Weyl and the kleinean tradition

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Introduction

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   - he considers this concept independently from its realizations,
   - he shows that a Riemann *surface* is a one-dimensional complex analytic *manifold* (§ 7).

(ii) Following HILBERT, he proves anew the *Dirichlet's principle* which is located at the core of RIEMANN's *Dissertation* (1851).

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In the preface, WEYL seems to reject categorically intuition and geometric representations in complex analysis because they lead to mistakes and confusions.

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- ANALYSIS SITUS is based on set theory (in fact, WEYL combines set topology and combinatorial topology),
- in the second part of his book, WEYL gives complete proofs of the main theorems belonging to the theory of Riemann surfaces (uniformization theorem, Riemann-Roch theorem, etc.)

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At that time, WEYL is *Privatdozent* at Göttingen. He seems mainly influenced by KOEBE and HILBERT.

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WEYL combines these two references : his proof of the uniformization theorem is based on Dirichlet's principle, in accordance with HILBERT's prescriptions (1909).

However, in his preface, WEYL doesn't consider rigour as an end in itself. Moreover, he criticizes a formalist conception of mathematics. But he doesn't explain his arguments which are expressed in a terse manner. They characterize a kind of *tacit knowledge* which must be described precisely.

WEYL and the arithmetizing of analysis

At first sight, WEYL continues the so-called « arithmetizing » of analysis. This expression is due to KLEIN in a famous address delivered at Göttingen in 1895 : « The arithmetizing of mathematics ».

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- Accordingly, WEYL builds up Riemann surfaces by generalizing WEIERSTRASS' analytic continuation,
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Contrary to KLEIN (course on Riemann surfaces at Leipzig in 1880-1881), WEYL doesn't refer to the intuitive image of a « continuous deformation » in order to characterize properties belonging to *analysis situs*.

A « modern demand for rigour »

WEYL, DIRF, p. V : « Die vorliegende Schrift gibt den Hauptinhalt einer (...) Vorlesung wieder, deren wesentliche Absicht war : die Grundideen der Riemannschen Funktionentheorie in einer Form zu entwickeln, die allen modernen Anforderungen an Strenge völlig genüge leistet. Eine solche strenge Darstellung, die namentlich auch bei Begründung der fundamentalen, in die Funktionentheorie hineinspielenden Begriffe und Sätze der *Analysis situs* sich nicht auf anschauliche Plausibilität beruft, sondern mengentheoretisch exakte Beweise gibt, liegt bis jetzt nicht vor ».

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- The category of *modernity* is employed by an actor (namely WEYL) to qualify a « demand for rigour ».
- In fact, WEYL paraphrases HILBERT. For instance, in his 23d problem which deals with calculus of variations, HILBERT refers to the « modern demand for rigour » [moderne Forderungen der Strenge] (cf. « mathematische Probleme », *Archiv der Mathematik und Physik*, 1901, p. 231).

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More generally, WEYL determines precisely the outline of *analysis situs* as a mathematical domain : « Derjenige Zweig der Mathematik, der es mit den Kontinuitätseigenschaften der zwei- (und mehr-) dimensionalen Mannigfaltigkeiten zu tun hat, wird als *Analysis Situs* oder *Topologie* bezeichnet. (...) Zwei Mannigfaltigkeiten müssen *im Sinne der Analysis Situs* als *äquivalent* betrachtet werden, wenn sie sich *Punkt für Punkt umkehrbar eindeutig und umkehrbar gebietsstetig* aufeinander abbilden lassen [=homeomorphism] ».

A scepticism against formalism

- On the one hand, in 1910-1913, he is fully convinced by ZERMELO's work in set theory and he believes that « continuity » can be formalized in a set-theoretical framework.
- On the other hand, he expresses scepticism about the « modern » formalization of mathematical knowledge : this process *can't* be an end in itself.

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Weyl, preface, p. VI : « Es kann nicht geleugnet werden : die Entdeckung der sich weit über alle unsere Vorstellungen hinausspannenden Allgemeinheit solcher Begriffe wie « Funktion », « Kurve », usw. auf der einen Seite, das Bedürfnis nach logischer Strenge auf der anderen, so erspriesslich, ja notwendig sie für unsere Wissenschaft waren, doch auch ungesunde Erscheinungen hervorgerufen. Ein Teil derjenigen mathematischen Produktion hat (...) den Zusammenhang mit dem lebendigen Strom der Wissenschaft verloren ».

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WEYL's viewpoint reminds us KLEIN's criticism of the arithmetizing of mathematics. Indeed, I will measure KLEIN's influence on WEYL more precisely by studying the conclusion of WEYL's *Habilitation* lecture which is entitled « On the definition of the fundamental notions of mathematics » (1910).

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Weyl's viewpoint reminds us Klein's criticism of the arithmetizing of mathematics. Indeed, I will measure Klein's influence on Weyl more precisely by studying the conclusion of Weyl's *Habilitation* lecture which is entitled « On the definition of the fundamental notions of mathematics » (1910).

In other words, we must not overestimate Hilbert's legacy in Weyl's monograph on Riemann surfaces. Klein's legacy plays simultaneously a central role in Weyl's practice of mathematics and in his epistemological reflections on mathematics in 1908-1913.

KLEIN's and HILBERT's legacy

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- The simultaneous reference to KLEIN and HILBERT leads to a « tension » which characterizes WEYL's implicit epistemology in his early works in complex analysis and in philosophy of mathematics.
- This tension explains why WEYL seems to contradict himself in the preface of *Die Idee der Riemannschen Fläche*.

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Hence, it is an oversimplification to think that WEYL just applies HILBERT's axiomatic method.

The complexity of KLEIN's « influence » on WEYL

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WEYL refers explicitly to

- KLEIN's book on algebraic functions of a complex variable entitled « Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale » (Teubner, 1882),

and implicitly to

- KLEIN's famous conference entitled « the arithmetizing of mathematics » (1895).

A kleinean conception of the unity of mathematics

KLEIN is also a key reference to explain WEYL's conception of the unity of mathematics during all his scientific career.

- WEYL often combines distinct methods and different domains in order to solve a same problem (cf. his book on Riemann surfaces or his article on complex semi-simple Lie groups).
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Moreover, Klein's main project consists in unifying « group theory » and « Riemann's geometric ideas ». Weyl continues tacitely this project during all his scientific career.

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- thirdly, it corresponds to an implicit conception of the unity of mathematics which has a great impact on WEYL's way of building up a theory,
- fourthly, it refers to a kind of « know how » in *analysis situs* which isn't immediately shared by CARTAN in 1925.

Plan of our presentation

First part : KLEIN's legacy in WEYL's early work. KLEIN's talk « The arithmetizing of mathematics » is an implicit source of inspiration for WEYL at an epistemological and a pedagogical level.

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Conclusion : CARTAN's early reception of WEYL's work on Lie groups. Because of his book on Riemann surfaces, Weyl is well acquainted with methods and concepts belonging to *analysis situs*. In 1925-1926, he *knows how* to extend these methods to Lie groups. On the other hand, in a 1925 letter to WEYL, CARTAN claims that it is difficult for him to understand these topological methods. This « lack of understanding » shows that WEYL's ability to apply topological methods to Lie groups is a *tacit knowledge* which is not shared by CARTAN yet.

First part : Klein's legacy in Weyl's early work

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- his introductory course in function theory during the winter-semester 1910-1911, recently published under the title *Einführung in der Funktionentheorie*.

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Let us recall that *Die Idee der Riemannschen Fläche* derives from an *advanced lecture course* given by Weyl during the winter-semester 1911-1912 at Göttingen.

WEYL's *Habilitation* lecture

In his *Habilitation* lecture, WEYL aims at classifying different ways of defining mathematical concepts. Two examples

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He also shows great interest in the set-theoretical foundation of mathematics — without any kind of scepticism. In particular, he refers to ZERMELO's axiomatization of set theory in 1908 (« Untersuchungen über die Grundlagen der Mengenlehre », *mathematische Annalen*, **65**). ZERMELO is at Göttingen between 1897 and 1910.

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The first part of WEYL's monograph on Riemann surfaces can be viewed as an application of his epistemological reflections concerning the definitions of mathematical concepts. He explicitly refers to *definitions by axioms* and *definitions by abstraction* in DIRF.

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« Aus dem Gegebenen unserer Empfindungswelt steigen wir durch gewisse geistige Prozesse der Abstraktion und Idealisation (...) zu gewissen, den Raum betreffenden Begriffen auf, die teils wie « Punkt », « Gerade », « Ebene », als Hinweis auf ideale Objekte, teils wie « liegen auf », « kongruent », « zwischen », als Hinweis ideale Beziehungen zwischen diesen Objekten zu verstehen sind. (...) Bei einer logischen Untersuchung der erhaltenen Sätze, welche die Geometrie ausmachen, stellt sich jedoch heraus, daß sie alle durch rein logische Schlüsse aus einer ziemlich geringen Anzahl von ihnen, die man als *Axiome* bezeichnet, hergeleitet werden können ».

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These conclusive remarks are not well-articulated with his preceding arguments on axiomatic method and set theory.

« Trotzdem erblicke ich den eigentlichen Wert und die eigentliche Bedeutung des so zustande kommenden Begriffssystems einer logisierten Mathematik doch darin, daß sich ihre Begriffe auch, ohne daß dabei die Wahrheit der auf sie bezüglichen Sätze Schaden leitet, *anschauungsmäßig* deuten lassen, und ich glaube der menschliche Geist kann auf keinem anderen Wege als durch Verarbeitung der gegebenen Wirklichkeit zu den mathematischen Begriffen aufsteigen ».

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« Die Anwendbarkeit unserer Wissenschaft erscheint dann nur als ein Symptom ihrer Bodenständigkeit, nicht als eigentlicher Wertmaßstab, und für die Mathematik, diesen stolzen Baum, der seine breite Krone frei im Äther entfaltet, aber seine Kraft zugleich mit tausend Wurzeln aus dem Erdboden wirklicher Anschauungen und Vorstellungen saugt, wäre es gleich verhängnisvoll [calamitous] wollte man ihn mit der Schere eines allzu engherzigen Utilitarismus beschneiden oder wollte man ihn aus dem Boden, dem er entsprossen ist, herausreißen ».

WEYL's *Habilitation* lecture

WEYL first assumes that, because of their vagueness, our intuitive representations must be analyzed by using well-defined concepts. But at the end of his *Habilitation* lecture, he claims that our concepts would be empty without a reference to real intuitions.

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How to explain such a contrast in WEYL's reasoning ? In fact, his concluding exhortations are disconnected from the rest of his Habilitation and they can be considered as *traces* of KLEIN's influence on WEYL (at an epistemological level) :

(i) In « the arithmetizing of mathematics », KLEIN shows that « intuition » and « logic » are complementary in mathematics,

(ii) he concludes his conference by comparing mathematical sciences to a « tree ». In other words, WEYL paraphrases KLEIN (up to a certain point) at the end of his *Habilitation* lecture.

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The simultaneous reference to HILBERT and KLEIN reveals a « tension » in WEYL's Habilitation.

« The Arithmetizing of mathematics » (1895)

KLEIN : « While I desire in every case the fullest logical working out of the material, yet I demand at the same time an intuitive grasp and investigation of the subject from all sides. Mathematical developments originating in intuition must not be considered actual constituents of the science till they have been brought into a strictly logical form. Conversely, the mere abstract statement of logical relations cannot satisfy us until the extent of their application to every branch of intuition is vividly set forth, and we recognize the manifold connections of the logical scheme, depending on the branch which we have chosen, to the other divisions of our knowledge ».

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We find exactly the same argument in Weyl's *Habilitation* lecture.

« The Arithmetizing of mathematics » (1895)

KLEIN's metaphor : « The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the branches as its essential part ? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts ».

« The Arithmetizing of mathematics » (1895)

KLEIN's metaphor : « The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the branches as its essential part ? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts ».

WEYL doesn't articulate the simultaneous reference to KLEIN and to HILBERT. This kind of « under-articulation » expresses a tacit knowledge which is of great importance for historians :

- WEYL's early work is not merely motivated by a « modern demand for rigour »,
- WEYL's implicit epistemology seems to be contradictory because he doesn't decide between two different viewpoints :
  (i) KLEIN, « The arithmetizing of mathematics »,
  (ii) HILBERT, *Die Grundlagen der Geometrie*.

KLEIN's work on Riemann surfaces

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KLEIN's ideas differ radically from a usual approach developed in particular by H. DURÈGE [*Elemente der Theorie der Functionen einer complexen veränderlichen Grösse*, first ed. 1864]

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« [KLEIN] is studying one of the most abstract questions of the theory of functions : to determine whether on a given Riemann surface there always exists a function admitting of given singularities. What does the celebrated German geometer do ? He replaces his Riemann surface by a metallic surface whose electric conductivity varies according to certain laws. He connects two of its points with two poles of a battery. The current, says he, must pass, and the distribution of this current on the surface will define a function whose singularities will be precisely those called for by enunciation.
Doubtless Professor KLEIN well knows he has given here only a sketch ; nevertheless he has not hesitated to publish it ; and he would probably believe he finds in it, if not a rigorous demonstration, at least a kind of moral certainty ».

KLEIN's work on Riemann surfaces

According to POINCARÉ, KLEIN's approach would be rejected by « logicians » because it lacks rigour. However, we can find very important results in KLEIN's book which will be reformulated and / or refined by HURWITZ and WEYL :

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(ii) In the article « über algebraische Gebilde mit Eindeutigen Transformationen in sich » (*mathematische Annalen*, **41**, 1893), HURWITZ proves that the number of automorphisms of a compact Riemann surface of genus $g \geqslant 2$ cannot exceed $84(g-1)$.

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(iii) WEYL, *Die Idee der Riemannschen Fläche*, § 21, p. 165 « Eine geschlossene Riemannsche Fläche vom Geschlechte $p > 1$ gestattet nur endlichviele konforme Abbildungen in sich » [last theorem formulated by WEYL in his monograph].

KLEIN's legacy in WEYL's monograph

At the beginning of his monograph, WEYL claims (in full accordance with KLEIN) that Riemann surfaces must be described *per se*, before introducing functions on them. In other words, Riemann surfaces are logically prior to (multi-valued)-functions.

Klein's legacy in Weyl's monograph

At the beginning of his monograph, Weyl claims (in full accordance with Klein) that Riemann surfaces must be described *per se*, before introducing functions on them. In other words, Riemann surfaces are logically prior to (multi-valued)-functions.

On the other hand, Weyl doesn't describe Riemann surfaces by using geometric and physical representations because of a « modern demand for rigour ». Does it mean that Weyl rejects completely Klein's intuitive approach in function theory ?

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At first sight, our answer would be « yes » :

- Weyl builds up Riemann surfaces by using a generalization of Weierstrass' analytic continuation (§§ 1-3),
- he defines axiomatically a topological surface and a Riemann surface (§§ 4 - 7).

WEYL's lecture course in complex analysis (1910-1911)

However, we have to keep in mind that WEYL's monograph results from an *advanced* course (1911-1912) which is preceded by an *introductory* course (1910-1911) in complex analysis. Does he ascribe the same functions to intuitive representations in these two different lecture courses ?

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In other words, it seems to us that it is meaningless to comment WEYL's monograph without referring to a kind of *tacit knowledge*, which is related to his *practice of teaching*.

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In other words, it seems to us that it is meaningless to comment WEYL's monograph without referring to a kind of *tacit knowledge*, which is related to his *practice of teaching*.

Precisely, in his lecture course in complex analysis, WEYL ascribes great importance to geometric and physical representations.

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(5) theory of multi-valued analytic functions and Riemann surfaces.

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KLEIN's influence on WEYL is very significant for three reasons :

- In accordance with KLEIN's approach (1882), WEYL regularly refers to geometric and physical representations,
- There is a link between pure and applied mathematics in WEYL's lecture course. KLEIN's « idée fixe » consists precisely in developing connections between mathematics and its applications at universities (especially Göttingen),
- in this course, WEYL rarely uses axiomatic method to define mathematical concepts. Their construction is preceded by intuitive representations and analysis of concrete examples.

Klein and the teaching of mathematics

In a « Bericht an die Breslauer Naturforscherversammlung über den Stand des mathematischen und physikalischen Unterrichts an den höheren Schulen » (22. September 1904), Klein enumerates a series of prescriptions concerning the teaching of mathematics (particularly at universities) :

KLEIN and the teaching of mathematics

In a « Bericht an die Breslauer Naturforscherversammlung über den Stand des mathematischen und physikalischen Unterrichts an den höheren Schulen » (22. September 1904), KLEIN enumerates a series of prescriptions concerning the teaching of mathematics (particularly at universities) :

- development of a *constructive method* based on the study of specific mathematical objets,
- importance of the so-called « Raumanschauung » [spatial intuition],
- description of the various connections between a mathematical theory and its applications to natural sciences,
- the « logical element » must be introduced progressively (the modern demand for rigour can't be immediately and fully required in an introductory course).

KLEIN and the teaching of mathematics

In 1890-1910, KLEIN is deeply involved in the organization of mathematical teaching in Gymnasien, Realschulen, Universities, etc. For instance, he strongly advocates for the so-called Meraner reform (1905), = development of a functional thinking at school (i.e. thinking in variations and functional dependencies).

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- More precisely, during the winter-semester 1909-1910, KLEIN organizes a seminar entitled « Mathematik und Psychologie » at Göttingen.
- In particular, he discusses some psychological and philosophical aspects of mathematical knowledge.

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- More precisely, during the winter-semester 1909-1910, KLEIN organizes a seminar entitled « Mathematik und Psychologie » at Göttingen.
- In particular, he discusses some psychological and philosophical aspects of mathematical knowledge.
- During the second session, WEYL makes a review on some articles from *L'enseignement mathématique* 1905-1908.

KLEIN and the teaching of mathematics

The volumes of *L'enseignement mathématique* WEYL refers to contain for instance :

- a translation of KLEIN's article devoted to the « Meraner Reform »,
- a presentation of different systems of teaching (in mathematics and natural sciences) in Europe and in the United States.

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All this tacit background is important to understand Weyl's lecture courses in *complex analysis* and on *Riemann surfaces*.

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All this tacit background is important to understand WEYL's lecture courses in *complex analysis* and on *Riemann surfaces*.

Moreover, in his Habilitation and in DIRF, WEYL claims that a definition based on an equivalence relation « has its psychological roots in our mind's capability for abstraction » This sentence could be interpreted as a trace of WEYL's involvement in KLEIN's seminar on mathematics and psychology.

KLEIN and the teaching of mathematics

As we have seen before

- WEYL's introductory course in complex analysis is based on a *constructive method* and on *geometric intuitions*,
- On the contrary, his advanced course on Riemann surfaces is mainly based on *axiomatic method* and it is motivated by a « modern demand for rigour ».

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This way of organizing « elementary » and advanced lecture courses corresponds exactly to a series of requirements formulated by KLEIN in two different writings :

- « lectures on mathematics » delivered before members of the congress of mathematics in connection with the world's fair in Chicago (1893),
- « the arithmetizing of mathematics » (1895).

KLEIN's lectures on mathematics in Chicago

An apparent contradiction in the teaching of analysis :

« Now, just here a practical difficulty presents itself in the teaching of mathematics, let us say of the elements of the differential and integral calculus. The teacher is confronted with the problem of harmonizing two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to the practical applications ; on the other, his conscientiousness as a teacher and man of science would seem to compel him to detract in nowise from perfect mathematical rigour and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics ».

KLEIN's lectures on mathematics in Chicago

The « counter-example » of JORDAN's *Cours d'analyse* (second edition) :

« In recent years the university instruction, at least in Europe, has been tending more and more in the latter direction (...). The second edition of the *Cours d'Analyse* of Camille Jordan may be regarded as an example of this extreme refinement in laying the foundations of the infinitesimal calculus. To place a work of this character in the hands of a beginner must necessarily have the effect that at the beginning a large part of the subject will remain unintelligible, and that, at a later stage, the student will not have gained the power of making use of the principles in the simple cases occurring in the applied sciences ».

Back to « the arithmetizing of mathematics »

KLEIN (1893) : « It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them ».

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KLEIN (1895) : « Among the university professors of our subject (...) intuition is frequently not only undervalued, but as much as possible ignored. This is doubtless a consequence of the intrinsic importance of the arithmetizing tendency in modern mathematics. But the result reaches far beyond the mark. It is high time to assert openly once for all that this implies, not only a false pedagogy, but also a distorted view of the science ».

Back to « the arithmetizing of mathematics »

KLEIN (1895) : « I gladly yield the utmost freedom to the preferences of individual academic teachers, and have always discouraged the laying-down of general rules for higher mathematical teaching, but this shall not prevent me from saying that two classes at least of mathematical lectures must be based on intuition ; the elementary lectures which actually introduce the beginner to higher mathematics — for the scholar must naturally follow the same course of development on a smaller scale, that the science itself has taken on a larger — and the lectures which are intended for those whose work is largely done by intuitive methods, namely, natural scientists and engineers ».

Back to « the arithmetizing of mathematics »

Precisely, WEYL's course in complex analysis consists of « elementary lectures which actually introduce the beginner to higher mathematics ». Accordingly, it is based on geometric and physical representations.

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Moreover, in 1910-1912 WEYL follows tacitely the historical development of function theory in accordance with KLEIN's prescriptions :

(a) 1910-1911
  - CAUCHY-RIEMANN equations,
  - CAUCHY's integral theorem and CAUCHY's integral formula,
  - multi-valued functions and Riemann surfaces associated to them,

(b) 1911-1912
  - Riemann surfaces considered *per se*,
  - construction of Riemann surfaces based on a generalization of analytic continuation,
  - axiomatization of topological and Riemann surfaces.

Relativization of the « modern demand for rigour »

To conclude, WEYL's monograph (1913) is generally considered as the first modern and rigorous presentation of RIEMANN's geometric ideas in function theory.

- Under this assumption, it is just characterized by one fact : making RIEMANN's and KLEIN's intuitive representations more *explicit* by using a generalization of WEIERSTRASS' analytic continuation and HILBERT's axiomatic method.

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This standpoint must be relativized because it doesn't take into account the fact that WEYL's monograph derives from a lecture course. Accordingly, this book must be analyzed in function of WEYL's *teaching practice* which is deeply influenced by KLEIN.

- WEYL uses *formal definitions* only in advanced courses. Consequently, he satisfies a series of pedagogical prescriptions due to KLEIN.

Relativization of the « modern demand for rigour »

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- In the preface of DIRF and in the conclusive remarks of his habilitation, he alludes to a kleinean viewpoint in epistemology which is quite disconnected from the rest of his own philosophical thoughts,
- the way he organizes his lecture courses has to be related to KLEIN's reflections on the teaching of mathematics.

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Our question is not only « How does WEYL define a Riemann surface ? » or « For which reason does he use the axiomatic method to define concepts — modern demand for rigour and generality, etc. — ? » but also « When does he mainly refer to definitions by axioms in a lecture course ? » The answer is « kleinean » : especially in advanced courses.

Second part : KLEIN and the unity of mathematics

WEYL's viewpoint on KLEIN's works in mathematics

In order to describe how WEYL perceives KLEIN's contributions in (pure) mathematics, we would like to mention especially two writings :

- « Felix Kleins Stellung in der mathematischen Gegenwart » (*Die Naturwissenschaften*, 1930),
- « Axiomatic versus constructive procedures in mathematics » (after 1953, first published by Tito Tonietti in *the mathematical intelligencer*, **7**, 1985).

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WEYL refers to

- KLEIN's way of unifying mathematics by combining different domains,
- KLEIN's main project which consists in constructing a series of connections between group theory and RIEMANN's geometric ideas.

WEYL's homage to KLEIN (1930)

In 1930, WEYL becomes professor at Göttingen after HILBERT's retirement. Shortly before, WEYL gives a famous address on the occasion of the inauguration of the Mathematics Institute at Göttingen (3. December 1929). This talk is merely an homage to Felix KLEIN.

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During this conference, WEYL doesn't focus

- on KLEIN's reflections concerning the teaching of mathematics,
- on KLEIN's institutional role in order to develop mathematical sciences and their practical applications at Göttingen.

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WEYL aims at describing KLEIN's contributions in pure mathematics and his underlying conception of the unity of mathematics.

WEYL's homage to KLEIN (1930)

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Weyl's homage to Klein (1930)

Although Klein advocates for the development of connections between mathematics and its applications, he mainly devotes himself to pure mathematics.

« *Im Zentrum seiner wissenschaftlichen Persönlichkeit war* Klein *reiner Mathematiker und ist es stets geblieben* ».

- Weyl's viewpoint is half-true, because Klein was deeply interested in the formalization of special and general relativity in his own research.

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- WEYL's viewpoint is half-true, because KLEIN was deeply interested in the formalization of special and general relativity in his own research.

For WEYL, KLEIN's work in (pure) mathematics is mainly characterized by the combination between separated « disciplines », distinct theories and different methods.

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Weyl mentions two examples to confirm this assumption :

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(i) Klein's *Erlanger Program* (1872).

- reference to the theory of substitutions and to invariant theory,
- classification of different geometries based on the concept of (continuous) group of transformations,
- generalization of Klein's own work on non-euclidean geometries (Cayley-Klein projective viewpoint in geometry),
- connection between Plücker's line geometry and Lie's sphere geometry.

(ii) Klein's work on the Icosahedron (1875-1884).

- finite groups of motions,
- resolution of the general quintic equation,
- development of the theory of modular functions,
- connection with Riemann surfaces.

WEYL's homage to KLEIN (1930)

KLEIN's way of unifying mathematics is based on two ingredients : (1) intuition and (2) group theory. (1) « Das Hauptorgan von KLEINs mathematischer Methodik war das *intuitive, die Zusammenhänge erschauende Verstehen* ».

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We can find exactly the same assumption in WEYL's late writings on constructive and axiomatic procedures in mathematics (after 1953) : « The chief organ of KLEIN's own productivity was this intuitive perception of interconnections and relations between separate fields (...) In the time of KLEIN's productivity (which had passed when I entered the University of Göttingen in 1904) the intuitive realization of inner connections between various domains had been the most characteristic feature of his achievements. Typical is his book on the Icosahedron in which geometry, algebra, function- and group-theory blend in polyphonic harmony ».

WEYL's homage to KLEIN (1930)

(2) The *group* concept plays a central role in the unification of mathematical theories and mathematical domains. WEYL underlines this fact in his homage to KLEIN :

- « *Die Gruppe blieb seit jener Zeit der beherrschende Gesichtspunkt von Kleins mathematischen Schaffen* ».

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- « *Die Gruppe blieb seit jener Zeit der beherrschende Gesichtspunkt von Kleins mathematischen Schaffen* ».
- KLEIN generalizes the group concept to various areas of research. He refers to (continuous) groups in his « Erlanger Programm », to finite and infinite discrete groups in his work on modular functions, etc.
- In other words, group theory plays a central role in the unification of mathematics in KLEIN's work.

Weyl's homage to Klein (1930)

In fact, Weyl doesn't comment neutrally Klein's work in pure mathematics. He implicitly refers to an *epistemic value* concerning the fruitfulness of a research. A mathematical production is all the more fruitful since it implies several new connections between separate domains. This *tacit knowledge* guides Weyl in his practice of mathematics.

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Accordingly, when WEYL claims for instance that NOETHER is an « algebraist », his judgement is a little bit pejorative.

Weyl's homage to Klein (1930)

In fact, Weyl doesn't comment neutrally Klein's work in pure mathematics. He implicitly refers to an *epistemic value* concerning the fruitfulness of a research. A mathematical production is all the more fruitful since it implies several new connections between separate domains. This *tacit knowledge* guides Weyl in his practice of mathematics.

Accordingly, when Weyl claims for instance that Noether is an « algebraist », his judgement is a little bit pejorative.

In other words, Weyl continues a kleinean tradition, which consists in producing mathematics by combining very different domains. Moreover, it becomes a criterium in order to value productions due to other mathematicians (Artin, Noether, etc.), cf. his talk on topology and algebra (1931), in which he takes a polemic tone against the so-called « algebraists ».

WEYL's homage to KLEIN (1930)

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Two examples :

- KLEIN's contributions in the study of modular functions,
- KLEIN's and POINCARÉ's uniformization theorem for algebraic functions (1881-1882).

Riemann's geometric ideas and group theory

On the occasion of the centenary of the discovery of hyperbolic geometry by Lobachevski in 1925, Weyl writes a forty pages text in which he describes Riemann's geometric ideas in *analysis situs*, *function theory*, *differential geometry*, etc. He also shows how these ideas can be connected to group theory in a wide sense. To this end, he refers successively to

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- his solution to the so-called « space problem » (1921-1923),
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During the 1920s, CARTAN and WEYL combine systematically RIEMANN's geometric ideas with group theory in differential geometry.

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In *analysis situs* and in function theory (1913-1916)

- In DIRF (§ 9), WEYL defines a *covering surface* $\overline{\mathfrak{F}}$ of a Riemann surface $\mathfrak{F}$. He considers the group Γ of automorphisms [Gruppe der Deck-Transformationen] of $\overline{\mathfrak{F}}$ (relatively to the base surface $\mathfrak{F}$).
- Let $\widetilde{\mathfrak{F}}$ be the universal covering surface of $\mathfrak{F}$, then Γ is an « analysis-situs-invariant » of $\mathfrak{F}$.
- In 1916, he constructs an analogy between the classification of covering surfaces (in *analysis situs*) and the classification of field extensions (in Galois theory). The group concept plays a central role in these two classifications.

RIEMANN's geometric ideas and group theory

In *differential geometry* (1921-1923)

(i) WEYL builds up a first version of the problem of space in his commentary to RIEMANN's *Habilitationsvortrag*,

(ii) WEYL's problem of space (second version 1921-1923) consists in the characterization of the so-called « infinitesimally pythagorean manifolds » (differentiable manifolds with a metric structure defined by a non-degenerate quadratic form).

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(iii) There exists a unique affine connection compatible with their metric structure.

(iv) To prove this assumption, WEYL rewrites his problem in the theoretical framework of (linear) Lie groups and Lie algebras. In other words, he solves this problem by using group-theoretical methods.

RIEMANN's geometric ideas and group theory

Conversely, in his article on Lie groups (1925-1926), WEYL uses RIEMANN's « geometric ideas » in *analysis situs* to prove the complete reducibility theorem (for complex semi-simple Lie group). On this occasion, he generalizes the theory of covering surfaces to Lie groups.

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(i) On the one hand, group-theoretical methods can be used to gain a better understanding of RIEMANN's geometric ideas.

(ii) on the other hand, topological concepts which were first introduced in the theory of Riemann surfaces can be applied to continuous groups. According to this viewpoint, RIEMANN's geometric ideas (in *analysis situs*) are an important tool in the study of Lie groups and their representations.

RIEMANN's geometric ideas and group theory

From the perspective of the conceptual history of mathematics, WEYL's series of papers on Lie groups « mark the birthdate of the systematic global theory of Lie groups » [cf. Armand BOREL, 2003].

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From the perspective of the conceptual history of mathematics, Weyl's series of papers on Lie groups « mark the birthdate of the systematic global theory of Lie groups » [cf. Armand Borel, 2003].

- In fact, Weyl aims at combining two viewpoints in the study of continuous groups,
  (i) Cartan's algebraic and local method,
  (ii) Schur's and Hurwitz's integral method which implies to describe the topological properties of a given Lie group (Is it compact ? simply connected ? etc.)

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Moreover, in his article on Lie groups, WEYL is tacitely influenced by KLEIN's way of unifying mathematical knowledge. (Combination of different domains, connections between group theory and Riemann's geometric ideas).

RIEMANN's geometric ideas and group theory

In 1913, WEYL develops the theory of covering surfaces to clarify the foundations of the uniformization theorem. In 1925, he *knows how* to generalize this theoretical framework to Lie groups.

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- In fact, he uses exactly the same terminology in *Die Idee der Riemannschen Fläche* and in his article on Lie groups :

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"Aus einer Darstellung der infinitesimalen Gruppe [$\mathfrak{su}(n)$] (...) erhält man durch Integration nach Lie die zugeordnete Matrix T für alle diejenigen t von [$\mathrm{SU}(n)$], welche einer gewissen *Umgebung* des Einheitselements e angehören. Aber wählt man ein t_0 in dieser Umgebung, so kann man die Darstellung fortsetzen aud diejenige Umgebung von t_0, in welche die erste Umgebung durch die Translation von e nach t_0 übergeht. Der zu iterierende *Prozeß der Fortsetzung* stößt offenbach niemals gegen eine Grenze".

KLEIN's legacy : a short synthesis

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(iii) WEYL is impressed by KLEIN's way of unifying mathematics

- KLEIN combines progressively different mathematical domains,
- he develops a project which consists in relating RIEMANN's geometric ideas to group theory,

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(iii) Weyl is impressed by Klein's way of unifying mathematics
  - Klein combines progressively different mathematical domains,
  - he develops a project which consists in relating Riemann's geometric ideas to group theory,

(iv) Weyl continues this project in different papers all along is career. He also aims at combining different domains (mainly topology and algebra).
  - However, Weyl is simultaneously attached to an hilbertian tradition. In particular, he regularly defines mathematical concepts by using a list of axioms (cf. his monograph on Riemann surfaces 1913, *Raum, Zeit, Materie* 1918-1923, his article on Lie groups 1925-1926, etc.)

conclusion

CARTAN and WEYL

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By contrast, Cartan's early reception of Weyl's article on Lie groups indicates that they do not share the same methods on continuous groups.

For instance, in a letter to Weyl (march 1925), Cartan sketches a proof of the complete reducibility theorem (for complex semi-simple Lie groups). He avoids intentionally "Weyl's idea of introducing the universal covering group and proving its compactness" (Hawkins).

CARTAN and WEYL

- According to CARTAN, it is "always delicate" to use considerations resulting from *analysis situs*.
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CARTAN doesn't mean that WEYL's proof lacks rigour. He admits that is not prepared to understand WEYL's technical approach which requires a long training.

In fact, CARTAN's opinion is provisional : already in 1927, CARTAN studies Lie groups from a global and a topological viewpoint. CARTAN's book *La théorie des groupes finis et continus et l'analysis situs* (1930) confirms this argument.

Cartan and Weyl

Cartan becomes progressively aware of the effectiveness of topological methods in the theory of Lie groups.

- Thus, Weyl's paper will have a deep impact on Cartan's own research in the framework of Lie groups,
- But at first, we can observe a distance between Cartan's and Weyl's approach. To explain this fact, we must refer to the notion of tacit knowledge.

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- But at first, we can observe a distance between CARTAN's and WEYL's approach. To explain this fact, we must refer to the notion of tacit knowledge.

Contrary to CARTAN, WEYL *knows how* to apply the theory of covering spaces to Lie groups, because he has in mind his early book on Riemann surfaces. Hence, CARTAN's « lack of understanding » shows that WEYL's ability to apply topological methods to Lie groups is a *tacit knowledge* which is not shared by CARTAN yet.

Cartan and Weyl

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« I am not so *well-versed* [we underline] in Lie theory as to dare to take more from it than the [fact] that form the infinitesimal group, by integration, the neighborhood of the identity element of the continuous group can be constructed ; the entire group I obtain first by a process of "continuation" and I orient myself about its connectivity relations by means of topological consideration. Incidentally, this consideration of analysis situs is very simple and applies to all semi-simple groups without distinguishing cases. This approach lies closer to my *whole way of thinking* than your more algebraic method, which at the moment I only half understand ».

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- conversely, WEYL is not acquainted with CARTAN's algebraic method which also requires an intense training,
- He aims at combining an algebraic and a topological viewpoint in the theory of Lie groups,
- The expression « my whole way of thinking » refers to a *tacit knowledge*