots & \text{if } x \in B^\epsilon \end{cases},$$

- It works as $|B^\epsilon| = O(\epsilon^2 s_\epsilon^2)$, and $\mathbf{u}_\epsilon$ has the good profile in $\Gamma_{i,j}^\epsilon$...

Multiphase Allen Cahn equation

- Multi Cahn Hilliard energy : if $\mathbf{u} \in \Sigma$

$$P_\epsilon(\mathbf{u}) = \frac{1}{2} \sum_i P_\epsilon(u_i)$$

- The L^2 gradient flow of M_ϵ reads

$$\partial_t u_i = \frac{1}{2}\left(\Delta u_i - \frac{1}{\epsilon^2} W'(u_i)\right) + \lambda(x) \tag{1}$$

where $\lambda(x)$ is a Lagrange multiplier associated to the constraint $\mathbf{u} \in \Sigma$ and satisfies

$$\lambda(x) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\epsilon^2} W'(u_i).$$

Application to image segmentation

- Data : Image $I : \Omega \to \mathbb{R}$, color coefficient $c = (c_1, c_2, \cdots c_N)$
- Image segmentation model : Minimize

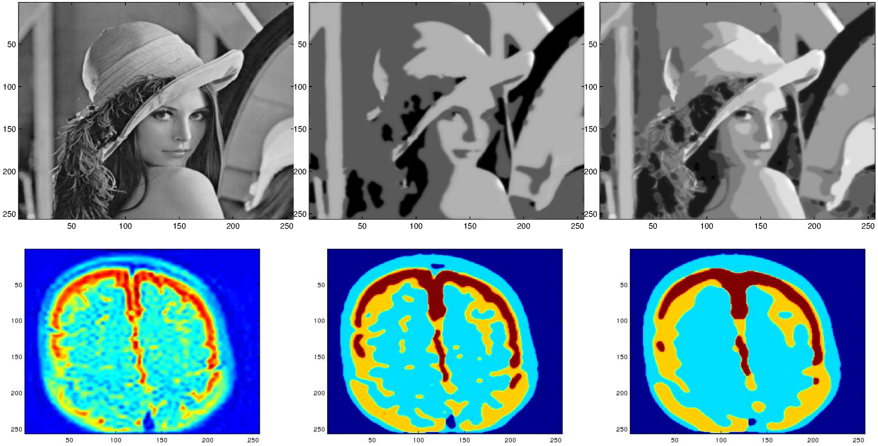
$$J(\Omega_1, \Omega_2, \cdots, \Omega_N) = \sum_{i=1}^{N} \left(\frac{1}{2\alpha} \int_{\Omega_i} (I(x) - c_i)^2 dx + P(\Omega_i) \right),$$

on the set of all partition $\{\Omega_i\}_{i=1:N}$ of Ω.

- Phase field approximation

$$J_\epsilon(\mathbf{u}) = \frac{1}{2\alpha} \int_{\Omega} (I(x) - c \cdot \mathbf{u})^2 dx + \frac{1}{c_W} P_\epsilon(\mathbf{u}).$$

Numerical experiments for different values of α and N



Additional constraint on the volume of each phase

- Minimization of the Cahn Hilliard energy

$$P_\epsilon(\mathbf{u}) = \frac{1}{2}\sum_i P_\epsilon(u_i),$$

under the constraint $\mathbf{u} \in \Sigma$ and

- The L^2 gradient flow of M_ϵ reads

$$\partial_t u_i = \Delta u_i - \frac{1}{\epsilon^2} W'(u_i) + \mu_i \sqrt{2W(u_i)} + \lambda(x),$$

where λ and μ_i are respectively the Lagrange multipliers associated to the constraint $\mathbf{u} \in \Sigma$ and $\int u_i = V_i$ for all $i = 1 : N$.

Additional constraint on the volume of each phase

- One degree of freedom : $\overline{\lambda} = \int \lambda(x)dx$.
- Integrating the Allen Cahn equation over Ω gives

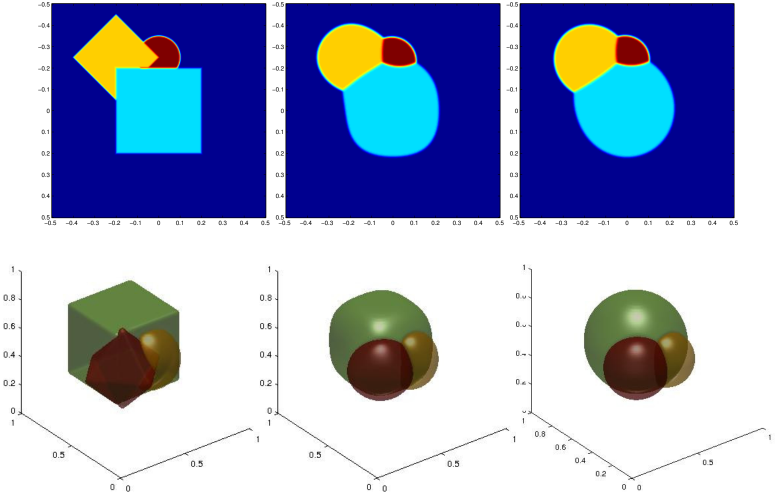
$$\mu_i = \frac{\frac{1}{\epsilon^2}\int W'(u_i)dx - \overline{\lambda}}{\int \sqrt{2W(u_i)}dx}$$

- Summing the Allen Cahn equations gives

$$\begin{aligned}\lambda(x) &= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\epsilon^2}W'(u_i) - \mu_i\sqrt{2W(u_i)}-\right)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\epsilon^2}W'(u_i) - \frac{\frac{1}{\epsilon^2}\int W'(u_i)dx}{\int \sqrt{2W(u_i)}dx}\sqrt{2W(u_i)}\right) + \overline{\lambda}\sum_{i=1}^{N}\frac{\sqrt{2W(u_i)}}{\int \sqrt{2W(u_i)}dx}\end{aligned}$$

- In practice, choose $\overline{\lambda} = 0$.

Numerical experiment : Evolution of three bubbles in 2D and 3D



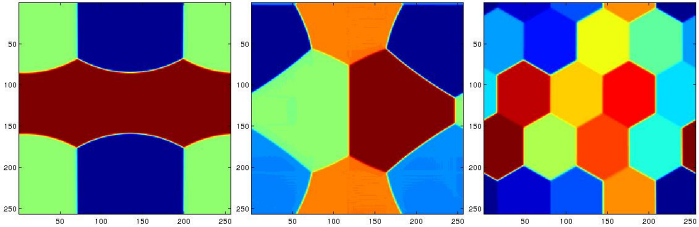


fig : Optimal partition in 2D with respectively $N = 3$, $N = 5$ and $N = 16$ phases

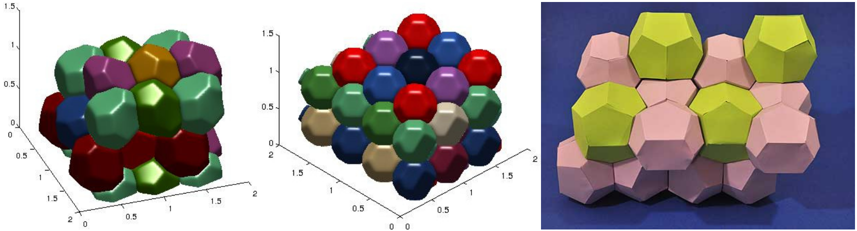


fig : Optimal partition in 2D with respectively $N = 8$ and $N = 16$ phases. Right : Weaire and Phelan structure

Willmore Flow

- Willmore Energy

$$\mathcal{W}(\Omega) = \frac{1}{2}\int_{\partial\Omega} H^2 d\mathcal{H}^{\mathrm{n}-1},$$

- L^2 gradient flow

$$Vn = \Delta_S H + |A|^2 H - \frac{1}{2}H^3,$$

where $|A|^2 = \sum \kappa_i^2$.

- In dimension 2 :

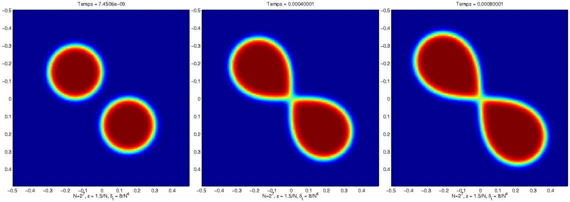
$$Vn = \Delta_S H + \frac{1}{2}H^3,$$

- In dimension 3 :

$$Vn = \Delta_S H + \frac{1}{2}H(H^2 - 4G),$$

Existence and regularity of Willmore Flow,

- Long time existence : single curve – [Dziuk Kuwert Schatzle-2002],
- Long time existence : higher dimension (small energy) [Kuwert Schatzle-2001]
- But in general, singularities in finite time !



Example of Willmore flow in dimension three

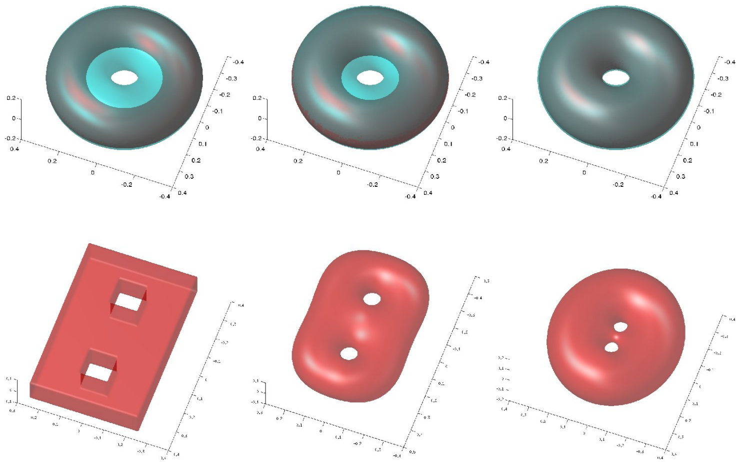


fig : Two smooth evolutions by Willmore flow ; a Clifford's torus and a Lawson-Kusner surface

Classical approximation of Willmore energy

- Phase field approximation

$$u_\epsilon = q\left(\frac{dist(x,\Omega)}{\epsilon}\right), \quad \text{with} \quad q'(s) = -\sqrt{2W(q(s))}.$$

- Remarks that

$$\begin{aligned}\frac{1}{\epsilon}\left(\epsilon\Delta u_\epsilon - \frac{1}{\epsilon}W'(u_\epsilon)\right) &= (\Delta dist(x,\Omega))^2\,\frac{1}{\epsilon}q'\left(\frac{dist(x,\Omega)}{\epsilon}\right)^2 \\ &\to H(x)^2 c_W \delta_{\partial\Omega}\end{aligned}$$

- Then, at least for smooth set Ω, we have

$$\mathcal{W}_\epsilon(u_\epsilon) = \frac{1}{2\epsilon}\int_{\mathbb{R}^d}\left(\epsilon\Delta u_\epsilon - \frac{1}{\epsilon}W'(u_\epsilon)\right)^2 dx \underset{\epsilon\to 0}{\Longrightarrow} c_W\frac{1}{2}\int_{\partial\Omega} H^2 d\mathcal{H}^{n-1}$$

De Giorgi conjecture : Γ-convergence of W_ϵ ?

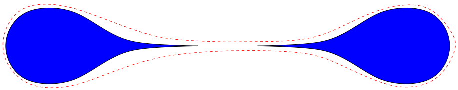
Definition (Classical approximation of Willmore energy)

$$\mathcal{W}_\epsilon(u) = \frac{1}{2\epsilon}\int_{\mathbb{R}^d}\left(\epsilon\Delta u - \frac{1}{\epsilon}W'(u)\right)^2 dx$$

Gamma-convergence of W_ϵ to $c_W\mathcal{W}$?
Ok in the case of C^2 Set adding a perimeter term [Röger,Schätzle 2006],[Nagase,Tonegawa 2007],

$$\Gamma - \lim_{\epsilon\to 0}(W_\epsilon + P_\epsilon) = c_W(\mathcal{W} + P)$$

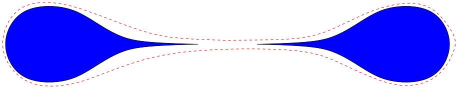
But $\mathcal{W}$ is not lower semi-continuous !



A relaxation of Willmore energy

- Semi-continuous envelope of $\overline{\mathcal{W}}$ for L^1-topology of set

$$\overline{\mathcal{W}}(\Omega) = \inf\{\liminf \mathcal{W}(\Omega_h),\ \partial\Omega_h \in \mathrm{C}^2,\ \Omega_h \to \Omega \text{ in } \mathrm{L}^1(\Omega)\}.$$



- Characterization of finite relaxed Willmore energy in dimension 2 : [Bellettini,Maso and Paolini 1993],[Bellettini,Mugnai,2004]

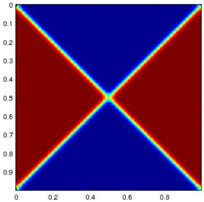
  if $\overline{\mathcal{W}}(E) < +\infty$, then a non oriented tangent must exist everywhere on the boundary of E.

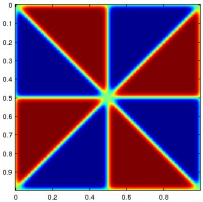
Γ-convergence of $\mathcal{W}_\epsilon$ to $c_W \overline{\mathcal{W}}$?

- Existence of Allen Cahn solutions [Dang Fife Peletier 92],[Kowalczyk Pacard 2012]

$$\Delta u_\epsilon - \frac{1}{\epsilon^2} W'(u_\epsilon) = 0,$$

  such as $u_\epsilon \to \chi_E$ with $\overline{\mathcal{W}}(E) = +\infty$
- Example of Allen Cahn solutions





- Find another relaxation (see varifold) but requirement of a classification of all Allen Cahn solutions !

Others approximations of Willmore Energy

Definition (Bellettini's approximation in dimension $N \geq 2$)

$$\mathcal{W}_\epsilon^B(u) = \frac{1}{2}\int \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)^2 \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{W(u)}{\epsilon}\right) dx$$

if $u_\epsilon = q\left(\frac{dist(x,\Omega)}{\epsilon}\right)$, then

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla u_\epsilon}{|\nabla u|}\right)^2 \left(\frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \frac{W(u_\epsilon)}{\epsilon}\right) &= (\Delta dist(x,\Omega))^2 \frac{1}{\epsilon} q'\left(\frac{dist(x,\Omega)}{\epsilon}\right)^2 \\ &\to H(x)^2 c_W \delta_{\partial\Omega}\end{aligned}$$

Then, at least for smooth set Ω, we have

$$\mathcal{W}_\epsilon(u_\epsilon) = \frac{1}{2}\int \operatorname{div}\left(\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}\right)^2 \left(\frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \frac{W(u_\epsilon)}{\epsilon}\right) dx \underset{\epsilon\to 0}{\Longrightarrow} c_W \frac{1}{2}\int_{\partial\Omega} H^2 d\mathcal{H}^{n-1}$$

Gamma-convergence of Bellettini's approximation

- Control of the mean curvature of the isolevel surfaces

$$\begin{aligned}\mathcal{W}_\epsilon^B(u_\epsilon) &\geq \frac{1}{2}\int_{\{|\nabla u_h|\neq 0\}} |\nabla u_\epsilon| \sqrt{2W(u_\epsilon)}\,\mathrm{div}\left(\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}\right)^2 dx \\ &\geq \frac{1}{2}\int_0^1 \sqrt{2W(t)} \int_{\{u_\epsilon=t\}\cap\{|\nabla u_\epsilon|\neq 0\}} \mathrm{div}\left(\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}\right)^2 d\mathcal{H}^{n-1}\,dt \\ &\geq c_W \overline{W}(\Omega),\end{aligned}$$

  if $u_\epsilon \to \chi_\Omega$.
- Γ-convergence of $W_\epsilon^B + P_\epsilon$ to $c_W(\overline{W} + P)$[Bellettini 1997]

Mugnai's approximation of Willmore Energy in dimension

Mugnai's approximation in dimension $n = 2$

- We have $H^2 = |A|^2$ in dimension 2
- if $u_\epsilon = q\left(\frac{dist(x,\Omega)}{\epsilon}\right)$, then

$$\begin{aligned}\frac{1}{\epsilon}\left|\epsilon\nabla^2 u_\epsilon - \frac{1}{\epsilon}W'(u)\frac{\nabla u}{|\nabla u|}\otimes\frac{\nabla u}{|\nabla u|}\right|^2 &= |\nabla^2 dist(x,\Omega)|^2\frac{1}{\epsilon}\,q'\left(\frac{dist(x,\Omega)}{\epsilon}\right)^2 \\ &\to |A(x)|^2 c_W \delta_{\partial\Omega}\end{aligned}$$

Definition (Mugnai's approximation in dimension $N = 2$)

$$W_\epsilon^M(u) = \frac{1}{2\epsilon}\int\left|\epsilon\nabla^2 u - \frac{1}{\epsilon}W'(u)\frac{\nabla u}{|\nabla u|}\otimes\frac{\nabla u}{|\nabla u|}\right|^2 dx$$

Gamma-convergence of Mugnai's approximation

- As for the Bellettini approximation, we have a control of the mean curvature of the isolevel surfaces which is given by the following inequality

$$|\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right| \leq \frac{1}{\epsilon} \left| \epsilon \nabla^2 u_\epsilon - \frac{1}{\epsilon} W'(u) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right|$$

- Γ-convergence of $W_\epsilon^M + P_\epsilon$ to $c_W(\overline{W} + P)$ [Mugnai 2010],[Bellettini, Mugnai 2010] in dimension 2.

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   - Classical phase field approximation of Willmore energy
   - Gradient flow and asymptotic expansion
   - About numerical scheme
   - Application to the optimal shape of red cells
   - Application in image processing

Approximating the Willmore flow with the classical approach

- Willmore energy

$$\mathcal{W}_\epsilon(u) = \frac{1}{2\epsilon}\int\left(\epsilon\Delta u - \frac{W'(u)}{\epsilon}\right)^2 dx$$

- Its L^2 gradient flow

$$\partial_t u = -\Delta\left(\Delta u - \frac{1}{\epsilon^2}W'(u)\right) + \frac{1}{\epsilon^2}W''(u)\left(\Delta u - \frac{1}{\epsilon^2}W'(u)\right),$$

or

$$\begin{cases}\epsilon^2\partial_t u = \Delta\psi - \frac{1}{\epsilon^2}W''(u)\psi\\ \psi = W'(u) - \epsilon^2\Delta u.\end{cases}$$

- Well-posedness and existence at fixed parameter ϵ : [Colli Laurencot-2011-2012] with volume and area constraints

Inner expansion

- Stretched variable

$$z = dist(x, \Omega_\epsilon(t))/\epsilon = d(x,t)/\epsilon.$$

- Inner expansions of u_ϵ and μ_ϵ :

$$U_\epsilon(x,t) = U_0(x,z,t) + \epsilon U_1(x,z,t) + \epsilon^2 U_2(x,z,t) + O(\epsilon^3),$$

$$\Psi_\epsilon(x,t) = \Psi_0(x,z,t) + \epsilon\Psi_1(x,z,t) + \epsilon^2\Psi_2(x,z,t) + \epsilon^3\Psi_3(x,z,t)O(\epsilon^3),$$

- Velocity of the front

$$V_\epsilon = -\partial_t d(x,t) = V_0 + \epsilon V_1 + O(\epsilon^2)$$

Formal asymptotic expansion

- About derivative of u

$$\begin{cases}\nabla u_\epsilon = \nabla_x U_\epsilon + \epsilon^{-1} m \partial_z U_\epsilon \quad \text{where } m = \nabla d(x,t) \\ \Delta u_\epsilon = \Delta_x U_\epsilon + \epsilon^{-1}\Delta d \partial_z U_\epsilon + \epsilon^{-2}\partial_{zz}^2 U_\epsilon \quad \text{as} \quad \nabla_x U \cdot m = 0 \\ \partial_t u_\epsilon = \partial_t U_\epsilon - \epsilon^{-1} V_\epsilon \partial_z U_\epsilon \end{cases}$$

- Geometric properties of the signed distance

$$\Delta d(x,t) = \sum \frac{\kappa_i}{1+\kappa_i d(x,t)} \Rightarrow \Delta d = H - \epsilon z |A|^2 + O(\epsilon^2),$$

where $|A|^2 = \sum \kappa_i^2$.

Formal asymptotic expansion

The phase field Willmore PDE

$$\begin{cases} \epsilon^2 \partial_t u & = \Delta\psi - \frac{1}{\epsilon^2} W''(u)\psi \\ \psi & = W'(u) - \epsilon^2 \Delta u. \end{cases}$$

implies (equation (1))

$$\epsilon^2 \left(\partial_t U_\epsilon - \frac{1}{\epsilon} V_\epsilon U_\epsilon \right) = \frac{1}{\epsilon^2} \partial^2_{zz} \Psi_\epsilon + \frac{1}{\epsilon} \Delta d \, \partial_z \Psi_\epsilon + \Delta_x \Psi_\epsilon - \frac{1}{\epsilon^2} W''(U_\epsilon) \Psi_\epsilon$$

and (equation (2))

$$\Psi_\epsilon = W'(U_\epsilon) - \partial^2_{zz} U_\epsilon - \epsilon \Delta d \, \partial_z U_\epsilon - \epsilon^2 \Delta_x U_\epsilon.$$

Order 0

- Order ϵ^{-2} in equation (1) and order 0 in equation (2) implies that

$$\begin{cases} 0 & = \partial^2_{zz}\Psi_0 - \frac{1}{\epsilon^2}W''(U_0)\Psi_0 \\ \Psi_0 & = W'(U_0) - \partial^2_{zz}U_0, \end{cases}$$

with the following boundary conditions (matching condition)

$$\lim_{z\to-\infty} U_0(x,z,t) = 1, \quad \lim_{z\to+\infty} U_0(x,z,t) = 0 \text{ and } \lim_{z\to\pm\infty} \Psi_0(x,z,t) = 0$$

- Then

$$\begin{cases} U_0(x,z,t) = q(z) \\ \Psi_0(x,z,t) = 0 \end{cases}$$

Order 1

- Order ϵ^{-1} in equation (1) and order 1 in equation (2) implies that

$$\begin{cases} 0 & = \partial^2_{zz}\Psi_1 - \frac{1}{\epsilon^2}W''(U_0)\Psi_1 \\ \Psi_1 & = W''(U_0)U_1 - \partial^2_{zz}U_1 - H(x)\partial_z U_0 \end{cases}$$

  with the following boundary conditions (matching condition)

$$\lim_{z\to\pm\infty} U_1(x,z,t) = 1, \quad \lim_{z\to 0} U_1(x,z,t) = 0 \text{ and } \lim_{z\to\pm\infty} \Psi_1(x,z,t) = 0.$$

- Then the first equation shows that $\Psi_1 = c(x,t)q'(z)$ and the second one that

$$\begin{cases} U_1(x,z,t) = 0 \\ \Psi_1(x,z,t) = -H(x)q'(z) \end{cases}.$$

Order 3

- Order ϵ^0 in equation (1) and order 2 in equation (2) implies that

$$\begin{cases} 0 & = \partial^2_{zz}\Psi_2 - \frac{1}{\epsilon^2}W''(U_0)\Psi_2 + H\partial_z\Psi_1 \\ \Psi_2 & = W''(U_0)U_2 - \partial^2_{zz}U_2 + |A|^2 z\partial_z U_0 \end{cases}$$

with the following boundary conditions (matching condition)

$$\lim_{z\to\pm\infty} U_2(x,z,t) = 1, \quad \lim_{z\to 0} U_2(x,z,t) = 0 \text{ and } \lim_{z\to\pm\infty} \Psi_2(x,z,t) = 0.$$

- The first equation shows that

$$\Psi_2 = c(x,t)q'(z) + H(x)^2\eta_2(z),$$

where the profile η_2 is defined as the solution of

$$\eta_2'' - W''(q)\eta_2 = q'', \quad \text{with } \lim_{z\to\pm\infty} \eta_2(z) = 0,$$

and satisfies $\eta_2(z) = \frac{1}{2}zq'(z)$.

Order 3

- The second equation reads now

$$\partial^2_{zz} U_2 - W''(U_0)U_2 = (|A|^2 - \frac{H^2}{2})zq'(z) + c(x,t)q'(z).$$

- Then, multiplying by q' and integrating over $\mathbb{R}$ leads to

$$c(x,t) = 0, \quad \text{and } U_2(x,z,t) = (|A(x)|^2 - \frac{H(x)^2}{2})\eta_1(z),$$

where the profile η_1 is defined as the solution of

$$\eta_1'' - W''(q)\eta_1 = zq', \quad \text{with } \lim_{z\to\pm\infty} \eta_1(z) = 0.$$

- To conclude,

$$\Psi_2(x,z,t) = H(x)^2\eta_2(z) \quad \text{and } U_2(x,z,t) = (|A(x)|^2 - \frac{H(x)^2}{2})\eta_1(z).$$

Order 4

- The first equation reads

$$\begin{aligned} -V_0 q'(z) &= [\partial_z^2 \Psi_3 - W''(q)\Psi_3] - W^{(3)}(q) U_2 \Psi_1 + (H\partial_z \Psi_2 - |A|^2 z \partial_z \Psi_1) + \Delta_x \Psi_1 \\ &= [\partial_z^2 \Psi_3 - W''(q)\Psi_3] + H(|A|^2 - H^2/2) W^{(3)}(q) q' \eta_1 \\ &\quad +(H^3/2 - \Delta_\Gamma H) q' + (H^3/2 + |A|^2 H) z q'' \end{aligned}$$

- Remark also that

$$\int_{\mathbb{R}} q'(s)^2 ds = c_W, \int_{\mathbb{R}} z q''(s) q'(z) dz = -\frac{1}{2} c_W, \text{ and } \int_{\mathbb{R}} W^{(3)}(q)(q')^2 \eta_1 dz = -\frac{1}{2} c_W,$$

as η_1 satisfies

$$\eta_1'' - W''(q)\eta_1 = zq' \quad \text{and } \eta_1''' - W''(q)\eta_1' - W^{(3)}(q) q' \eta_1 = (zq')',$$

- Then multiplying the equation by q' and integrating over $\mathbb{R}$ leads to

$$\begin{aligned} V_0 &= H(|A|^2 - H^2/2)/2 - (H^3/2 - \Delta_\Gamma H) + (H^3/2 + |A|^2 H)/2 \\ &= \Delta_\Gamma H + H|A^2| - \frac{1}{2} H^3 \end{aligned}$$

Asymptotic expansion in smooth case

- Formal asymptotic expansion in smooth case [Loreti March-2000]

$$\begin{cases} u_\epsilon(x,t) & \simeq q\left(\frac{d(x,\Omega^\epsilon(t))}{\epsilon}\right) + \epsilon^2\left(A^2 - \frac{1}{2}H^2\right)\eta_1\left(\frac{d(x,\Omega^\epsilon(t))}{\epsilon}\right) \\ \psi_\epsilon(x,t) & \simeq -\epsilon H q'\left(\frac{d(x,\Omega^\epsilon(t))}{\epsilon}\right) + \epsilon^2 H^2 \eta_2\left(\frac{d(x,\Omega^\epsilon(t))}{\epsilon}\right) \end{cases},$$

where

$$\begin{cases} \eta_1''(s) - W''(q(s))\eta_1(s) = sq'(s) \\ \eta_2''(s) - W''(q(s))\eta_2(s) = q''(s) \end{cases}$$

- Formal convergence

$$V^\epsilon = \Delta_S H + |A|^2 H - \frac{1}{2}H^3 + O(\epsilon)$$

- the velocity limit depends on the second term of order 2 in the asymptotic expansion of u_ϵ and μ_ϵ!

Approximating the Willmore flow with the Mugnai's model

- Willmore

$$\mathcal{W}_\epsilon^M(u) = \frac{1}{2\epsilon}\int \left| \epsilon D^2 u - \frac{1}{\epsilon} W'(u) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right|^2 dx.$$

- Its L^2 gradient flow

$$\begin{cases} \epsilon^2 \partial_t u = \Delta\psi - \frac{1}{\epsilon^2} W''(u)\psi + W'(u)B(u) \\ \psi = W'(u) - \epsilon^2 \Delta u, \end{cases}$$

where

$$B(u) = \operatorname{div}\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|}\right) - \operatorname{div}\left(\nabla\left(\frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|}\right).$$

- Well-posedness and existence at fixed parameter ϵ ? Requires presumably a regularization of the term $B(u)$ as done numerically

Asymptotic expansion in smooth case

- Formal asymptotic expansion in smooth case

$$\begin{cases} u_\epsilon(x,t) & \simeq q\left(\frac{d(x,\Omega^\epsilon(t))}{\epsilon}\right) + \epsilon^2 \frac{A^2}{2}\eta_1\left(\frac{d(x,\Omega^\epsilon(t)}{\epsilon}\right) \\ \psi_\epsilon(x,t) & \simeq -\epsilon H q'\left(\frac{d(x,\Omega^\epsilon(t)}{\epsilon}\right) + \epsilon^2 A^2 \eta_2\left(\frac{d(x,\Omega^\epsilon(t)}{\epsilon}\right), \end{cases}$$

- Formal convergence

$$V^\epsilon = \Delta_S H + B^3 - \frac{1}{2}H|A|^2 + O(\epsilon),$$

where $B^3 = \sum \kappa_i^3$.

- This corresponds to Willmore flow in dimension 2 and 3 !

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An implicit spectral scheme based on a fixed point iteration

- Phase field system to solve

$$\begin{cases} \partial_t u = \Delta\mu - \frac{1}{\epsilon^2} W''(u)\mu \\ \mu = \frac{1}{\epsilon^2} W'(u) - \Delta u. \end{cases}$$

- Implicit discretization in time

$$\begin{cases} u^{n+1} = \delta_t \left[\Delta\mu^{n+1} - \frac{1}{\epsilon^2} W''(u^{n+1})\mu^{n+1} \right] + u^n \\ \mu^{n+1} = \frac{1}{\epsilon^2} W'(u^{n+1}) - \Delta u^{n+1}, \end{cases}$$

- Computed with a Fixed-point iteration

$$\phi \begin{pmatrix} u^{n+1} \\ \mu^{n+1} \end{pmatrix} = \left(I_d + \delta_t \Delta^2 \right)^{-1} \begin{pmatrix} I_d & \delta_t \Delta \\ -\Delta & I_d \end{pmatrix} \begin{pmatrix} u^n - \frac{\delta_t}{\epsilon^2} W''(u^{n+1})\mu^{n+1} \\ \frac{1}{\epsilon^2} W'(u^{n+1}) \end{pmatrix}$$

- Used a Fourier discretization in space
- Stability :

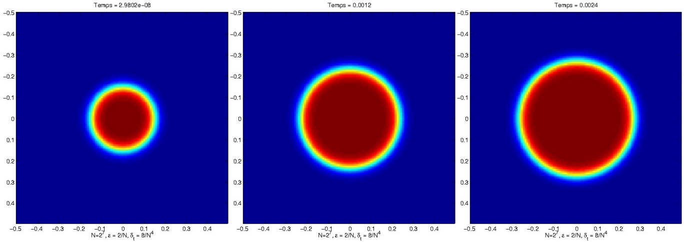
$$\delta_t \leq C \min\left\{ \epsilon^4, \delta_x^2 \epsilon^2 \right\}.$$

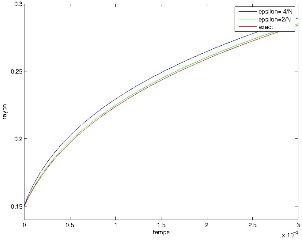
Matlab code

```
%%%%%%%%%%  initialization %%%%%%%%%%%%%
% U, T U_fourier,  w , epsilon, delta_t

%%%%%%%%%%% diffusion operator in Fourier space  %%%%%

K1 = ones(N,1)*[0:N/2,-N/2+1:-1];
M1 = exp(-4*pi^2*delta_t2*(K1.^2 + K1'.^2));
M=1./(1 + delta_t*16*pi^4*(K1.^2 + K1'.^2).^2);
M2 =  -4*pi^2*(K1.^2 + K1'.^2);


for i=1:T/delta_t,

Uk = U;
wk = w;
res = 1;

%%%%%% fixed point iteration  %%%%%%%%%%%%%

while res > 10^(-8);
potentiel_1 = (2*Uk.^3 - 3*Uk.^2 + Uk)  ;
potentiel_2 = (6*Uk.^2 - 6*Uk + 1);
temp1 = fft2(potentiel_1);
temp2 = fft2(potentiel_2.*wk);

Uk_plus = ifft2(M.*(U_fourier + delta_t/epsilon^2*(M2.*temp1 + temp2)));
wk = ifft2(M.*(M2.*(U_fourier + delta_t/epsilon^2*temp2) - 1/epsilon^2*temp1));

res = norm((Uk_plus-Uk));
Uk = Uk_plus;
end

w = wk;
U = Uk;
U_fourier = fft2(U);

end
```

Validation of this numerical method

- Willmore flow of a initial circle with radius equals to R_0 :
  $R(t) = \left(R_0^4 + 2t\right)^{1/4}$.





Validation of this numerical method

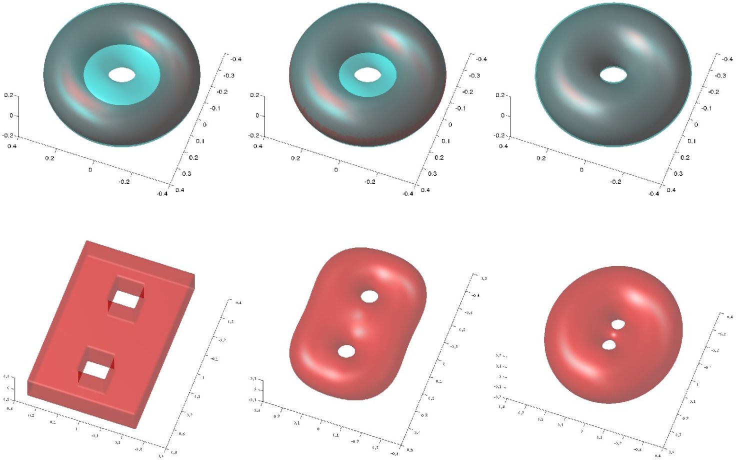
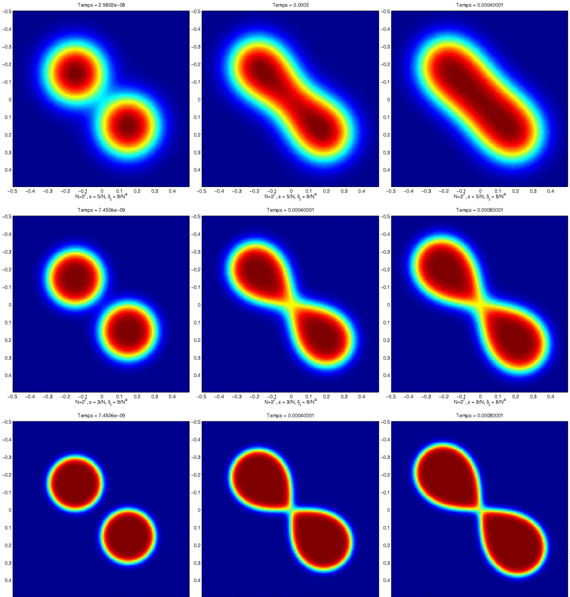
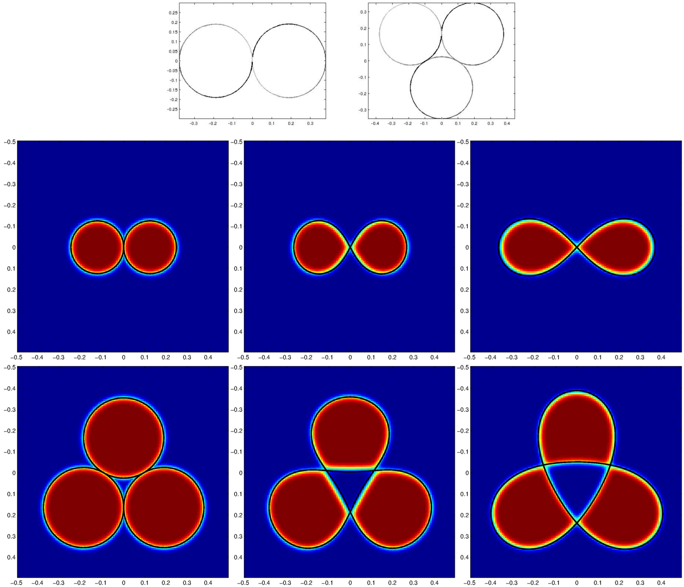


fig : Two smooth evolutions by Willmore flow ; a Clifford's torus and a Lawson-Kusner surface

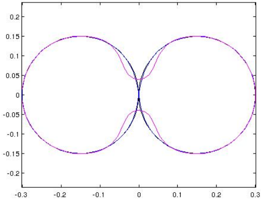
Union of two disjoint circles

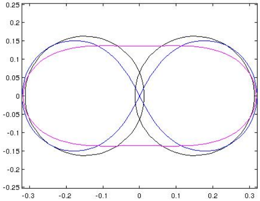


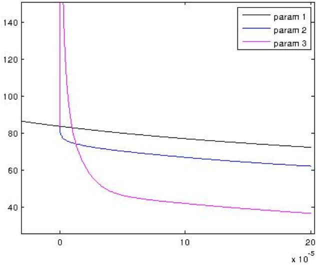
Comparison phase field // parametric [Dziuk-2008]



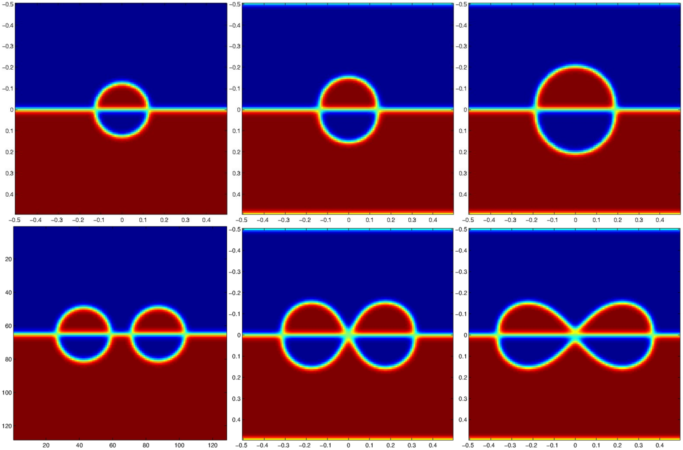
Parametric Willmore flow of two disjoint circles

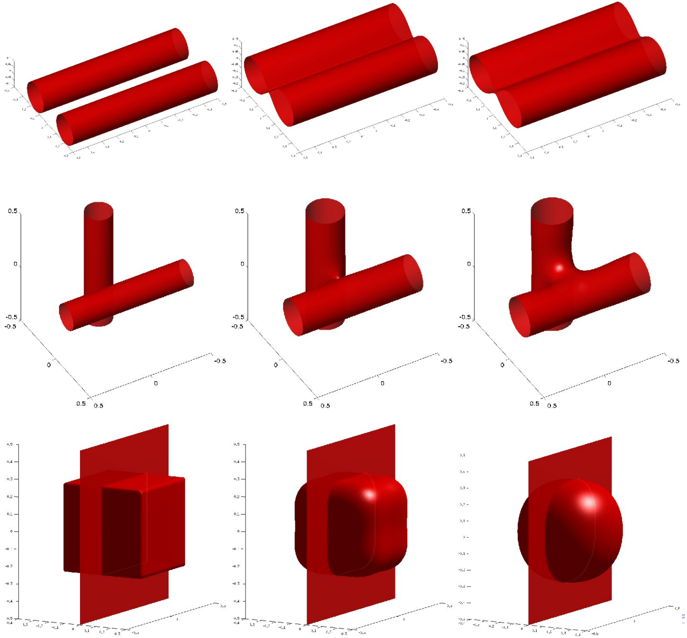






Other experiments




0.5
0
-0.5
-0.5
0
0.5
0.5
0
-0.5

An implicit spectral scheme based on a fixed point iteration

- An Implicit discretization in time, Fourier discretization in space and a fixed point iteration to solve

$$\begin{cases} \partial_t u = \frac{1}{\epsilon^2}\mu - \frac{1}{\epsilon^4} W''(u)\mu + \frac{1}{\epsilon^2} W'(u)B(u) \\ \mu = W'(u) - \epsilon^2 \Delta u. \end{cases}$$

  Where

$$B(u) = \left[\left(\left| \nabla \left(\frac{\nabla u}{|\nabla u|} \right) \right|^2 - \left| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right|^2 \right) - rot \left(rot \left(\frac{\nabla u}{|\nabla u|} \right) \right) \cdot \frac{\nabla u}{|\nabla u|} \right]$$

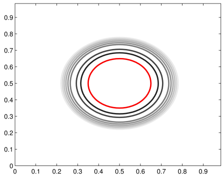
- In practice, use a regularization of $B(u)$:

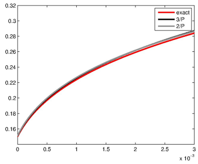
$$B_\sigma(u) = \left[\left(\left| \nabla \nu_{u,\sigma} \right|^2 - \left| \mathrm{div}\, \nu_{u,\sigma} \right|^2 \right) - rot \left(rot \left(\nu_{u,\sigma} \right) \right) \cdot \nu_{u,\sigma} \right]$$

  where $\nu_{u,\sigma} = \frac{\nabla u}{\sqrt{|\nabla u|^2 + \sigma^2}}$

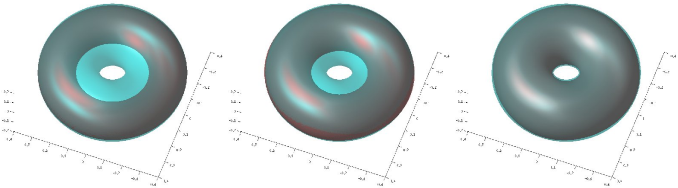
Validation of this numerical method

- Willmore flow of a initial circle with radius equals to R_0,

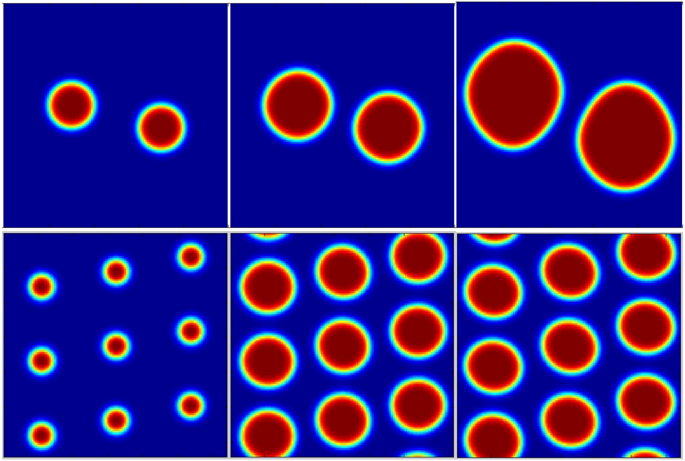




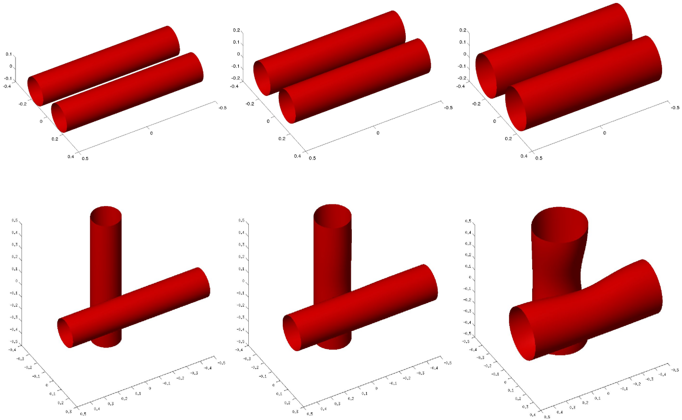
- Clifford's torus



Other experiments



Other experiments



Willmore flow with conservation of area and volume

- Minimization of Willmore energy

$$\mathcal{W}(\Omega) = \frac{1}{2}\int_{\partial\Omega} H^2 d\mathcal{H}^{\mathrm{n}-1},$$

under area and volume constraints

$$\int_{\Omega} 1 d\mathcal{H}^n = V_0, \quad \text{and} \int_{\partial\Omega} 1 d\mathcal{H}^{\mathrm{n}-1} = A_0.$$

- It's L^2 gradient flow reads as

$$V_n = \Delta_S H + |A|^2 H - \frac{1}{2}H^3 + \lambda_V + \lambda_A H,$$

where λ_V and λ_A are two Lagrange multipliers defined such as

$$\int_{\Omega} V_n d\mathcal{H}^n = 0, \quad \text{and} \quad \int_{\Omega} H V_n d\mathcal{H}^n = 0,$$

Phase field versus

- Minimization of

$$\mathcal{W}_\epsilon(u) = \frac{1}{2\epsilon}\int_{\mathbb{R}^d}\left(\epsilon\Delta u - \frac{1}{\epsilon}W'(u)\right)^2 dx,$$

under discrete area and volume constraints

$$\int_{\mathbb{R}^d} u\,dx = V_0, \quad \text{and} \quad \int_{\mathbb{R}^d}\left(\epsilon\frac{|\nabla u|^2}{2} + \frac{1}{\epsilon}W(u)\right)dx = c_W A_0.$$

- It's L^2 gradient flow reads as

$$\begin{cases} \epsilon^2\partial_t u = \Delta\mu - \frac{1}{\epsilon^2}W''(u)\mu + \epsilon\lambda_V + \lambda_A\mu, \\ \mu = W'(u) - \epsilon^2\Delta u. \end{cases}$$

where λ_V and λ_A are two Lagrange multipliers defined such as $\int_{\mathbb{R}^d} u_t dx = 0$ and $\int_{\mathbb{R}^d} \mu u_t dx = 0$.
- Well-posedness and existence at fixed parameter ϵ [Colli Laurencot-2011-2012]

A slightly variant

- A local Lagrange multiplier λ_V :

$$\begin{cases} \epsilon^2 \partial_t u = \Delta\mu - \frac{1}{\epsilon^2} W''(u)\mu + \epsilon\lambda_V \sqrt{2W(u)} + \lambda_A \mu, \\ \mu = W'(u) - \epsilon^2 \Delta u. \end{cases}$$

- Sharp interface limit ?

$$V_\epsilon = \Delta_S H + |A|^2 H - \frac{1}{2} H^3 + \lambda_V + \lambda_A H + O(\epsilon)$$

- Explicit expression of λ_A and λ_V :

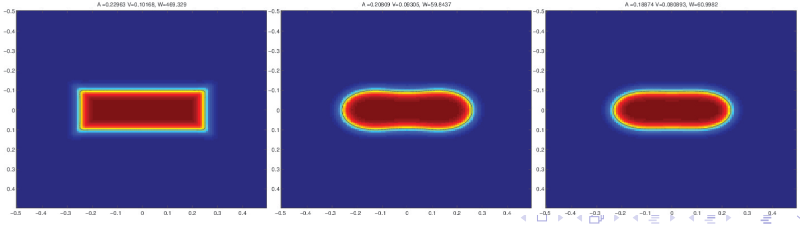
$$\begin{pmatrix} \lambda_V \\ \lambda_A \end{pmatrix} = -\begin{pmatrix} \int \epsilon \sqrt{2W(u)} dx & \int \mu dx \\ \int \epsilon \sqrt{2W(u)} \mu dx & \int \mu^2 dx \end{pmatrix}^{-1} \begin{pmatrix} \int \Delta\mu - \frac{1}{\epsilon^2} W''(u)\mu dx \\ \int \left(\Delta\mu - \frac{1}{\epsilon^2} W''(u)\mu\right) \mu dx \end{pmatrix}$$

About numerical scheme

- Implicit discretization in time

$$\begin{cases} u^{n+1} & = u^n + \frac{\delta_t}{\epsilon^2}\Big[\Delta\mu^{n+1} - \frac{1}{\epsilon^2}W''(u^{n+1})\mu^{n+1} \\ & \qquad + \epsilon\lambda_V^{n+1}\sqrt{2W(u^{n+1})} + \lambda_A^{n+1}\mu^{n+1}\Big] \\ \mu^{n+1} & = W'(u^{n+1}) - \epsilon^2\Delta u^{n+1}, \end{cases}$$

- Resolution with a fixed point iteration and Fourier space.
- But numerical experiments show some important losses on the volume and the area : $\Longrightarrow$ use a penalty formulation [Du,Liu,Wang 2006]



A variant approach : minimization and projection

- A splitting approach

$$\begin{cases} u^{n+1/2} & = u^n + \frac{\delta_t}{\epsilon^2}\left[\Delta\mu^{n+1/2} - \frac{1}{\epsilon^2}W''(u^{n+1/2})\mu^{n+1/2}\right. \\ \mu^{n+1/2} & = W'(u^{n+1/2}) - \epsilon^2\Delta u^{n+1/2}, \end{cases}$$

and

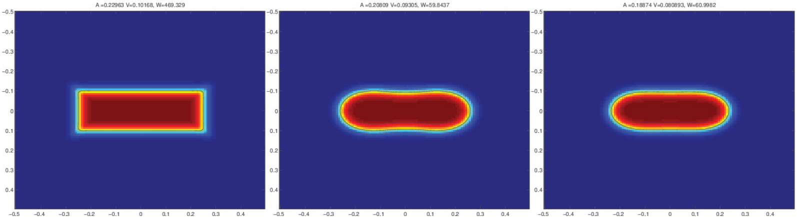
$$u^{n+1} = u^{n+1/2} + \frac{\delta_t}{\epsilon^2}\left(\epsilon\lambda_V\sqrt{2W(u^{n+1/2})} + \lambda_A\mu^{n+1/2}\right)$$

- The two Lagrange multiplier λ_V and λ_A are defined as the solution of $F(u^{n+1}) = 0$, with

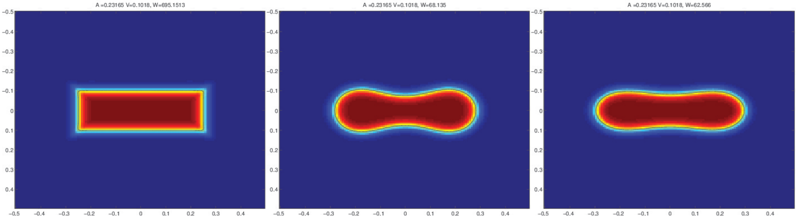
$$F(u) = \begin{pmatrix} \int u dx - V_0 \\ \int\left(\epsilon\frac{|\nabla u|^2}{2} + \frac{1}{\epsilon}W(u)\right)dx - c_W A_0 \end{pmatrix}$$

- In practice, we use a Newton method to obtain an approximation of λ_V and λ_A !

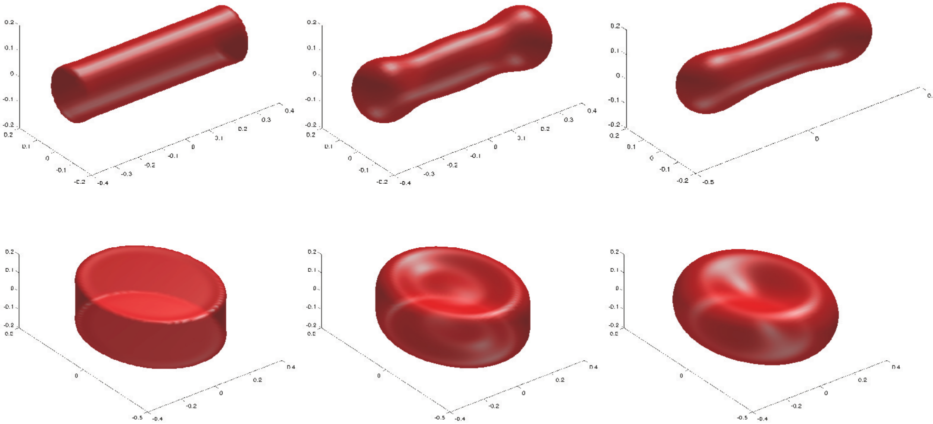
- With implicit discretization of the continuous PDE



- With splitting approach : minimization and projection

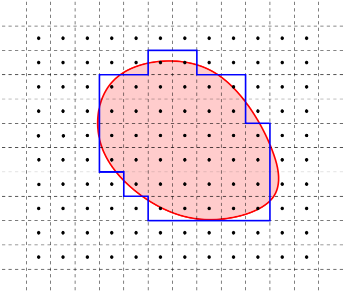


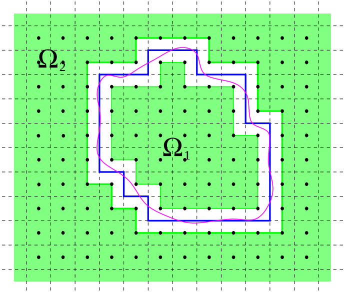
Numerical experiments in 3 dimension



1. Introduction
2. Phase field approximation of mean curvature flow
3. Conserved and multiphase mean curvature flow
4. Approximation of Willmore energy and flow:
   - Classical phase field approximation of Willmore energy
   - Gradient flow and asymptotic expansion
   - About numerical scheme
   - Application to the optimal shape of red cells
   - Application in image processing

Regularization of discrete contour by Willmore energy





Find the set Ω^* such as

$$\Omega^* = \underset{\Omega_1 \subset \Omega \subset \Omega_2^c}{\arg\min}\, \mathcal{W}(\Omega), \quad \text{with } \mathcal{W}(\Omega) = \int_{\partial\Omega} H^2 d\mathcal{H}^{n-1}$$

where Ω_1 and Ω_2 are two given set such as $\Omega_1 \subset \Omega_2^c$

Phase field versus

- Let us introduce $u_{1,\epsilon}$ and $u_{2,\epsilon}$ defined by

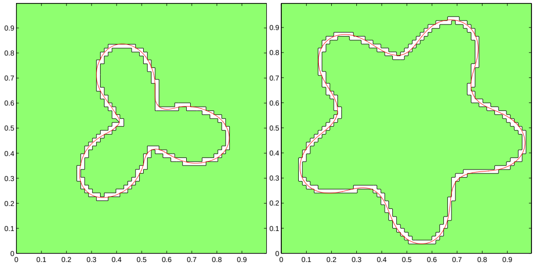
$$u_{1,\epsilon} = q\left(\frac{dist(x,\Omega_1)}{\epsilon}\right), \text{ and } u_{2,\epsilon} = q\left(\frac{dist(x,\Omega_2)}{\epsilon}\right)$$

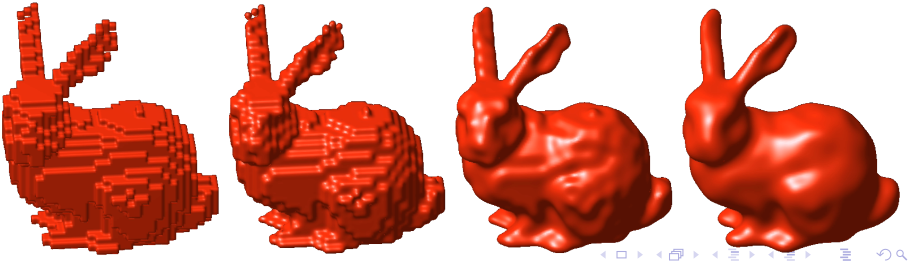
- Find the solution of

$$u^* = \underset{u_{1,\epsilon} \subset u_\epsilon \subset 1-u_{2,\epsilon}}{\arg\min} \mathcal{W}_\epsilon(u_\epsilon), \quad \text{with } \mathcal{W}_\epsilon(u) = \frac{1}{2\epsilon}\int\left(\epsilon\Delta u - \frac{W'(u)}{\epsilon}\right)^2 dx$$

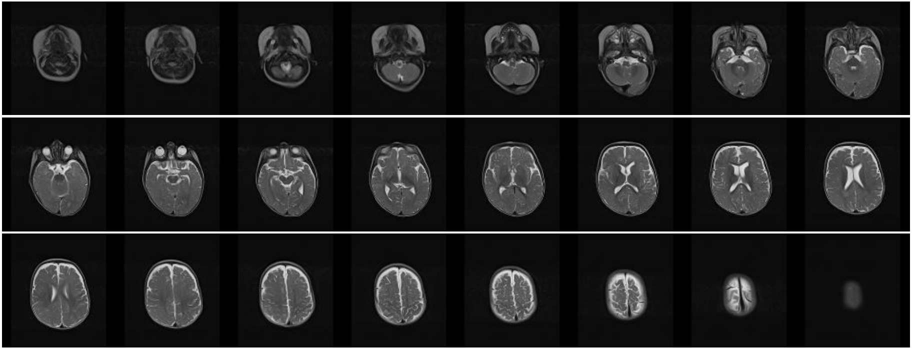
- Numerical scheme : an implicit Euler scheme based on a projection fixed point iteration.
- Remark that if $\Omega_1 = \Omega_2^c$, then $u^* = q\left(\frac{dist(x,\Omega_1)}{\epsilon}\right)$.

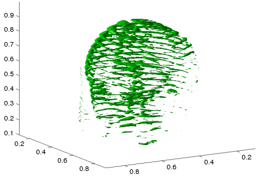
Numerical experiments

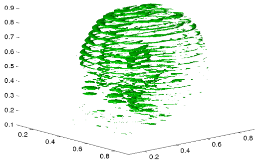


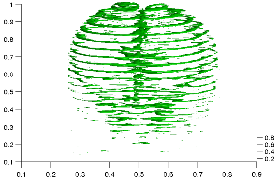


Motivation : Magnetic resonance Imaging





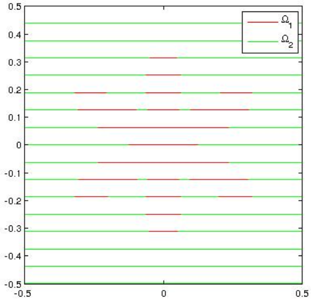




Mathematical formulation

- Find the set Ω^* as a minimizer of

$$\mathcal{W}_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} \mathcal{W}(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2^c \\ +\infty & \text{otherwise} \end{cases}$$



Penalized Willmore energy : $\mathcal{W}_{\Omega_1,\Omega_2}$

- Note that $\mathcal{W}_{\Omega_1,\Omega_2}$ is not lower semi-continuous

$$\mathcal{W}_{\Omega_1,\Omega_2}(\Omega) = \begin{cases} \mathcal{W}(\Omega) & \text{if } \Omega_1 \subset \Omega \subset \Omega_2^c \\ +\infty & \text{otherwise} \end{cases}$$

- Relaxation of the penalized Willmore energy ?
- Phase field approximation

$$\mathcal{W}_{\epsilon,u_{1,\epsilon},u_{2,\epsilon}}(u) = \begin{cases} \mathcal{W}_\epsilon(u) & \text{if } u_{1,\epsilon} \leq u \leq 1 - u_{2,\epsilon} \\ +\infty & \text{otherwise,} \end{cases}$$

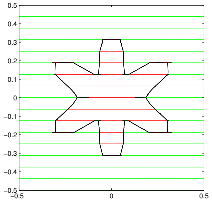
with $u_{i,\epsilon} = q\left(\frac{dist(x,\Omega_i)}{\epsilon}\right)$.

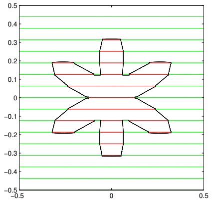
- About numerical scheme : an implicit Euler scheme based on a projection fixed point iteration.

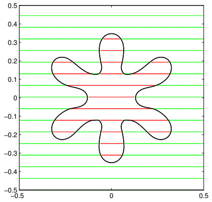
Matlab code

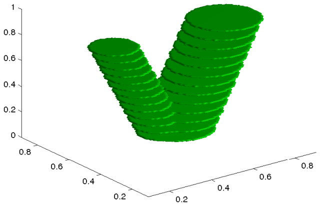
```
%%%%%%%%%%  initialization %%%%%%%%%%%%%
% U, T U_fourier,  w , epsilon, delta_t,U1,U2

%%%%%%%%%%% diffusion operator in Fourier space  %%%%%

K1 = ones(N,1)*[0:N/2,-N/2+1:-1];
M1 = exp(-4*pi^2*delta_t2*(K1.^2 + K1'.^2));
M=1./(1 + delta_t*16*pi^4*(K1.^2 + K1'.^2).^2);
M2 =  -4*pi^2*(K1.^2 + K1'.^2);


for i=1:T/delta_t,

Uk = U;
wk = w;
res = 1;

%%%%%% fixed point iteration  %%%%%%%%%%%%%

while res > 10^(-8);
potentiel_1 = (2*Uk.^3 - 3*Uk.^2 + Uk)  ;
potentiel_2 = (6*Uk.^2 - 6*Uk + 1);
temp1 = fft2(potentiel_1);
temp2 = fft2(potentiel_2.*wk);

Uk_plus = ifft2(M.*(U_fourier + delta_t/epsilon^2*(M2.*temp1 + temp2)));
wk = ifft2(M.*(M2.*(U_fourier + delta_t/epsilon^2*temp2) - 1/epsilon^2*temp1));
Uk_plus = max(min(1-U2,Uk_plus),U1);

res = norm((Uk_plus-Uk));
Uk = Uk_plus;
end

w = wk;
U = Uk;
U_fourier = fft2(U);

end
```

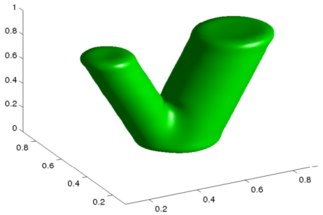
Numerical experiments











Numerical experiments