Self-Parametric Instability in Spatially Extended Systems

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We study the stability of almost homoclinic homogeneous limit cycles with respect to spatiotemporal perturbations. It is shown that they are generically unstable. The instability is either the phase instability or a finite wavelength period doubling instability.

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Bifurcations which occur in one parameter families of dynamical systems play an important role in the understanding of universal physical phenomena [1]. In this Letter we report on the existence of a new type of instability which arises in spatially extended systems. We name it "self-parametric" instability since it is the consequence of anharmonicity of a spatially homogeneous limit cycle which acts as a parametric forcing on itself. More precisely we consider a partial differential equation which possesses a spatially independent time-periodic solution. This solution is assumed to be stable with respect to homogeneous perturbations. We demonstrate that this solution is generically unstable in respect to inhomogeneous perturbations, when it approaches an Andronov homoclinic bifurcation [2]. The instability is either the Kuramoto phase instability [3] or a finite period doubling instability which occurs at finite wavelength. As for the Turing instability [4], the typical length which emerges in the latter case is intrinsic and not related to the particular geometry of the system.

Let $u = (u_1, u_2, \dots, u_n)$ be a set of n scalar fields which obey the following equation:

$$\partial_t u = f(u, \vec{\nabla}). \tag{1}$$

The physical system described by Eq. (1) is assumed to be space and time translationally invariant and space isotropic. Let $F(u) = f(u, 0)$ denote the vector field associated with the homogeneous solutions of the partial differential equation:

$$\partial_t u = F(u). \tag{2}$$

Let $u_h(t)$ be a given stable solution of Eq. (2). A small inhomogeneous perturbation $v(t, \vec{r})$ obeys the linear equation

$$\partial_t v = L(t)v + D(t, \vec{\nabla}^2)v, \tag{3}$$

where $L(t) = DF/Du_{u=u_h(t)}$ and the inhomogeneous linear operator $D(t, \vec{\nabla})$ [it satisfies $D(t, 0) = 0$] is defined as $D(t, \vec{\nabla}^2) = Df/DU_{u=u_h(t)} - L(t)$. At the leading order in the space derivatives (i.e., for long wavelength perturbations), the inhomogeneous term appears as a time-dependent generalized diffusion operator:

$$\partial_t v_k = L(t)v_k - k^2 D(t)v_k + O(k^4), \tag{4}$$

where the v_k is the amplitude of the Fourier mode characterized by a wave vector $\vec{k}$. Far from singularities, for small k, the stability of the homogeneous solution with respect to spatiotemporal fluctuations appears then as a regular perturbation of the stability of the homogeneous solution. The Lyapunov exponents σ_k of the homogeneous solution are a function of k^2 [σ_h are the Lyapunov exponents of $u_h(t)$ of Eq. (2)],

$$\sigma_k = \sigma_h + O(k^2). \tag{5}$$

(i) In the case where u_h is a stable homogeneous stationary solution, the perturbation cannot lead to an instability, unless either the operator L or D is singular. In the latter case, the instability is actually the Turing instability [4].

(ii) In the case where $u_h(t)$ is a stable homogeneous time-periodic solution, there exists a Goldstone mode, due to the time translation invariance. The existence of this neutral phase mode for the homogeneous solution is at the origin of the Kuramoto phase instability [3] for long wavelength perturbations. The criteria of instability simply depend on the sign of the coefficient of the k^2 term in the "phase" Floquet exponent [5].

We now focus our attention on the singular case where the limit cycle approaches a *saddle* fixed point (Andronov bifurcation [2]). Since this bifurcation occurs already for planar vector fields, we simplify our analysis by considering only two components scalar field $u = (u_1, u_2)$. We first analyze the homogeneous vector field close to the periodic solution $u_h(t) = (u_{h,1}(t), u_{h,2}(t))$, far from the Andronov bifurcation. Let $\Phi_0(t)$ be the flow of the equation $\partial_t v = L(t)v$ [i.e., $v(t) = \Phi_0(t)v(0)$] in the local frame [5] [$\dot{u}_h(t)$ and $R_{\pi/2}\dot{u}_h(t)$; here $R_{\pi/2}$ represents the rotation of $\pi/2$]. This flow takes the form

$$\Phi_0(t) = \begin{pmatrix} 1 & \alpha(t) \\ 0 & \beta(t) \end{pmatrix},$$

where the time-dependent functions $\alpha(t)$ and $\beta(t)$ are easily computed from the two by two matrix $L(t)$ and the solution $u_h(t)$. The monodromy matrix $M = \Phi_0(T)$, where T represents the period of the limit cycle, has two eigenvalues (Floquet multipliers) $\sigma_1 = 1$ and $\sigma_2 = \beta(T)$. The limit cycle is stable when $0 < \beta(T) < 1$. For small $|k|$, the Floquet multipliers become

$$\sigma_\phi = 1 + \sigma_\phi^{(2)} k^2 + O(k^4)$$

and

$$\sigma_a = \beta(T) + \sigma_a^{(2)} k^2 + O(k^4).$$

The phase instability occurs in the case $\sigma_\phi^{(2)} > 0$.

Close to the Andronov bifurcation the above analysis becomes singular. As shown by Andronov [2] the stability of the limit cycle can be studied by decomposing the monodromy map of the vector field into two parts (see Fig. 1). One, singular (Φ_0^S), close to the saddle fixed point, can be computed from the linear approximation of the equations and the other, regular (Φ_0^R), far from the limit cycle. We have

$$\Phi_0(T) = \Phi_0^S(\Phi_0^R). \tag{6}$$

The linear operator $L(t)$ when $u_h(t)$ is close to the saddle fixed point is

$$L(t) = \begin{pmatrix} 0 & [2e^{t(1+\lambda)}\delta\lambda\epsilon(1+\lambda)]/(\delta^2\lambda^2 + e^{2t(1+\lambda)}\epsilon^2) \\ 0 & (1+\lambda)(\delta^2\lambda^2 - e^{2t(1+\lambda)}\epsilon^2)/(\delta^2\lambda^2 + e^{2t(1+\lambda)}\epsilon^2) \end{pmatrix}.$$

Take the initial coordinates along the eigenvectors of the saddle fixed point, and up to a time rescaling, suppose that the expanding eigenvalue is 1 and denote by $-\lambda$ the contracting one. Denote by ϵ the bifurcation parameter, i.e., the distance between the stable manifold of the saddle fixed point and the limit cycle as it enters the "linear box" of size δ.

The condition $\lambda > 1$ guarantees the stability of the almost homoclinic homogeneous limit cycle. Elementary calculations allow us to estimate Φ_0^S and Φ_0^R. At the leading order in ϵ

$$\Phi_0^S = \begin{pmatrix} 1 & \lambda\delta/\epsilon \\ 0 & \lambda^2(\epsilon/\delta)^{\lambda-1} \end{pmatrix}$$

and

$$\Phi_0^R = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

Here a and b are functions of ϵ/δ which remain finite as $\epsilon \to 0$ for small but finite δ. The form of Φ_0^S deserves comment. Athough this matrix has distinct eigenvalues [1 and $\lambda^2(\epsilon/\delta)^{\lambda-1}$], it can hardly be put in a diagonal form because its two eigenvectors are almost parallel. An amplitude perturbation of the limit cycle when it enters the linear box transforms into a strong phase perturbation. Close to the Andronov bifurcation, there is a strong coupling between amplitude and phase perturbations, and even if one of the Floquet multipliers [$\lambda^2(\epsilon/\delta)^{\lambda-1}b$] tends to zero while the other remains finite, no dimensional reduction is possible.

The instability of the almost homoclinic limit cycle by respect to spatiotemporal perturbations is closely related to the nature of the singularity induced by the saddle fixed point. The perturbated monodromy map [$\Phi_k = \Phi_k^S(\Phi_k^R)$] is easily computed for small k. The perturbation inside the linear box is explicitly computed, while perturbation techniques are used outside the linear box where the flow is regular. The determinant of the monodromy map tends to zero as $\epsilon \to 0$, for δ small enough but finite while its trace diverges. Indeed, writing

$$\Phi_k^R = \Phi_0^R + k^2\begin{pmatrix} w & x \\ y & z \end{pmatrix}, \tag{7}$$

we see that $\mathrm{tr}(\Phi_k)$ reads

$$\mathrm{tr}(\Phi_k) = \mathrm{tr}(\Phi_0) + \frac{\delta}{\epsilon}k^2(y + \cdots). \tag{8}$$

The quantity y is generically nonzero and arbitrary large in modulus if δ is small enough, and the remaining terms "$\cdots$" are dominated by the term $\frac{\delta}{\epsilon}k^2y$ (see [6] for rigorous results). In particular, we see that if $k^2 > \frac{\delta}{\epsilon}$, then $\mathrm{tr}(\Phi_k)$ is large. This establishes the instability and shows that its nature is governed by the sign of y. For positive y as one of the Floquet multipliers diverges to $+\infty$, the instability of the limit cycle is the Kuramoto phase instability. For negative y, as one of the Floquet multipliers diverges to $-\infty$, the instability is a period doubling instability. Away from the bifurcation, the trace as a function of k^2 becomes a regular curve (see Fig. 2). Although the Kuramoto phase

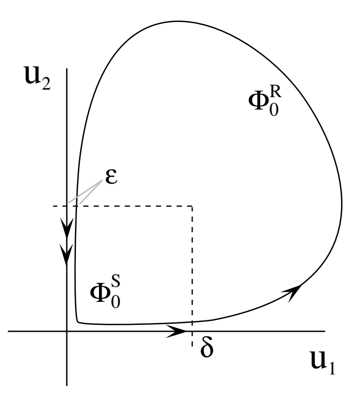

FIG. 1. Andronov decomposition of the flow.

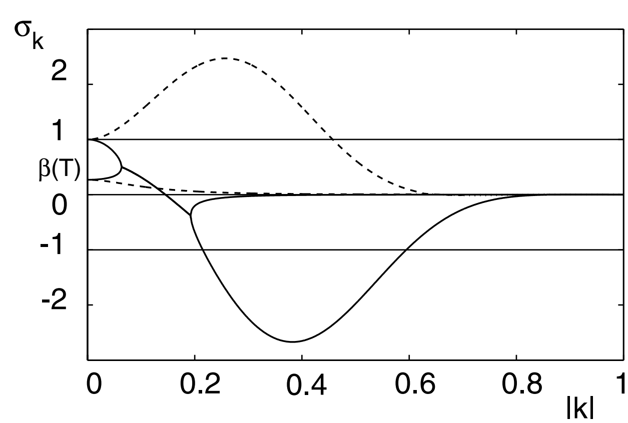

FIG. 2. Floquet multipliers as functions of $|k|$ for Eqs. (11) and (12) for $\mu = 0.1$. The dashed lines represent the Floquet multipliers in the case of the phase instability ($\beta = 1$). The solid lines represent the real parts of the Floquet multipliers in the case of the finite wave number period doubling instability ($\beta = -1$).

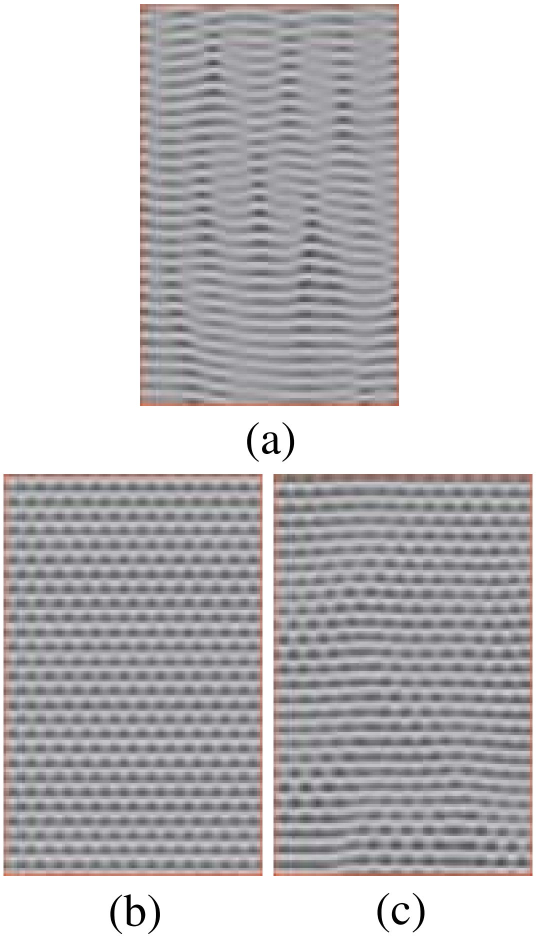

FIG. 3. x-t diagram of the component X of solutions of Eqs. (11) and (12) (size of the domain $L = 150$, $\mu = 5 \times 10^{-2}$). (a) Phase turbulence ($\beta = -1$); (b) regular period doubling finite wave vector pattern ($\beta = 1$); (c) irregular period doubling finite wave vector pattern ($\beta = 1$, but with different initial conditions).

instability arises at zero wavelength, the period doubling instability is characterized by a finite wave number.

The weakly nonlinear development of the latter instability involves two order parameters, the amplitude of the period doubling bifurcation, and the phase mode. The order parameter equations, close to the instability onset, read

$$\partial_t A = (\mu - \mu_c)A \pm |A|^2 A + \alpha \phi_{xx} A + \beta \phi_x^2 A + A_{xx}, \tag{9}$$

$$\partial_t \phi = \delta \phi_{xx} + \phi_x^2 + \eta |A|^2, \tag{10}$$

and

$$u(t) = u_h(t - \phi) + A \exp(ik_0 x)\zeta(t - \phi) + \text{c.c.} + \text{hot},$$

where $\zeta(t)$ is the Floquet eigenmode corresponding to the period doubling [$\zeta(t + T) = -\zeta(t)$, $\zeta(t + 2T) = \zeta(t)$], and k_0 is the wavelength of the instability. In order to illustrate the nonlinear development of the instability, let us consider the following simple example:

$$\partial_t X = Y + \nabla^2 X - \beta \nabla^2 Y, \tag{11}$$

$$\partial_t Y = (\mu - X)Y - X + X^2 + \nabla^2 Y + \beta \nabla^2 X. \tag{12}$$

The limit cyle appears when $\mu = 0$ and disappears for $\mu = 0.135$ through an Andronov homoclinic bifurcation. The matrix of the partial derivatives has been chosen in order to simplify the analysis of the instability. In particular, for such a matrix we have $y \propto \beta$; for positive β the instability of the limit cycle is thus the Kuramoto phase instability, and for negative β, it is the period doubling instability. Figure 3 shows the nonlinear evolution of the two different instabilities. The case $\beta = 0$ is nongeneric and corresponds to $y = 0$.

Although this instability has been observed in physical [7] and chemical systems [8] and model equations [9], no clear explanation of the genericity of the mechanism was proposed. We have related this instability to the self-parametric forcing of a homogeneous periodic solution. In two dimensional systems subcritical hexagons are ruled out near the onset by the symmetry induced by the period doubling which do not permit quadratic terms in the amplitude equations [10].

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