A SCHUR FUNCTION IDENTITY RELATED TO THE (−1)–ENUMERATION OF SELF-COMPLEMENTARY PLANE PARTITIONS

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Abstract. We give another proof for the (-1)–enumeration of self-complementary plane partitions with at least one odd side-length by specializing a certain Schur function identity. The proof is analogous to Stanley's proof for the ordinary enumeration. In addition, we obtain enumerations of 180°-symmetric rhombus tilings of hexagons with a barrier of arbitrary length along the central line.

1. Introduction

Plane partitions were first introduced by MacMahon (see Figure 1 for an example and Section 2 for a definition). He counted plane partitions contained in a given box [13, Art. 429, proof in Art. 494] (see Eq. (2)) and also investigated the number of plane partitions with certain symmetries.

In [15], Mills, Robbins and Rumsey introduced additional complementation symmetries giving six new combinations of symmetries which led to more conjectures all of which were settled in the 1980's and 90's (see [19, 10, 3, 22]). All these numbers can be expressed as nice product formulas typically involving rising factorials.

Many of these theorems come with q–analogs. Recall that in a q–analog of an enumeration result for a set M, each object is counted by a power of q, that is, one considers $f_M(q) = \sum_{x\in M} q^{\mathrm{stat}(x)}$, where stat assigns an integer to each object in M.

In the case of plane partitions, a typical example for $\mathrm{stat}(x)$ is the number of little cubes in the plane partition x. The closed forms for $f_M(q)$ now contain q–rising factorials instead of rising factorials (see [1, 2, 14]).

Interestingly, upon setting $q = -1$ in the various q–analogs, one consistently obtains enumerations of other objects, usually with additional symmetry constraints. This observation, dubbed the "(-1)–phenomenon" has been explained for many but not all cases by Stembridge (see [20] and [21]).

For the plane partitions with complementation symmetries, the aforementioned q-weights either give trivial results or are not well-defined. The one exception is the enumeration of self-complementary plane partitions which was settled by Stanley [19]

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using a Schur function identity. It gives rise to a q-analog via the principal specialization (that is, setting $x_i = q^i$ in the Schur functions, see Eq. (3)).

On the other hand, for all cases with complementation symmetries Kuperberg defined a $\pm 1-$ weight [9, p.26] (that is, one considers $\sum_{x\in M} w(x)$, where $w(x) = 1$ or $w(x) = -1$). It has been proved in Kuperberg's own paper and in [4] and [5] that this sum admits a nice closed product formula in almost all cases.

In accordance with the "(-1)-phenomenon" mentioned above, the results coincide with the enumerations of other similar objects, but in most cases an explanation in the spirit of Stembridge's papers is missing.

The purpose of the present paper is the explanation of another curious observation. One could expect to obtain a proof of the (-1)–enumeration for self-complementary plane partitions from [5] by setting $x_i = (-1)^i$ in the Schur function identity Stanley uses in [19]. However, the ± 1–weight on individual plane partitions arising in this way is different. Yet, for self-complementary plane partitions with at least one odd sidelength, setting $x_i = (-1)^i$ in Stanley's formulas yields exactly the same formulas as in [5].

We will see that this mystery can be explained by a similar Schur function identity which is actually a generalization of the one Stanley uses. The three sides of the box containing the plane partitions originally play a symmetrical role, but in the Schur function approach the symmetry is broken arbitrarily. In [19] (see also the erratum), this is done in a way to minimize complications. We will see below that a less straightforward approach produces the desired ± 1–weight. As a by-product, we obtain an additional result about certain subclasses of self-complementary plane partitions with a fixed line in the middle (see Figure 4 and Theorem 3). The Schur function identities given in Theorem 2 have already been obtained in [8, Cor 7.3] by Ishikawa, Okada, Tagawa and Zeng as a corollary to a very general Pfaffian identity whose entries are products of determinants. In this paper, we will give a different direct proof using the Littlewood-Richardson rule.

In Section 2, we will review the necessary definitions and properties of plane partitions and Schur functions. In Section 3, we state and prove the Schur function identity (see Theorem 2). Finally in Section 4, we explain the connection with plane partitions and draw the conclusions for the enumeration of self-complementary plane partitions with a fixed line in the middle in Theorem 3 and the (-1)–enumeration of self-complementary plane partitions with at least one odd sidelength in Theorem 4. The enumeration expressed as a product formula in Theorem 3 can also be expressed as a Pfaffian using the methods from [5], which immediately leads to the evaluation of the Pfaffian stated in the subsequent corollary. In fact, the structure of the Pfaffian evaluation was first observed by computer experiments and led backwards to the right Schur function identity.

We remark that our results do not explain why the enumeration of self-complementary plane partitions factors into terms corresponding to the enumeration of ordinary ones, but an explanation should probably include the case of self-complementary plane partitions with a fixed line in the middle because the result has exactly the same structure.

2. Definitions and basic properties

A plane partition P can be defined as a finite set of points (i, j, k) with integers $i, j, k > 0$ and if $(i, j, k) \in P$ and $1 \le i' \le i$, $1 \le j' \le j$, $1 \le k' \le k$ then $(i', j', k') \in P$. We interpret these points as midpoints of cubes and represent a plane partition by

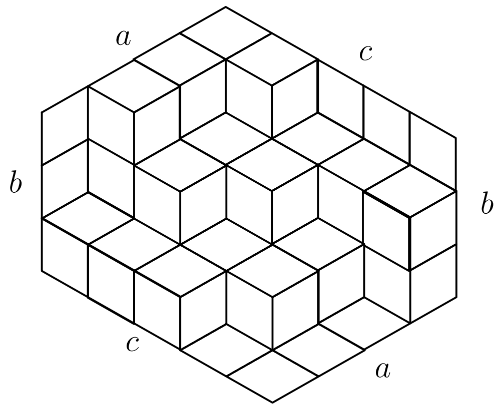


FIGURE 1. A self–complementary plane partition

stacks of cubes (see Figure 1). If we have $i \leq a$, $j \leq c$ and $k \leq b$ for all cubes of the plane partition, we say that the plane partition is contained in a box with sidelengths a, b, c.

Another way to represent a plane partition is by writing down the number of cubes above a certain place in the xy–plane giving an array of numbers with weakly decreasing rows and columns. The plane partition of Figure 1 corresponds to the array

$$\begin{matrix} 3 & 3 & 2 & 2 & 2 \\ 3 & 2 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{matrix} \tag{1}$$

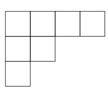
This representation gives the connection to semi-standard tableaux and Schur functions which was used in Stanley's proof of the ordinary enumeration of self–complementary plane partitions (see Section 4).

A plane partition P contained in an $a \times b \times c$–box is called *self–complementary* if whenever $(i, j, k) \in P$ then $(a+1-i, c+1-j, b+1-k) \notin P$ for $1 \leq i \leq a$, $1 \leq j \leq c$, $1 \leq k \leq b$ (see Figure 1).

The corresponding array of numbers has the property that an entry and the corresponding entry in the array rotated by $180°$ add up to b (see the array in (1) with $b = 3$).

Now, we define the ± 1-weight for self-complementary plane partitions. The weight changes sign if we remove one cube and add the opposite one (see Figure 2). The half-full plane partition has weight 1 (see Figure 3). Since every self-complementary plane partition can be reached from the half-full plane partition by moving cubes as described above, this defines the weight uniquely.

A partition λ is a sequence of integers $\lambda_1 \geq \cdots \geq \lambda_l$ which can be represented by a Ferrers diagram as an array of squares with λ_i squares in row i. For example, the partition $(4, 2, 1)$ is represented by the following Ferrers diagram.



A partition μ is contained in a partition λ if $\mu_i \leq \lambda_i$ for all i. λ/μ denotes the set of squares in the Ferrers diagram of λ that are not in the Ferrers diagram of μ. If this set

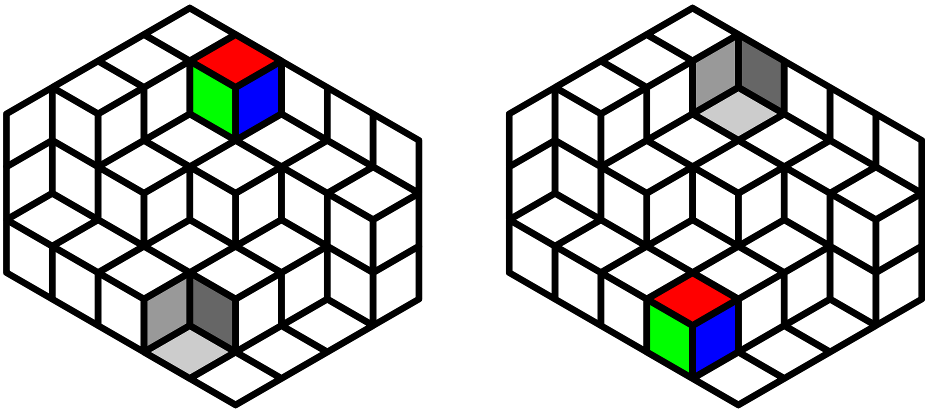


FIGURE 2. Two plane partitions with different ± 1–weight

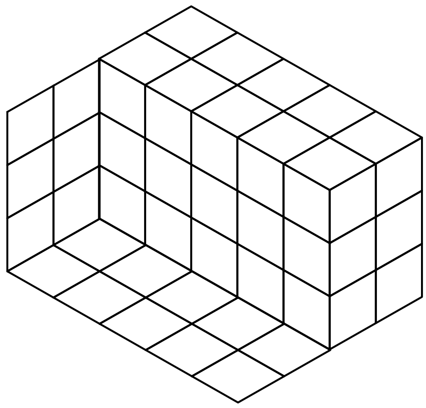


FIGURE 3. A plane partition of weight 1.

of squares does not contain two squares in the same column, it is called a horizontal strip. By $|\lambda/\mu|$, we denote the number of squares in λ/μ.

A semistandard tableau of shape λ (or λ/μ) is a filling of the squares of the Ferrers diagram of λ (or λ/μ) with positive integers which is increasing along the rows and strictly increasing along the columns. If there are ν_1 entries 1, ν_2 entries 2, etc., then $\nu = (\nu_1, \ldots, \nu_k)$ is called the content of the tableau T.

A Schur function is the generating function of semistandard tableaux of a given shape, namely $s_\lambda(x_1, \ldots, x_n) = \sum_T x^T$ where T runs over the semistandard tableaux of shape λ and $x^T = x_1^{\#\text{ 1's in } T} x_2^{\#\text{ 2's in } T} \ldots$.

We will use the following results about plane partitions. The first result was obtained by MacMahon in [13, Art. 429,$x \to 1$, proof in Art. 494]:

The number of all plane partitions contained in a box with sidelengths a, b, c equals

$$B(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \prod_{i=1}^{a} \frac{(c+i)_b}{(i)_b}, \tag{2}$$

where $(a)_n := a(a+1)(a+2)\ldots(a+n-1)$ *is the rising factorial.*

Stanley's proof of the ordinary enumeration of self–complementary plane partitions gives the following result:

Theorem 1 (Stanley [19]). *The number $SC(a,b,c)$ of self–complementary plane partitions contained in a box with sidelengths a,b,c can be expressed in terms of $B(a,b,c)$ in the following way:*

$$\begin{array}{rl} B\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)^2 & \textit{for } a,b,c \textit{ even,} \\ B\left(\frac{a}{2},\frac{b+1}{2},\frac{c-1}{2}\right)B\left(\frac{a}{2},\frac{b-1}{2},\frac{c+1}{2}\right) & \textit{for } a \textit{ even and } b,\ c \textit{ odd,} \\ B\left(\frac{a+1}{2},\frac{b}{2},\frac{c}{2}\right)B\left(\frac{a-1}{2},\frac{b}{2},\frac{c}{2}\right) & \textit{for } a \textit{ odd and } b,\ c \textit{ even.} \end{array}$$

The result is obtained as the case $x_i = 1$ of the expansions in Schur functions of the products

$$\begin{array}{ll} s_{s^r}(x_1,x_2,\dots,x_{t+r})^2 & \text{for the } 2r\times 2s\times 2t\text{–box,} \\ s_{s^r}(x_1,x_2,\dots,x_{t+r})s_{(s+1)^r}(x_1,x_2,\dots,x_{t+r}) & \text{for the } 2r\times(2s+1)\times 2t\text{–box,} \\ s_{s^{r+1}}(x_1,x_2,\dots,x_{t+r+1})s_{s^r}(x_1,x_2,\dots,x_{t+r+1}) & \text{for the } (2r+1)\times 2s\times(2t+1)\text{–box.} \end{array}$$

Note that for $x_i = q^i$, $i = 1,\dots,n$, a Schur function corresponding to a rectangular partition can be expressed by a product formula as a special case of Stanley's hook-content formula ([17]):

$$s_{\gamma^\alpha}(q,q^2,\dots,q^n) = q^{\gamma\alpha(\alpha+1)/2}s_{\gamma^\alpha}(1,q,\dots,q^{n-1}) = q^{\gamma\alpha(\alpha+1)/2}\prod_{i=1}^{\alpha}\prod_{k=0}^{\gamma-1}\frac{1-q^{i+n-\alpha+k}}{1-q^{i+k}}. \quad (3)$$

Furthermore, it is straightforward to obtain from this equation that $s_{c^a}(1,1,\dots,1)$ (with $a+b$ arguments of 1) is $B(a,b,c)$ and $s_{c^a}(1,-1,1,\dots,(-1)^{a+b-1})$ (with $a+b$ arguments in the Schur function) is $SC(a,b,c)$ (see Section 2 of [20]).

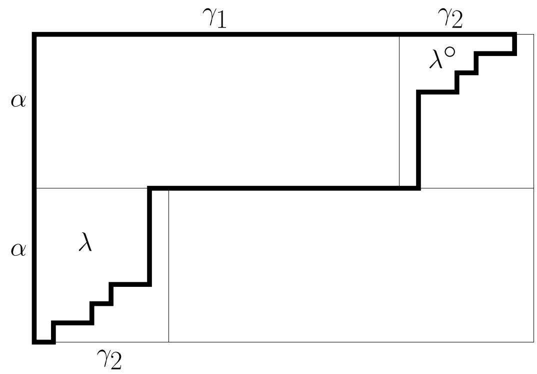
3. The Schur function identity

In this section, we state and prove a Schur function identity which implies a closed form for the (−1)-enumeration of self-complementary plane partitions (see Theorem 4).

Theorem 2 (Ishikawa, Okada, Tagawa, Zeng [8]). *Let $\gamma_1 \geq \gamma_2$. Then we have the following two identities:*

$$\begin{aligned} &s_{(\gamma_1^\alpha)}(x_1,\dots,x_n)s_{(\gamma_2^\alpha)}(x_1,\dots,x_{n+1}) \\ &\quad = \sum_{\lambda\subseteq(\gamma_2^\alpha)}\sum_{\substack{\pi\subseteq\lambda \\ \lambda/\pi \textit{ horiz. strip}}} x_{n+1}^{|\lambda/\pi|}\cdot s_{(\gamma_1+\gamma_2-\lambda_\alpha,\gamma_1+\gamma_2-\lambda_{\alpha-1},\dots,\gamma_1+\gamma_2-\lambda_1,\pi_1,\dots,\pi_\alpha)}(x_1,\dots,x_n) \\ &\quad = \sum_{\lambda\subseteq(\gamma_2^\alpha)}\sum_{T} x^T, \end{aligned} \quad (4)$$

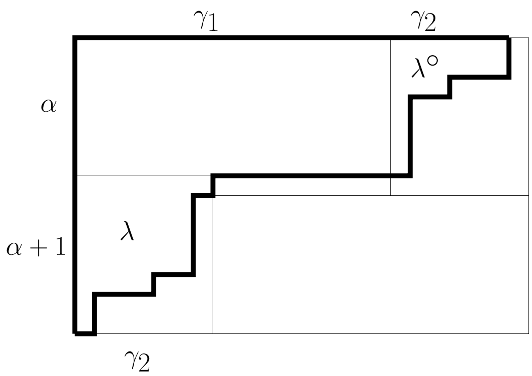
where T is a semistandard tableau of the following shape:



Here, entries $n+1$ can only occur in the region of shape λ and all other entries are smaller. λ° is the shape obtained by taking the complement of λ in the rectangle γ_2^α and rotating by $180°$. x^T is short-hand notation for the monomial $x_1^{\#\text{ 1's in }T} x_2^{\#\text{ 2's in }T} \dots$.

$$
\begin{aligned}
&s_{(\gamma_1^\alpha)}(x_1,\dots,x_n)s_{(\gamma_2^{\alpha+1})}(x_1,\dots,x_{n+1})\\
&\quad = \sum_{\substack{\lambda\subseteq(\gamma_2^{\alpha+1})\\ \lambda_1=\gamma_2}} \sum_{\substack{\pi\subseteq\lambda\\ \lambda/\pi \text{ horiz. strip}}} x_{n+1}^{|\lambda/\pi|}\cdot s_{(\gamma_1+\gamma_2-\lambda_{\alpha+1},\gamma_1+\gamma_2-\lambda_\alpha,\dots,\gamma_1+\gamma_2-\lambda_2,\pi_1,\dots,\pi_{\alpha+1})}(x_1,\dots,x_n)\\
&\quad = \sum_{\substack{\lambda\subseteq(\gamma_2^{\alpha+1})\\ \lambda_1=\gamma_2}} \sum_T x^T,
\end{aligned} \tag{5}
$$

where T is a semistandard tableau of the following shape:



Entries $n+1$ can again only occur in the region of shape λ.The first row of λ is γ_2, and λ° is the rotated complement of λ in the rectangle $\gamma_2^{\alpha+1}$.

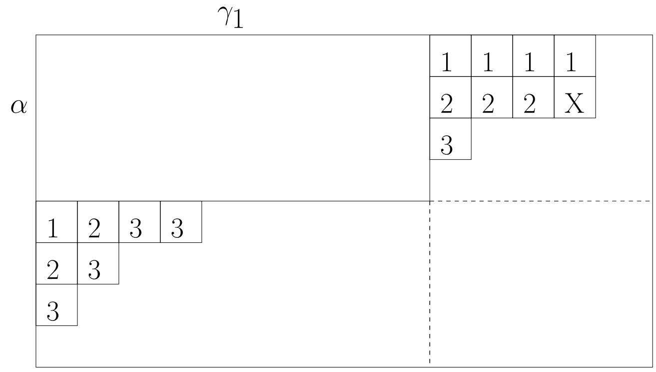
Remark. *The case $x_{n+1}=0$ and $\gamma_1=\gamma_2$ of (4) and (5) gives the identities used by Stanley. Note also that both sides of the identities are zero if $n<\alpha$.*

Proof. We want to expand the product on the left-hand side of the first equation. The tableaux in the second factor can contain entries $n+1$, but only in the last row. Therefore, we can split the term into summands in the following way:

$$s_{(\gamma_1^\alpha)}(x_1,\ldots,x_n)\cdot s_{(\gamma_2^\alpha)}(x_1,\ldots,x_{n+1})$$
$$= \sum_{k=0}^{\gamma_2} x_{n+1}^k \cdot s_{(\gamma_1^\alpha)}(x_1,\ldots,x_n)\cdot s_{(\gamma_2^{\alpha-1},\gamma_2-k)}(x_1,\ldots,x_n).$$

This product of two Schur functions can be expanded by the Littlewood–Richardson rule (see [12]), i.e. $s_\mu(x_1,\ldots,x_n)\cdot s_\nu(x_1,\ldots,x_n) = \sum_\rho c^\rho_{\mu\nu} s_\rho(x_1,\ldots,x_n)$, where $c^\rho_{\mu\nu}$ is the number of semistandard tableaux of shape ρ/μ and content ν such that reading each row from right to left starting with the first row we always have $\#1's \geq \#2's \geq \#3's \geq \ldots$.

We apply the Littlewood-Richardson rule with $\mu = (\gamma_1^\alpha)$ and $\nu = (\gamma_2^{\alpha-1}, \gamma_2 - k)$. There are not many possibilities for the shape ρ which has to contain the shape μ and must have $|\mu| + |\nu|$ boxes.



The left upper corner of this picture is the shape $\mu = (\gamma_1^\alpha)$ which has been taken out of ρ. The right lower corner is also empty since the partition $\nu = (\gamma_2^{\alpha-1}, \gamma_2 - k)$ encoding the content tells us that the highest possible entry in the Littlewood-Richardson tableau of shape ρ/μ is α.

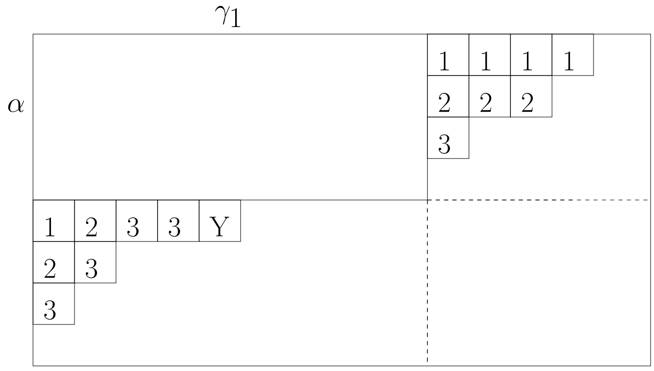
It turns out that row i in the right upper block consists only of entries i. Consider for example the position X in the picture. It cannot contain an entry 3 (or higher) because the word $11113222\ldots$ would have a 3 before the first 2. Since the number of 1's is given by $\nu_1 = \gamma_2$, we can see that the right upper corner is inside the rectangle $\alpha \times \gamma_2$, and there is exactly one admissible filling of each partition λ° inside this rectangle.

Now, how many possibilities are there to place the remaining λ_α entries 1, $\lambda_{\alpha-1}$ entries 2, …, λ_2 entries $\alpha - 1$ and $\lambda_1 - k$ entries α into the left lower corner?

We claim that for the entries from 1 to $\alpha - 1$ there is only the following possibility: Write all 1's in the first row, then write a 2 under each 1 and place the remaining 2's in the first row, then write a 3 under each 2 and place the remaining 3's in the first row, and so on up to $\alpha - 1$.

Suppose we have already filled in the entries from 1 to i in this way (in particular, every column ends with i at this point and we can write entries $i+1$ either in the first row or underneath an entry i). Now assume that we would not write $i+1$ under each of the $\lambda_{\alpha-i+1}$ entries i.

Then we read the Littlewood-Richardson word up to and including the entries $i+1$ of the first row of the left lower corner (if there is no entry $i+1$ in the first row, we just read the right upper corner). In this word, we have $\lambda_i^\circ = \gamma_2 - \lambda_{\alpha-i+1}$ entries i and at least $\gamma_2 - \lambda_{\alpha-i+1} + 1$ entries $i+1$ (all but those in the lower left corner under an entry i) which clearly contradicts the Littlewood-Richardson condition.



Consider for example the position Y in the picture. It cannot contain an entry 3 because we would read four 3's before the fourth 2. Therefore, we have to write a 3 underneath each 2.

Note that this argument also proves that the first row of the partition in the left lower corner cannot be bigger than γ_2.

In the case $k = 0$, the same argument holds for the entries α and the shape in the left lower corner is λ with a unique filling. If $k \neq 0$, then we certainly cannot have more α's in the first row, so we simply have to remove some of the α's which means that we remove a horizontal strip from the shape λ. Since the filling is unique for each choice of the horizontal strip all the coefficients $c^{\rho}_{\mu\nu}$ are 1.

This leads immediately to the first equality in identity (4). We get the second equality by filling the missing horizontal strip with entries $n+1$ in the resulting tableaux of shape ρ occuring in the Schur function. This gives tableaux of the shape T in the figure where entries $n+1$ occur only in a certain region.

The second identity is proved analogously. We just have to make sure that there can be no $\alpha+1$ in λ° which follows from the fact that the first row of the shape in the left lower corner cannot be bigger than γ_2 and $\gamma_2 \leq \gamma_1$. $\square$

4. Connection to self-complementary plane partitions

In this section we will explain how to get the (-1)–enumeration for self-complementary plane partitions from Theorem 2 for the case of a box with at least one odd side and why this does not work with the Schur function identity from Stanley's proof.

Furthermore, we will explain what kind of self-complementary plane partitions are counted by the general case $\gamma_1 \neq \gamma_2$.

We will prove the following two theorems.

Theorem 3. *The enumeration of self-complementary plane partitions in an $a \times b \times (c_1 + c_2)/2$–box where the corresponding rhombus tiling contains a fixed "middle line" of length $(c_1 - c_2)/2$ parallel to $(c_1 + c_2)/2$ (see Figure 4) equals*

$$\begin{aligned}
B\left(\tfrac{a}{2}, \tfrac{b}{2}, \tfrac{c_1}{2}\right) B\left(\tfrac{a}{2}, \tfrac{b}{2}, \tfrac{c_2}{2}\right) &\quad \textit{for } a, b, c_1, c_2 \textit{ even,}\\
B\left(\tfrac{a+1}{2}, \tfrac{b}{2}, \tfrac{c_1}{2}\right) B\left(\tfrac{a-1}{2}, \tfrac{b}{2}, \tfrac{c_2}{2}\right) &\quad \textit{for } a \textit{ odd and } b,\ c_1, c_2 \textit{ even,}\\
B\left(\tfrac{a-1}{2}, \tfrac{b+1}{2}, \tfrac{c_1}{2}\right) B\left(\tfrac{a+1}{2}, \tfrac{b-1}{2}, \tfrac{c_2}{2}\right) &\quad \textit{for } a, b \textit{ odd and } c_1,\ c_2 \textit{ even.}
\end{aligned}$$

where $B(a,b,c)$ is given in (2).

In all cases the "middle line" lies at equal distance from the border of the hexagon in the direction of the line and orthogonal to the line (see Figure 4).

The "middle line" is just a line of length $\frac{c_1-c_2}{2}$ if a,b is even, it is a strip of $\frac{c_1-c_2}{2}$ rhombi if a is even and b is odd and it is a line of length $\frac{c_1-c_2}{2}-1$ with two attached triangles if a is odd and b is even.

In terms of the corresponding integer arrays $a \times \frac{c_1+c_2}{2}$ with weakly decreasing rows and columns where opposite entries add up to b, we have the condition that the entry in row $a/2$ and column $c_1/2$ has to be at least $b/2$ in the first case. In the second case, the condition is that the entries in row $(a+1)/2$ and columns $c_2/2+1,\dots,c_1/2$ are filled with entries $b/2$. In the third case, the places in row $(a+1)/2$ and columns $c_2/2+1,\dots,c_1/2$ are empty, the entries above them are at least $(b-1)/2$, the entries below them are at most $(b+1)/2$ and the entry to the left of them is at least $(b+1)/2$.

Theorem 4. *The (-1)–enumeration of self-complementary plane partitions in an $a\times b\times c$–box equals up to sign*

$$SC\left(\tfrac{a}{2},\tfrac{b+1}{2},\tfrac{c-1}{2}\right)SC\left(\tfrac{a}{2},\tfrac{b-1}{2},\tfrac{c+1}{2}\right) \qquad \text{for } a \text{ even and } b,\,c \text{ odd}$$
$$SC\left(\tfrac{a+1}{2},\tfrac{b}{2},\tfrac{c}{2}\right)SC\left(\tfrac{a-1}{2},\tfrac{b}{2},\tfrac{c}{2}\right) \qquad \text{for } a \text{ odd and } b,\,c \text{ even}$$

where $SC(a,b,c)$ is given by Theorem 1.

Remark. *In the case of three even sidelengths, the (-1)–enumeration of self-complementary plane partitions is $B(a/2,b/2,c/2)$ (see [4]), so we cannot expect to prove it with the help of an identity involving the product of two Schur functions.*

Similarly to the proof of the (-1)–enumeration in [5], the enumerations in Theorem 3 can be expressed by Pfaffians via families of non-intersecting lattice paths. Therefore, we have the following corollary.

Corollary. *The following Pfaffians evaluate to the respective expressions in Theorem 3.*

$$\operatorname{Pf}_{1\le i,j\le a}\left(\sum_{k=1}^{\frac{a+b}{2}}\left(\binom{(b+c_1)/2}{b+i-k}\binom{(b+c_2)/2}{j+k-a-1}-\binom{(b+c_1)/2}{b+j-k}\binom{(b+c_2)/2}{i+k-a-1}\right)\right)$$
$$\text{for } a,b,c_1,c_2 \text{ even}$$

$$(-1)^{\frac{a-1}{2}}\operatorname{Pf}_{1\le i,j\le a+1}\left(\begin{array}{c|c}\displaystyle\sum_{k=1}^{\frac{a+b-1}{2}}\left(\binom{\frac{b+c_1}{2}}{b+i-k}\binom{\frac{b+c_2}{2}}{j+k-a-1}-\binom{\frac{b+c_1}{2}}{b+j-k}\binom{\frac{b+c_2}{2}}{i+k-a-1}\right) & \binom{\frac{b+c_1}{2}}{b+i-\frac{a+b+1}{2}}\\ \hline -\binom{\frac{b+c_1}{2}}{b+j-\frac{a+b+1}{2}} & 0\end{array}\right)$$
$$\text{for } a \text{ odd and } b,c_1,c_2 \text{ even,}$$

$$(-1)^{\frac{a-1}{2}}\operatorname{Pf}_{1\le i,j\le a+1}\left(\begin{array}{c|c}\displaystyle\sum_{k=1}^{\frac{a+b}{2}}\left(\binom{\frac{b+c_2-1}{2}}{b+i-k}\binom{\frac{b+c_1+1}{2}}{j+k-a-1}-\binom{\frac{b+c_2-1}{2}}{b+j-k}\binom{\frac{b+c_1+1}{2}}{i+k-a-1}\right) & -\binom{\frac{b+c_2-1}{2}}{b+i-\frac{a+b}{2}-1}\\ \hline \binom{\frac{b+c_2-1}{2}}{b+j-\frac{a+b}{2}-1} & 0\end{array}\right)$$
$$\text{for } a,b \text{ odd and } c_1,c_2 \text{ even,}$$

The second and the third matrix should be read as $a \times a$–matrices with an extra row and column.

Proof of Theorem 3. We will see that the self-complementary plane partitions with this additional constraint are in bijection with the semi-standard tableaux appearing in the Schur function identities (4) and (5).

Let us start with the case a, b, c_1, c_2 even.

As stated in Section 2, self-complementary plane partitions can be represented by rectangular $a \times (c_1 + c_2)/2$–arrays of positive integers with decreasing rows and columns with the additional condition that entries related by a 180°-rotation add up to b. For example, the self-complementary plane partition in Figure 4a corresponds to the array

$$\begin{array}{cccccccc} 4 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 3 & 2 & 1 & \overline{1} & \overline{1} & 1 & 1 & 0 \end{array}$$

The fixed bold line in the picture becomes the constraint that array entries above the horizontal middle line of length $(c_1 - c_2)/2$ are greater or equal to $b/2$.

To switch from decreasing to increasing and to make the columns *strictly* increasing, we rotate the array by 180° degree and add i to row i and get

$$\begin{array}{cccccccc} 1 & 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & \overline{5} & \overline{5} & 5 & 5 & 6 \end{array} \tag{6}$$

Now, we have a semi-standard tableau where entries related by a 180°-rotation add up to $a + b + 1$ which happens to be odd in this case. The entries above the middle line of length $(c_1 - c_2)/2$ are now smaller or equal to $(a + b)/2$.

If we now look only at the subtableau consisting of the entries $\leq (a + b + 1)/2$, we get shapes ρ with $\rho = \rho^\circ$ who are above the middle line. This is exactly the shape of the tableau T in (4), but without the condition on the entries $n + 1$.

Therefore, we can count these plane partitions by setting $\gamma_1 = c_1/2$, $\gamma_2 = c_2/2$, $\alpha = a/2$, $n = (a + b)/2$, $x_1 = x_2 = \cdots = x_n = 1$ and $x_{n+1} = 0$.

(The case $\gamma_1 = \gamma_2$ is an instance of the original proof of Stanley.)

The closed form for this enumeration follows at once from the left-hand side of (4) and identity (3) with $q = 1$.

Similarly, we get the second case a odd and b, c_1, c_2 even by setting $x_1 = x_2 = \cdots = x_{n+1} = 1$, $n = (a + b - 1)/2$, $\gamma_1 = c_1/2$, $\gamma_2 = c_2/2$ and $\alpha = (a - 1)/2$ in the second part of Theorem 2 with an example shown in Figure 4b.

The corresponding tableau is

$$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 2 & 4 & 4 & \boxed{4} & \boxed{4} & 4 & 4 & 6 \\ 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 \end{array}$$

After substracting i from row i and rotating, we obtain an array with weakly decreasing rows and columns where the boxed entries become $b/2$. This translates to a strip of $(c_1 - c_2)/2$ rhombi in the middle of the hexagon (see Figure 4b).

The last case a, b odd and c_1, c_2 even is obtained by setting $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = 0$, $n = (a + b)/2$, $\gamma_1 = c_1/2$, $\gamma_2 = c_2/2$ and $\alpha = (a - 1)/2$ in the second part of Theorem 2 with an example shown in Figure 4c.

Since there are now no entries $n+1$, we can complete the semi-standard tableau to a rectangle where corresponding entries add up to $2n+1$. This means that there are no integers that properly fit in the center part of the middle row and we can think of the entries as $n+1/2$ to preserve all monotony restrictions.

In the example, the corresponding "tableau" with strictly increasing columns is

$$\begin{array}{ccccc} 1 & 2 & 3 & 3 & 4 \\ 2 & \boxed{3.5} & \boxed{3.5} & \boxed{3.5} & 5 \\ 3 & 4 & 4 & 5 & 6 \end{array}$$

After substracting i from row i and rotating, we obtain an array with weakly decreasing rows and columns where the entries in row $(a-1)/2$ and columns $c_2/2+1, c_2/2+2, \ldots, c_1/2$ are at least $(b-1)/2$ and the corresponding entries in row $(a+3)/2$ are at most $(b+1)/2$. Furthermore, the entry in row $(a+1)/2$ and column $c_2/2$ is at least $(b+1)/2$.

If we represent this by stacks of cubes and treat the non-integer entries as empty, we obtain a bijection to rhombus tilings with 180°-rotational symmetry of hexagons with sides $a, b, (c_1+c_2)/2$ and a middle line of length $(c_1-c_2)/2-1$ with two triangles attached to the end (see Figure 4).

□

Remark. *Obviously, we could also look at the specialization involving x_{n+1} in the second part, which works, but gives plane partitions which are a reflection of the ones obtained in the second case above. The c_1, c_2 are always even, but this is no restriction on the parity of $(c_1+c_2)/2$, so indeed all possible sidelengths of hexagons are treated.*

Proof of Theorem 4. We want to set $x_i = (-1)^i$ in Theorem 2 to obtain our result.

Let us look again at the example of a tableau corresponding to a self-complementary plane partition.

$$\begin{array}{cccccccc} 1 & 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & 5 & 5 & 5 & 5 & 6 \end{array}$$

Clearly, the action on the tableau corresponding to removing and adding a cube in a plane partition as shown in Figure 2 consists of adding one to an entry and subtracting one of the entry related by a 180°-rotation. If this action changes two entries 2 and 5 to 3 and 4, the subtableau of entries ≤ 3 loses an entry 2 and gains an entry 3. Therefore, the contribution of these entries to the terms in the Schur function changes from x_2 to x_3 which gives the desired sign change for $x_i = (-1)^i$.

On the other hand, if we exchange two entries 3 and 4, the weight obviously remains unchanged, so in this case, we do not obtain the desired ± 1-weight.

In this example, the maximal possible entry $a+b$ in the array is even and the only case leading to problems are two entries $(a+b)/2$ and $(a+b)/2+1$ swapping place leaving the weight in the Schur function unchanged while the ± 1–weight changes.

Since the parameters a, b, c played a symmetrical in the definition of self-complementary and of the ± 1–weight, we can assume that $a+b$ is odd if not all of a, b, c are even.

Now, the only problematic case involves two entries $(a+b+1)/2$ changing to $(a+b+1)/2+1$ and $(a+b+1)/2-1$. It is clear that exactly one of the entries $(a+b+1)/2$

must be in the left lower region where it contributes to the weight, so one factor in the weight changes from $x_{(a+b+1)/2}$ to $x_{(a+b+1)/2-1}$ which coincides with the change of the ± 1–weight for $x_i = (-1)^i$.

So, we assume that a and b have different parity and it only remains to set $\alpha = a/2$, $\gamma_1 = (c+1)/2$, $\gamma_2 = (c-1)/2$ and $n = (a+b-1)/2$ in (4) for the case a even and b, c odd and $\alpha = (a-1)/2$, $\gamma_1 = \gamma_2 = c/2$ and $n = (a+b-1)/2$ in (5) for the case a odd and b, c even. (Note that the condition imposed by the middle-line of length 1 is automatically fulfilled in a self-complementary plane partition and therefore does not change the enumeration.)

The left-hand side in both cases lead to the case $q = -1$ of (3) which gives the desired result. □

Proof of the Corollary. The original proof of Theorem 4 used a bijection between plane partitions and families of non-intersecting lattice paths which gives a Pfaffian for both the weighted and the unweighted case which can then be evaluated.

Since this is the case $c_1 = c_2$ in Theorem 3, we can go through the proof backwards and ask which Pfaffians correspond to the plane partitions in Theorem 3 for general c_1, c_2. Of course, then we know that they evaluate to the nice factored expression of Theorem 3.

By exactly the method of [4], we can find a bijection to families of non-intersecting lattice paths shown in Figure 5, express this number as a sum of determinants using the main theorem on non-intersecting lattice paths (see Lindström, [11, Lemma 1] or Gessel and Viennot [6, Theorem 1])) and finally express this as a Pfaffian using a theorem by Ishikawa and Wakayama ([7]). □

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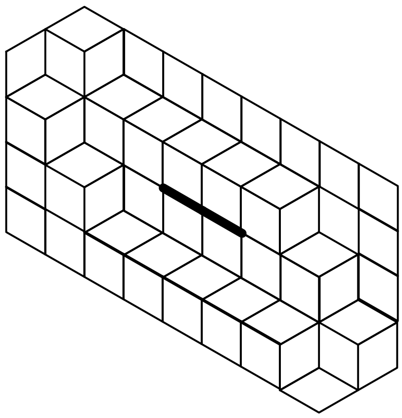
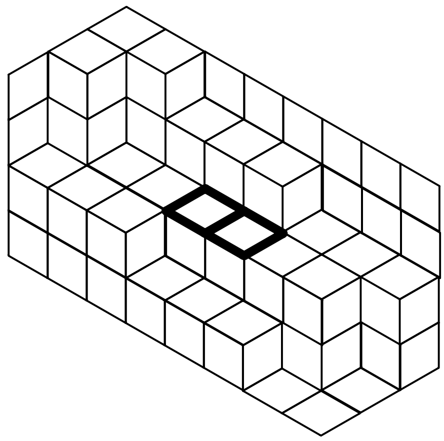
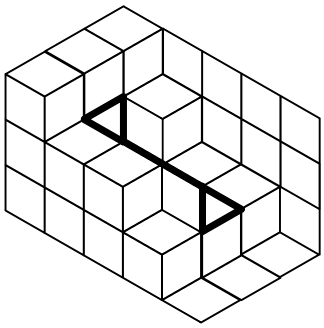


a. The case a, b even.



b. The case a odd, b even.



c. The case a, b odd.

FIGURE 4. Self-complementary plane partitions with a fixed line in the middle.

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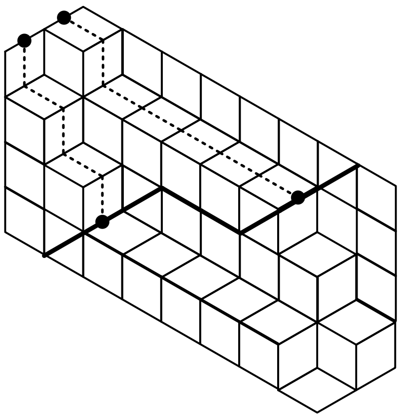
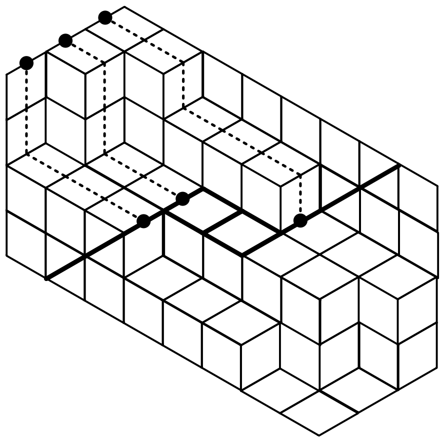
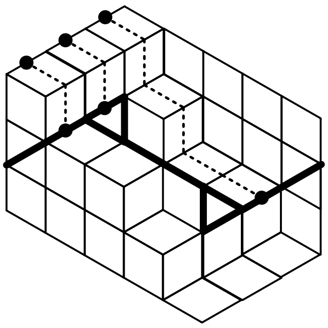


a. The paths for a, b even.



b. The paths for a odd, b even.



c. The paths for a, b odd.

FIGURE 5. The path families.