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Spatial Unfolding of Elementary Bifurcations

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We consider solutions of a partial differential equation which are homogeneous in space and stationary or periodic in time. We study the stability with respect to large wavelength perturbations and the weakly nonlinear behavior around these solutions, especially when they are close to bifurcations for the ordinary differential equation governing the homogeneous solutions of the PDE. We distinguish cases where a spatial parity symmetry holds. All bifurcations occurring generically for two-dimensional ODES are treated. Our main result is that for almost homoclinic periodic solutions instability is generic.

KEY WORDS: Spatially homogenous solution; codimension one bifurcations; spatial unfolding; phase and period-doubling instabilities; parity symmetry; amplitude equation.

1. INTRODUCTION

Codimension one bifurcations of simple solutions (fixed points and limit cycles) of ordinary differential equation have been extensively studied, in particular by the Russian school.[(4)] They are frequently observed in Physical, Chemical and Biological systems. This is indeed one of the great merit of the Poincaré qualitative theory[(12)] and the Andronov[(1)] subsequent work to provide a language in order to describe the behavior of complex systems when some external parameters are varied, particularly in situations where the equations governing those systems are not exactly known (coarse systems). In this paper we address the problem of the stability of simple solutions of ODEs and the robustness of their bifurcations in the frame of spatial unfoldings.

We consider PDEs of the form

$$\partial_t u = F(u, \partial_x) \tag{1}$$

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i.e., invariant with respect to translations of time (autonomous) and space. Here u is in $\mathbf{R}^d$, $d \geqslant 1$, and for simplicity the space variable x is one-dimensional ($x \in \mathbf{R}$). Spatially homogeneous solutions of this PDE are solutions of the equation

$$\frac{du}{dt} = F(u, 0) = f(u) \tag{2}$$

(we write $f(u)$ for $F(u, 0)$), which is an autonomous ordinary differential equation in dimension d.

Among the solutions of Eq. (2), of prime interest are those which correspond to an asymptotic behavior, in particular attractive fixed points and attractive periodic orbits. Consider a solution $t \mapsto u_h(t)$ of Eq. (2) which is an attractive (linearly stable) fixed point or periodic orbit. The corresponding homogeneous solution for the PDE (1) is thus stable with respect to homogeneous perturbations. We adress the question of the behavior of inhomogeneous perturbations. A small inhomogeneous perturbation $u(x, t)$ of $u_h(t)$ obeys the equation

$$\partial_t u = DF(u_h(t), \partial_x)\, u \tag{3}$$

This equation being linear, it reduces in Fourier coordinates to

$$\partial_t \hat{u}(k) = DF(u_h(t), ik)\, \hat{u}(k) \tag{4}$$

which is just an ordinary differential equation parametrized by k, with constant (resp. periodic) coefficients if $u_h(t)$ is a stationary (resp. periodic) solution.

Now the behavior of inhomogeneous perturbations of $u_h(t)$ is, without further hypotheses, by far a too general problem. It is thus necessary to specify, particularize this problem in order to be able to provide significative results.

An interesting way to particularize the problem is to look at it close to a bifurcation. Indeed, bifurcation theory tells us that this simplifies greatly the problem, and at the same time preserves its generality: normal forms or unfoldings of bifurcations are both "particular" and "universal" examples. Thus we will suppose that F, f, and u_h depend on a parameter μ, and that a bifurcation occurs at $\mu = 0$ for the solution u_h of the differential equation (2). We will suppose that μ is close to 0, and we will moreover restrict ourselves to large wavelength perturbations (we will suppose that the wavenumber k is small). We call this approach *spatial unfolding of bifurcations.*

At the leading order in space derivatives, Eq. (4) reads

$$\partial_t \hat{u}(k) = (L(t) + ikC(t) - k^2 D(t) + \mathcal{O}(|k|^3))\, \hat{u}(k) \tag{5}$$

where $L(t)$, $C(t)$, $D(t)$ are $d \times d$ real matrices which are constant (resp. periodic) in time if u_h is a constant (resp. periodic) solution (remark that $L(t)$ is nothing else than $Df(u_h(t))$). This differential equation depends on two small parameters μ and k, and we know that, when $k = 0$, a bifurcation occurs at $\mu = 0$. The main question is: are the spatial effects destabilizing or not? in other words, is the homogeneous bifurcation anticipated (for $k \neq 0$, before $\mu = 0$) by another bifurcation due to the spatial effects?

It is often the case that the system described by the PDE (1) admits an additional parity symmetry with respect to the space variable x. In this case, there exists a linear involution I of $\mathbf{R}^d$ such that the equation is invariant by the transformation $(x, u) \mapsto (-x, Iu)$.

We will say in the following that the problem considered here (the local study around u_h) is $(x \leftrightarrow -x)$-*invariant* if the PDE admits a symmetry $(x, u) \mapsto (-x, Iu)$, and if moreover vector coordinates of $u_h(t)$ vanish identically (i.e., $Iu_h = u_h$). If this is the case, the linear PDE (3) around u_h involves only derivatives of even order, and in Eq. (5), the "convective" matrix $C(t)$ vanishes identically. We will distinguish along the paper between cases where $(x \leftrightarrow -x)$-invariance holds or not for our problem.

Fixed points are treated in Section 2, and periodic orbits in Section 3. We have considered all codimension one bifurcations that occur generically in dimension $d \leqslant 2$ (when Eq. (2) admits no additional symmetry). These are, for fixed points: saddle-node and Hopf bifurcations, and for periodic orbits: Hopf, saddle-node of points, saddle-node of cycles, and homoclinic bifurcations.

The main new result, which is stated and justified in Subsection 3.3, concerns homoclinic bifurcations: in this case, spatial effects are always destabilizing, in other words almost homoclinic periodic orbits are always unstable with respect to inhomogeneous perturbations. This result was conjectured in ref. 2 and announced in refs. 3 and 13 (but only in the $(x \leftrightarrow -x)$-invariant case).

All the other results are rather elementary and many of them are classical. Besides, some of them may be surprising to the reader, for instance the fact that, when the $(x \leftrightarrow -x)$-invariance does not hold, fixed points close to a Hopf bifurcation and periodic orbits close to a saddle-node of cycles are always unstable with respect to inhomogeneous perturbations. Nevertheless, we have found it interesting to present all these results in a unified way. The paper thus takes, up to Subsection 3.2, the form of a survey, followed by a more technical part in Subsection 3.3. Paper ends up with a short conclusion.

2. FIXED POINT

We suppose that u_h is a fixed point for Eq. (2), which is linearly stable for $\mu < 0$, and for which a bifurcation occurs at $\mu = 0$. Generically (if f has no particular symmetry), this can happen in two ways: a saddle-node or a Hopf bifurcation.

2.1. Saddle-Node Bifurcation

We place ourselves at the bifurcation, i.e., at $\mu = 0$. We suppose that $u_h = 0$ and 0 is an eigenvalue of multiplicity 1 of $Df(0)$, all the other eigenvalues having strictly negative real parts. Up to a linear change of coordinates, the matrix $L = Df(0)$ reads

$$L = \begin{pmatrix} 0 & 0 \\ 0 & L_- \end{pmatrix}$$

where L_- is a $(d-1) \times (d-1)$-matrix with eigenvalues having strictly negative real parts.

For k close to 0, the eigenvalues of the matrix $M(k) = L + ikC - k^2 D$ are close to those of the matrix L. Denote by $\lambda(k)$ the one which is close to 0, and denote by $\varepsilon(k)$ the (unique) vector in the corresponding eigendirection having its first coordinate equal to 1. Write $\varepsilon(k) = (1, y(k))$, where $y(k)$ is a $(d-1)$-vector, and write $\lambda(k) = k\lambda_1 + k^2\lambda_2 + \cdots$ and $y(k) = ky_1 + k^2 y_2 + \cdots$. Finally, denote by C_h the horizontal $(d-1)$-vector $(C_{1,2},..., C_{1,d})$, and denote by C_v the vertical vector $(C_{2,1},..., C_{d,1})$.

At order one in k, $M(k)\, \varepsilon(k) = \lambda(k)\, \varepsilon(k)$ gives

$$\lambda_1 = iC_{1,1} \qquad \text{and} \qquad L_- y_1 + ikC_v = 0$$

and at order two in k, we have

$$\lambda_2 = -D_{1,1} + iC_h y_1 = -D_{1,1} + C_h L_-^{-1} C_v$$

(a) $(x \leftrightarrow -x)$-invariant case. In this case $C = 0$ and thus $\lambda_1 = 0$ and $\lambda_2 = -D_{1,1}$. Thus, if $D_{1,1} < 0$ (resp. > 0), then the spatial effects destabilize (resp. do not destabilize) the homogeneous solution u_h; thus for $\mu < 0$ close to 0, this solution is unstable (resp. stable) with respect to large wavelength perturbations.

The amplitude equation describing the nonlinear development of the instability reads

$$s_T = \mu - s^2 + \nu s_{XX} \tag{6}$$

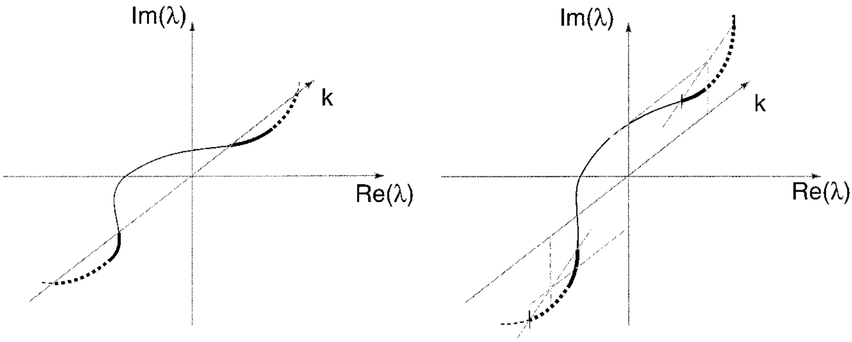

Fig. 1. Growth rate versus wave number, for $\mu < 0$ and $D_{1,1} < 0$; left corresponds to case (a) and right to case (b) (dotted lines are for terms of order higher than k^2).

with $u = u_h + s(X, T)\,\phi_0 + \cdots$ where ϕ_0 is the eigenvector of L corresponding to the eigenvalue 1 and $v = D_{1,1}$. For negative but small v the next order term (s_{XXXX}) can provide saturation if it comes with a negative coefficient.

For $v < 0$, the spatially extended solution looses its stability before the saddle-node bifurcation (Fig. 1 left). A wavelength is selected and a spatial periodic pattern is expected to occur. It is the well known Turing instability.[14]

(b) Non ($x \leftrightarrow -x$)-invariant case. In this case, if $D_{1,1} < C_h L_-^{-1} C_v$ (resp. $D_{1,1} > C_h L_-^{-1} C_v$), then the spatial effects destabilize (resp. do not destabilize) the homogeneous solution u_h; thus for $\mu < 0$ close to 0, this solution is unstable (resp. stable) with respect to large wavelength perturbations.

This case is not very different from the ($x \to -x$)-invariant case. The instability will appear as sketched on Fig. 1 (right) and will give rise to a wave propagating with the velocity $C_{1,1}$. A balance between C and D may give rise to an instability even if D is proportional to identity and positive. This situation can occur for example in chemical reactions in the presence of electric field.[9]

2.2. Hopf Bifurcation

Again, we place ourselves at the bifurcation, i.e., at $\mu = 0$. We suppose that $u_h = 0$ and that $Df(0)$ has two purely imaginary eigenvalues $\pm i\omega$, each of multiplicity one, all the other eigenvalues having strictly negative real parts. Up to a linear change of coordinates, the matrix $L = Df(0)$ reads

$$L = \begin{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} & 0 \\ 0 & L_- \end{pmatrix}$$

where L_- is a $(d-2)\times(d-2)$-matrix with eigenvalues having strictly negative real parts. For k close to 0, the eigenvalues of $L+ikC-k^2 D$ are close to those of L. Denote by $\lambda_+(k)$ (resp. $\lambda_-(k)$) the one which is close to $i\omega$ (resp. $-i\omega$), and write $\lambda_+(k)=i\omega+k\lambda_{1,+}+k^2\lambda_{2,+}+\cdots$ and $\lambda_-(k)=-i\omega+k\lambda_{1,-}+k^2\lambda_{2,-}+\cdots$. Let

$$P=\begin{pmatrix}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix} & 0\\ 0 & \mathrm{Id}_{\mathbf{R}^{d-2}}\end{pmatrix}$$

and write $L'=P^{-1}LP$, $C'=P^{-1}LP$, $D'=P^{-1}DP$. We have

$$L'=\begin{pmatrix}\begin{pmatrix}i\omega & 0\\ 0 & -i\omega\end{pmatrix} & 0\\ 0 & L_-\end{pmatrix}$$

At order one in k, we have $\lambda_{1,+}=iC'_{1,1}$ and $\lambda_{1,-}=iC'_{2,2}$. Write $\alpha=(C_{1,1}+C_{2,2})/2$ and $\beta=(C_{2,1}-C_{1,2})/2$; then, $C'_{1,1}=\alpha+i\beta$ and $C'_{2,2}=\alpha-i\beta$; thus,

$$\lambda_{1,+}=i\alpha-\beta \qquad \text{and} \qquad \lambda_{1,-}=i\alpha+\beta$$

If $C=0$, then $\lambda_{1,+}=\lambda_{1,-}=0$ and, at order two in k, we have $\lambda_{2,+}=-D'_{1,1}$ and $\lambda_{2,-}=-D'_{2,2}$. Write $\gamma=(D_{1,1}+D_{2,2})/2$ and $\delta=(D_{2,1}-D_{1,2})/2$; then,

$$\lambda_{2,+}=-\gamma-i\delta \qquad \text{and} \qquad \lambda_{2,-}=-\gamma+i\delta$$

(a) $(x\leftrightarrow -x)$-invariant case. In this case $C=0$ and, according to the expressions of $\lambda_{2,+}$ and $\lambda_{2,-}$ above, if $D_{1,1}+D_{2,2}<0$ (resp. >0), then the spatial effects destabilize (resp. do not destabilize) the homogeneous solution u_h; thus for $\mu<0$ close to 0, this solution is unstable (resp. stable) with respect to large wavelength perturbations.

With $v_r=D_{1,1}+D_{2,2}$, the amplitude equation for this instability reads:

$$\partial_\tau A=\mu A-(1+i\alpha_i)\,|A|^2\,A+(v_r+i\beta_i)\,A_{XX}+\text{h.o.t} \tag{7}$$

with $u=A(X,\tau)\,e^{i\omega t}\phi_\omega+\text{c.c.}$, where ϕ_ω is the eigenvector of L corresponding to the eigenvalue $i\omega$.

In the case where v_r is negative but small the instability arise as sketched on Fig. 2 (left). Left and right traveling waves patterns are

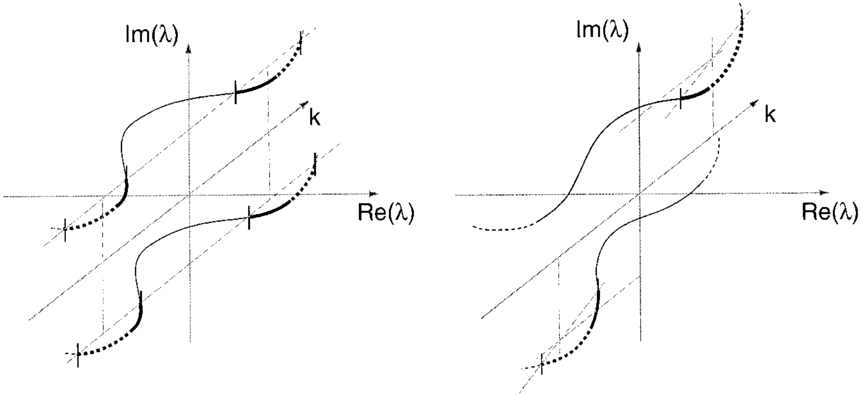

Fig. 2. Growth rate vs wave number for Hopf instability with a first order spatial derivative term (Dotted line are for higher order terms). Figure on the left corresponds to case (a), when $D_{1,1}+D_{2,2}<0$, and figure on the right corresponds to case (b), when $C_{2,1}-C_{1,2}<0$.

expected to occur.[6] Moreover in Eq. (7), higher order terms are necessary for saturation of the instability. It includes terms A_{XXXX}, $|A|^4 A$, $|A_X|^2 A$, all with complex coefficients.

(b) Non $(x \leftrightarrow -x)$-invariant case. In this case, if $C_{2,1}-C_{1,2}$ is non zero (which occurs generically), then we see that one of the quantities $\lambda_{1,+}$ and $\lambda_{1,-}$ has a positive real part. Thus, spatial effects are *always* destabilizing, and, for $\mu<0$ close to 0, u_h is *always* unstable with respect to large wavelength perturbations.

The growth rate of the perturbation is shown on Fig. 2 (right) and a wavenumber is selected. The amplitude equation writes:

$$\partial_\tau A = \mu A + (c + i\rho)\, A_X - (1 + i\alpha_i)\, |A|^2 A + (v_r + i\beta)\, A_{XX} + \text{h.o.t.} \qquad (8)$$

with $u = A(X, \tau)\, e^{i\omega t}\phi_\omega + \text{c.c.}$, where ϕ_ω is the eigenvector of L corresponding to the eigenvalue $i\omega$, and $\rho = C_{2,1} - C_{1,2}$.

Once more, the instability of extended solutions will appear before the homogeneous Hopf bifurcation, even for positive v_r. In the case where v_r is negative but small, additional terms have to be included in the amplitude equation. The symmetry between right and left traveling wave is lost, which can lead to a new bifurcation, with one of those waves disappearing.

The stability of the limit cycle arising through Hopf bifurcation in the $(x \leftrightarrow -x)$-invariant case when v_r is equal to 1 is given by the celebrated Benjamin–Feir–Kuramoto[5, 10] criterion. We note that even in the non $(x \leftrightarrow -x)$-invariant case, the transformation of variable $A = Be^{i(\rho/2)X}$ leads to the same criterion.

3. PERIODIC ORBIT

3.1. Preliminaries

We now suppose that $t \mapsto u_h(t)$ is a periodic solution of the homogeneous equation (2) (denote by T its period). Thus $L(t)$, $C(t)$, and $D(t)$ are now T-periodic $d \times d$ real matrices. During this paragraph 3.1, we forget that u_h depends on the parameter μ and is close to a bifurcation, and we recall basic computations that will be used later.

Denote by $\Phi_k(t)$ the flow of the differential equation (5) over one period T. For $k=0$, $\Phi_0(T)$ is a first return (monodromy) map for the differential equation (2) around $u_h(.)$, thus one of its eigenvalue is always equal to 1 (it corresponds to phase translation, in the direction of the flow). We suppose that u_h is linearly stable with respect to homogeneous perturbations, i.e., that all the other eigenvalues of $\Phi_0(T)$ are strictly inside the unit circle.

For k close to 0, the eigenvalues of $\Phi_k(T)$ are close to those of $\Phi_0(T)$, let us denote by $\lambda(k)$ the one which is close to 1. The stability with respect to the wavenumber k depends on the position of $|\lambda(k)|$ with respect to 1. Write

$$\lambda(k) = 1 + k\lambda_1 + k^2\lambda_2 + \mathcal{O}(|k|^3)$$

The coefficient λ_2 is real, while λ_1 has a vanishing real part.

(a) $(x \leftrightarrow -x)$-invariant case. In this case $C(.) \equiv 0$, λ_1 vanishes, and the stability with respect to large wavelength perturbations is given by the

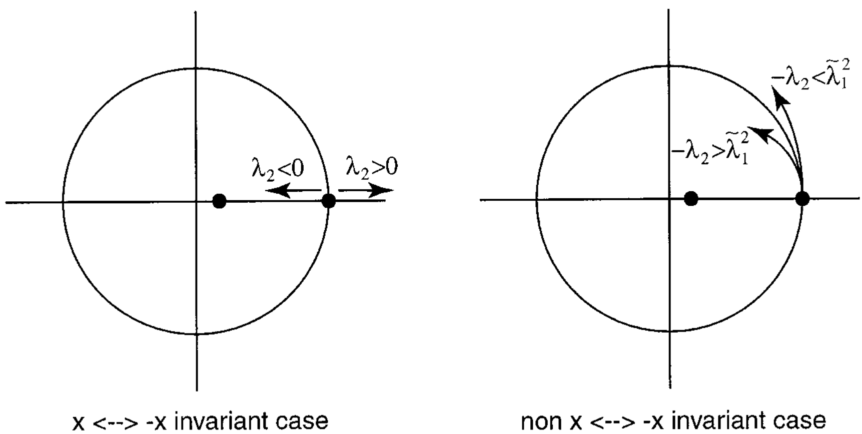

Fig. 3. Behavior of $\lambda(k)$ for $|k|$ small.

sign of λ_2. If $\lambda_2 < 0$, then $u_h(.)$ is phase stable, while it exhibits the well-known phase Kuramoto instability[10] if $\lambda_2 > 0$.

(b) Non $(x \leftrightarrow -x)$-invariant case. In this case $C(.)$ is not identically vanishing, thus λ_1 is generically nonvanishing and purely imaginary. Write $\lambda_1 = i\tilde{\lambda}_1$. The stability with respect to small k now depends on both coefficients $\tilde{\lambda}_1$ and λ_2. More precisely, if $-\lambda_2 > \tilde{\lambda}_1^2$ (resp. $-\lambda_2 < \tilde{\lambda}_1^2$), then $u_h(\cdot)$ is stable (resp. unstable) with respect to sufficiently small wavenumbers k.

Formal Computation of λ_1 and λ_2. For simplicity, we now restrict ourselves to the case $d = 2$, although the following computations of λ_1 and λ_2 could be carried out in higher dimension.

Let $e_1(t) = f(u_h(t))$ and $e_2(t) = \mathrm{Rot}_{\pi/2}(e_1(t))$, $t \in \mathbf{R}$. This defines a local frame $(e_1(\cdot), e_2(\cdot))$ along the periodic solution $u_h(\cdot)$. Let us formulate the differential equation (5) using coordinates in this local frame. It takes the form

$$\partial_t \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + ik \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} - k^2 \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} + \mathcal{O}(|k|^3) \right) \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

where a, b, c_j, and d_j are real and T-periodic. Write

$$B_s^t = e^{\int_s^t b(v)\, dv}$$

The quantity B_0^T is equal to the second eigenvalue of $\Phi_0(T)$ (the first one being equal to 1). By hypothesis (linear stability), it belongs to $]0; 1[$.

For k close to 0, denote by ε_k the (unique) vector belonging to the eigendirection of $\Phi_k(T)$ corresponding to the eigenvalue $\lambda(k)$ and having first coordinate equal to 1. Write $\varepsilon_k = (1, y_k)$. For $k = 0$, we have $y_0 = 0$. For $t \in \mathbf{R}$, write $\varepsilon_k(t) = \Phi_k(t)\, \varepsilon_k$, write $\varepsilon_k(t) = (x_k(t), y_k(t))$, and write

$$x_k(t) = 1 + kx_1(t) + k^2 x_2(t) + \cdots \qquad \text{and} \qquad y_k(t) = ky_1(t) + k^2 y_2(t) + \cdots$$

The relation $\varepsilon_k(T) = \lambda(k)\, \varepsilon_k(0)$ yields

$$\lambda_1 = x_1(T), \quad y_1(t) = y_1(0), \quad \lambda_2 = x_2(T), \quad \text{and} \quad y_2(T) = y_2(0) + \lambda_1 y_1(0)$$

and the differential equation (9) reads, at the first order in k,

$$\frac{dx_1}{dt} = ay_1 + ic_1$$

$$\frac{dy_1}{dt} = by_1 + ic_3$$

and, at the second order in k,

$$\frac{dx_2}{dt} = ay_2 + i(c_1 x_1 + c_2 y_1) - d_1$$

$$\frac{dy_2}{dt} = by_2 + i(c_3 x_1 + c_4 y_1) - d_3$$

(a) $(x \leftrightarrow -x)$-invariant case. In this case $C(\cdot) \equiv 0$. Thus, $x_1 \equiv 0$, $y_1 \equiv 0$, $y_2(\cdot)$ is the unique T-periodic, solution of the differential equation $(dY/dt) = bY - d_3$, i.e., $y_2(t) = B_0^t y_2(0) - \int_0^t B_s^t d_3(s)\, ds$, where $y_2(0) = -(1 - B_0^T)^{-1} \int_0^T B_s^T d_3(s)\, ds$, and we obtain

$$\lambda_1 = 0 \qquad \text{and} \qquad \lambda_2 = \int_0^T (ay_2 - d_1)(s)\, ds$$

(b) Non $(x \leftrightarrow -x)$-invariant case. Here $C(\cdot) \not\equiv 0$, $y_1(\cdot)$ is the unique T-periodic solution of the differential equation $(dY/dt) = bY + ic_3$, $x_1(t) = \int_0^t (ay_1 + ic_1)(s)\, ds$, $y_2(\cdot)$ is the unique solution of the differential equation $(dY/dt) = bY + i(c_3 x_1 + c_4 y_1) - d_3$ satisfying $y_2(T) = y_2(0) + \lambda_1 y_1(0)$, and λ_1 and λ_2 read

$$\lambda_1 = x_1(T), \qquad \text{and} \qquad \lambda_2 = \int_0^T (ay_2 + i(c_1 x_1 + c_2 y_1) - d_1)(s)\, ds$$

In both cases (a) and (b), we can explicitly write down expressions of λ_1 and λ_2 depending on a, b, c_j, $j = 1 \cdots 4$, d_1 and d_3.

3.2. Bifurcating Periodic Orbit

Now we suppose again that the periodic solution u_h depends on the parameter $\mu \leqslant 0$, and that a bifurcation occurs at $\mu = 0$ for this solution. We know since Andronov that, in dimension $d = 2$, bifurcations which occur generically in one parameter families of periodic orbits are of the four following types: Hopf, saddle-node of points, saddle-node of cycles, and homoclinic. Another bifurcation of primary importance is the period-doubling bifurcation. In this section, we consider the saddle-node of points, saddle-node of cycles, and period-doubling bifurcations. Hopf bifurcation was treated in paragraph 2.2, and the homoclinic bifurcation will be treated in the next section. Dimension d is any.

3.2.1. Saddle-Node of Points

In this case, at the bifurcation ($\mu = 0$), a saddle point appears, we have seen in Subsection 2.1 that spatial effects could destabilize or not this saddle point. For $\mu < 0$ close to 0, the periodic orbit $u_h(\cdot)$ spends almost all its time close to this saddle point. Thus, spatial effects will have the same kind of influence (destabilizing or not) on the periodic solution for $\mu < 0$ close to 0 as on the saddle point for $\mu = 0$. If, for $\mu = 0$, they destabilize (resp. do not destabilize) this saddle point, then, for $\mu < 0$ close to 0, they will similarly destabilize (resp. not destabilize) the periodic solution.

3.2.2. Saddle-Node of Cycles

We place ourselves at the bifurcation ($\mu = 0$), thus we suppose that 1 is an eigenvalue of multiplicity 2 for the monodromy map $\Phi_0(T)$. Thus $B_0^T = 1$ and, writing $\Phi_0(T) = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$, we have, with the previous notations, $\alpha = \int_0^T B_0^s\, a(s)\, ds$.

For k close to 0, let $\lambda(k)$ denote one of the two eigenvalues of $\Phi_k(T)$, and let ε_k belong to the corresponding eigendirection and having first coordinate equal to 1. Write $\lambda(k) = 1 + \lambda_1(k)$ and $\varepsilon_k = (1, y_k)$; thus $\lambda_1(k)$ and y_k are small. Write $\varepsilon_k(t) = \Phi_k(t)\, \varepsilon_k$ and $\varepsilon_k(t) = (1 + x_1(t), y_1(t))$. We thus have

$$x_1(T) = \lambda_1(k) \qquad \text{and} \qquad y_1(T) = y_1(0) + \lambda_1(k)\, y_1(0)$$

On the other hand, we derive from (9) that

$$y_1(T) = y_1(0) + ik \int_0^T B_s^T c_3(s)\, ds + \cdots$$

This shows that $|y_1(0)| \gg |k|$, and thus that $y_1(t) = B_0^t\, y_1(0) + \cdots$. Thus, we get from (9) $\lambda_1(k) = x_1(T) = y_1(0)\, \alpha + \cdots$, and finally

$$\lambda_1(k)^2 = ik\alpha \int_0^T B_s^T c_3(s)\, ds + \cdots \tag{10}$$

(a) $(x \leftrightarrow -x)$-invariant case. In this case $C \equiv 0$ and a computation similar to the previous one yields

$$\lambda_1(k)^2 = -k^2 \alpha I + \cdots \qquad \text{where} \quad I = \int_0^T B_s^T\, d_3(s)\, ds$$

Thus, if $-\alpha I > 0$, then the eigenvalues of $\Phi_k(T)$ get away from 1 by real values, and the spatial effects destabilize the solution u_h (for $\mu < 0$ close to 0,

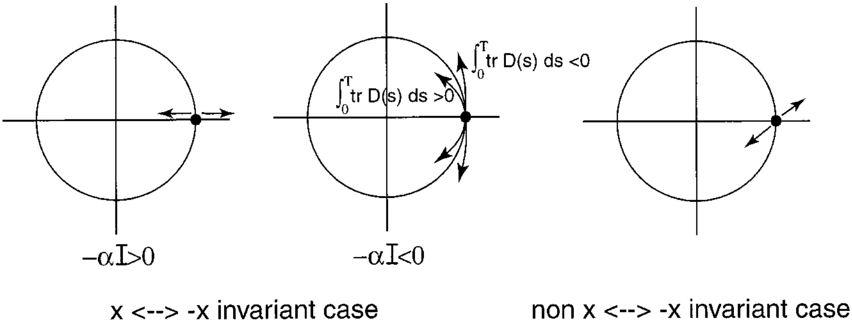

Fig. 4. Behavior of the two eigenvalues of $\Phi_k(T)$ for $|k|$ small, at a saddle-node of cycles.

this solution is unstable with respect to large wavelength perturbations), see Fig. 4.

If $-\alpha I < 0$, then the eigenvalues of $\Phi_k(T)$ get away from 1 by (at first order) purely imaginary values; as $\Phi_k(T)$ is real, they are complex conjugated, and their modulus is given by the determinant of $\Phi_k(T)$, i.e., at first order in k^2 by $\exp(-k^2 \int_0^T tr\, D(s)\, ds)$. Thus, if $\int_0^T tr\, D(s)\, ds < 0$ (resp. > 0), then the spatial effects destabilize (resp. do not destabilize) the solution u_h; for $\mu < 0$ close to 0, this solution is unstable (resp. stable) with respect to large wavelength perturbations, see Fig. 4.

(b) Non $(x \leftrightarrow -x)$-invariant case. In this case, expression (10) shows that spatial effects (generically) *always* destabilize the solution u_h (for $\mu < 0$ close to 0, this solution is *always* unstable with respect to large wavelength perturbations), see Fig. 4.

3.2.3. Period-Doubling Bifurcation

Here the dimension d is of course higher than 2. At the bifurcation, the monodromy map $\Phi_0(T)$ has two neutral eigenvalues, one equal to 1, and the other one equal to -1. Thus $\Phi_0(2T)$ has a double eigenvalue which is equal to 1, but there is no Jordan bloc corresponding to it. The two neutral eigendirection are thus linearly decoupled[4] and the influence of spatial coupling can be studied for each of it independently; the problem thus reduces to the (general) case treated in Subsection 3.1.

3.3. Andronov Homoclinic Bifurcation

Now we suppose that the bifurcation occurring at $\mu = 0$ is a homoclinic bifurcation. We denote by $u \mapsto f_0(u)$ the function $u \mapsto f(u)$ when $\mu = 0$. We place ourselves close to the bifurcation, i.e., we suppose

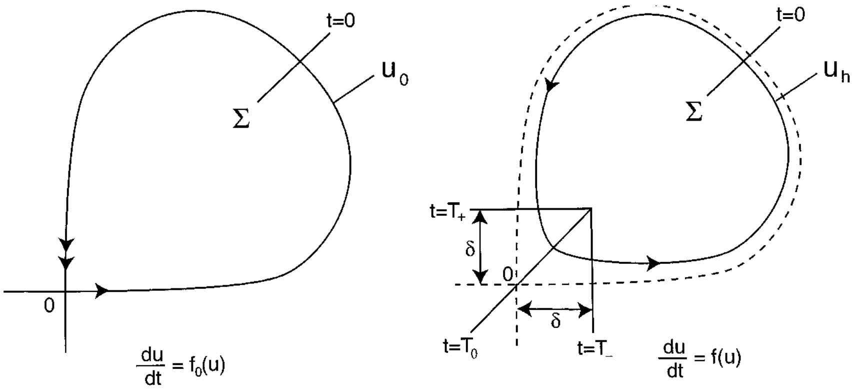

Fig. 5. Homoclinic and quasi-homoclinic orbits.

that μ is close to 0 but strictly negative; thus, $f(\cdot)$ is close to (but different from) $f_0(\cdot)$.

We suppose that $f_0(0)=0=f(0)$, that $Df_0(0)$ reads $\left(\begin{smallmatrix} b_+ & 0 \\ 0 & b_- \end{smallmatrix}\right)$, where $0<b_+<-b_-$, and that the differential equation $(du/dt)=f_0(u)$ admits a solution $u_{h,0}(\cdot)$ homoclinic to the fixed point 0, and we suppose that the trajectories of $u_h(\cdot)$ and $u_{h,0}(\cdot)$ are close.

We are going to show that, for $\mu<0$ close enough to 0, the periodic solution $u_h(\cdot)$ is (generically) always linearly unstable with respect to inhomogeneous perturbations. In the $(x\leftrightarrow -x)$-invariant case, this result was conjectured in ref. 2 and announced in refs. 3 and 13. A complete mathematical proof (in the $(x\leftrightarrow -x)$-invariant case, but in any dimension d) can be found in ref. 13.

Consider the differential equation (9) (in the local frame along $u_h(\cdot)$) where the origin of times is fixed on a section Σ transverse to some point of the trajectory of $u_{h,0}(\cdot)$ (see the figure). We can write down a similar differential equation in the local frame of the homoclinic solution $u_{h,0}(\cdot)$; let us write it

$$\partial_t \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 0 & a_0 \\ 0 & b_0 \end{pmatrix} + ik \begin{pmatrix} c_{1,0} & c_{2,0} \\ c_{3,0} & c_{4,0} \end{pmatrix} - k^2 \begin{pmatrix} d_{1,0} & d_{2,0} \\ d_{3,0} & d_{4,0} \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix}$$

This differential equation is not periodic any more, its coefficients are limits (on any bounded time interval) of the coefficients of (9) when $\mu\to 0$. We have $b_0(t)\to b_+-b_-$ when $t\to+\infty$, $b_0(t)\to b_--b_+$ when $t\to-\infty$, $a_0(t)\to 0$ when $t\to\pm\infty$, and each coefficient $c_{j,0}(t)$ (resp. $d_{j,0}(t)$) admits limits when $t\to+\infty$ and $t\to-\infty$, say $c_{j,0,+}$ and $c_{j,0,-}$ (resp. $d_{j,0,+}$ and $d_{j,0,-}$).

Following Andronov's classical idea, we are going to decompose the periodic solution $u_h(\cdot)$ into two parts, one close to and the other one far from the fixed point 0. Let δ be a small positive parameter. Denote by T_+ (resp. T_-) the first positive time when $u_h(\cdot)$ enters (resp. escapes) the box of size 2δ centered in 0. Denote by T_0 the time between T_+ and T_- where $u_h(\cdot)$ belongs to the diagonal $x=y$ (see the figure). The quantities $T_- - T_0$ and $T_0 - T_+$ are both large, but $T_- - T_0$ is larger than $T_0 - T_+$ (more precisely, the ratio $T_- T_0/(T_0 - T_+)$ is close to $(|b_-|/b_+) > 1$).

For $t \in [T_-; T_+]$, i.e., when $u_h(t)$ lies inside the small box, we have

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \mathrm{Rot}_{-2\theta(t)} \begin{pmatrix} 0 \\ b_- - b_+ \end{pmatrix} + \cdots$$

where $\theta(t)$ is the angle $(\varepsilon_1, e_1(t))$ (here $\varepsilon_1 = (1, 0)$ and $e_1(t) = f(u_h(t))$ is the speed vector), arid the remaining terms "$\cdots$" are small if δ and μ are close to 0. The angle $\theta(t)$ is close to $-\pi/2$ for $t - T_0 \ll 0$, close to 0 for $t - T_0 \gg 0$, and jumps between these two values during a bounded time interval around T_0. Thus the qualitative behaviors of $a(t)$ and $b(t)$ for $t \in [T_+; T_-]$ are as follows (see Fig. 6): the coefficient $b(\cdot)$ is close to $b_+ - b_-$ for $t - T_0 \ll 0$, close to $b_- - b_+$ for $t - T_0 \gg 0$, and jumps between these two values during a bounded time interval around T_0; the coefficient $a(\cdot)$ is close to 0 except during this bounded time interval, where it takes finite positive values between 0 and $b_+ - b_-$.

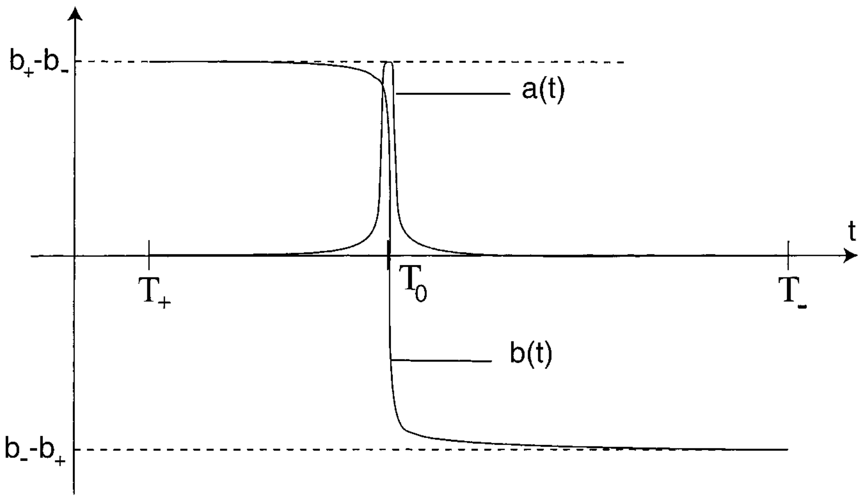

Fig. 6. Behavior of $a(t)$ and $b(t)$ for $t \in [T_+; T_-]$.

3.3.1. $(x \leftrightarrow -x)$-invariant Case

(a) Phase stability. We are first going to estimate λ_2. We know that y_2 is the unique T-periodic solution of the differential equation $(dy/dt) = by - d_3$. Let us consider the limit when $\mu \to 0$ of this equation, i.e.,

$$\frac{dy}{dt} = b_0 y - d_{3,0}$$

The asymptotic behavior of b_0 and $d_{3,0}$ show that this equation admits a unique solution $y_-(\cdot)$ (resp. $y_+(\cdot)$) which is bounded when $t \to -\infty$ (resp. when $t \to +\infty$). Generically, these two solutions are different and the sign of $y_-(\cdot) - y_+(\cdot)$ is constant (remark that the only case where this genericity result does not hold is when the matrix $D(t)$ is proportional to the identity: in this case the coefficient $d_3(\cdot)$ vanishes identically, and so do y_-, and y_+; the periodic solution $u_h(\cdot)$ is then stable, because the coupling has only a trivial stabilizing effect).

The sign of $y_-(\cdot) - y_+(\cdot)$ governs the nature of the instability. Indeed, when $t \to +\infty$, $|y_-(\cdot)| \to +\infty$ and has the sign of $y_-(\cdot) - y_+(\cdot)$. The fact that $T_- - T_0 > T_0 - T_+$ shows that the behavior of $y_2(\cdot)$ is the following: on $[T_- - T; T_+]$ (i.e., when u_h is "far" from 0), it is very close to $y_-(\cdot)$; on $[T_+; T_0]$, $|y_2(\cdot)|$ grows exponentially and y_2 has the same sign as $y_- - y_+$; $|y_2(\cdot)|$ takes a maximal value around $t = T_0$, and decreases exponentially afterwards. Thus. the main contribution in the expression $\lambda_2 = \int_0^T (ay_2 - d_1)(s)\, ds$ is the integral of ay_2 on a bounded interval around $t = T_0$. As $a > 0$ on this interval, we see that λ_2 is large and has the sign of $y_-(\cdot) - y_+(\cdot)$.

The conclusion is that, if $y_-(\cdot) - y_+(\cdot) > 0$, then the periodic solution $u_h(\cdot)$ is phase unstable close to the homoclinic bifurcation.

(b) Period doubling instability. If $y_-(\cdot) - y_+(\cdot) < 0$, then $u_h(\cdot)$ is phase stable, i.e., stable with respect to values of k (very) close to 0. We are going to see, however, that in this case another instability holds, with respect to small but finite values of k.

Denote by Φ_k^R (resp. Φ_k^S) the flow of the differential equation (9) between the times $T_- - T$ and T_+, i.e., outside of the small box of size 2δ (resp. between the times T_+ and T_-, i.e., inside the small box of size 2δ). The composition $\Phi_k^S \circ \Phi_k^R$ represents the flow over one period.

Let us write $\Phi_0^R = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & \beta \end{smallmatrix}\right)$. The flow Φ_k^R is a perturbation of Φ_0^R which remains non-singular when one approaches the homoclinic bifurcation; we can thus write

$$\Phi_k^R = \Phi_0^R + k^2 \begin{pmatrix} w & x \\ y & z \end{pmatrix} + \mathcal{O}(k^2)$$

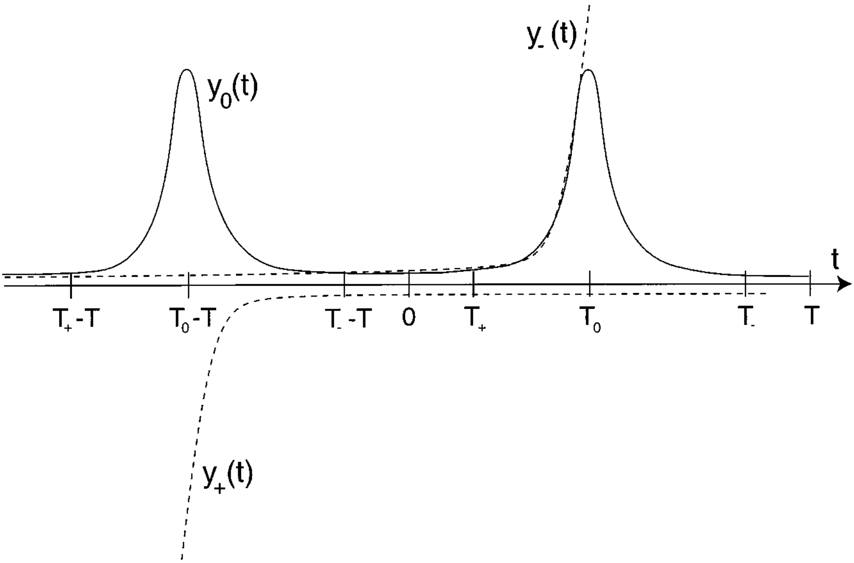

Fig. 7. Behaviors of $y_0(\cdot)$, $y_+(\cdot)$, and $y_-(\cdot)$ on **R**.

Remark that the quantity y in this expression of Φ_k^R is the value at time T_+ of the solution of the differential equation $(dy/dt) = by - d_3$, with initial condition 0 at tune $T_- - T$. If δ is sufficiently small, the behavior of this solution is once again governed by the behavior of y_-; it grows exponentially, and has the sign of $y_- - y_+$ (thus negative in the case considered here) when t approaches T_+. Thus, the value y at time T_+ is negative and arbitrarily large if δ is sufficiently small.

Let us write $\Phi_0^S = \begin{pmatrix} 1 & \eta \\ 0 & \zeta \end{pmatrix}$. If $-\mu$ is the distance between $u_h(T_+)$ (at the entrance of the small box) and the stable manifold of 0, and denoting $\delta^{-1}\,|\mu|$ by ε and $|b_-|/b_+$ by γ, we find that

$$\eta = \gamma(1 + \mathcal{O}(\delta))\,\varepsilon^{-1+\mathcal{O}(\delta)} \qquad \text{and} \qquad \zeta = \gamma^2(1 + \mathcal{O}(\delta))\,\varepsilon^{\gamma - 1 + \mathcal{O}(\delta)}$$

The very singular form of Φ_0^S deserves comments. Although this linear map has distinct eigenvalues (namely 1 and η), it can hardly be put in a diagonal form because its two eigenvectors are almost parallel. An amplitude perturbations of the limit cycle when it enters the small box transforms into a strong phase perturbation. There is thus a strong coupling between amplitude and phase perturbations, and even if one of the Floquet multipliers (ζ) tends to zero while the other remains finite, no dimensional reduction is possible.

The flow Φ_k^S is a perturbation of Φ_0^S which becomes singular when one approaches the homoclinic bifurcation; nevertheless, we have the following estimate:

$$\Phi_k^S = q_k(\Phi_0^S + \eta k^2 \mathcal{O}(1))$$

where $q_k = \varepsilon^{\mathcal{O}(k^2)}$. Denote by T_k the trace of $\Phi_k^S \circ \Phi_k^R$. The previous expressions of Φ_k^R and Φ_k^S show that

$$T_k = q_k(T_0 + \eta k^2(y + \mathcal{O}(1)))$$

If y is sufficiently large (i.e., if δ is sufficiently small), and if ηk^2 is large (i.e., if $k^2 \gg \varepsilon$), we see that the dominant term in this expression of T_k is the term $q_k \eta k^2 y$; it is large and negative. As on the other hand the determinant of $\Phi_k^S \circ \Phi_k^R$ is small, we see finally that, when k^2 is small but $k^2 \gg \varepsilon$, this first return map has two real eigenvalues, one close to 0, and the other one large negative. This proves the instability in this case.

3.3.2. Non $(x \leftrightarrow -x)$-invariant Case

In this case, the same kind of computation as in the previous paragraph (period doubling instability) can be achieved. It shows that, generically, for $|k|$ small but $|k| \gg \varepsilon$, the trace of the monodromy map $\Phi_k^S \circ \Phi_k^R$ has a large modulus (and an argument close to $\pm\pi/2$), which proves already the instability. Nevertheless, we want to be more precise and show that the instability occurs for arbitrarily small values of k, i.e., that the phase instability criterion $-\lambda_2 < \tilde{\lambda}_1^2$ holds.

Write $y_1 = i\tilde{y}_1$ and $x_1 = i\tilde{x}_1$. Then $\tilde{y}_1$ is defined as the unique T-periodic solution of the differential equation $(dy/dt) = by + c_3$. Consider the corresponding limit differential equation:

$$\frac{dy}{dt} = b_0 y + c_{3,0}$$

Again, this differential equation admits a unique solution $y_-(\cdot)$ (resp. $y_+(\cdot)$) which is bounded when $t \to -\infty$ (resp. when $t \to +\infty$). Generically, these two solutions are different, and the sign of $y_-(\cdot) - y_+(\cdot)$ is constant.

As in the previous paragraph, the behaviour of $|\tilde{y}_1(\cdot)|$ is the following: it grows exponentially on $[T_+; T_0]$, and decreases exponentially on $[T_0; T_-]$ (on these two intervals $\tilde{y}_1(\cdot)$ has the sign of $y_-(\cdot) - y_+(\cdot)$). The quantity $|\tilde{y}_1(t)|$ thus takes a maximal value when t is around T_0. We deduce from this the behavior of $|\tilde{x}_1(t)| = |\int_0^t (a\tilde{y}_1 + c_1)(s)\, ds|$: it grows very

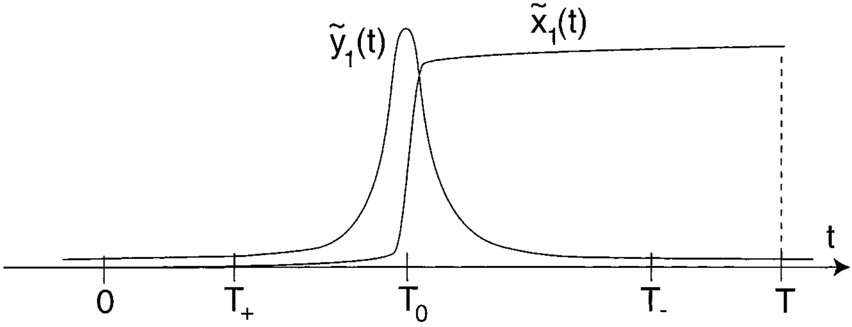

Fig. 8. Behaviors of $\tilde{x}_1(t)$ and $\tilde{y}_1(t)$ for $t \in [0; T]$.

fast when t belongs to a bounded interval around T_0, and varies much slower outside of this interval. We have

$$\frac{dy_2}{dt} = by_2 - c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3$$

The constraint $y_2(T) = y_2(0) - \tilde{x}_1(T)\,\tilde{y}_1(0)$ yields

$$y_2(0)(1 - B_0^T) = -\tilde{x}_1(T)\,\tilde{y}_1(0) - \int_0^T B_s^T(-c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3)(s)\,ds$$

Now, according to the behaviors of b and a, we have, for δ sufficiently small, on one hand

$$\tilde{y}_1(0) = (1 - B_0^T)^{-1} \int_0^T B_s^T c_3(s)\,ds \simeq \int_{T_-}^T B_s^T c_3(s)\,ds$$

and on the other hand

$$\int_0^T B_s^T(-c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3)(s)\,ds \simeq \tilde{x}_1(T) \int_{T_-}^T B_s^T c_3(s)\,ds$$

Finally, we obtain $|y_2(0)| \ll |\tilde{x}_1(T)|$, which shows that $\max_{t \in [0;\,T]} |y_2(t)| \ll \tilde{x}_1(T)^2$, and finally that

$$|\lambda_2| \ll \tilde{\lambda}_1^2$$

This proves the instability.

3.3.3. Nonlinear Behavior

In summary, close to an Andronov bifurcation, a limit cycle is always unstable with respect to spatially inhomogeneous perturbations. Depending on the form of the coupling, this instability is either the phase instability or a period doubling instability.

For the phase instability, amplitude equation with $u(t, x) = u_h(t - \phi)$ is the well known Kuramoto–Sivashinsky equation:[10]

$$\partial_\tau \phi = \mu \phi_{XX} + \alpha \phi_X^2 - \phi_{XXXX} \tag{11}$$

For the period doubling instability in the $(x \leftrightarrow -x)$-invariant case, the nonlinear amplitude equation with $u(t, x) = u_h(t - \phi) + A e^{ik_0 x} \zeta(t - \phi) + \text{c.c.} + \cdots$ reads:

$$\partial_\tau A = \mu A \pm \alpha |A|^2 A + \alpha \phi_{XX} A + \beta \phi_X^2 + A_{XX} \tag{12}$$

$$\partial_\tau \phi = \delta \phi_{XX} + \phi_X^2 + \eta |A|^2 \tag{13}$$

where $\zeta(t)$ is the Floquet eigenvector corresponding to the period doubling.

We show numerical computation of the following equations:

$$u_t = v + \gamma u_x + u_{xx} - \beta v_{xx} \tag{14}$$

$$v_t = (\mu - u) v - u + u^2 + \beta u_{xx} + v_{xx} \tag{15}$$

The homogeneous part of this equation admits a stable periodic solution for $0 < \mu < 0.135$ which disappears via an Andronov bifurcation.

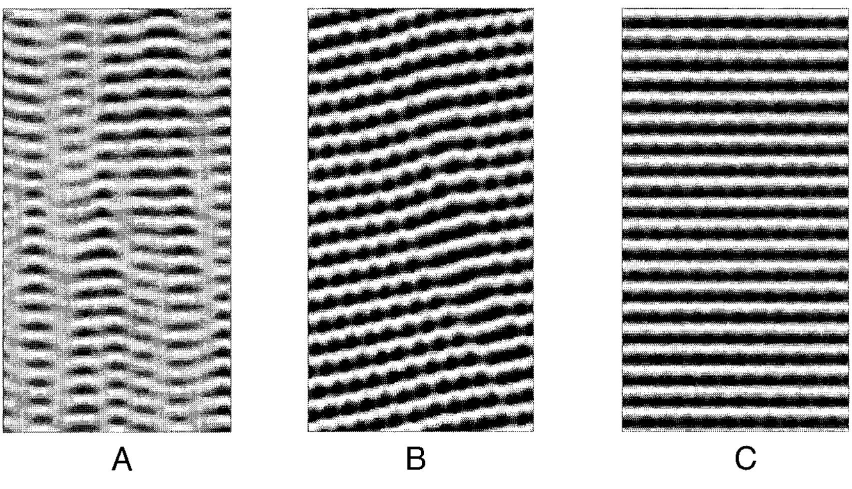

Fig. 9. Numerical simulation of Eqs. (14) and (15).

Results appear on Fig. 9 where abcissa is for the spatial coordinate and ordinate is for time. The intensity of gray corresponds to the value of u.

1. For A the parameters are $\mu = 0.075$, $\beta = 1$, $\gamma = 0$. For these values of parameter the system exhibits a phase instability. There is no wavenumber selected.
2. For B the parameters are $\mu = 0.075$, $\beta = -1$, $\gamma = 0$. The system exhibits this stable pattern after a bifurcation from the homogeneous state. It clearly shows period doubling, and a wave length is selected.
3. For C the parameters are $\mu = 0.075$, $\beta = -1$, $\gamma = 0.35$. The case is more involved and the limit cycle is unstable for small wavenumber (phase instability) and is also unstable for a finite wavenumber with a negative real part of the Floquet multiplier. On this diagram there are traces of phase instability and of period doubling instability.

4. CONCLUSION

We have considered together a partial differential equation and the ordinary differential equation governing the homogeneous solutions of this PDE. We have studied the stability (with respect to large wavelength perturbations) and the nonlinear behavior around solutions of the PDE which are homogeneous in space and stationary or periodic. in time, in particular when these solutions were close to bifurcations for the homogeneous problem. We call this approach *spatial unfolding of bifurcations* since we considered only large wavelength (small wavenumber) perturbations. We have considered all generic bifurcations in dimension up to 2, and we have distinguished cases where the PDE admits or not an additional parity symmetry ($x \leftrightarrow -x$) with respect to the space variable.

The main results are the following. First, a coupling involving only first order derivatives may already (in all cases) yield an instability. Then, a fixed point or a periodic orbit close to a Hopf bifurcation or a periodic orbit close to a saddle-node of cycles are always (generically) unstable with respect to inhomogeneous perturbations if the parity symmetry ($x \leftrightarrow -x$) is broken. Finally, almost homoclinic periodic orbits are always (generically) unstable with respect to inhomogeneous perturbations; more precisely,

- if ($x \leftrightarrow -x$)-invariance holds, there are two possibilities: either the classical phase instability, or a period doubling instability at a finite (intrinsic) wavenumber.
- if ($x \leftrightarrow -x$)-invariance does not hold, instability is always a phase instability.

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