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Axisymmetric capillary surfaces as a dynamical system

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Abstract
Equations governing the shape of a static axisymmetric capillary surface can be viewed as a dynamical system. This point of view enables us to recover the existence of pendent drops having arbitrarily many bumps and to depict the asymptotic shape of long drops (displaying a large number of bumps). We also recover the existence of a singular pendent drop of infinite length, as a limit of pendent drops of large length. Finally we provide a classification of all solutions according to their dynamical behaviour.

Mathematics Subject Classification: 34A34, 34C27, 34C29, 76B45, 76D45

1. Introduction

Since the inauguration of the mathematical theory of meniscus equilibria by Laplace [10] and Young [15], most of the quantitative work on the subject has been devoted to the numerical calculation of meniscus profiles. Lord Kelvin [9] produced a number of profiles of pendent drops using a geometrical construction which is essentially a first-order graphical integration method applied to a second-order equation. He calculated a remarkable particular solution of a parametric form of the pendent drop shape equation which may possess several bumps. As proved by Pitts [13], these drops are unstable under weak perturbations (we refer to the constant volume—not the constant pressure—instability): only drops having one bump and one neck and a volume not exceeding a certain maximal volume are stable (see figure 1).

From a more mathematical point of view, there exists a considerable body of literature on the subject (for an extensive list of references, see [6]). A number of major contributions are due to Concus and Finn. In [3, 4], they proved the existence of a singular solution, corresponding to an axisymmetric drop of infinite length, together with partial results towards

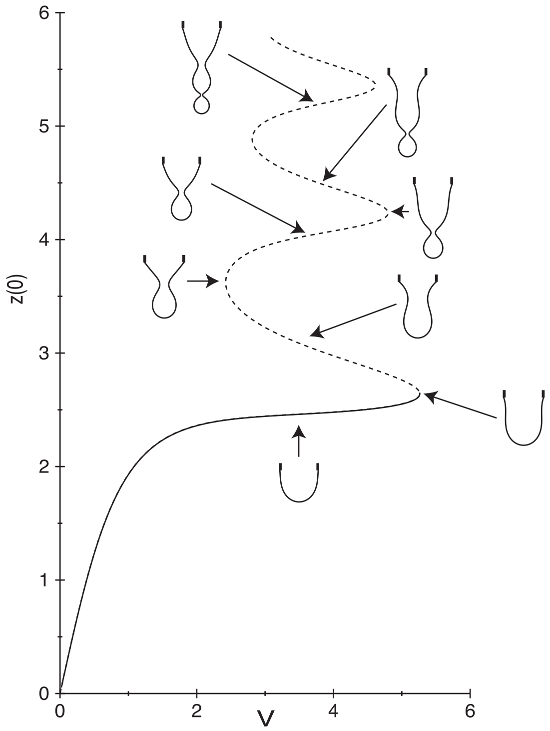

Figure 1. Numerical solution of the set of ODEs showing the repartition of multi-bump drops depending on the initial conditions. The dashed line corresponds to unstable drops (with respect to the constant volume instability).

its uniqueness, and in [5] they proved the existence of solutions corresponding to axisymmetric drops having an arbitrarily large number of bumps. Further contributions were made by Bidaut-Veron [1, 2] and by Finn [7], in extending previous results to more general gravity fields, and obtaining the singular solution as a limit of regular solutions when the length goes to infinity. Finally, the full proof of the uniqueness of this singular solution was recently obtained by Nickolov [12].

In this paper, we shall attack this old problem from a new point of view. We shall view the equations governing the shape of an axisymmetric capillary surface as a three-dimensional autonomous differential equation, and study its dynamical properties. Using classical dynamical system techniques (stable and unstable manifolds, Lyapunov functions, asymptotics, perturbation of integrable systems and averaging of perturbations), we shall recover the existence of pendent drops of any length, and give a precise description of their asymptotic shape when the length becomes large. We shall also recover Concus and Finn's result of existence of a singular solution corresponding to a pendent drop of infinite length, and the connection to pendent drops of large lengths. Finally, we shall provide a systematic study of the dynamical properties of this differential equation. By doing so, we hope to shed new light on a classical problem which received so much attention from earlier authors. A short proof of the uniqueness of the singular drop of infinite length was recently obtained by the second author [14], by the same dynamical approach.

2. Preliminaries

2.1. Setup

We consider a static capillary surface in $\mathbb{R}^3$, i.e. an interface at equilibrium between two fluids of different densities. At any point of the surface, the difference ΔP between the pressures on each side has to be balanced by the surface tension, which is proportional to the mean curvature of the surface. Namely we have

$$\Delta P = \Gamma \chi, \tag{1}$$

where Γ is the surface tension coefficient and χ denotes the mean curvature (with respect to an adequate orientation of the surface). Let $\rho > 0$ denote the difference between the densities of the two fluids (we shall refer to the phase of largest density as the 'liquid phase'). We have

$$\Delta P(z) = \Delta P(z = 0) + \rho g z,$$

where g denotes the gravitation (the positive direction of the vertical coordinate z is chosen downward). For convenience, we shall choose the origin of z in such a way that $\Delta P(z) = \rho g z$. Now, up to changing the length unit, we can suppose that $\Gamma = \rho g$, thus equation (1) reads

$$z = \chi. \tag{2}$$

In this paper, we shall consider the case where the capillary surface is axisymmetric (with respect to the z-axis). In this case equation (2) becomes

$$z = -\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\cos\theta}{r},$$

where the variables r, θ, z, and s are defined in figure 2 (the sign of θ is chosen according to the orientation opposite to the usual trigonometric orientation). By definition of r, θ, z, and s, we have $\mathrm{d}r = \sin\theta\,\mathrm{d}s$ and $\mathrm{d}z = -\cos\theta\,\mathrm{d}s$. Thus we obtain the following three-dimensional autonomous differential equation:

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\cos\theta}{r} - z, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\cos\theta, \tag{3}$$

the arc-length variable s playing the role of the time (taking the vertical coordinate z as the time, one would obtain a two-dimensional non-autonomous differential equation). Our strategy is to study this equation from a dynamical systems point of view.

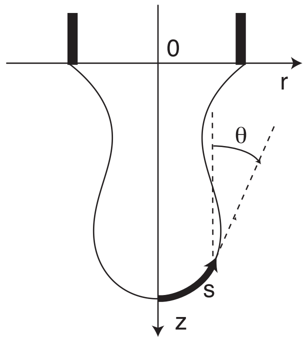

Figure 2. Definition of the variables r, θ, z, and s.

2.2. Preliminary remarks

In order to get rid of the singularity at $\{r = 0\}$, we can perform the change of time $\mathrm{d}s = r\,\mathrm{d}u$. Equation (3) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}u} = r\sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}u} = \cos\theta - rz, \qquad \frac{\mathrm{d}z}{\mathrm{d}u} = -r\cos\theta. \tag{4}$$

The plane $\{r = 0\}$ is now invariant by the flow, and equation (4) has the following symmetries:

$$\theta \to \theta + 2\pi,$$

$$r \to -r, \qquad \theta \to \pi - \theta,$$

$$z \to -z, \qquad \theta \to \pi - \theta.$$

In particular, it is sufficient to study this equation in the domain $\{r \geqslant 0\}$.

The set of equilibrium points of this equation is

$$\left\{(r, \theta, z) \,\middle|\, r = 0, \theta = \pm\frac{\pi}{2}[2\pi], z \in \mathbb{R}\right\}$$

(it is included in the invariant plane $\{r = 0\}$). The spectrum at each of these equilibrium points is $\{-1, 0, 1\}$ (computations will be carried out in the following). Thus each of these equilibrium points has a one-dimensional stable manifolds and a one-dimensional unstable manifold (the z-direction is neutral). These stable and unstable manifolds play an important role: they correspond to profiles with a top (resp. a tip) at the height z of the equilibrium. The dynamics in the neighbourhood of the plane $\{r = 0\}$ is shown in figure 3.

Equation (4) admits interesting asymptotic forms in the two limits $z \to +\infty$ and $r \to +\infty$, which are studied below. The study of these two limits is a useful preliminary for the understanding of the dynamics of the full equation (4) (which will appear as a 'mixture' of the dynamics in these two limits).

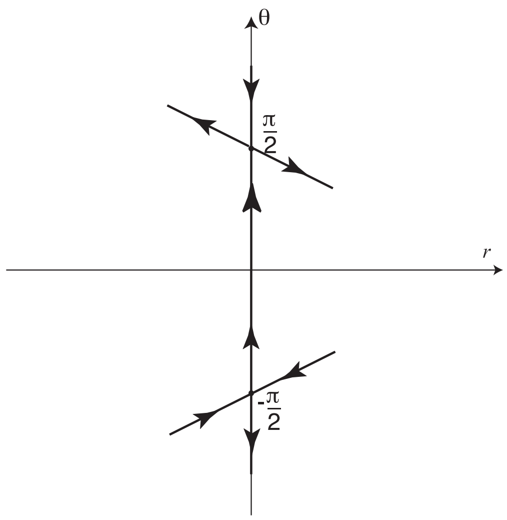

Figure 3. Dynamics of equation (4) in a neighbourhood of the invariant plane {r=0} (the picture is for $z = 1$).

2.3. Asymptotics when $z \to \infty$

When $z \to \infty$, the pressure goes to infinity, but above all it becomes asymptotically constant. Thus, after an appropriate rescaling, the limit when $z \to \infty$ corresponds to the constant pressure field case. The expression of $\mathrm{d}\theta/\mathrm{d}s$ in equation (3) suggests the following change of variables:

$$R = zr, \qquad \mathrm{d}t = z(s)\mathrm{d}s, \qquad Z = \frac{z^2}{2}.$$

Then equation (3) becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta - \frac{R}{2Z}\cos\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\cos\theta}{R} - 1, \qquad \frac{\mathrm{d}Z}{\mathrm{d}t} = -\cos\theta. \tag{5}$$

This change of variables has a natural geometrical interpretation: it corresponds to making, at height z in figure 2, a blowup of factor z (both in the directions of r and z), the new variable t corresponding to the new curvilinear abscissa after this blowup. However, we shall mostly use the set of variables $(R, \theta, z, \mathrm{d}t)$.

Asymptotically, when $Z \to +\infty$, equation (5) reads as

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\cos\theta}{R} - 1, \qquad \frac{\mathrm{d}Z}{\mathrm{d}t} = -\cos\theta. \tag{6}$$

This last equation corresponds to the constant pressure field case (the equation governing (R, θ) does not depend on Z any more). It admits the following conserved quantity:

$$H(R, \theta) = R\left(\cos\theta - \frac{R}{2}\right).$$

We have $\mathrm{d}t = z\,\mathrm{d}s = R\,\mathrm{d}u$. If we perform the change of time $\mathrm{d}t = R\,\mathrm{d}u$, equation (6) gives

$$\frac{\mathrm{d}R}{\mathrm{d}u} = R\sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}u} = \cos\theta - R, \tag{7}$$

which is nothing but the Hamiltonian equation derived from $H(.,.)$ and the usual symplectic form $(-\mathrm{d}\theta, \mathrm{d}R)$ of $\mathbb{R}^2$.

Thus equation (5) can be viewed as a perturbation of the asymptotic integrable equation (6) (the perturbation becoming small when Z becomes large). This point of view will be used extensively in the following.

Let us briefly describe the phase portrait of equation (7), which is drawn in figure 4 (more precise computations and justifications follow in the next sections). It has the symmetries $\theta \to \theta + 2\pi$, $(\theta, u) \to (-\theta, -u)$, and $(R, \theta) \mapsto (-R, \pi - \theta)$. The equilibria are elliptic equilibria at $(R, \theta) = (1, 2k\pi)$ and $(-1, (2k + 1)\pi)$ and saddle equilibria at $(0, k\pi)$, $k \in \mathbb{Z}$. The level curve $\{H(R, \theta) = 0\} = \{R = 0\} \cup \{R = 2\cos\theta\}$ is made of the stable and unstable manifolds of the saddle equilibria. This level curve is the boundary of bounded invariant domains in which the trajectories are closed curves, among which the domain

$$\mathcal{D} = \left\{(R, \theta) \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ and } 0 < R < 2\cos\theta\right\}.$$

Inside $\mathcal{D}$, we have $0 < H(R, \theta) \leqslant 1/2$, trajectories are level sets of $H_{|\mathcal{D}}$, and, except for the elliptic equilibrium $(1, 0)$, they are closed curves. The maximal value $1/2$ is reached at the point $(1, 0)$. The profiles in the (R, Z)-plane corresponding to some solutions of this equation are shown in figure 5.

The profile corresponding to the heteroclinic solution connecting the equilibria $(R, \theta) = (0, \pm(\pi/2)\,[2\pi])$ (along which $H = 0$, $R = 2\cos\theta$) is nothing but a sphere (of radius 2), as one can easily check. The elliptic equilibrium $(R, \theta) = (1, 0\ [2\pi])$ corresponds to a straight vertical cylinder, and periodic solutions inside the domain $\mathcal{D}$ (i.e. $0 < H < 1/2$)

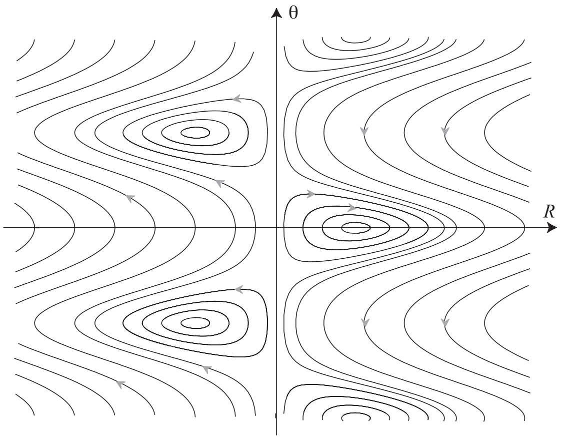

Figure 4. Phase portrait of equation (6).

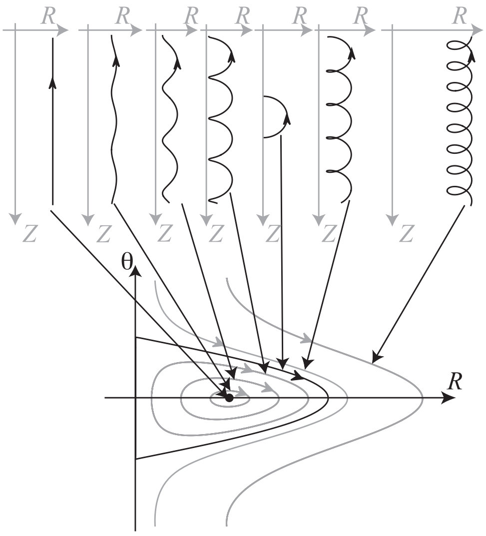

Figure 5. Numerical computations of profiles corresponding to solutions of equation (7) (compare with figure 17 of [5] and figures of chapter 4 of [6]). The arrows along the profiles correspond to the sense of increasing u (the liquid phase is lying at the 'left' of the profiles with respect to the orientation given by these arrows). The border of the domain $\mathcal{D}$ is drawn in black.

give oscillatory profiles, for which z is monotonic (therefore, displaying no self-intersection). Outside $\mathcal{D}$ (i.e. for $H < 0$), θ varies monotonically; therefore the profiles display infinitely many self-intersections.

2.4. *Asymptotics when $r \to \infty$*

In the limit $r \to \pm\infty$, the expressions of $\mathrm{d}\theta/\mathrm{d}s$ and $\mathrm{d}z/\mathrm{d}s$ become

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -z, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\cos\theta, \tag{8}$$

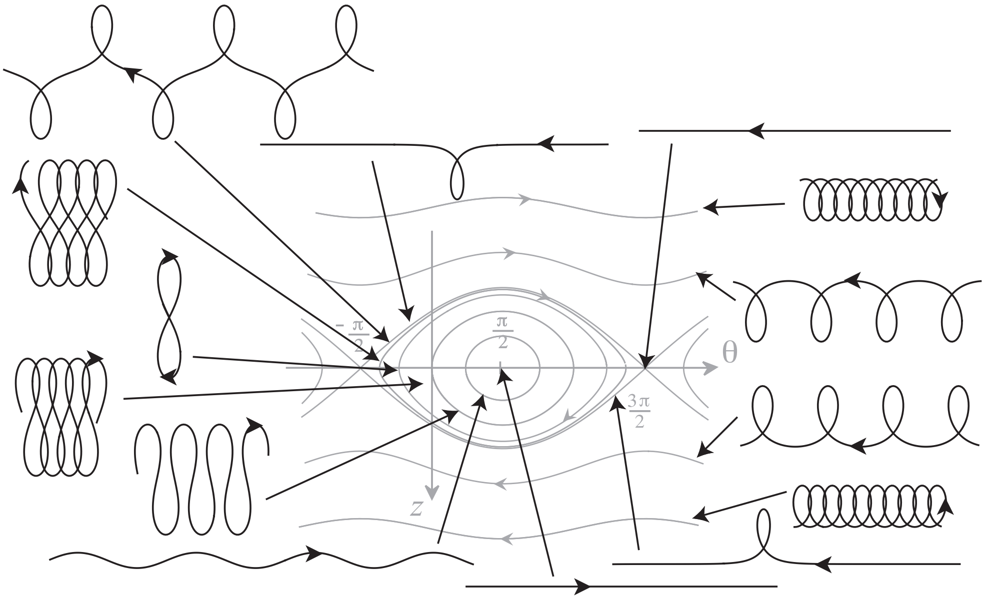

Figure 6. Numerical computations of profiles corresponding to solutions of equation (8). The arrows along the profiles correspond to the sense of increasing s (the liquid phase is lying at the 'left' of the profiles with respect to the orientation given by these arrows).

and rotational invariance converges to translational invariance. Thus the solutions of equation (8) correspond to capillary surfaces which are invariant under translations (in the direction orthogonal to the (r, z)-plane). Equation (8) yields the second-order equation $\mathrm{d}^2\theta/\mathrm{d}s^2 = \cos\theta$, i.e. a conservative oscillator in the potential $-\sin\theta$ (the usual pendulum without damping). The energy

$$E(\theta, z) = -\sin\theta + \frac{z^2}{2}$$

is a conserved quantity for equation (8).

Some profiles given by the solutions of this equation are shown in figure 6, for various values of E. The elliptic (resp. hyperbolic) equilibrium at $E = -1$ (resp. $E = +1$) corresponds to a plane horizontal interface at $z = 0$, the liquid phase being above (resp. below) the interface. Small oscillations around the elliptic equilibrium give a slightly oscillating interface (with an intrinsic wavelength), the liquid phase being above the interface. Profiles corresponding to solutions homoclinic to the hyperbolic equilibrium $-(\pi/2)[2\pi]$ display a single self-intersection, while for nearby solutions an infinity of self-intersections occur. If $E > 1$, then θ varies monotonically and the profiles display an infinity of self-intersections.

2.5. *Structure induced by the asymptotically conserved quantities*

One can expect that the dynamics of equation (4) reflects both limits $z \to \infty$ and $r \to \infty$. It will appear in the rest of this paper that this dynamics can be understood as a 'mixture' of the dynamics in these two limits. The conserved quantities H and E appearing in these two limits will play a crucial role. Using these quantities, we can already make the important following remarks.

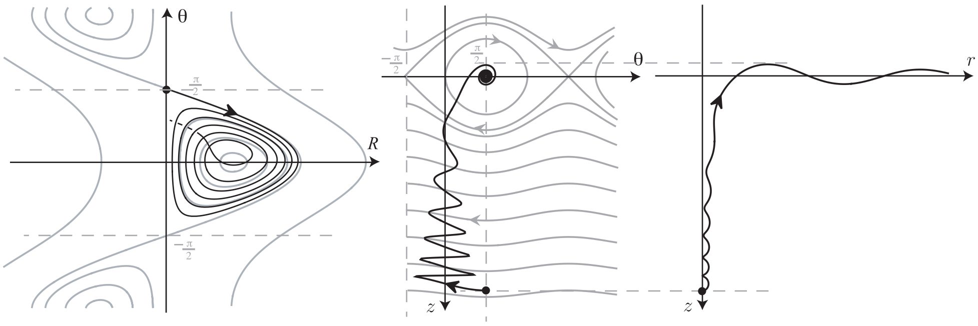

Figure 7. Numerical computations of the trajectory corresponding to a finite pendent drop (with $\bar{z} > 0$), respectively in the $(R = zr, \theta)$-, (θ, z)-, and (r, z)-planes (in the (R, θ)-plane, only some part of the trajectory, before the first time where $z = 0$, is drawn). In each figure the black point corresponds to the tip of the drop (i.e. $r = 0, \theta = \pi/2, z = \bar{z}$), and the arrow along the trajectory indicates the sense of increasing u. The (R, θ)-plane illustrates the fact that the domain $\mathcal{D}$ is positively invariant. The (θ, z)-plane illustrates the decrease of the Lyapunov function E, and the convergence towards $(\theta, z) = (\pi/2, 0)$ when $u \to +\infty$.

If $u \mapsto (r, \theta, z)$ is a solution of equation (4), let us still denote by H the map $u \mapsto H(zr, \theta)$ and by E the map $u \mapsto E(\theta, z)$. Then we have, along the solution,

$$\frac{\mathrm{d}H}{\mathrm{d}u} = r^2 \cos\theta (zr - \cos\theta), \qquad \frac{\mathrm{d}E}{\mathrm{d}u} = -\cos^2\theta. \tag{9}$$

Thus, on the one hand, the following holds:

$$r \neq 0 \qquad z \neq 0 \qquad \text{and} \qquad H = 0 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}u} > 0 \tag{10}$$

(in other words, unless $z = 0$, the domain where $H > 0$ is positively invariant), and, on the other hand, E is a (decreasing) Lyapunov function for equation (4) (see figure 7).

3. Statement of the main results and organization of the paper

3.1. Preliminaries

Our main objective is to study the shape of pendent drops of finite length. The following proposition states that for any height $\bar{z}$ there exists exactly one solution corresponding to a drop having its tip at height $\bar{z}$. It also gives a few properties of this solution, illustrated in figure 7. We postpone its proof to section 7.

Proposition 1. *For any $\bar{z} > 0$, there exists a unique solution $u \mapsto (r, \theta, z)$ of equation (4) such that $r > 0$ and $(r, \theta, z) \to (0, \pi/2, \bar{z})$ when $u \to -\infty$ (uniqueness is up to time translation). It has the following properties:*

- *this solution eventually crosses the plane $\{z = 0\}$, and, if u_0 denotes the time where the first crossing occurs (i.e. $z(u_0) = 0$ and $z(u) > 0$ for $u < u_0$), then, for $u < u_0$, we have $(zr, \theta) \in \mathcal{D}$ (and therefore $H(zr, \theta) > 0$, $-\pi/2 < \theta < \pi/2$, and $z'(u) < 0$);*
- *we have $(r, \theta, z) \to (+\infty, \pi/2, 0)$ when $u \to +\infty$.*

This solution will be referred to in the following as 'the solution corresponding to a pendent drop of finite length $\bar{z}$'.

3.2. Asymptotics when $z \to \infty$

According to the previous proposition, solutions corresponding to pendent drops of (large) finite height lie in the domain $\mathcal{D}$ before the first time where z vanishes. Inside this domain and for z large, they remain on bounded time intervals close to those of the asymptotic equation (6). They are thus almost periodic, and this almost periodicity explains the existence of the 'bumps' in the shape of the profiles.

Let us still denote by H the function $(r, \theta, z) \mapsto H(rz, \theta)$, and by $H(u)$ the values of this function along a solution of (4). We will estimate for z large the variations of the quantities H and z during one 'period' (i.e. one 'bump'). They read at first order as

$$\delta H = \frac{I(H)}{z^2} \qquad \text{and} \qquad \delta z = -\frac{J(H)}{z},$$

where $I(\cdot)$ and $J(\cdot)$ are defined as certain integrals computed along the periodic trajectories of the asymptotic differential equation (6). This suggests that the variations of H with z should be related to the solutions of the averaged differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}z} = -\frac{1}{z}\frac{I(H)}{J(H)}. \tag{11}$$

We will see that the quantities I and J are positive. We will show that the averaged differential equation (11) admits a unique ('universal') solution $z \mapsto \eta(z)$ defined on $]0; 1[$ and having a continuous extension satisfying $\eta(0) = 1/2$ and $\eta(1) = 0$, and that any other solution can be deduced from η by a linear change of time.

Take any $\bar{z} > 0$ and consider the solution $u \mapsto (r, \theta, z)$ corresponding to a pendent drop of length $\bar{z}$. According to proposition 1, the part of the trajectory before the first crossing with the plane $\{z = 0\}$ can be parametrized by z, it is the graph of a map $z \mapsto (r_{\bar{z}}(z), \theta_{\bar{z}}(z))$, defined on $[0; \bar{z}]$. Let $z \mapsto H_{\bar{z}}(z)$ denote the map corresponding to the values of H along the same part of the trajectory. Let us define the function $h_{\bar{z}}$, by

$$h_{\bar{z}}(\kappa) = H_{\bar{z}}(\kappa\bar{z}), \qquad \kappa \in [0; 1].$$

Our main result is the following.

Theorem 1. *The function $h_{\bar{z}}$ converges towards η when $\bar{z} \to +\infty$, uniformly on any compact subset of* $]0; 1]$. *More precisely, for any $\varepsilon > 0$, the following estimate holds:*

$$\sup_{\varepsilon<\kappa<1} |h_{\bar{z}}(\kappa) - \eta(\kappa)| < \frac{C}{\bar{z}},$$

where $C > 0$ is a constant which depends only on ε.

This result explains the global shape of the drop in terms of level sets of the function H in the domain $\mathcal{D}$ and of the universal function η (see figure 8): if we zoom by a factor $\kappa\bar{z}$ on the profile of the drop at height $\kappa\bar{z}$, the resulting picture converges when $\bar{z} \to +\infty$, and the asymptotic picture is given by the periodic solution of equation (6) (constant pressure field case) corresponding to the level curve $\{H = \eta(\kappa)\}$.

3.3. Pendent and singular drops

We shall recover Concus and Finn's result [3, 4] of existence of a singular solution corresponding to a drop of infinite length.

Theorem 2. *There exists a solution $u \mapsto (r, \theta, z)$ of equation (4) satisfying $z \to +\infty$ when $u \to -\infty$. Moreover, any such solution satisfies $rz \to 1$ when $u \to -\infty$.*

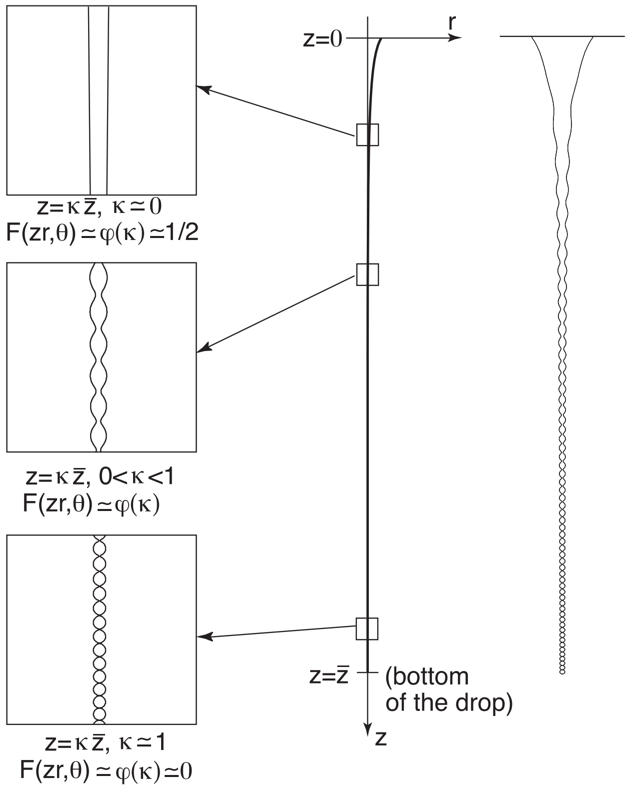

Figure 8. Shape of drops of very large length.

This singular solution is actually unique. This was proved recently by Nickolov [12]. A short proof of this uniqueness (via the same dynamical systems approach) was recently obtained by the second author [14].

Following Finn, we shall investigate the connection between finite drops of large lengths and the drop of infinite length, and recover Finn's result ([7], see also [5]) that profiles of pendent drops converge, when their length goes to $+\infty$, towards the profile of the infinite drop. For this purpose, for any $\bar{z} > 0$, consider the map $z \mapsto (r_{\bar{z}}(z), \theta_{\bar{z}}(z))$ defined in the previous paragraph. The point $(r_{\bar{z}}(0), \theta_{\bar{z}}(0))$ corresponds to the 'first intersection' (of the profile of the pendent drop of length $\bar{z}$) with the horizontal plane $\{z = 0\}$. It defines a (canonical) smooth map:

$$\iota\colon\ ]0;\infty[\ \to \mathbb{R}_+^* \times \left]-\frac{\pi}{2};\frac{\pi}{2}\right[, \qquad \bar{z} \mapsto (r_{\bar{z}}(0), \theta_{\bar{z}}(0))$$

(indeed, since $H > 0$ for any $z \in\ ]0;\bar{z}[$, we must have $\theta_{\bar{z}}(0) \in\ ]-\pi/2;\pi/2[$). This map ι is injective.

As is the case for drops of finite lengths, the singular solution corresponding to the drop of infinite length eventually intersects the plane $\{z = 0\}$, and the part of the trajectory before this first intersection can be parametrized by a map $z \mapsto (r_\infty(z), \theta_\infty(z))$, defined on $[0;+\infty[$ (with $-\pi/2 < \theta_\infty(z) < \pi/2$).

We shall prove the following result (it assumes as known the uniqueness of the infinite drop).

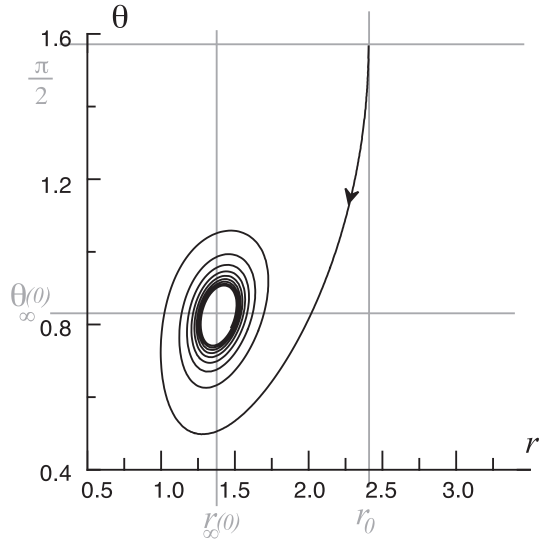

Figure 9. Numerical computation of the set $\iota(]0;+\infty[)$ (in the (r,θ)-plane). The arrow along the curve corresponds to the sense of increasing $\bar{z}$. We can see that this set $\iota(]0;+\infty[)$ is a submanifold (no self-intersection) and that $\iota(\bar{z})$ converges when $\bar{z}\to 0$ and $\bar{z}\to+\infty$ (actually, the convergence when $\bar{z}\to+\infty$ is so slow that the end of the spiral appears as a black splash; therefore, only the set $\iota(]0;\bar{z}[)$ for some $\bar{z}$ large positive is drawn).

Theorem 3. *The set $\iota(]0;+\infty[)$ is bounded and defines a submanifold of $\mathbb{R}^2$ (it does not accumulate on itself), and we have*

$$\iota(\bar{z}) = (r_{\bar{z}}(0),\theta_{\bar{z}}(0)) \to (r_\infty(0),\theta_\infty(0)) \qquad \textit{when } \bar{z}\to+\infty;$$

$$\iota(\bar{z}) \to \left(r_0,\frac{\pi}{2}\right) \qquad \textit{when } \bar{z}\to 0 \textit{ (for a certain } r_0>0\textit{)}.$$

As a consequence of the limit $\iota(\bar{z})\to(r_\infty(0),\theta_\infty(0))$ when $\bar{z}\to+\infty$ and of the continuity of the solutions of a differential equation with respect to the initial conditions, this theorem yields the following corollary.

Corollary 1. *The maps $z\mapsto(r_{\bar{z}}(z),\theta_{\bar{z}}(z))$ converge towards the map $z\mapsto(r_\infty(z),\theta_\infty(z))$ when $\bar{z}\to+\infty$, uniformly on any compact subset of* $[0;+\infty[$ *(for any C^k-topology).*

A numerical computation of the set $\iota(]0;+\infty[)$ is shown in figure 9. In this figure one can see that this set resembles a trajectory of a solution of a planar differential equation, converging towards a focus (or a centre) when $\bar{z}\to+\infty$ (we have no mathematical explanation of this behaviour).

3.4. Asymptotics when $r\to\infty$

Finally we shall study the shape of pendent drops when $u\to+\infty$, or equivalently when $r\to+\infty$. Take any $\bar{z}>0$ and let $u\mapsto(r,\theta,z)$ denote the solution corresponding to a pendent drop of length $\bar{z}$. We know from proposition 1 that $(r,\theta,z)\to(+\infty,\pi/2,0)$ when $u\to+\infty$. Using the time variable s, we see that $\mathrm{d}u/\mathrm{d}s\to 1$ when $s\to+\infty$, and that $\theta(s)$ and $z(s)$ are asymptotically governed by the dynamics of the conservative oscillator (8) (the pendulum) around the elliptic equilibrium ($\theta=\pi/2$, $z=0$) (see figure 7). For r large enough (or equivalently for u large enough), r varies monotonically with u (or s), and θ and z can be viewed as functions of r. The following result provides more precision.

Theorem 4.

(a) The set $U = \{u \in \mathbb{R} | z(u) = 0\}$ *is infinite and discrete, namely we have* $U = \{u_j | j \in \mathbb{N}\}$*, with, for any* $j \in \mathbb{N}$, $u_j < u_{j+1}$, $z < 0$ *on* $]u_{2j}; u_{2j+1}[$ *and* $z > 0$ *on* $]u_{2j+1}; u_{2j+2}[$.

(b) Let us write $r_j = r(u_j)$*. Then* $r_{j+1} - r_j \to \pi$ *when* $j \to +\infty$.

(c) The rate of convergence towards 0 *of* $|(\theta - \pi/2, z)|$ *is that of* $r^{-1/2}$*. More precisely, there exists a constant* $K > 0$ *(depending on the solution) such that, if we write*

$$\alpha = \frac{\theta - \pi/2}{K\sqrt{r}} \qquad \text{and} \qquad \zeta = \frac{z}{K\sqrt{r}},$$

then the maps $r \mapsto (\alpha(r_{2j} + r), \zeta(r_{2j} + r))$ *converge towards the map* $r \mapsto (-\cos r, -\sin r)$*, when* $j \to +\infty$*, uniformly on any compact subset of* $\mathbb{R}$.

In other words, if we consider the profile of the solution in the (r, z)-plane, and if we make on this profile, at abscissa r, a zoom of factor $1/\sqrt{r}$ in the vertical direction z, then the resulting picture consists, asymptotically when $r \to +\infty$, of regular oscillations.

3.5. Organization of the paper

Section 4 is devoted to the proof of theorem 1. In section 4.1, we briefly comment and give a quantitative estimate on the dynamics of the asymptotic differential equation (6) in the domain $\mathcal{D}$. Computations of δH and δz during one 'period' (up to the first order in $1/z$) are carried out in section 4.2. In section 4.3, we provide semi-explicit formulae for I and J, in particular we prove their positiveness. Averaged differential equation (11) is studied in section 4.4, and finally theorem 1 is proved in section 4.5. Theorems 2 and 3 are proved in section 5. The proof essentially relies on the estimates of section 4 on δH and δz. Theorem 4 is proved in section 6, by a classical perturbation method. Proposition 1 is proved in section 7. Actually, we state in this section a much more complete result (theorem 5) providing a classification of the solutions of equation (4) according to their asymptotic dynamical behaviour (when $u \to \pm\infty$).

Some supplementary remarks and results are collected in section 8. In section 8.1, we provide an asymptotic expansion up to any order of the profile at the tip of a pendent drop or at the top of a sessile drop. In section 8.2, we compute the asymptotic shape of a pendent drop whose height converges to 0 (this computation is necessary for the proof of assertion 1 in theorem 3). We briefly comment on the local shape of long drops in section 8.3. In section 8.4, we discuss the problem of the existence of self-intersections for the profiles in the (r, z)-plane of the solutions. We show that for a pendent drop whose length is either very small or very large, the whole profile displays no self-intersection.

Although we concentrated on the physically relevant case of a drop in $\mathbb{R}^3$ (its boundary is a two-dimensional surface), it is very likely that all our results extend *mutatis mutandis* to higher dimensions. In section 8.5, we briefly indicate what the main computations become in higher dimensions.

4. Asymptotics when $z \to \infty$

The aim of this section is to prove theorem 1. For this purpose, we shall have to estimate the variations of H and z during one 'oscillation', along the solutions of equation (4), for z large and $H > 0$. The main difficulty is to get some uniformity in these estimates, up to the level set

$\{H = 0\}$ (the border of the domain $\mathcal{D}$). We found it more convenient to use the set of variables (R, θ, z, t), thus the differential equation takes the form:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta - \frac{R}{z^2}\cos\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\cos\theta}{R} - 1, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{1}{z}\cos\theta \tag{12}$$

and the limit when $z \to +\infty$ reads as

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\cos\theta}{R} - 1. \tag{13}$$

4.1. Estimates on the asymptotic flow in the domain $\mathcal{D}$

The aim of this section is to provide some information on the dynamics of the asymptotic differential equation (13) in the domain $\mathcal{D}$. Here are some elementary observations: for $(R, \theta) \in \mathcal{D}$, we have $0 < H(R, \theta) \leqslant 1/2$ and $H(R, \theta) = 1/2 \Leftrightarrow (R, \theta) = (1, 0)$. The point $(1, 0)$ is the only critical point of $H(., .)$ and the only equilibrium point of the differential equation (13) in $\mathcal{D}$. The matrix of the linearization at the point $(1, 0)$ of this differential equation reads as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{14}$$

(this point is thus an 'elliptic' point).

It is not difficult to prove by hand that all the solutions of (13) which lie inside the invariant domain $\mathcal{D}$ are periodic (except the constant solution at $(1, 0)$); however, the computations below (required for the next sections) will provide a formal proof of this fact. The phase space of equation (13) is represented in figure 10.

For $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we shall denote by $\arg(x, y)$ the argument (in $\mathbb{R}/2\pi\mathbb{Z}$) of the complex number $x + \mathrm{i}y$.

Recall that $\mathcal{D} = \{(R, \theta) \,|\, -\pi/2 < \theta < \pi/2 \text{ and } 0 < R < 2\cos\theta\}$, let $\mathcal{C} = \{(R, \theta) \,|\, -\pi/2 < \theta < \pi/2 \text{ and } R = 2\cos\theta\}$, and let $\mathcal{E} = (\mathcal{D} \cup \mathcal{C}) \setminus \{(1, 0)\}$. For $(R, \theta) \in \mathcal{E}$, let

$$\alpha(R, \theta) = \arg\left(\sin\theta, \frac{\cos\theta}{R} - 1\right) - \arg(R - 1, \theta).$$

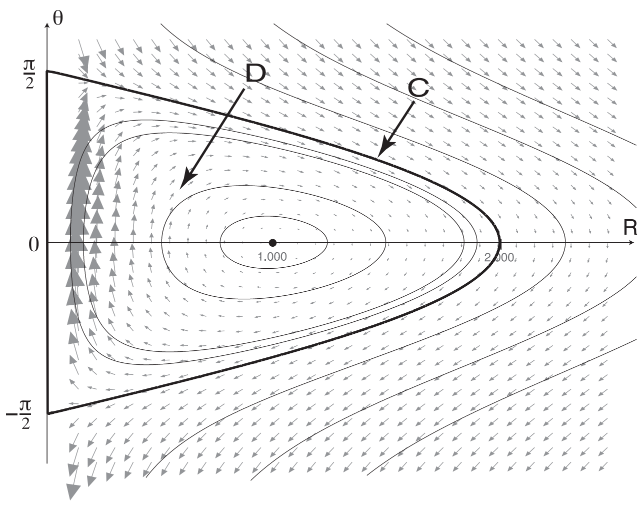

Figure 10. Phase space of equation (13).

Lemma 1. *There exists* $\varepsilon_1 > 0$ *such that, for any* $(R, \theta) \in \mathcal{E}$*, we have* $\alpha(R, \theta) \in [-\pi + \varepsilon_1; -\varepsilon_1] \bmod 2\pi$.

Proof. Take any $(R, \theta) \in \mathcal{E}$ and denote by Λ the wedge product $(R-1, \theta) \wedge (\sin\theta, \cos\theta/R - 1)$. We have

$$-\Lambda = \theta \sin\theta + \frac{R-1}{R}(R - \cos\theta).$$

According to this expression, if $R \geqslant 1$, or if $R \leqslant \cos\theta$, then $-\Lambda > 0$ (indeed the two terms are non-negative, and they cannot vanish at the same time). If, on the other hand, $\cos\theta < R < 1$, then we can write

$$-\Lambda = (\theta \sin\theta + R - 1) - \frac{R-1}{R}\cos\theta$$

and thus

$$-\Lambda \geqslant \theta \sin\theta + R - 1 \geqslant \sin^2\theta + R - 1 = R - \cos^2\theta \geqslant R - \cos\theta > 0.$$

This proves that, for any $(R, \theta) \in \mathcal{E}$, $\alpha(R, \theta) \in \,]-\pi; 0[\bmod 2\pi$.

The function $(R, \theta) \mapsto \alpha(R, \theta)$ is continuous on $\mathcal{E}$. Moreover, according to (14), it converges towards $-\pi/2$ when $(R, \theta) \to (0, 0)$. Thus, in order to complete the proof of the lemma, it suffices to show that $\alpha(R, \theta)$ cannot become arbitrarily close to 0 or π when $R \to 0$.

If $R \to 0$ and θ does not converge towards $\pm\pi/2$, then $\arg(\sin\theta, \cos\theta/R - 1) \to \pi/2$ and the accumulation points of $\arg(R - 1, \theta)$ belong to $[\pi - \arctan(\pi/2); \pi + \arctan(\pi/2)]$; therefore, the accumulation points of $\alpha(R, \theta)$ belong to $[-\pi/2 - \arctan(\pi/2); -\pi/2 + \arctan(\pi/2)] \subset \,]-\pi; 0[$.

On the other hand, if $R \to 0$ and, say, $\theta \to \pi/2$, then $\arg(R-1, \theta) \to \pi - \arctan(\pi/2)$ and the accumulation points of $\arg(\sin\theta, \cos\theta/R - 1)$ belong to $[\arctan(-1/2); \pi/2]$; therefore, the accumulation points of $\alpha(R, \theta)$ belong to $[\arctan(-1/2) - \pi + \arctan(\pi/2); -\pi/2 + \arctan(\pi/2)] \subset \,]-\pi; 0[$, and the proof follows. $\square$

We deduce from this lemma and from (14) the following corollary.

Corollary 2. *There exists* $\varepsilon_2 > 0$ *such that, if* $t \mapsto (R(t), \theta(t))$ *is any solution in* $\mathcal{D}$ *of the differential equation (13), and if* $(R(t), \theta(t)) \not\equiv (1, 0)$*, then, for any* $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \arg(R - 1, \theta)(t) < -\varepsilon_2.$$

Finally, as $R \mapsto H(R, 0) = R - R^2/2$ is strictly increasing on $[0; 1]$, we obtain the following corollary.

Corollary 3. *All solutions* $t \mapsto (R(t), \theta(t))$ *of the differential equation (13) in* $\mathcal{D}$ *(except the constant solution* $(1, 0)$*) are periodic, and the periods are bounded from above and from below by positive constants. There is a one-to-one correspondence between these (periodic) solutions and the values in* $]0; 1/2[$ *taken by* $H(.,.)$ *on their trajectories.*

Notation. For $a \in \,]0; 1/2[$, if $t \mapsto (R(t), \theta(t))$ denotes the solution of equation (13) in $\mathcal{D}$ on which $H(.,.)$ takes the value a (this solution is unique up to time translation and periodic), let us write

$$I(a) = \int R\cos\theta(R - \cos\theta)\,\mathrm{d}t, \qquad J(a) = \int \cos\theta\,\mathrm{d}t,$$

these integrals being taken over one period of the solution.

4.2. Variation of H and z during one oscillation

The following lemma is the main technical step of this paper. Consider any smooth map $\varphi : \mathbb{R}^2 \to \mathbb{R}$, satisfying $\varphi(0, \pm\pi/2) = 0$.

Lemma 2. *For any $z_0 > 0$ sufficiently large, and for any $(R_0, \theta_0) \in \mathcal{D}$ satisfying*

$$|(R_0 - 1, \theta_0)| \geqslant \frac{1}{z_0}$$

($|\cdots|$ denotes the Euclidean norm in $\mathbb{R}^2$), if $t \mapsto (R(t), \theta(t), z(t))$ denotes the solution of the differential equation (12) with initial condition (R_0, θ_0, z_0) at time $t = 0$, and if $t \mapsto (\bar{R}(t), \bar{\theta}(t))$ denotes the solution of the asymptotic differential equation (13) with initial condition (R_0, θ_0) at time $t = 0$ (let us denote by $\bar{T}$ its period), then the following holds:

(a) there is a unique $T > 0$ such that the map $\alpha : t \mapsto \arg(R(t)-1, \theta(t))$ is strictly decreasing on $[0, T]$ and satisfies $\alpha(t) \neq \alpha(0)$ if $t \in]0; T[$ and $\alpha(T) = \alpha(0)$ (in $\mathbb{R}/2\pi\mathbb{Z}$);

(b) we have

$$\left| \int_0^T \varphi(R(t), \theta(t))\, \mathrm{d}t - \int_0^{\bar{T}} \varphi(\bar{R}(s), \bar{\theta}(s))\, \mathrm{d}s \right| \leqslant \frac{C}{z_0},$$

where C is a positive constant depending only on φ.

Proof. According to the expression of $\mathrm{d}z/\mathrm{d}t$ in (12), we have, for z_0 large enough, and for $t \in [0; 4\pi/\varepsilon_2]$, $z(t) > 0$ and $(R(t), \theta(t)) \in \mathcal{D}$ (the constant ε_2 is the one introduced in corollary 2). Moreover, since the differential equation (12) is a perturbation of order $1/z_0^2$ of the asymptotic differential equation (13), and according to corollary 2 and to the expression (14) of the linearization at $(1, 0)$ of this asymptotic differential equation, we see that, for z_0 large enough, we have, for any $t \in [0; 4\pi/\varepsilon_2]$,

$$|(R(t), \theta(t))| > \frac{1}{2z_0} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \arg(R - 1, \theta)(t) < -\frac{\varepsilon_2}{2}.$$

This proves the first assertion of the lemma.

Let us denote by Δ the term on the left-hand side of the inequality of the second assertion. Let us fix $\varepsilon > 0$ small (for instance $\varepsilon = 1/100$) and let us distinguish three cases depending on $H(R_0, \theta_0)$.

First case ($\varepsilon < H(R_0, \theta_0) < 1/2 - \varepsilon$). In this case, because differential equation (12) is a perturbation of order $1/z_0^2$ of the asymptotic differential equation (13), we have

$$|(R(t) - \bar{R}(t), \theta(t) - \bar{\theta}(t))| \leqslant \frac{C}{z_0^2}, \qquad t \in [0; T], \tag{15}$$

and $|T - \bar{T}| \leqslant C/z_0^2$, where C is a positive constant depending on ε. We thus get the estimate $\Delta \leqslant C/z_0^2$ (which is stronger than the one stated in the lemma).

Second case ($H(R_0, \theta_0) \geqslant 1/2 - \varepsilon$). In this case, (15) still holds, but because $(R - 1, \theta)$ can be of the order of $1/z_0$, we only have $|T - \bar{T}| \leqslant C/z_0$; anyway, this gives the inequality stated in the lemma.

Third case ($H(R_0, \theta_0) \geqslant 1/2 - \varepsilon$). This is the most intricate case, because of the singularity at $\{R = 0\}$ of the differential equations (12) and (13), and because of the points $(0, \pm\pi/2)$, where ∇H vanishes. Nevertheless, this case will be required in the following; thus, we have to treat it.

Let $\alpha_0 = -\arg(R_0 - 1, \theta_0)$. For any $\alpha \in [\alpha_0; \alpha_0 + 2\pi[$, there exists a unique $t \in [0; T[$ and a unique $s \in [0; \bar{T}[$ such that $-\alpha = \arg(R(t) - 1, \theta(t)) = \arg(\bar{R}(s) - 1, \bar{\theta}(s))$. This defines one-to-one correspondences between variables t, s, and θ, and, according to the implicit function theorem, smooth diffeomorphisms between the intervals $[0; T[$, $[0; \bar{T}[$, and $[\alpha_0; \alpha_0 + 2\pi[$; these can be extended to smooth diffeomorphisms between neighbourhoods of $[0; T]$, $[0; \bar{T}]$, and $[\alpha_0; \alpha_0 + 2\pi]$ in $\mathbb{R}$.

According to (9), we have $\mathrm{d}H/\mathrm{d}t = R\cos\theta(R - \cos\theta)/z^2$; thus $\mathrm{d}H/\mathrm{d}t$ is of the order of $1/z_0^2$. As a consequence, we have

$$|H(R(t), \theta(t)) - H(\bar{R}(s), \bar{\theta}(s))| \leqslant \frac{C}{z_0^2}, \qquad t \in [0; T], \quad s \in [0\bar{T}].$$

Write $\alpha_\pm = \arg(-1, \pm\pi/2)$, and let $I_\pm = [\alpha_\pm - 1/z_0; \alpha_\pm + 1/z_0]$ (in $\mathbb{R}/2\pi\mathbb{Z}$).

Claim. For any $\alpha \in \mathbb{R}/2\pi\mathbb{Z} \setminus (I_+ \cup I_-)$, we have

$$|(R(t(\alpha)) - \bar{R}(s(\alpha)), \theta(t(\alpha)) - \bar{\theta}(s(\alpha)))| \leqslant \frac{C}{z_0},$$

where C is a positive constant.

Indeed, according to lemma 1, we have

$$\begin{aligned}&|H(R(t(\alpha)), \theta(t(\alpha))) - H(\bar{R}(s(\alpha)), \bar{\theta}(s(\alpha)))| \\ &\qquad \geqslant m|\sin\varepsilon_1|\, |(R(t(\alpha)) - \bar{R}(s(\alpha)), \theta(t(\alpha)) - \bar{\theta}(s(\alpha)))|\end{aligned}$$

(ε_1 is the constant introduced in lemma 1), with

$$m = \min|\nabla H(R, \theta)| \geqslant \frac{C}{z_0}$$

(because we have excluded I_- and I_+), and the claim follows.

We have $\Delta \leqslant \Delta_1 + \Delta_2$, with

$$\Delta_1 = \left| \int_{\alpha_0}^{\alpha_0+2\pi} (\varphi(R(t(\alpha)), \theta(t(\alpha))) - \varphi(\bar{R}(s(\alpha)), \bar{\theta}(s(\alpha)))) \frac{\mathrm{d}t}{\mathrm{d}\alpha}\, \mathrm{d}\alpha \right|,$$

$$\Delta_2 = \left| \int_{\alpha_0}^{\alpha_0+2\pi} \varphi(\bar{R}(s(\alpha)), \bar{\theta}(s(\alpha))) \left(\frac{\mathrm{d}t}{\mathrm{d}\alpha} - \frac{\mathrm{d}s}{\mathrm{d}\alpha} \right) \mathrm{d}\alpha \right|.$$

By definition of α, we have

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \frac{|(\bar{R} - 1, \bar{\theta})|^2}{(\sin\bar{\theta}, \cos\bar{\theta}/\bar{R} - 1) \wedge (\bar{R} - 1, \bar{\theta})},$$

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{|(R - 1, \theta)|^2}{(\sin\theta - (R/z^2)\cos\theta, \cos\theta/R - 1) \wedge (R - 1, \theta)}.$$

According to lemma 1, the denominators in these two expressions are bounded from below by positive constants. According to the claim above, this yields $\Delta_1 \leqslant C/z_0$, for some constant $C > 0$ (depending on the Lipschitz constant of φ).

It remains to estimate Δ_2. We have to be careful with the terms $\cos\theta/R$ and $\cos\bar{\theta}/\bar{R}$, occurring in the expressions of $\mathrm{d}t/\mathrm{d}\alpha$ and $\mathrm{d}s/\mathrm{d}\alpha$, since these terms are unbounded for R or $\bar{R}$ close to 0. Remark that the denominators, in the expressions of $\mathrm{d}t/\mathrm{d}\alpha$ and $\mathrm{d}s/\mathrm{d}\alpha$, are of the order of $1 + |\cos\theta/R|$ and (resp.) $1 + |\cos\bar{\theta}/\bar{R}|$. Therefore, we can write

$$\left|\left(\frac{\mathrm{d}t}{\mathrm{d}\alpha} - \frac{\mathrm{d}s}{\mathrm{d}\alpha}\right)(\alpha)\right| \leqslant \left|\left[|(R-1,\theta)|^2\left(\sin\bar{\theta}, \frac{\cos\bar{\theta}}{\bar{R}} - 1\right) \wedge (\bar{R}-1,\bar{\theta})\right.\right.$$
$$-|(\bar{R}-1,\bar{\theta})|^2\left(\sin\theta - \frac{R}{z^2}\cos\theta, \frac{\cos\theta}{R} - 1\right) \wedge (R-1,\theta)\bigg]$$
$$\left.\times\left[\left(1 + \left|\frac{\cos\theta}{R}\right|\right)\left(1 + \left|\frac{\cos\bar{\theta}}{\bar{R}}\right|\right)\right]^{-1}\right|.$$

According to the claim above, for $\alpha \notin I_- \cup I_+$, we have

$$\left|\left(\frac{\mathrm{d}t}{\mathrm{d}\alpha} - \frac{\mathrm{d}s}{\mathrm{d}\alpha}\right)(\alpha)\right| \leqslant \frac{C}{z_0} + \frac{|\cos\bar{\theta}/\bar{R} - \cos\theta/R|}{(1 + |\cos\theta/R|)(1 + |\cos\bar{\theta}/\bar{R}|)}.$$

The last term in this inequality reads

$$\frac{1}{\bar{R} + |\cos\bar{\theta}|}\,\frac{|R(\cos\bar{\theta} - \cos\theta) + \cos\theta(R - \bar{R})|}{R + |\cos\theta|} \leqslant \frac{1}{\bar{R} + |\cos\bar{\theta}|}\,\frac{C}{z_0}$$

(still for $\alpha \notin I_- \cup I_+$), where C is a positive constant. Finally, the hypothesis that $\varphi(0, \pm\pi/2) = 0$ (which was not used up to now) shows that the quantity $|\varphi(\bar{R}, \bar{R})|/(\bar{R} + |\cos\bar{\theta}|)$ is bounded. This yields the desired estimate $\Delta_2 \leqslant C/z_0$, and finishes the proof. □

Corollary 4. *With the same notation, we have*

$$\left|H(R(T), \theta(T)) - H(R_0, \theta_0) - \frac{1}{z_0^2} I(H(R_0, \theta_0))\right| \leqslant \frac{C}{z_0^3},$$
$$\left|z(T) - z(0) + \frac{1}{z_0} J(H(R_0, \theta_0))\right| \leqslant \frac{C}{z_0^2},$$

where C is a positive constant.

Proof. We have

$$H(R(T), \theta(T)) - H(R_0, \theta_0) = \int_0^T \frac{1}{z^2} R\cos\theta(R - \cos\theta)\,\mathrm{d}t;$$

thus, according to the expression of $\mathrm{d}z/\mathrm{d}t$ in (12),

$$H(R(T), \theta(T)) - H(R_0, \theta_0) = \frac{1}{z_0^2}\int_0^T R\cos\theta(R - \cos\theta)\,\mathrm{d}t + \mathcal{O}\left(\frac{1}{z_0^4}\right),$$

and the first inequality follows from the previous lemma applied to the function $\varphi(R, \theta) = R\cos\theta(R - \cos\theta)$. The proof of the second inequality is similar. □

4.3. Study of the functions $I(\cdot)$ and $J(\cdot)$

Given a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$, let us consider the integral

$$K(a) = \int \psi(R, \cos\theta)\,\mathrm{d}t,$$

where $a \in]0; 1/2[$, $t \mapsto (R(t)\theta(t))$ denotes the solution of equation (13) corresponding to the value a for $H(.,.)$ (this solution is unique up to time translation and periodic), and the integral is taken over one period.

The identity $H(R,\theta) = a$ yields $R = \cos\theta \pm \sqrt{\cos^2\theta - 2a}$. We have $K(a) = 2(K_2 - K_1)$, where

$$K_i = \int_{\arccos\sqrt{2a}}^{0} \psi(R_i, \cos\theta)\frac{\mathrm{d}t}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

and

$$R_i = \cos\theta + \epsilon_i\sqrt{\cos^2\theta - 2a}, \qquad \epsilon_1 = -1 \quad \text{and} \quad \epsilon_2 = +1.$$

The change of variable $u = \cos\theta$ yields

$$K_i = \int_{\sqrt{2a}}^{1} \psi(R_i, u)\frac{\mathrm{d}t}{\mathrm{d}u}\,\mathrm{d}u;$$

thus, according to the expression of $\mathrm{d}\theta/\mathrm{d}t$,

$$K_i = \int_{\sqrt{2a}}^{1} \psi(R_i, u)\frac{R_i}{(R_i - u)\sqrt{1-u^2}}\,\mathrm{d}u.$$

As $R_i - u = \epsilon_i\sqrt{u^2 - 2a}$, we find

$$K(a) = 2\int_{\sqrt{2a}}^{1} \frac{\psi(R_2, u)R_2 + \psi(R_1, u)R_1}{\sqrt{u^2-2a}\sqrt{1-u^2}}\,\mathrm{d}u.$$

In particular,

$$I(a) = 2\int_{\sqrt{2a}}^{1} \frac{u(R_2^2 - R_1^2)}{\sqrt{1-u^2}}\,\mathrm{d}u = 8\int_{\sqrt{2a}}^{1} \frac{u^2\sqrt{u^2-2a}}{\sqrt{1-u^2}}\,\mathrm{d}u$$

and

$$J(a) = 2\int_{\sqrt{2a}}^{1} \frac{u(R_1 + R_2)}{\sqrt{u^2-2a}\sqrt{1-u^2}}\,\mathrm{d}u = 4\int_{\sqrt{2a}}^{1} \frac{u^2}{\sqrt{u^2-2a}\sqrt{1-u^2}}\,\mathrm{d}u.$$

Let us make the change of variables: $[\sqrt{2a}; 1] \to [0; 1]$, $u \mapsto v$, with $u^2 - 2a = (1-2a)v^2$, and thus $u\,\mathrm{d}u = (1-2a)v\,\mathrm{d}v$. We obtain

$$I(a) = 8\int_0^1 \frac{\sqrt{2a + (1-2a)v^2}(1-2a)v^2}{\sqrt{1-v^2}}\,\mathrm{d}v,$$

$$J(a) = 4\int_0^1 \frac{\sqrt{2a + (1-2a)v^2}}{\sqrt{1-v^2}}\,\mathrm{d}v.$$

These expressions show that $I(\cdot)$ and $J(\cdot)$ are smooth on $]0; 1/2[$ and admit continuous extensions to the closed interval $[0; 1/2]$ (let us keep the same notation for these continuous extensions). Elementary computations give

$$I(\cdot) > 0 \quad \text{and} \quad J(.) > 0 \qquad \text{on } \left]0; \tfrac{1}{2}\right[,$$

$$I(0) = \tfrac{16}{3}, \qquad I\left(\tfrac{1}{2}\right) = 0, \qquad J(0) = 4, \qquad J\left(\tfrac{1}{2}\right) = 2\pi.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}a}\sqrt{2a + (1-2a)v^2} = \frac{1-v^2}{\sqrt{2a + (1-2a)v^2}},$$

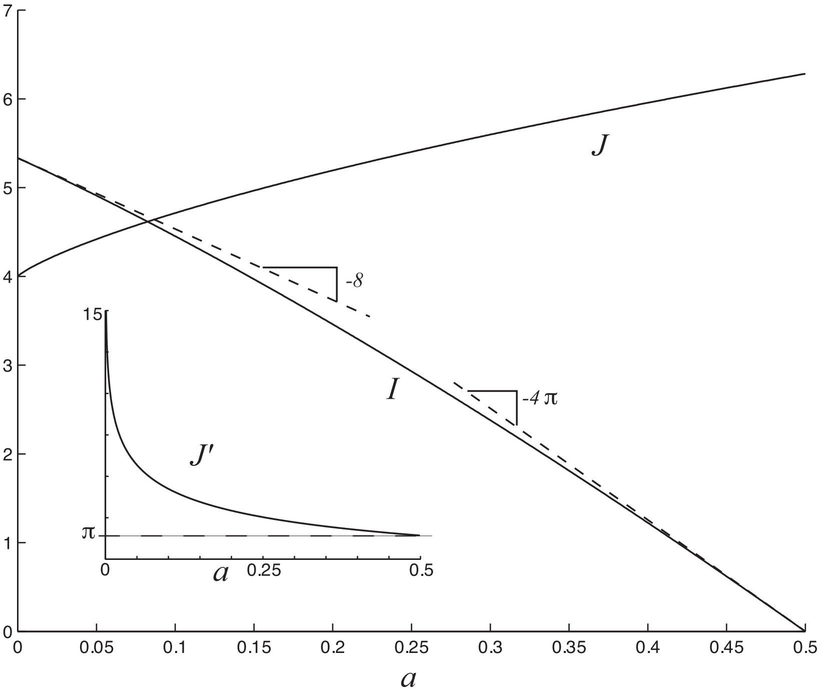

Figure 11. Numerical computations of the graphs of $I(\cdot)$ and $J(\cdot)$ with the slope of $I(\cdot)$ when a tends to 0 and 0.5. The inset plot shows $J'(\cdot)$ and its limits at $a = 0$ and $a = 0.5$.

and thus,

$$I'(a) = -16\int_0^1 \frac{\sqrt{2a+(1-2a)v^2}\; v^2}{\sqrt{1-v^2}}\,\mathrm{d}v + 8\int_0^1 \frac{\sqrt{1-v^2}(1-2a)v^2}{\sqrt{2a+(1-2a)v^2}}\,\mathrm{d}v,$$

$$J'(a) = 4\int_0^1 \frac{\sqrt{1-v^2}}{\sqrt{2a+(1-2a)v^2}}\,\mathrm{d}v. \tag{16}$$

Elementary computations and considerations give

$$I'(a) \to -8 \quad \text{when } a \to 0, \qquad I'(a) \to -4\pi \quad \text{when } a \to \tfrac{1}{2},$$

$$J'(\cdot) > 0 \quad \text{and} \quad J''(\cdot) < 0 \qquad \text{on } \left]0; \tfrac{1}{2}\right[,$$

$$J'(a) \to +\infty \quad \text{when } a \to 0, \qquad J'(a) \to \pi \quad \text{when } a \to \tfrac{1}{2}.$$

Numerical computations of the graphs of $I(\cdot)$ and $J(\cdot)$ are shown in figure 11.

4.4. Averaged differential equation for H

Corollary 4 indicates that the behaviour of the quantity $H(\cdot)$ along the solutions of the differential equation (12) should be related to the following 'averaged' differential equation:

$$\frac{\mathrm{d}a}{\mathrm{d}z} = -\frac{1}{z}\frac{I(a)}{J(a)}, \qquad a \in \left[0; \tfrac{1}{2}\right]. \tag{17}$$

This section is devoted to the study of the solutions of this equation.

According to the computations of the previous section, the functions $I(\cdot)$ and $J(\cdot)$ are locally (but unfortunately not uniformly) Lipschitz on $]0; 1/2]$. This ensures existence and uniqueness of solutions to the differential equation (17) on the domain $(z, a) \in]0; \infty[\times]0; 1/2[$. These solutions are all equivalent up to linear transforms of z; indeed, if $z \mapsto a(z)$ is a solution, then, for any $\lambda > 0$, $z \mapsto a(\lambda z)$ is also a solution.

Lemma 3. *For any $z_0 > 0$ and $a_0 \in]0; 1/2[$, the solution $z \mapsto a(z)$ with initial condition $a(z_0) = a_0$ is defined on a maximal time interval of the form $]0; z_{\max}[$, with $0 < z_{\max} < +\infty$, and satisfies*

$$a(z) \to 0 \quad \textit{when } z \to z_{\max}^- \qquad \textit{and} \qquad a(z) \to \tfrac{1}{2} \quad \textit{when } z \to 0^+.$$

Proof. Let $]z_{\min}; z_{\max}[$ denote the maximal interval of existence of the solution (with $0 \leqslant z_{\min} < z_{\max} \leqslant +\infty$). The function $a(\cdot)$ is decreasing on this interval; thus it admits a limit a_- (resp. a_+) when $z \to z_{\min}^+$ (resp. when $z \to z_{\max}^-$), with $0 \leqslant a_+ < a_- \leqslant 1/2$. Because $I(a_+) > 0$, and because the function $z \mapsto 1/z$ is not integrable in a neighbourhood of $+\infty$, we see that, necessarily, we have $z_{\max} < +\infty$ and $a_+ = 0$.

On the other hand, as $I(1/2) = 0$, and because of the Lipschitz properties of $I(\cdot)$ and $J(\cdot)$ in a left-hand neighbourhood of $1/2$, we see that we cannot have $z_{\min} > 0$ (or else we would have $a_- = 1/2$) and this would be in contradiction with the uniqueness of solutions of a differential equation, when a Lipschitz condition is fulfilled. Thus $z_{\min} = 0$. Now, because the function $z \mapsto 1/z$ is not integrable in a right-hand neighbourhood of 0, we see that, necessarily, $a_- = 1/2$. □

Let $z \mapsto \eta(z)$ denote the unique solution of the differential equation (17) for which $]0; 1[$ is the maximal interval of existence; according to the previous lemma, the function η admits a continuous extension (still denoted by η) to the closed interval $[0; 1]$, defined by $\eta(0) = 1/2$ and $\eta(1) = 0$. We have

$$\eta'(z) = -\frac{1}{z}\frac{I}{J}, \qquad \eta''(z) = \frac{1}{z^2}\frac{I}{J}\left(1 + \frac{I'J - IJ'}{J^2}\right),$$

which shows, according to the previous computations, that $\eta'(\cdot) < 0$ on $]0; 1[$, and that

$$\eta'(z) \to -\tfrac{4}{3} \quad \text{and} \quad \eta''(z) \to -\infty \qquad \text{when } z \to 1^-.$$

For z close to 0, the change of variable $\varphi = 1/2 - \delta$ ($\delta > 0$ small) gives (according to the previous computations)

$$\delta' = \frac{1}{z}\frac{I(1/2 - \delta)}{J(1/2 - \delta)} \sim \frac{2\delta}{z},$$

which shows that $\delta/z^2 \to 1$ when $z \to 0$. This shows that first and second right-hand derivatives of η can be defined at $z = 0$, by

$$\eta'(0) = 0 \qquad \text{and} \qquad \eta''(0) = -2.$$

A numerical computation of the graph of $\eta(.)$ is shown in figure 12.

4.5. *Global shape of long drops*

Take any $\bar{z} > 0$ and, for $z \in [0; \bar{z}]$, let us consider the quantities $r_{\bar{z}}(z)$, $\theta_{\bar{z}}(z)$, and $H_{\bar{z}}(z)$, as defined in section 3.2. As in section 3.2, we define the function $h_{\bar{z}} :]0; 1] \to [0; 1/2]$, by : $h_{\bar{z}}(\kappa) = H_{\bar{z}}(\kappa\bar{z})$, $\kappa \in]0; 1]$.

In the following, since $\bar{z}$ is fixed, we shall omit the indices $\bar{z}$, and simply write $r(z)$, $\theta(z)$, $H(z)$, and $h(\kappa)$ for these quantities. Let $R(z) = zr(z)$.

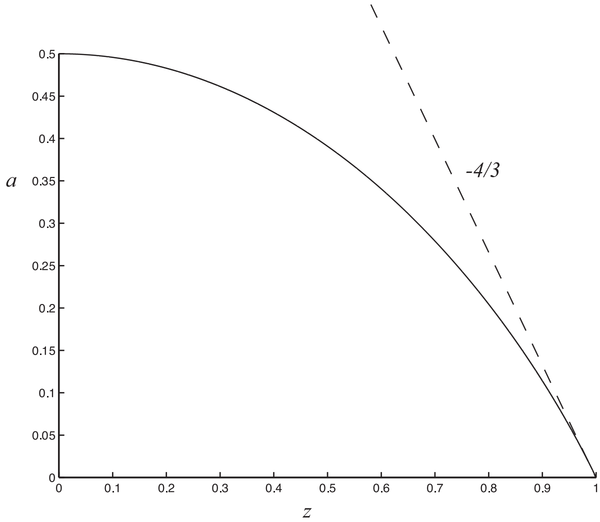

Figure 12. Numerical computation of the graph of $\eta(\cdot)$, the slope when $z \to 0$ is represented by the dashed line.

Proof of theorem 1. Take any $\varepsilon > 0$, $\varepsilon < 1$. Let

$$\underline{x} = \inf \left\{ x \in \left] \frac{\varepsilon}{2}; 1 \right] \,\middle|\, \text{ for any } \ y \in [x; 1], h(y) < \eta\left(\frac{\varepsilon}{2}\right) \right\}$$

(remark that $\eta(\varepsilon/2) < 1/2$). According to lemma 2, for $\bar{z}$ large enough, there is a unique integer n and unique points $z_0 = \bar{z} > z_1 > \cdots > z_{n-1} > \underline{x}\bar{z} \geqslant z_n > 0$ such that, for any $z \in [z_n; \bar{z}]$, we have

$$\arg(R(z) - 1, \theta(z)) = \arg\left(-1, \frac{\pi}{2}\right) \Leftrightarrow \text{ there is } i \in \{0, \ldots, n\} \text{ such that } z = z_i.$$

For $\bar{z}$ large enough, we can apply the estimates of corollary 4. They give, for any $i \in \{0, \ldots, n-1\}$,

$$\left| H(z_{i+1}) - H(z_i) - \frac{1}{z_i^2} I(H(z_i)) \right| \leqslant \frac{C}{\bar{z}^3},$$

$$\left| z_{i+1} - z_i + \frac{1}{z_i} J(H(z_i)) \right| \leqslant \frac{C}{\bar{z}^2},$$

where $C > 0$ is a constant depending only on ε (for $i = 0$, these estimates hold by passing to the limit in corollary 4).

For $i \in \{0, \ldots, n\}$, let us write $x_i = z_i/\bar{z}$. Then the two previous estimates can be rewritten as follows:

$$\left| h(x_{i+1}) - h(x_i) - \frac{1}{\bar{z}^2 x_i^2} I(h(x_i)) \right| \leqslant \frac{C}{\bar{z}^3},$$

$$\left| x_{i+1} - x_i + \frac{1}{\bar{z}^2 x_i} J(h(x_i)) \right| \leqslant \frac{C}{\bar{z}^3},$$

and finally we obtain

$$\left| \frac{h(x_{i+1}) - h(x_i)}{x_{i+1} - x_i} + \frac{1}{x_i} \frac{I(h(x_i))}{J(h(x_i))} \right| \leqslant \frac{C}{\bar{z}}. \tag{18}$$

Consider the function $\tilde{h} : [x_n; 1] \to [0; 1/2[$ defined by

$$\tilde{h}(x) = h(x_i) + \frac{h(x_{i+1}) - h(x_i)}{x_{i+1} - x_i}(x - x_i), \qquad x \in [x_{i+1}; x_i], \quad i \in \{0, \ldots, n-1\}$$

(it is continuous, affine by pieces, and coincides with h on $\{x_0, \ldots, x_n\}$). For any $i \in \{0, \ldots, n-1\}$ and $x \in]x_{i+1}; x_i[$, we can write

$$|\tilde{h}'(x) - \eta'(x)| \leqslant \frac{C}{\bar{z}} + A_1 + A_2,$$

where

$$A_1 = \left| \frac{1}{x_i} \frac{I(h(x_i))}{J(h(x_i))} - \frac{1}{x} \frac{I(h(x))}{J(h(x))} \right|, \qquad A_2 = \frac{1}{x} \left| \frac{I(h(x))}{J(h(x))} - \frac{I(\eta(x))}{J(\eta(x))} \right|.$$

Recall that $J(\cdot)$ is not uniformly Lipschitzian on $[0; 1/2]$. Nevertheless, proceeding as in the proof of lemma 2, one gets

$$\left| \frac{I(h(x_i))}{J(h(x_i))} - \frac{I(h(x))}{J(h(x))} \right| \leqslant \frac{C}{\bar{z}},$$

(where C is a positive constant depending only on ε). As, on the other hand, $|x - x_i| \leqslant C/\bar{z}^2$, this yields $A_1 \leqslant C/\bar{z}$.

To estimate A_2, we have to take into account the fact that $J'(a) \to \infty$ when $a \to 0$. According to the expression (16), we have, for a small enough,

$$J'(a) \leqslant 4 \int_0^1 \frac{\mathrm{d}v}{\sqrt{2a + v^2/2}} = 4\sqrt{2} \int_0^{1/\sqrt{4a}} \frac{\mathrm{d}w}{\sqrt{1 + w^2}} = 4\sqrt{2}\, \mathrm{argsh} \frac{1}{\sqrt{4a}} \sim 4\sqrt{2} \log \frac{1}{\sqrt{a}}.$$

In particular, the function $a \mapsto J'(a)$ is integrable on $]0; 1/2]$.

According to the computations of sections 4.3 and 4.4, we know that $\eta(x) \geqslant (1 - x)/C$, $x \in [0; 1/2]$. On the other hand, (18) shows that $|\tilde{f}'(x)|$ is bounded away from 0 for x close to 1; thus, we have $\tilde{h}(x) \geqslant (1 - x)/C$. Together with the facts that $I'(\cdot)$ is bounded and $J'(\cdot)$ is integrable, this gives

$$A_2 \leqslant \frac{\psi(x)}{x} |\tilde{h}(x) - \eta(x)|,$$

where ψ denotes a certain positive integrable function on $]0; 1/2]$ (independent of ε). Finally, we obtain

$$|\tilde{f}'(x) - \eta'(x)| \leqslant \frac{C}{\bar{z}} + \frac{\psi(x)}{x} |\tilde{h}(x) - \eta(x)|.$$

Let $g(x) = |\tilde{h}(x) - \eta(x)|$, $x \in [x_n; 1]$. At any point $x \in [x_n; 1] \backslash \{x_0, \ldots, x_n\}$ such that $g(x) \neq 0$, this function is derivable and satisfies

$$|g'(x)| \leqslant \frac{C}{\bar{z}} + \frac{\psi(x)}{x} g(x).$$

Integrating this differential inequation on $[x_n; 1]$, we obtain, for any $x \in [\underline{x}; 1]$,

$$|\tilde{h}(x) - \eta(x)| \leqslant \frac{C}{\bar{z}},$$

for a certain constant $C > 0$ depending on ε (recall that $\underline{x} \geqslant \varepsilon/2$). Since $|\tilde{f} - f| \leqslant C/\bar{z}^2$, the same estimate holds for $|h(x) - \eta(x)|$. Finally, we see by definition of $\underline{x}$ that, for $\bar{z}$ large enough, we have $\underline{x} < \varepsilon$, and this finishes the proof. □

5. Pendent and singular drops

The aim of this section is to prove theorems 2 and 3. The proofs are mainly based on the estimates of the previous section.

5.1. Flow with respect to z in the domain $H > 0$

For any $z > 0$, let

$$\mathcal{D}_z = \{(r, \theta) \in \mathbb{R}^2 | (zr, \theta) \in \mathcal{D}\},$$

and, for any $a \in [0; 1/2[$, let

$$\mathcal{D}_{z,a} = \{(r, \theta) \in \mathcal{D}_z | H(zr, \theta) > a\}.$$

Finally, let $\mathcal{D}_0 =]0; +\infty[\times]-\pi/2; \pi/2[$.

For $z_0 \geqslant 0$ and $(r_0, \theta_0) \in \mathcal{D}_{z_0}$, let us consider the solution $u \mapsto (r, \theta, z)$ of equation (4) with initial condition (r_0, θ_0, z_0) at time $u = 0$. Let $]u_-; u_+[$ denote the largest interval containing 0 on which $H(rz, \theta) > 0$. Then we have $u_+ < +\infty$, $u_- \geqslant -\infty$, $z(u_+) = 0$, and, for any $u \in]u_-; u_+[$, $z(u) > 0$ and $\mathrm{d}z/\mathrm{d}u(u) < 0$ (for more detailed justifications, see section 7). Let $z_{\max} \leqslant +\infty$ denote the limit of $z(u)$ when $u \to u_-$. Then, for any $z_1 \in]0; z_{\max}[$, let us write $\Phi_{z_0}^{z_1}(r_0, \theta_0) = (r(u_1), \theta(u_1))$, where u_1 is the unique point of $]u_-; u_+[$ such that $z(u_1) = z_1$. This defines a map $\Phi_{z_0}^{z_1} : \mathcal{D}_{z_0} \to \mathcal{D}_{z_1}$ (the flow with respect to the variable z), which is defined on the whole domain $\mathcal{D}_{z_0}$ if $z_1 \leqslant z_0$, and only on a subset of it if $z_1 \geqslant z_0$.

Lemma 4. *For any $a \in]0; 1/2[$, there exists $\Gamma > 0$ such that, for any $z_0 \geqslant \Gamma$, one can find $\Delta > 0$ such that the following inclusion holds:*

$$\Phi_{z_0+\Delta}^{z_0}(\mathcal{D}_{z_0+\Delta}) \subset \mathcal{D}_{z_0,a}.$$

Proof. Take any $a \in]0; 1/2[$ and $z_0 > 0$, and $(r_0, \theta_0) \in \mathcal{D}_{z_0} \backslash \mathcal{D}_{z_0,a}$. For any $z \in]0; z_{\max}[$ (with the above notation), let us write $(r(z), \theta(z)) = \Phi_{z_0}^{z}(r_0, \theta_0)$ and $R(z) = zr(z)$. According to lemma 2, if z_0 is sufficiently large (depending on a), then one can define uniquely $n \in \mathbb{N}^* \cup \{+\infty\}$ (which will turn out to be necessarily finite) and, writing $\mathcal{N} = \{i \in \mathbb{N}^* | i < n\}$, a family $(z_i)_{i \in \mathcal{N}}$ of points of $]z_0; z_{\max}[$ satisfying

- $z_{i-1} < z_i$ for any $i \in \mathcal{N}$;
- for any $z \in]z_0; z_{\max}[$,

  $\arg(R(z) - 1, \theta(z)) = \arg(R_0 - 1, \theta_0) \Leftrightarrow$ there is $i \in \mathcal{N}$ such that $z = z_i$.

Let us write $h(z) = H(R(z), \theta(z))$, $z \in]0; z_{\max}[$. Estimates of corollary 4 read, for any $i \in \mathcal{N}$

$$\left| h(z_i) - h(z_{i-1}) + \frac{1}{z_i^2} I(h(z_i)) \right| < \frac{C}{z_i^3},$$

$$\left| z_i - z_{i-1} - \frac{1}{z_i} J(h(z_i)) \right| < \frac{C}{z_i^2},$$

where C is a positive constant depending on a. Let $z_\infty = \sup_{i \in \mathcal{N} \cup \{0\}} z_i$ (the set $\mathcal{N}$ could be empty), and consider the function $\tilde{h} : [z_0; z_\infty[\to]0; 1/2]$, defined by

$$\tilde{h}(z) = h(z_{i-1}) + \frac{h(z_i) - h(z_{i-1})}{z_i - z_{i-1}}(z - z_{i-1}), \qquad z \in [z_{i-1}; z_i], \quad i \in \mathcal{N}.$$

This function is continuous, affine on any interval $[z_{i-1}; z_i]$, and the estimates above yield

$$\left|\tilde{f}'(z) + \frac{1}{z_i}\frac{I(h(z_i))}{J(h(z_i))}\right| < \frac{C}{z_i^2}, \qquad z \in [z_{i-1}; z_i], \quad i \in \mathcal{N}.$$

Since $z_i - z_{i-1}$ is of the order of $1/z_i^2$, this yields $\tilde{f}'(z) < -1/Cz$, $z \in [z_0; z_\infty[$, for a certain constant $C > 0$ depending on a.

Since $\tilde{h}(z_0) < 1/2$ and $\tilde{h}(z_i) > 0$ for any $i \in \mathcal{N}$, this implies $z_\infty < Cz_0$, for some constant $C > 1$ depending on a (and the set $\mathcal{N}$ is actually finite). Finally, $z_{\max} - z_\infty < 4\pi/\varepsilon_2$ if z_0 is large enough, which completes the proof. □

5.2. *Proof of theorems 2 and 3*

To begin with, we shall not assume as known the uniqueness of the infinite drop, and we shall prove the following assertions:

1. we have $\iota(\bar{z}) \to (r_0, \pi/2)$ when $\bar{z} \to 0$, for a certain $r_0 > 0$;
2. the set $\iota(]0; +\infty[)$ is bounded and defines a submanifold of $\mathbb{R}^2$ (it does not accumulate on itself);
3. for any point (r_0, θ_0) belonging to the (non-empty) compact set

$$\iota_\infty = \bigcap_{\bar{z}>0} \overline{\iota([\bar{z}; +\infty[)},$$

the solution of equation (4) with initial condition $(r_0, \theta_0, 0)$ corresponds to a drop of infinite length (it satisfies $z \to +\infty$ when $u \to -\infty$).

Assertion 1 is a direct consequence of proposition 3 of section 8.2 (asymptotics when $z \to 0$).

Proof of assertion 2. To prove that $\iota(]0; +\infty[)$ is bounded, we shall apply lemma 4. Let Γ_1 denote the constant Γ given by this lemma, corresponding to say $a = 1/4$, and let Δ_1 denote the constant Δ given by this lemma, corresponding to $a = 1/4$, $\Gamma = \Gamma_1$, and $z_0 = \Gamma_1$. Then, according to this lemma, we have $\iota(]z_0 + \Delta_1; +\infty[) \subset \Phi^0_{z_0}(\mathcal{D}_{z_0,1/4})$. Since $\mathcal{D}_{z_0,1/4}$ is relatively compact in $\mathcal{D}_{z_0}$, and since $\Phi^0_{z_0} : \mathcal{D}_{z_0} \to \mathcal{D}_0$ is continuous, this show that the set $\iota(]z_0 + \Delta_1; +\infty[)$ is relatively compact in $\mathcal{D}_0$, therefore bounded. Thus, since $\iota(z)$ converges when $z \to 0$, the set $\iota(]0; +\infty[)$ is bounded.

For any $0 \leqslant z < \bar{z}$, let us write $\iota_z(\bar{z}) = (r_{\bar{z}}(z), \theta_{\bar{z}}(z))$ (we use the notation of section 3.2).

Let us prove that the set $\iota(]0; +\infty[)$ is a submanifold of $\mathcal{D}_0$, i.e. that it does not accumulate on itself. Let us proceed by contradiction and suppose that the converse is true. Then there exists $z_\infty > 0$ and a sequence $z_n \to +\infty$ such that $\iota(z_n) \to \iota(z_\infty)$ when $n \to +\infty$. Keeping the above notation, we can suppose that $z_0 > z_\infty$. For n large enough, we have $z_n > z_0 + \Delta_1$; therefore $\iota_{z_\infty}(z_n)$ belongs to the set $\Phi^{z_\infty}_{z_0}(\mathcal{D}_{z_0,1/4})$, which is relatively compact in $\mathcal{D}_{z_\infty}$. Up to extracting a subsequence, one can suppose that the points $\iota_{z_\infty}(z_n)$ converge, when $n \to +\infty$, towards a point $y_\infty \in \mathcal{D}_{z_\infty}$. Then, by continuity of the map $\Phi^0_{z_\infty}$, the points $\iota(z_n) = \Phi^0_{z_\infty}(\iota_{z_\infty}(z_n))$ converge towards $\Phi^0_{z_\infty}(y_\infty)$ when $n \to +\infty$. As a consequence we have $\iota(z_\infty) = \Phi^0_{z_\infty}(y_\infty)$, which is impossible, thus a contradiction. This finishes the proof of assertion 2. □

Proof of assertion 3. Take any $(r_\infty, \theta_\infty) \in \iota_\infty$, and let $u \mapsto (r, \theta, z)$ denote the solution of equation (4) with initial condition $(r_\infty, \theta_\infty, 0)$ at time $u = 0$. We have $H(0) = 0$ and $\mathrm{d}H/\mathrm{d}u(0) < 0$, thus $H(u) > 0$ for $u < 0$ close to 0.

Claim (for any $u < 0$, we have $H(u) > 0$). Let us suppose that the converse is true, and let $u_- = \sup\{u < 0 | H(u) = 0\}$. Then $\mathrm{d}z/\mathrm{d}u < 0$ on $]u_-; 0]$; thus, $z(u_-) > 0$. According to (10), this yields $\mathrm{d}H/\mathrm{d}u(u_-) > 0$; therefore, $H(u) < 0$ for $u < u_-$ close to u_-. This is in contradiction with the fact that $(r_\infty, \theta_\infty)$ is the limit of a sequence of initial conditions such that the corresponding solutions satisfy $H > 0$ on $]-\infty; 0[$ (continuity of the solutions of a differential equation with respect to initial conditions). The claim is proved. As a consequence, we have $z(u) > 0$ and $\mathrm{d}z/\mathrm{d}u(u) < 0$ for all $u < 0$.

Claim (we have $z(u) \to +\infty$ when $u \to -\infty$). If the converse was true, then $z(u)$ would have a finite limit $\bar{z}$ when $u \to -\infty$, and as a consequence we would have $(r_\infty, \theta_\infty) = \iota(\bar{z})$, which is in contradiction with assertion 2. Assertion 3 is proved. □

Now we wil prove theorems 2 and 3. It immediately follows from lemma 4 that, if a solution corresponding to an infinite drop exists, then it necessarily satisfies $(R, \theta) \to (1, 0)$ when $u \to -\infty$. Thus, since the set $\iota_\infty = \bigcap_{\bar{z}>0} \overline{\iota([\bar{z}; +\infty[)}$ is non-empty, theorem 2 follows from assertion 3.

If moreover we assume the uniqueness of the drop of infinite length, then we can say that the set ι_∞ is a singleton, and theorem 3 immediately follows from assertions 1, 2 and 3. □

6. Asymptotics when $r \to \infty$

The aim of this section is to prove theorem 4.

We consider a solution $u \mapsto (r, \theta, z)$ of equation (4), whose trajectory lies in the invariant subspace $\{r > 0\}$, and satisfying $(r, \theta, z) \to (+\infty, \pi/2, 0)$ when $u \to +\infty$. Since $\mathrm{d}s = r\,\mathrm{d}u$, we have $s \to +\infty$ when $u \to +\infty$. Let us write $\theta = \pi/2 + \beta$. We have $\beta \to 0$ and $r \to +\infty$ when $s \to +\infty$. Equation (3) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \cos\beta, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}s} = -z - \frac{\sin\beta}{r}, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \sin\beta,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\beta\\ z\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\beta\\ z\end{pmatrix} + \mathcal{O}(|\beta|^3) + \mathcal{O}\left(\frac{|\beta|}{r}\right).$$

Let us suppose that we are not in the particular explicit case $(\beta, z) \equiv (0, 0)$. Thus, $z = 0$ yields $\beta \neq 0[\pi]$ and thus $\mathrm{d}z/\mathrm{d}s \neq 0$. As a consequence, the set $S = \{s \in \mathbb{R} \mid z(s) = 0\}$ is infinite and discrete, and the sign of $\mathrm{d}z/\mathrm{d}s$ is alternatively positive and negative at successive points of S. Take any $s_0 \in S$ (say such that $\mathrm{d}z/\mathrm{d}s(s_0) < 0$), and let $s_0 < s_1 < s_2 < \cdots$ denote the points of $S \cap [s_0; +\infty[$. We have $s_{j+1} - s_j \to \pi$ when $j \to +\infty$. Moreover we have $E(s_0) < 1$; thus, for any $s \geqslant s_0$, we have $E(s) < 1$; thus, $\beta(s) \neq \pi\ [2\pi]$, and actually $\beta(s) \in\ ]-\pi; \pi[$.

Let $F = E + 1$; we have

$$F = (1 - \cos\beta) + \frac{z^2}{2}, \qquad \frac{\mathrm{d}E}{\mathrm{d}s} = -\frac{\sin^2\beta}{r}.$$

For $j \in \mathbb{N}$, let us write r_j, β_j, F_j for $r(s_j)$, $\beta(s_j)$, $F(s_j)$ (we have $\beta_{2j} < 0$ and $\beta_{2j+1} > 0$). We have $F_{2j} \sim \beta_{2j}^2$ and, for j large positive,

$$F_{2j+2} - F_{2j} = -\frac{1}{r_{2j}} I_j + \mathcal{O}\left(\frac{1}{r_{2j}^2}\right), \qquad I_j = \int_{s_{2j}}^{s_{2j+2}} \sin^2\beta\,\mathrm{d}s.$$

For $s \in \mathbb{R}$, let us write $(\beta, z) = |(\beta, z)|(-\cos\alpha, -\sin\alpha)$. We have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{(\beta, z) \wedge (\mathrm{d}\beta/\mathrm{d}s, \mathrm{d}z/\mathrm{d}s)}{|(\beta, z)|^2} = \frac{1}{\beta^2 + z^2}\left(z^2 + \beta\sin\beta + \frac{z\sin\beta}{r}\right) = 1 + \mathcal{O}(r_{2j}^{-1}) + \mathcal{O}(F_{2j}).$$

Thus

$$I_j = \int_0^{2\pi} \sin^2\beta \frac{\mathrm{d}s}{\mathrm{d}\alpha}\,\mathrm{d}\alpha = \left(\int_0^{2\pi} \beta^2\,\mathrm{d}\alpha\right)(1 + \mathcal{O}(r_{2j}^{-1}) + \mathcal{O}(F_{2j}))$$
$$= 2\pi F_{2j}(1 + \mathcal{O}(r_{2j}^{-1}) + \mathcal{O}(F_{2j})).$$

Finally we obtain

$$\frac{F_{2j+2} - F_{2j}}{F_{2j}} \sim -\frac{2\pi}{r_{2j}} \sim -\frac{1}{j},$$

and thus

$$\log F_{2j} \sim -\log j \sim -\log r_{2j}.$$

For s large enough, we have $\mathrm{d}r/\mathrm{d}s > 0$ (actually $\mathrm{d}r/\mathrm{d}s \simeq 1$); thus we can view F as a function of r. The last estimate shows that $\log F \sim -\log r$ for s large.

Now we have

$$\frac{1}{rF}\frac{\mathrm{d}(rF)}{\mathrm{d}s} = \frac{1}{rF}(F\cos\beta - \sin^2\beta) = \frac{1}{rF}\left(\frac{z^2 - \beta^2}{2} + \mathcal{O}(\beta^4)\right).$$

For $s \in [s_{2j}; s_{2j+2}]$, the quantity $|1/(r(s)F(s)) - 1/(r(s_{2j})F(s_{2j}))|$ is of the order of $r^{-1+\mathrm{o}(1)}$. Thus, proceeding as above, we see that the quantity

$$\int_{s_{2j}}^{s_{2j+2}} \frac{1}{rF}\frac{\mathrm{d}(rF)}{\mathrm{d}s}\,\mathrm{d}s$$

is of the order of $r^{-2+\mathrm{o}(1)}$ (the main contribution vanishes in the mean). This shows that $\log(rF)$ has a limit when $s \to +\infty$. Thus $\log(\sqrt{r}|(\beta, z)|)$ has a limit when $s \to +\infty$, and theorem 4 follows.

7. Classification of the solutions

The aim of this section is to list the possible behaviour, when $u \to \pm\infty$, of the solutions of equation (4), and to give a classification according to this behaviour (proposition 1 will follow from these results). Let us mention that the results in this section do not depend on the ones of sections 4–6, but only on two ingredients which were given in section 2.5: the fact that the domain where $H > 0$ is positively invariant unless $z = 0$ (assertion (10)), and the fact that E is a (decreasing) Lyapunov function for equation (4).

7.1. Statement of the classification results

For any solution $u \mapsto (r, \theta, z)$ of equation (4), the expression for $\mathrm{d}r/\mathrm{d}u$ shows that the quantity $|r(u)|$ cannot blow up in finite time (neither in the past nor in the future); according to the expression of $\mathrm{d}z/\mathrm{d}u$, the same holds for $|z(u)|$, and thus for $|\theta(u)|$. Thus the flow of equation (4) is defined globally in time.

The dynamics on the invariant plane $\{r = 0\}$ is simple: z is constant and θ varies monotonically from $-(\pi/2)$ $[2\pi]$ to $(\pi/2)$ $[2\pi]$ or from $(3\pi/2)$ $[2\pi]$ to $(\pi/2)$ $[2\pi]$. Apart from this plane, the possible behaviour can be split into five categories, which are listed by the following theorem (illustrated by figure 13). We tried to find simple meaningful denominations for these categories. As is usual in dynamical systems, we use the letter α (resp. ω) to refer to the behaviour when $u \to -\infty$ (resp. when $u \to +\infty$).

Theorem 5.

(a) *For any $\bar{z} \in \mathbb{R}$ and $k \in \mathbb{Z}$, there exists a unique solution $u \mapsto (r, \theta, z)$ of equation (4) (uniqueness is up to translation of u), lying in the invariant subspace $\{r > 0\}$, and satisfying $(r, \theta, z) \to (0, \pi/2 + 2k\pi, \bar{z})$ when $u \to -\infty$. These solutions satisfy $(r, \theta, z) \to (+\infty, \pi/2+2k\pi, 0)$ when $u \to +\infty$ (with the same integer k). These solutions will be called 'of type α-bounded'.*

(b) *For any $\bar{z} \in \mathbb{R}$ and $k \in \mathbb{Z}$, there exists a unique solution $u \mapsto (r, \theta, z)$ of equation (4) (uniqueness is up to translation of u), lying in the invariant subspace $\{r > 0\}$, and satisfying $(r, \theta, z) \to (0, -\pi/2 + 2k\pi, \bar{z})$ when $u \to +\infty$. If $\bar{z} = 0$, then we have $(\theta, z) \equiv (-\pi/2 + 2k\pi, 0)$. If $\bar{z} > 0$ (resp. $\bar{z} < 0$), then these solutions satisfy $(\theta, z) \to (+\infty, +\infty)$ (resp. $(\theta, z) \to (-\infty, -\infty)$) when $u \to -\infty$. These solutions will be called 'of type ω-bounded'.*

(c) *Any solution $u \mapsto (r, \theta, z)$ of equation (4) lying in the invariant subspace $\{r > 0\}$ and which does not belong to the two classes described above belongs to one of the three following (mutually exclusive) classes.*
   1. *There exists $k \in \mathbb{Z}$ such that $(r, \theta, z) \to (0, 2k\pi, +\infty)$ or $(r, \theta, z) \to (0, (2k+1)\pi, -\infty)$ when $u \to -\infty$, and such that $(r, \theta, z) \to (+\infty, \pi/2 + 2k\pi, 0)$ when $u \to +\infty$ (with the same integer k). These solutions will be called 'of type α-singular'.*
   2. *We have $(\theta, z) \to (+\infty, +\infty)$ or $(\theta, z) \to (-\infty, -\infty)$ when $u \to -\infty$, and there exists $k \in \mathbb{Z}$ such that $(r, \theta, z) \to (+\infty, \pi/2 + 2k\pi, 0)$ when $u \to +\infty$. These solutions will be called 'of type α-θ-infinite'.*
   3. *We have $r \to +\infty$ when $u \to -\infty$, and there exists $k \in \mathbb{Z}$ such that (θ, z) converges in mean towards the set $\{(-\pi/2 + 2k\pi, 0), (-\pi/2 + (2k+1)\pi, 0)\}$ when $u \to -\infty$. These solutions will be called 'of type α-r-infinite'.*

The following proposition provides further precisions.

Then, according to the previous theorem, the sign of $z(u)$ is constant for u sufficiently large and negative, and, according to the symmetry $(\theta, z) \to (\pi - \theta, -z)$, we can suppose without loss of generality that this sign is positive. Then the following proposition holds.

Proposition 2. *Let $u \mapsto (r, \theta, z)$ denote a solution of equation (4), lying in the invariant subspace $\{r > 0\}$, which is not of type α-r-infinite, and which is not one of the particular explicit solutions satisfying $(\theta, z) \equiv ((\pi/2)[\pi], 0)$. We suppose (without loss of generality) that $z > 0$ for u large and negative. Then the following holds.*

(a) *If the solution is of type α-bounded or α-singular, then its trajectory eventually crosses the plane $\{z = 0\}$. If u_0 denotes the time where the first crossing occurs (i.e. $z(u_0) = 0$ and $z(u) > 0$ for $u < u_0$), then, for $u < u_0$, we have*

$$z > 0 \qquad \frac{\mathrm{d}z}{\mathrm{d}u} < 0 \qquad \textit{and} \qquad H > 0.$$

(b) *If the solution is of type ω-bounded, then, for any $u \in \mathbb{R}$, we have $z > 0$, $H < 0$ and $\mathrm{d}\theta/\mathrm{d}u < 0$ (thus θ decreases from $+\infty$ to $-\pi/2 + 2k\pi$).*

(c) *If the solution is of type α-θ-infinite, then there exists $u_0 \in \mathbb{R}$ such that, for any $u < u_0$, we have $z > 0$, $H < 0$ and $\mathrm{d}\theta/\mathrm{d}u < 0$ (thus θ is a decreasing function of u for u sufficiently large and negative).*

Numerical simulations of solutions belonging to each of the types enumerated in theorem 5 are shown in figure 13. Let us briefly comment on this figure and the above results.

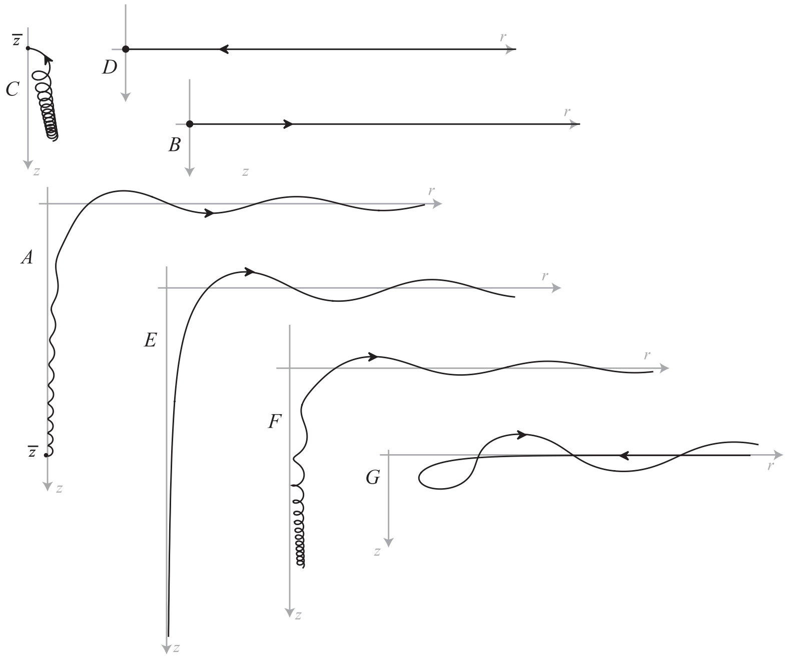

Figure 13. Profiles corresponding to the various dynamical behaviour distinguished by theorem 5 (compare figure 17 of [5] and figures in chapter 4 of [6]). The arrows along the profiles correspond to the sense of increasing u (the liquid phase is lying at the 'left' of the profiles with respect to the orientation given by these arrows). Profile A represents a finite pendent drop (type α-bounded) with $\bar{z} > 0$ (profile B is the limit case $\bar{z} = 0$); profile C represents a sessile drop (type ω-bounded) with $\bar{z} > 0$ (profile D is the limit case $\bar{z} = 0$); profile E represents the singular drop of infinite length; profiles F and G correspond, respectively, to the 'α-θ-infinite' and 'α-r-infinite' cases.

Solutions of type α-bounded with $\bar{z} \geqslant 0$ correspond to pendent drops of finite length (the case $\bar{z} < 0$ is physically less relevant, since in this case the liquid phase lies above the capillary surface, and the pressure is negative at the top of this surface at height $\bar{z}$).

Solutions of type ω-bounded with $\bar{z} \geqslant 0$ correspond to sessile drops (again, the case $\bar{z} < 0$ is physically less relevant). The asymptotics $\theta \to -\infty$ and $z \to +\infty$ when $u \to -\infty$ (provided that $\bar{z} > 0$) show that the profile in the (r, z)-plane of such a solution displays an infinity of self-intersections. But the part of the solution corresponding to u larger than some u_0 gives the surface of a sessile drop under suitable boundary conditions (say a sessile drop lying on a horizontal surface).

Solutions of type α-singular correspond to pendent drops having an infinite length. As discussed in section 3.3, there is exactly one solution of this type.

The two remaining classes of solutions (solutions of types α-θ-infinite and α-r-infinite) seem to be physically less relevant (although pieces of the corresponding profiles may appear as possible interfaces under suitable boundary conditions). As for solutions of type ω-bounded, the profile in the (r, z)-plane of a solution of type α-θ-infinite displays an infinity of self-intersections.

7.2. Stable and unstable manifolds of equilibria

Equilibrium points of equation (4) are the points of the set $\{(r, \theta, z) | r = 0, \theta = \pm(\pi/2)[2\pi], z \in \mathbb{R}\}$. At each equilibrium point of the form $(0, (\pi/2)[2\pi], z)$

(resp. $(0, -(\pi/2)[2\pi], z)$), the matrix of the linearization of (4) reads

$$\begin{pmatrix} 1 & 0 & 0 \\ -z & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left(\text{resp. } \begin{pmatrix} -1 & 0 & 0 \\ -z & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

These two matrices have the same eigenvalues $(1, -1, 0)$. According to the centre manifold theorem [8], any equilibrium has a one-dimensional stable (resp. unstable) manifold. The eigendirections of these matrices are given by the following eigenvectors: for the first matrix, $(1, -z/2, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, and for the second matrix, $(0, 1, 0)$, $(1, z/2, 0)$ and $(0, 0, 1)$. Thus the stable (resp. unstable) manifolds of equilibria of the form $(0, (\pi/2)[2\pi], z)$ (resp. $(0, -(\pi/2)[2\pi], z)$) lie in the invariant plane $\{r = 0\}$ (see figure 3). For any equilibrium of the form $(0, (\pi/2)[2\pi], z)$ (resp. $(0, -(\pi/2)[2\pi], z)$), there exists a unique (up to translation of u) solution lying in the invariant subspace $\{r > 0\}$ and converging towards this equilibrium when $u \to -\infty$ (resp. when $u \to +\infty$).

7.3. Behaviour when $u \to +\infty$

Let us consider any solution $u \mapsto (r, \theta, z)$ of equation (4), lying in the invariant subspace $\{r > 0\}$, and let us discuss its asymptotic behaviour in relation to the Lyapunov function E. The following lemma describes the two possible types of behaviour when $u \to +\infty$.

Lemma 5. *One of the two following (mutually exclusive) cases occurs:*

- *'ω-bounded'—there exists $\bar{z} \in \mathbb{R}$ such that $(r, \theta, z) \to (0, -(\pi/2)[2\pi], \bar{z})$ when $u \to +\infty$;*
- *'ω-r-infinite'—we have $(r, \theta, z) \to (+\infty, (\pi/2)[2\pi], 0)$ when $u \to +\infty$.*

Proof. Since E is decreasing and bounded from below, it must converge towards a limit $E_{+\infty}$, and $|z(u)|$ is bounded when $u \to +\infty$. Let us distinguish three cases.

Case 1 ($r(u)$ is bounded when $u \to +\infty$). In this case, $|\mathrm{d}\theta/\mathrm{d}u|$ is bounded when $u \to +\infty$; thus, the expression for $\mathrm{d}E/\mathrm{d}u$ shows that $\cos\theta \to 0$, and thus that $\theta \to \pm(\pi/2)[2\pi]$. Since r is bounded, the expression for $\mathrm{d}r/\mathrm{d}u$ shows that actually $\theta \to -(\pi/2)[2\pi]$. Thus $r \to 0$ when $u \to +\infty$ and the rate of convergence is exponential. Then expression of $\mathrm{d}z/\mathrm{d}u$ shows that $z(u)$ has a limit when $u \to +\infty$. Thus the solution lies in the stable manifold of an equilibrium of the form $(0, -(\pi/2)[2\pi], \bar{z})$ (this gives the 'ω-bounded' case).

Case 2 ($r(u)$ is unbounded but does not diverge to $+\infty$ when $u \to +\infty$). In this case there exists $r_0 > 0$ and a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \to +\infty$ when $n \to +\infty$, such that $r(u_n) = r_0$. Since $|(\mathrm{d}\theta/\mathrm{d}u)(u_n)|$ is bounded, the expression for $\mathrm{d}E/\mathrm{d}u$ shows that $\cos\theta(u_n) \to 0$, and thus that $\theta(u_n) \to \pm(\pi/2)[2\pi]$, when $n \to +\infty$. Moreover, we can suppose that $(\mathrm{d}r/\mathrm{d}u)(u_{2n}) \geqslant 0$ and that $(\mathrm{d}r/\mathrm{d}u)(u_{2n+1}) \leqslant 0$, which yields $\theta(u_{2n}) \to +\pi/2\ [2\pi]$ and $\theta(u_{2n+1}) \to -\pi/2\ [2\pi]$ when $n \to +\infty$. Moreover we must have $|(\mathrm{d}\theta/\mathrm{d}u)(u_n)| \to 0$, and thus $z(u_n) \to 0$ when $n \to +\infty$. This shows that $E(u_{2n}) \to -1$ and that $E(u_{2n+1}) \to +1$ when $n \to +\infty$, which is impossible since E decreases with u. Thus, this case cannot occur.

Case 3 ($r(u) \to +\infty$ when $u \to +\infty$). To treat this case we shall use the variable s instead of u. Since $\mathrm{d}s = r\,\mathrm{d}u$, we see that $s \to +\infty$ when $u \to +\infty$. Expressions in (3) for $\mathrm{d}\theta/\mathrm{d}s$ and $\mathrm{d}z/\mathrm{d}s$ are small perturbations of the integrable equation (8) (the equation of the pendulum). We have

$$\frac{\mathrm{d}E}{\mathrm{d}s} = -\frac{\cos^2\theta}{r}.$$

Since r grows at most linearly with s when $s \to +\infty$, we see (on the phase space of equation (8)) that the only possible values for $E_{+\infty}$ are ± 1 (indeed, for the other values of $E_{+\infty}$, the corresponding solutions of the asymptotic equation (8) are periodic, and the mean over one period of $\cos^2\theta$ is non-zero). We cannot have $E_{+\infty} = +1$, or else θ would converge (at least in mean) towards $-(\pi/2)\ [2\pi]$; thus, r would converge towards 0. Thus $E_{+\infty} = -1$, and thus $\theta \to (\pi/2)\ [2\pi]$ and $z \to 0$ when $u \to +\infty$ (we are in the case 'ω-r-infinite'). The lemma is proved. □

7.4. *Behaviour when $u \to -\infty$*

The next lemma describes the four possible types of behaviour when $u \to -\infty$.

Lemma 6. *One of the four following (mutually exclusive) cases occurs:*

- *'α-bounded'—there exists $\bar{z} \in \mathbb{R}$ such that $(r, \theta, z) \to (0, (\pi/2)[2\pi], \bar{z})$ when $u \to -\infty$;*
- *'α-r-infinite'—we have $r \to +\infty$ when $u \to -\infty$, and there exists $k \in \mathbb{Z}$ such that (θ, z) converges in mean towards the set $\{(-\pi/2 + 2k\pi, 0), (-\pi/2 + (2k+1)\pi, 0)\}$ when $u \to -\infty$;*
- *'α-singular'—we have $(r, z) \to (0, +\infty)$ when $u \to -\infty$ and there exists $k \in \mathbb{Z}$ such that $\theta \in\ ]-\pi/2 + 2k\pi\,;\, \pi/2 + 2k\pi[$ for u large enough;*
- *'α-θ-infinite'—we have $(z, \theta) \to (+\infty, +\infty)$ when $u \to -\infty$, and, for u large enough, we have $\mathrm{d}\theta/\mathrm{d}u < 0$.*

Proof. Let us denote by $E_{-\infty}$ the limit (in $\mathbb{R} \cup \{-\infty\}$) of E when $u \to -\infty$, and let us distinguish two cases.

Case A ($E_{-\infty} > -\infty$). In this case the situation resembles the one treated in the proof of the previous lemma, and we shall again distinguish among three cases.

Case A1 ($r(u)$ is bounded when $u \to -\infty$). In this case, arguing as above, we see that $\cos\theta \to 0$, thus $\theta \to \pm\pi/2$, and the fact that $r(u)$ is bounded yields $\theta \to +\pi/2$ when $u \to -\infty$. Thus $r(u) \to 0$, the rate of convergence is exponential, and z has a limit when $u \to -\infty$. Thus the solution lies in the unstable manifold of an equilibrium of the form $(0, (\pi/2)[2\pi], \bar{z})$, and we are in the case 'α-bounded'.

Case A2 ($r(u)$ is unbounded but does not diverge to $+\infty$ when $u \to -\infty$). The same arguments as in the proof of the previous lemma show that this case cannot occur.

Case A3 ($r(u) \to +\infty$ when $u \to -\infty$). Again, we shall use in this case the variable s instead of u. We have $s \to -\infty$ when $u \to -\infty$. Arguing as above, we see that $E_{-\infty}$ necessarily equals ± 1, and, since $r(u) \to +\infty$, that $E_{-\infty} = +1$. This shows that (θ, z) converges in mean towards $(-(\pi/2)[2\pi], 0)$. Moreover, since E decreases, we have either $E \equiv 1$ (this corresponds to the particular solution $(\theta, z) \equiv (-(\pi/2)[2\pi], 0)$) or $E(u) < 1$ for all $u \in \mathbb{R}$. In this last case, we see that there exists $k \in \mathbb{Z}$ such that $\theta(u)$ is confined to the interval $]-\pi/2 + 2k\pi\,;\, -\pi/2 + 2(k+1)\pi[$ (it converges in mean towards the boundary of this interval). This corresponds to the case 'α-r-infinite'.

Let us mention that, in this case, it may happen that (θ, z) actually converges towards a certain limit $(-\pi/2+2k\pi, 0)$, and it may also happen that (θ, z) 'oscillates' indefinitely between the two points $(-\pi/2 + 2k\pi, 0)$ and $(-\pi/2 + 2(k+1)\pi, 0)$, when $u \to -\infty$ (approaching the heteroclinic cycle connecting these two equilibria in the phase space of equation (8)).

Case B ($E_{-\infty} = -\infty$). In this case we have $z \to \pm\infty$ when $u \to -\infty$, and, according to the symmetry $(\theta, z) \mapsto (\pi - \theta, -z)$, we can suppose without loss of generality that $z \to +\infty$ when $u \to -\infty$.

To treat this case, we shall use the quantity H, which is asymptotically invariant when $z \to \pm\infty$. Recall (assertion (10)) that

$$H = 0 \qquad r \neq 0 \qquad \text{and} \qquad z \neq 0 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}u} > 0. \tag{19}$$

Going back to our solution, there exists $u_0 \in \mathbb{R}$ such that $z(u) > 0$ for any $u < u_0$. The previous remark leads us to distinguish the two following cases.

Case B1 (for any $u < u_0$, we have $H(u) > 0$). This yields (for $u < u_0$) $\cos\theta > zr/2 > 0$; thus, $\theta \in]-\pi/2; \pi/2[\ [2\pi]$, and the map $u \mapsto z(u)$ is strictly decreasing on $]-\infty; u_0[$. Since $z(u) \to +\infty$, we necessarily have $r(u) \to 0$ when $u \to -\infty$. Thus we are in the case 'α-singular' (in this case, we also have $\theta \to 0\ [2\pi]$ when $u \to -\infty$, but this will be proved only in section 5).

Case B2 (there exists $u_1 < u_0$ such that $H(u_1) \leqslant 0$). In this case, (19) shows that, for any $u < u_1$, we have $H(u) < 0$. Thus (for $u < u_1$), we have $\cos\theta < zr/2$, and thus $\mathrm{d}\theta/\mathrm{d}u < 0$. We claim that $\theta(u) \to +\infty$ when $u \to -\infty$. To prove this claim, let us proceed by contradiction and suppose that $\theta(u)$ converges towards a finite limit θ_∞ when $u \to -\infty$. Then, since $z \to +\infty$ when $u \to -\infty$, we must have $\cos\theta_\infty \geqslant 0$. On the other hand, we must have $\sin\theta_\infty \geqslant 0$, or else $r(u)$ would converge towards 0 at an exponential rate; thus, $z(u)$ would have a finite limit when $u \to -\infty$, thus a contradiction. But we must also have $\sin\theta_\infty \leqslant 0$, or else $r(u)$ would diverge to $+\infty$, and $\mathrm{d}\theta/\mathrm{d}u$ would diverge to $-\infty$, which is impossible. Thus we necessarily have $\theta_\infty = 0\ [2\pi]$; since $\cos\theta < zr/2$, we have $\mathrm{d}\theta/\mathrm{d}u < -\cos\theta$, which is impossible; the claim is proved, we are thus in the case 'α-θ-infinite'. □

7.5. Classification

We still consider a solution $u \mapsto (r, \theta, z)$ of equation (4), lying in the invariant subspace $\{r > 0\}$. Not all combinations of the ω-behaviour listed in lemma 5 and the α-behaviour listed in lemma 6 are possible, as the following lemma shows.

Lemma 7. *If the behaviour when $u \to +\infty$ is of type ω-bounded, then either we are in the particular explicit case $(\theta, z) \equiv (-(\pi/2)[2\pi], 0)$, or the behaviour when $u \to -\infty$ is of type α-θ-infinite.*

Proof. Suppose that the behaviour when $u \to +\infty$ of the solution is of type ω-bounded. Then we have $E_{+\infty} \geqslant 1$. If $E_{+\infty} = 1$, then $\bar{z} = \lim_{u\to+\infty} z(u) = 0$, and the solution is one of the particular explicit solutions satisfying $(\theta, z) \equiv (-(\pi/2)[2\pi], 0)$.

Let us suppose that $E_{+\infty} > 1$. As a consequence $E(u) > 1$ (and thus $z(u) \neq 0$) for any $u \in \mathbb{R}$; thus, the sign of $z(u)$ is the same for all $u \in \mathbb{R}$. Without loss of generality, we can suppose that $z(u) > 0$ for all $u \in \mathbb{R}$.

Now, since the stable manifold is tangent to the stable direction at the point $(0, -\pi/2+2k\pi, \bar{z})$ (the limit when $u \to +\infty$), we must have $(\theta(u)+\pi/2-2k\pi)/r(u) \to \bar{z}/2$ when $u \to +\infty$. Thus $\cos\theta \sim \bar{z}r/2$, and thus, according to (9), $\mathrm{d}H/\mathrm{d}u \sim \bar{z}^2r^4/4$ when $u \to +\infty$; in particular $\mathrm{d}H/\mathrm{d}u > 0$ for u large positive. Thus, since $H \to 0$ when $u \to +\infty$, we see that $H < 0$ for u large enough, and, since $z(u)$ never vanishes, according to (19), $H < 0$ for any $u \in \mathbb{R}$.

If the behaviour when $u \to -\infty$ of the solution were of type α-bounded, then we would have $(r, \theta, z) \to (0, \pi/2, \underline{z})$ when $u \to -\infty$, for a certain $\underline{z} \geqslant 0$, and actually $\underline{z} > 0$ (or else we would be in the particular explicit case $(\theta, z) \equiv ((\pi/2)[2\pi], 0)$). Then, because the unstable manifold is tangent to the unstable direction at the point $(0, (\pi/2)[2\pi], \underline{z})$, we would have (proceeding as above) $\mathrm{d}H/\mathrm{d}u > 0$ for u large and negative, and since we would also have $\lim_{u\to-\infty} H = 0$, this would force $H > 0$ for u large and negative, which is in contradiction with the assertion above.

Thus the behaviour when $u \to -\infty$ of the solution cannot be of type α-bounded. Also, it cannot be that of type α-r-infinite, or else we would have $E(u) \leqslant 1$ for any $u \in \mathbb{R}$, which is impossible. Since $H(u) < 0$ for any $u \in \mathbb{R}$, it cannot be of type α-singular. It is thus of type α-θ-infinite, and the lemma is proved. $\square$

As a consequence, any solution whose behaviour when $u \to -\infty$ is not of type α-θ-infinite necessarily satisfies: $(r, \theta, z) \to (+\infty, (\pi/2)[2\pi], 0)$ when $u \to +\infty$ (ω-r-infinite type). This finishes proving the main assertions of theorem 5 (the only assertion that remains to be proved is that, for solutions of type α-singular, $\theta \to 0$ when $u \to -\infty$; it will be proved in section 5).

7.6. Proof of proposition 2

Let us consider a solution $u \mapsto (r, \theta, z)$ as in proposition 2, i.e. lying in the invariant subspace $\{r > 0\}$, which is not of type α-r-infinite, which does not satisfy $(\theta, z) \equiv ((\pi/2)[\pi], 0)$, and such that $z(u) > 0$ for u large and negative.

If this solution is of type ω-bounded, then we have seen in the proof of lemma 7 that $z > 0$ and $H < 0$ (and thus $\mathrm{d}\theta/\mathrm{d}u < 0$) for any $u \in \mathbb{R}$, and this proves assertion 2 of the proposition. If this solution is of type α-θ-infinite, then assertion 3 of the proposition follows from the proof of lemma 6 (we are in the case B2).

It remains to prove assertion 1. Suppose the solution is of type α-bounded or α-singular. We know that $(r, \theta, z) \to (+\infty, (\pi/2)[2\pi], 0)$ when $u \to +\infty$, and that we never have $(\theta, z) \equiv ((\pi/2)[2\pi], 0)$. The equation governing (θ, z) is a small perturbation of the integrable system (8), in the neighbourhood of an elliptic point, and this makes the trajectory cross the plane $\{z = 0\}$ an infinity of times (the precise asymptotics is described by theorem 4).

Thus there exists $u_0 \in \mathbb{R}$ such that $z(u_0) = 0$ and $z(u) > 0$ for $u < u_0$, and we can suppose without loss of generality (up to a translation in the variable u) that $u_0 = 0$. Then according to (19), if $H(u_1) > 0$ for some $u_1 < 0$, then $H(u) > 0$ for any $u \in [u_1; 0[$. If the solution is of type α-singular, then we have $H(u) > 0$ on $]-\infty; 0[$. If the solution is of type α-bounded, then, we have seen in the proof of lemma 7 that $H(u) > 0$ for u large and negative, thus $H(u) > 0$ on $]-\infty; 0[$. In both cases, $H > 0$ yields $\cos\theta > 0$ and $\mathrm{d}z/\mathrm{d}u > 0$ for $u \in]-\infty; 0[$, which proves assertion 1. $\square$

8. Miscellaneous

8.1. Shape at the tip of pendent drops and at the top of sessile drops

There is a large body of literature on asymptotic expansions in capillarity (see, e.g., [11]). For the sake of completeness, we rapidly recall below how to compute up to any order the asymptotic expansion of the profile of a pendent drop at its tip or of a sessile drop at its top. For the singular drop of infinite length, an asymptotic expansion of z in powers of r (when $r \to 0, z \to +\infty$) was given by Concus and Finn [3] (for an asymptotic expansion of r in powers of z^{-1} see [14]).

Tip of a pendent drop. Let us consider a solution $u \mapsto (r, \theta, z)$ of equation (4), lying in the invariant subspace $\{r > 0\}$, and satisfying $(r, \theta, z) \to (0, \pi/2, \bar{z})$ when $u \to -\infty$ with $\bar{z} > 0$ (pendent drop of finite length).

Close to the equilibrium $(0, \pi/2, \bar{z})$ (corresponding to the tip of the drop), the unstable manifold of this equilibrium (containing the trajectory of the solution we are considering) can be parametrized by the variable r; this defines smooth functions $r \mapsto z(r)$ and $r \mapsto \theta(r)$ on a neighbourhood of 0. According to equation (3), these functions satisfy

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\cot\theta}{r} - \frac{z}{\sin\theta}, \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = -\cot\theta. \tag{20}$$

Let us write $\theta = \pi/2 - \beta$. Equation (20) becomes

$$\frac{\mathrm{d}\beta}{\mathrm{d}r} = -\frac{\tan\beta}{r} + \frac{z}{\cos\beta}, \tag{21i}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\tan\beta. \tag{21ii}$$

Let

$$\sum_{k=0}^{+\infty} \beta_k r^k \qquad \text{and} \qquad \sum_{k=0}^{+\infty} z_k r^k$$

denote the asymptotic expansions of the functions $r \mapsto z(r)$ and $r \mapsto \beta(r)$ at $r = 0$. We have $\beta_0 = 0$ and $z_0 = \bar{z}$. These asymptotic expansions are determined by equations (21i) and (21ii), and can be computed by induction.

At order 0, equation (21i) yields $\beta_1 = \bar{z}/2$ and equation (21ii) yields $z_1 = 0$. At order 1, equation (21i) yields $\beta_2 = 0$ and equation (21ii) yields $z_2 = -\bar{z}/4$. Let us suppose that $\beta_1, \ldots, \beta_k$ and $z_1, \ldots, z_k$ have been computed this way. Then β_{k+1} (resp. z_{k+1}) is uniquely determined by equation (21i) (resp. equation (21ii)) at order k.

In particular, we have

$$z(r) = \bar{z} - \frac{\bar{z}}{4} r^2 + \mathcal{O}(r^3)$$

(the curvature radius of the profile at $r = 0$ equals $2\bar{z}$). This is in coherence with the observation made in the introduction that the the shape of the drop close to its tip converges, when $\bar{z} \to +\infty$, and after a blowup of factor $\bar{z}$, towards a sphere of radius 2.

Top of a sessile drop. Let us suppose that the solution $u \mapsto (r, \theta, z)$ corresponds to a sessile drop, and that (r, θ, z) converges towards $(0, -\pi/2, \bar{z})$ when $u \to +\infty$, with $\bar{z} > 0$. Again, we can locally parametrize by r the stable manifold of the equilibrium $(0, -\pi/2, \bar{z})$. Writing $\theta = -\pi/2 + \beta$, equation (20) becomes

$$\frac{\mathrm{d}\beta}{\mathrm{d}r} = -\frac{\tan\theta}{r} + \frac{z}{\cos\beta}, \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = \tan\beta.$$

Again, the asymptotic expansions in r of β and z are uniquely determined by these equations, and can be computed by induction. This time we find

$$z(r) = \bar{z} + \frac{\bar{z}}{4} r^2 + \mathcal{O}(r^3).$$

8.2. Asymptotic shape of pendent drops when the height converges to 0

For any $\bar{z} \in \mathbb{R}$, there exists a unique (up to translation of u) solution $u \mapsto (r, \theta, z)$ of equation (4), lying in the invariant subspace $\{r > 0\}$, and satisfying $(r, \theta, z) \to (0, \pi/2, \bar{z})$ when $u \to -\infty$. The aim of this section is to study the asymptotic shape of this solution when $\bar{z} \to 0$.

For this solution we have $E(u) \to -1 + \bar{z}^2/2$ when $u \to -\infty$. We suppose that $\bar{z}$ is small. Then, since E cannot be smaller than -1, we see that $E(u)$ is close to -1 for all $u \in \mathbb{R}$, and as a consequence $(\theta, z) \simeq (\pi/2, 0)$ for all $u \in \mathbb{R}$. Thus we have $\mathrm{d}r/\mathrm{d}u > 0$ for all $u \in \mathbb{R}$, and the trajectory of our solution can be parametrized by r. Let $r \mapsto (\theta_{\bar{z}}(r), z_{\bar{z}}(r))$ denote the function $[0; +\infty[\to \mathbb{R}^2$ whose graph is this trajectory.

Let us write $\bar{z} = \varepsilon$, $\theta = \pi/2 + \varepsilon\beta$, and $z = \varepsilon\zeta$. Equation (4) gives

$$\frac{\mathrm{d}r}{\mathrm{d}u} = r\cos(\varepsilon\beta), \qquad \frac{\mathrm{d}\beta}{\mathrm{d}u} = -r\zeta - \frac{\sin(\varepsilon\beta)}{\varepsilon}, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}u} = r\frac{\sin(\varepsilon\beta)}{\varepsilon}. \tag{22}$$

This system converges, when $\varepsilon \to 0$, towards the system

$$\frac{\mathrm{d}r}{\mathrm{d}u} = r, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}u} = -r\zeta - \beta, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}u} = r\beta. \tag{23}$$

The set of equilibrium points of the limit system (23) is the set $\{(r, \beta, \zeta) | r = \beta = 0\}$. At any point of this set, the matrix of the linearization of the system reads

$$\begin{pmatrix} 1 & 0 & 0 \\ -\zeta & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $(-1, 0, 1)$. Thus, according to the centre-stable manifold theorem [8], any equilibrium point has a one-dimensional stable and a one-dimensional unstable manifold. The unstable direction is given by the vector $(1, -\zeta/2, 0)$; it is transverse to the invariant plane $\{r = 0\}$, and since $\mathrm{d}r/\mathrm{d}u \neq 0$ outside this plane, the unstable manifold of any equilibrium point can be parametrized by r.

Let $r \mapsto (\beta_1(r), \zeta_1(r))$ denote the function whose graph is the unstable manifold of the equilibrium point $(0, 0, 1)$ for the system (23). The following proposition gives the desired asymptotics.

Proposition 3. *The maps* $r \mapsto ((\theta_{\bar{z}}(r) - \pi/2)/\bar{z}, z_{\bar{z}}(r)/\bar{z})$ *converge towards the map* $r \mapsto (\beta_1(r), \zeta_1(r))$ *when* $\bar{z} \to 0$*, uniformly on any compact subset of* $[0; +\infty[$ *(convergence holds for any* C^k*-topology).*

Proof. Adding the equation $\mathrm{d}\varepsilon/\mathrm{d}u = 0$ to the system (22), we obtain a smooth differential equation on $\mathbb{R}^4$. Each point of the plane $\{r = 0, \beta = 0\}$ is an equilibrium point, having a one-dimensional stable (resp. unstable manifold), for this differential equation. The quantity ε is constant along any solution, and, for ε small enough, the unstable manifold of these equilibrium points can be parametrized by r. Viewed as functions of r, they depend smoothly (for the topology of uniform convergence over compact subsets of $\mathbb{R}$) on the equilibrium point to which they are attached (this follows from the centre-unstable manifold, [8]). The proposition follows. □

8.3. Local shape of long drops

According to theorem 1, if we consider the profile of a pendent drop of height $\bar{z}$, and if we zoom by a factor $\kappa\bar{z}$ (with $0 < \kappa \leqslant 1$) on a point of this profile situated at height $z = \kappa\bar{z}$, then the resulting picture converges, when $\bar{z} \to +\infty$, towards the profile corresponding to the periodic solution of equation (6) on the level set $\{H = \eta(\kappa)\}$. See the introduction for pictures displaying various such profiles, for different values of H.

8.4. The problem of self-intersections

We do not know if the profiles in the (r, z)-plane of finite pendent drops may display self-intersections. What we can say is that, if we consider the profile of a finite pendent drop, and if we cut this profile into two parts, corresponding to the parts of the trajectory before (resp. after) the first intersection with the plane $\{z = 0\}$, then each of these two parts, taken separately, does not display any self-intersection. Indeed, according to the results of section 7, the first part of the profile can be parametrized by z, while the second part can be parametrized by r.

It was proved by Concus and Finn [5] (see also Bidaut-Veron's paper [1]), that the profile of the (unique) singular infinite drop displays no self-intersection (more precisely the entire profile can be parametrized by r). We were not able to recover this result by the 'dynamical systems' approach. Nevertheless, using this result and the remark above, one can easily prove that the profiles of finite drops of sufficiently large length ($\bar{z}$ large) also display no self-intersection (we leave the details to the reader). On the other hand, we saw in section 8.2 that the profiles of pendent drops of sufficiently small length ($\bar{z}$ small) can be entirely parametrized by r, therefore again display no self-intersection.

8.5. Higher-dimensional case

In this section we briefly mention what the main definitions and computations stated before become in higher dimension. It is very likely that all previous results extend, up to these minor modifications, to the higher-dimensional case.

We consider an axisymmetric capillary hypersurface in $\mathbb{R}^{n+1}$, with $n \geqslant 2$ (the physical case treated in the previous sections corresponds to $n = 2$). We define the variables r, θ, z, and s as in section 2.1 (this time we have $r = \sqrt{x_1^2 + \cdots + x_n^2}$, where $(x_1, \ldots, x_n)$ denote the 'horizontal' coordinates). We still have $\mathrm{d}r = \sin\theta\,\mathrm{d}s$ and $\mathrm{d}z = \cos\theta\,\mathrm{d}s$. The balance between pressure and surface tension reads

$$\Delta P = \Gamma\left(-\frac{\mathrm{d}\theta}{\mathrm{d}s} + (n-1)\frac{\cos\theta}{r}\right).$$

We have $\Delta P(z) = \Delta P(z = 0) + \rho g z$, and an adequate choice of the origin of z and of the length scale leads to $\Delta P(z) = \Gamma z$ and to the system

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = (n-1)\left(\frac{\cos\theta}{r} - z\right), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\cos\theta.$$

The change of time $\mathrm{d}s = r\,\mathrm{d}u$ gives

$$\frac{\mathrm{d}r}{\mathrm{d}u} = r\sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}u} = (n-1)(\cos\theta - rz), \qquad \frac{\mathrm{d}z}{\mathrm{d}u} = -r\cos\theta. \tag{24}$$

Equilibrium points of this equation are the points (r, θ, z) satisfying $r = 0$ and $\theta = \pm(\pi/2)$ $[2\pi]$. Each of these points has a one-dimensional stable (resp. unstable) manifold. When $r \to \pm\infty$, expressions governing $\mathrm{d}\theta/\mathrm{d}s$ and $\mathrm{d}z/\mathrm{d}s$ converge towards

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -(n-1)z, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\cos\theta,$$

which yields the second-order equation

$$\frac{\mathrm{d}^2\theta}{(n-1)^2\mathrm{d}s^2} = \frac{\cos\theta}{n-1};$$

thus the quantity

$$E = \frac{z^2}{2} - \frac{\sin\theta}{n-1}$$

is asymptotically conserved when $r \to \pm\infty$. Along a solution of equation (24), we have

$$\frac{\mathrm{d}E}{\mathrm{d}u} = -\cos^2\theta.$$

The change of variable and time $R = zr$ and $\mathrm{d}t = z\,\mathrm{d}s$ gives

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta - \frac{R}{z^2}\cos\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = (n-1)\left(\frac{\cos\theta}{R} - 1\right), \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{\cos\theta}{z}.$$

The asymptotic differential equation (when $z \to +\infty$) governing R and θ reads

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sin\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = (n-1)\left(\frac{\cos\theta}{R} - 1\right) \tag{25}$$

and admits the conserved quantity

$$H(R,\theta) = R^{n-1}\left(\cos\theta - \frac{n-1}{n}R\right).$$

Let us still denote by H the function $(r,\theta,z) \mapsto H(zr,\theta)$, and by $H(u)$ its values along a solution $u \mapsto (r,\theta,z)$ of equation (24). Then we have

$$H = (zr)^{n-1}\left(\cos\theta - \frac{n-1}{n}zr\right) \qquad \text{and} \qquad \frac{\mathrm{d}H}{\mathrm{d}u} = (n-1)r^n z^{n-2}\cos\theta(zr - \cos\theta).$$

In particular,

$$r > 0 \qquad z > 0 \qquad \text{and} \qquad H = 0 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}u} > 0.$$

Let

$$\mathcal{D} = \left\{(R,\theta) | -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{and} \quad 0 < R < \frac{n}{n-1}\cos\theta\right\}.$$

For $(R,\theta) \in \mathcal{D}$, we have $0 < H(R,\theta) < 1/n$ and $H(R,\theta) = 1/n \Leftrightarrow (R,\theta) = (1,0)$. The point $(1,0)$ is the only critical point of $H(.,.)$ and the only equilibrium point of the differential equation (25) in $\mathcal{D}$. The matrix of the linearization at the point $(1,0)$ of this differential equation reads

$$\begin{pmatrix} 0 & 1 \\ -(n-1) & 0 \end{pmatrix}$$

(this point is thus an 'elliptic' point).

All solutions of equation (25) in $\mathcal{D}$ are periodic (except the fixed point $(1,0)$) and they are parametrized by the value in $]0; 1/n[$ taken by $H(.,.)$ on their trajectories. For $a \in\,]0; 1/n[$, let us write

$$I_n(a) = \int (n-1)R^n\cos\theta(R - \cos\theta)\,\mathrm{d}t, \qquad J_n(a) = \int \cos\theta\,\mathrm{d}t,$$

where the integrals are taken over one period of the periodic solution of (25) corresponding to the value a for $H(.,.)$. One finds that

$$I_n(a) = 2\int_{(na)^{1/n}}^{1} \frac{(R_2^n - R_1^n)u}{\sqrt{1-u^2}}\,\mathrm{d}u,$$

where $R_1 < R_2$ denote the two roots of the equation $R^{n-1}(u - ((n-1)/n)R) = a$. This equation shows that $I_n(a) > 0$ for any $a \in\,]0; 1/n[$. The averaged differential equation for H reads

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{1}{z}\frac{I_n(H)}{J_n(H)}.$$

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References

[1] Bidaut-Veron M-F 1986 Global existence and uniqueness results for singular solutions of the capillarity equation *Pacific J. Math.* **125** 317–33

[2] Bidaut-Veron M-F 1996 Rotationally symmetric hypersurfaces with prescribed mean curvature *Pacific J. Math.* **173** 29–67

[3] Concus P and Finn R 1975 A singular solution of the capillary equation: I. Existence *Invent. Math.* **29** 143–8

[4] Concus P and Finn R 1975 A singular solution of the capillary equation: II. Uniqueness *Invent. Math.* **29** 149–60

[5] Concus P and Finn R 1979 The shape of a pendent liquid drop *Phil. Trans. R. Soc.* A **292** 307–40

[6] Finn R 1986 Equilibrium capillary surfaces *Grundlehren der Mathematischen Wissenschaften* vol 284 (New York: Springer)

[7] Finn R 1992 Green's identities and pendent liquid drops, I *Dev. in Part. Diff. Equ. and Appl. to Math. Phys.* ed G Buttazzo *et al* (New York: Plenum)

[8] Kelley A 1967 The stable, center stable, center, center unstable and unstable manifolds *J. Diff. Eqns* **3** 546–70

[9] Lord Kelvin (Sir W Thomson) 1986 *Popular Lectures and Addresses, I* (London: The Royal Institution) p 1

[10] Laplace P S 1805 *Méchanique Céleste* suppl. au livre X (Paris: Courcier)

[11] Miersemann E 1994 Asymptotic formulas for small sessile drops *Z. Anal. Anwend.* **13** 209–31

[12] Nickolov R 2001 Uniqueness of the singular solution to the capillary equation *Indiana Univ. Math. J.* **50** 4

[13] Pitts E 1976 *J. Inst. Math. Appl.* **17** 387

[14] Risler E 2001 Uniqueness of the pendent drop of infinite length *Preprint* INLN

[15] Young T 1805 *Phil. Trans. R. Soc.* **95** 65