Global behaviour of bistable solutions for hyperbolic gradient systems in one unbounded spatial dimension

Emmanuel RISLER*

Univ Lyon, INSA de Lyon,
CNRS UMR 5208, Institut Camille Jordan,
20 avenue Albert Einstein, F-69621 Villeurbanne CEDEX, France

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This paper is concerned with damped hyperbolic gradient systems of the form

$$\alpha u_{tt} + u_t = -\nabla V(u) + u_{xx},$$

where spatial domain is the whole real line, state-parameter $u$ is multidimensional, $\alpha$ is a positive quantity, and the potential $V$ is coercive at infinity. For such systems, under generic assumptions on the potential, the asymptotic behaviour of every bistable solution — that is, every solution close at both ends of space to spatially homogeneous stable equilibria — is described. Every such solutions approaches, far to the left in space a stacked family of bistable fronts travelling to the left, far to the right in space a stacked family of bistable fronts travelling to the right, and in between relaxes towards stationary solutions. In the absence of maximum principle, the arguments are purely variational. This extends previous results obtained in companion papers about the damped wave equation or parabolic gradient systems, in the spirit of the program initiated in the late seventies by Fife and MacLeod about the global asymptotic behaviour of bistable solutions.

*http://math.univ-lyon1.fr/~erisler/
1 Introduction

This paper deals with the global dynamics of nonlinear hyperbolic systems of the form

\[ \alpha u_{tt} + u_t = -\nabla V(u) + u_{xx}, \]

where time variable \( t \) and space variable \( x \) are real, spatial domain is the whole real line, the function \( (x,t) \mapsto u(x,t) \) takes its values in \( \mathbb{R}^n \) with \( n \) a positive integer, \( \alpha \) is a positive quantity, and the nonlinearity is the gradient of a scalar potential function \( V: \mathbb{R}^n \to \mathbb{R} \), which is assumed to be regular (of class at least \( C^3 \)) and coercive at infinity (see hypothesis (H\_coerc) in subsection 2.1 on the following page).

The aim of this paper is to extend to hyperbolic systems of the form (1) the results describing the global asymptotic behaviour of bistable solutions obtained in [14, 15] for parabolic systems of the form

\[ u_t = -\nabla V(u) + u_{xx}. \]

As was already observed by several authors, the long-time asymptotics of solutions of the two systems (1) and (2) present strong similarities, see [5] and references therein. The common feature of these two systems that will be extensively used in this paper is the existence – at least formally – of an energy functional, not only for solutions considered in the laboratory frame (at rest), but also for solutions considered in every frame travelling at a constant speed.

If \((v, w)\) is a pair of vectors of \( \mathbb{R}^n \), let \( v \cdot w \) and \( |v| = \sqrt{v \cdot v} \) denote the usual Euclidean scalar product and the usual Euclidean norm, respectively, and let us write simply \( v^2 \) for \( |v|^2 \). If \( (x,t) \mapsto u(x,t) \) is a solution of system (1), the (formal) energy of the solution reads

\[ \mathcal{E}[u(\cdot, t)] = \int_{\mathbb{R}} \left( \alpha \frac{u_t(x,t)^2}{2} + \frac{u_x(x,t)^2}{2} + V(u(x,t)) \right) dx, \]

and its time derivative reads, at least formally,

\[ \frac{d}{dt} \mathcal{E}[u(\cdot, t)] = -\int_{\mathbb{R}} u_t(x,t)^2 \, dx \leq 0. \]

In the parabolic case \( \alpha = 0 \), the same properties hold with the same expression for the energy (the inertial term involving \( \alpha \) vanishes); by the way, an additional feature in this case is the fact that the parabolic system (2) is nothing but the (formal) gradient of energy functional (3) (this does not hold for hyperbolic system (1)).

A striking feature of both systems (1) and (2) is the fact that a formal (Lyapunov) energy functional exists not only in the laboratory frame, but also in every frame travelling at a constant speed (see sub-subsection 3.3.2 on page 14 and specifically equality (14)).

In the parabolic case, this is known for long and was in particular used by P. C. Fife and J. MacLeod tot prove global convergence towards bistable fronts and to study the global behaviour of bistable solutions in the scalar case \( n = 1 \), [2,4]. More recently, this property received a detailed attention from several authors [6,9,13], and it was shown that...
this structure is sufficient (in itself, that is without the use of the maximum principle) to prove results of global convergence towards travelling fronts. In the hyperbolic case, a similar strategy was successfully applied by Th. Gallay and R. Joly in the scalar case \( n = 1 \) to prove global stability of travelling fronts for a bistable potential. \[5\].

By similar arguments, a full description of the global asymptotic behaviour of every bistable solution was recently obtained for parabolic systems, \[14, 15\]. Roughly speaking, such a solution must approach:

- far to the right a stacked family of fronts travelling to the right,
- far to the left a stacked family of fronts travelling to the left,
- in between a pattern made of bistable stationary solutions (possibly a single homogeneous stable equilibrium) getting slowly away from one another.

The aim of this paper is to extend this result to the case of hyperbolic systems of the form \( (1) \) (Theorem 1 on page 10). This will also provide an extension of the global stability result obtained by Gallay and Joly in the scalar case \( n = 1 \), \[5\].

2 Assumptions, notation, and statement of the results

2.1 Semi-flow in uniformly local Sobolev space and coercivity hypothesis

Let us assume that the potential function \( V : \mathbb{R}^n \to \mathbb{R} \) is of class \( C^k \) where \( k \) is an integer not smaller than 3, and that this potential function is strictly coercive at infinity in the following sense:

\[(H_{\text{coerc}}) \lim_{R \to +\infty} \inf_{|u| \geq R} \frac{u \cdot \nabla V(u)}{|u|^2} > 0 \]

(or in other words there exists a positive quantity \( \varepsilon \) such that the quantity \( u \cdot \nabla V(u) \) is larger than \( \varepsilon |u|^2 \) as soon as \( |u| \) is sufficiently large).

System \( (1) \) defines a local semi-flow on the uniformly local energy space \( H^1_{ul}(\mathbb{R}, \mathbb{R}^n) \times L^2_{ul}(\mathbb{R}, \mathbb{R}^n) \), and, according to hypothesis \( (H_{\text{coerc}}) \), this semi-flow is actually global (see Proposition 2 on page 12).

2.2 First generic hypothesis on the potential: critical points are nondegenerate

The results of this paper require several generic hypotheses on the potential \( V \). The simplest of those hypotheses is:

\[(H_{\text{non-deg}}) \quad \text{Every critical point of } V \text{ is nondegenerate.} \]
In other words, for all \( u \) in \( \mathbb{R}^n \), if \( \nabla V(u) \) vanishes, then the Hessian \( D^2V(u) \) possesses no vanishing eigenvalue. As a consequence, in view of hypothesis (H\textsubscript{coerc}), the number of critical points of \( V \) is finite. Everywhere in this paper, the term “minimum point” denotes a point where a function — namely the potential \( V \) — reaches a local or global minimum.

**Notation.** Let \( \mathcal{M} \) denote the set of (nondegenerate, local or global) minimum points of \( V \):
\[
\mathcal{M} = \{ u \in \mathbb{R}^n : \nabla V(u) = 0 \text{ and } D^2V(u) \text{ is positive definite} \}.
\]

### 2.3 Bistable solutions

Our targets are bistable solutions, let us recall their definition already stated in \([15]\).

**Definition.** A solution \( (x,t) \mapsto u(x,t) \) of system (1) is called a bistable solution if there are two (possibly equal) points \( m^- \) and \( m^+ \) in \( \mathcal{M} \) such that the quantities:
\[
\limsup_{x \to -\infty} |u(x,t) - m^-| \quad \text{and} \quad \limsup_{x \to +\infty} |u(x,t) - m^+|
\]
both approach 0 when time approaches \(+\infty\). More precisely, such a solution is called a bistable solution connecting \( m^- \) to \( m^+ \) (see figure 1).

![Figure 1: A bistable solution connecting \( m^- \) to \( m^+ \).](image)

### 2.4 Stationary solutions and travelling fronts: definition and notation

Let \( c \) be a real quantity. A function
\[
\phi : \mathbb{R} \to \mathbb{R}^n, \quad \xi \mapsto \phi(\xi)
\]
is the profile of a wave travelling at speed \( c \) (or is a stationary solution if \( c \) vanishes) for the parabolic system (2) if the function \( (x,t) \mapsto \phi(x - ct) \) is a solution of this system, that is if \( \phi \) is a solution of the differential system
\[
\phi'' = -c\phi' + \nabla V(\phi).
\]

In this case, for every real quantity \( x_0 \), the function
\[
(x,t) \mapsto \phi(\sqrt{1 + \alpha c^2 x - ct - x_0})
\]
is a solution of the hyperbolic system (1), more precisely a wave travelling at the physical speed $s$ related to the parabolic speed $c$ by:

$$s = \frac{c}{\sqrt{1 + \alpha c^2}} \iff c = \frac{s}{\sqrt{1 - \alpha s^2}}.$$ 

System (5) can be viewed as a damped oscillator (or a conservative oscillator if $c$ vanishes) in the potential $-V$, the speed $c$ playing the role of the damping coefficient.

**Notation.** If $u_-$ and $u_+$ are critical points of $V$ (and $c$ is a real quantity), let $\Phi_c(u_-, u_+)$ denote the set of nonconstant solutions of system (5) connecting $u_-$ to $u_+$. With symbols,

$$\Phi_c(u_-, u_+) = \{ \phi : \mathbb{R} \to \mathbb{R}^n : \phi \text{ is a nonconstant solution of system (5)}$$

and $\phi(\xi) \xrightarrow{\xi \to -\infty} u_-$ and $\phi(\xi) \xrightarrow{\xi \to +\infty} u_+ \}$. 

### 2.5 Breakup of space translation invariance for stationary solutions and travelling fronts

Due to space translation invariance, nonconstant solutions of system (5) go by one-parameter families. For various reasons, it is convenient to pick a single “representative” in each of these families. This is done through the next definitions.

Let $\lambda_{\text{min}}$ ($\lambda_{\text{max}}$) denote the minimum (respectively, maximum) of all eigenvalues of the Hessian matrices of the potential $V$ at (local) minimum points. In other words, if $\sigma(D^2V(u))$ denotes the spectrum of the Hessian matrix of $V$ at a point $u$ in $\mathbb{R}^n$,

$$\lambda_{\text{min}} = \min_{m \in \mathcal{M}} \min \sigma(D^2V(m)) \quad \text{and} \quad \lambda_{\text{max}} = \max_{m \in \mathcal{M}} \max \sigma(D^2V(m))$$

(recall that the set $\mathcal{M}$ is finite). Obviously,

$$0 < \lambda_{\text{min}} \leq \lambda_{\text{max}} < +\infty.$$ 

**Notation.** For the remaining of this paper, let us fix a positive quantity $d_{\text{Esc}}$, sufficiently small so that, for every (local) minimum point $m$ of $V$ and for all $u$ in $\mathbb{R}^n$ satisfying $|u - m| \leq d_{\text{Esc}}$, every eigenvalue $\lambda$ of $D^2V(u)$ satisfies:

$$\frac{\lambda_{\text{min}}}{2} \leq \lambda \leq 2\lambda_{\text{max}}.$$ 

It is well known (see for instance [13, 15] for a proof of this elementary result) that very nonconstant stationary solution of system (5), connecting two points of $\mathcal{M}$, “escapes” at least at distance $d_{\text{Esc}}$ from each of these two points (whatever the value of the speed $c$ and even if these two points are equal) at some position of space (see figure 2). Thus, for $c$ in $\mathbb{R}$ and $(m_-, m_+)$ in $\mathcal{M}^2$, we may consider the set of normalized bistable fronts/stationary solutions connecting $m_-$ to $m_+$ (see figure 3):

$$\Phi_{c,\text{norm}}(m_-, m_+) = \{ \phi \in \Phi_c(m_-, m_+) : |\phi(0) - m_+|_D = d_{\text{Esc}}$$

and $|\phi(\xi) - m_+|_D < d_{\text{Esc}}$ for all $\xi > 0 \}$. 

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Figure 2: Every function in $\Phi_c(m_-, m_+)$ (that is, stationary in a frame travelling at a zero or nonzero speed and connecting two minimum points and nonconstant) escapes at least at distance $d_{\text{Esc}}$ of these minimum points.

Figure 3: Normalized stationary solution.
2.6 Additional generic hypotheses on the potential

The main result of this paper (Theorem 1 below) requires additional generic hypotheses on the potential $V$, that will now be stated. A formal proof of the genericity of these hypotheses is scheduled (work in progress by Romain Joly and the author).

($\text{H}_{\text{bist}}$) Every front travelling at a nonzero speed and invading a stable equilibrium (a minimum point of $V$) is bistable.

In other words, for every minimum point $m_+$ in $\mathcal{M}$, every critical point $u_-$ of $V$, and every positive quantity $c$, if the set $\Phi_c(u_-, m_+)$ is nonempty, then $u_-$ must belong to $\mathcal{M}$. As a consequence of this hypothesis, only bistable travelling fronts will be involved in the asymptotic behaviour of bistable solutions.

The statement of the two remaining hypotheses requires the following notation.

Notation. If $m_+$ is a point in $\mathcal{M}$ and $c$ is a positive quantity, let $\Phi_c(m_+)$ denote the set of fronts travelling at speed $c$ and “invading” the equilibrium $m_+$ (note that according to hypothesis ($\text{H}_{\text{bist}}$) all these fronts are bistable), and let us define similarly $\Phi_{c,\text{norm}}(m_+)$. With symbols,

$$\Phi_c(m_+) = \bigcup_{m_- \in \mathcal{M}} \Phi_c(m_-, m_+)$$

and

$$\Phi_{c,\text{norm}}(m_+) = \bigcup_{m_- \in \mathcal{M}} \Phi_{c,\text{norm}}(m_-, m_+).$$

The two additional generic hypotheses that will be made on $V$ are the following.

($\text{H}_{\text{disc-c}}$) For every $m_+$ in $\mathcal{M}$, the set:

$$\{ c \in (0, +\infty) : \Phi_c(m_+ \neq \emptyset) \}$$

has an empty interior.

($\text{H}_{\text{disc-}\Phi}$) For every minimum point $m_+$ in $\mathcal{M}$ and every positive quantity $c$, the set

$$\{(\phi(0), \phi'(0)) : \phi \in \Phi_{c,\text{norm}}(m_+)\}$$

is totally discontinuous — if not empty — in $\mathbb{R}^{2n}$. That is, its connected components are singletons. Equivalently, the set $\Phi_{c,\text{norm}}(m_+)$ is totally disconnected for the topology of compact convergence (uniform convergence on compact subsets of $\mathbb{R}$).

In these two last definitions, the subscript “disc” refers to the concept of “discontinuity” or “discreteness”.

Finally, let us define the following “group of generic hypotheses”:

($\text{G}$) ($\text{H}_{\text{non-deg}}$) and ($\text{H}_{\text{bist}}$) and ($\text{H}_{\text{disc-c}}$) and ($\text{H}_{\text{disc-}\Phi}$).
Figure 4: Propagating terrace of (bistable) fronts travelling to the right ($\sigma_i$ denotes the “physical” speed corresponding to $c_i$, that is: $\sigma_i = c_i / \sqrt{1 + \alpha c_i^2}$).

2.7 Propagating terraces of bistable solutions

This subsection is devoted to the next definition. Its purpose is to enable a compact formulation of the main result of this paper (Theorem 1 below). Some comments on the terminology and related references are given at the end of this subsection.

Definition (propagating terrace of bistable fronts, figure 4). Let $m_-$ and $m_+$ be two minimum points of $V$ (satisfying $V(m_-) \leq V(m_+)$). A function $T : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$, $(x,t) \mapsto T(x,t)$ is called a propagating terrace of bistable fronts travelling to the right, connecting $m_-$ to $m_+$, if there exists a nonnegative integer $q$ such that:

1. if $q$ equals 0, then $m_- = m_+$ and, for every real quantity $x$ and every nonnegative time $t$,
   $$T(x,t) = m_- = m_+ ;$$

2. if $q$ equals 1, then there exist
   - a positive quantity $c_1$
   - and a function $\phi$ in $\Phi_{c_1}(m_-, m_+)$ (that is, the profile of a bistable front travelling at parabolic speed $c_1$ and connecting $m_-$ to $m_+$)
   - and a $C^1$-function $t \mapsto x_1(t)$, defined on $\mathbb{R}_+$, and such that $x_1'(t)$ approaches the quantity $c_1 / \sqrt{1 + \alpha c_1^2}$ (the corresponding physical speed) when $t$ approaches $+\infty$

   such that, for every real quantity $x$ and every nonnegative time $t$,
   $$T(x,t) = \phi\left[\sqrt{1 + \alpha c_1^2}(x - x_1(t))\right] ;$$
3. If $q$ is not smaller than 2, then there exists $q - 1$ minimum points $m_1, \ldots, m_{q-1}$ of $V$, satisfying (if we denote $m_+$ by $m_0$ and $m_-$ by $m_q$)

$$V(m_0) > V(m_1) > \cdots > V(m_q),$$

and there exist $q$ positive quantities $c_1, \ldots, c_q$ satisfying:

$$c_1 \geq \cdots \geq c_q,$$

and for each integer $i$ in $\{1, \ldots, q\}$, there exist:

- a function $\phi_i$ in $\Phi_{c_i}(m_i, m_{i-1})$ (that is, the profile of a bistable front travelling at parabolic speed $c_i$ and connecting $m_i$ to $m_{i-1}$)
- and a $C^1$-function $t \mapsto x_i(t)$, defined on $\mathbb{R}_+$, and such that $x_i'(t)$ approaches the quantity $c_i/\sqrt{1 + \alpha c_i^2}$ (the corresponding physical speed) when $t$ approaches $+\infty$,

such that, for every integer $i$ in $\{1, \ldots, q-1\}$,

$$x_{i+1}(t) - x_i(t) \to +\infty \quad \text{when} \quad t \to +\infty,$$

and such that, for every real quantity $x$ and every nonnegative time $t$,

$$T(x, t) = m_0 + \sum_{i=1}^{q} \left( \phi_i \left[ \sqrt{1 + \alpha c_i^2} (x - x_i(t)) \right] - m_{i-1} \right).$$

Obviously, item 2 may have been omitted in this definition, since it fits with item 3 with $q$ equals 1.

A propagating terrace of bistable fronts travelling to the left may be defined similarly.

The terminology “propagating terrace” was introduced by A. Ducrot, T. Giletti, and H. Matano in [1] (and subsequently used by P. Poláčik, [10–12]) to denote a stacked family (a layer) of travelling fronts in a (scalar) reaction-diffusion equation. This led the author to keep the same terminology in the present context. This terminology is convenient to denote objects that would otherwise require a long description. It is also used in the companion papers [14, 16]. We refer to [14] for additional comments on this terminological choice.

To finish, observe that in the present context terraces are only made of bistable fronts, by contrast with the propagating terraces introduced and used by the authors cited above; that (still in the present context) terraces are approached by solutions but are (in general) not solutions themselves; and that a propagating terrace may be nothing but a single stable homogeneous equilibrium (when $q$ equals 0) or may involve a single travelling front (when $q$ equals 1).

### 2.8 Main result (approach to terraces of bistable fronts and relaxation behind)

The following theorem, illustrated by figure 5 is the main result of this paper.
Theorem 1 (approach to propagating terraces of bistable fronts and relaxation behind). Assume that \( V \) satisfies the coercivity hypothesis (\( H_{\text{coerc}} \)) and the generic hypotheses (G). Then, for every bistable solution \((x,t) \mapsto u(x,t)\) of system \([1]\) there exist:

- a propagating terrace \( T_{\text{left}} \) of bistable fronts travelling to the left,
- a propagating terrace \( T_{\text{right}} \) of bistable fronts travelling to the right,

such that, for every sufficiently small positive quantity \( \varepsilon \), the following limits hold:

\[
\sup_{x \in (-\infty, -\varepsilon t]} |u(x,t) - T_{\text{left}}(x,t)| \to 0
\]

and
\[
\sup_{x \in [\varepsilon t, +\infty)} |u(x,t) - T_{\text{right}}(x,t)| \to 0
\]

and
\[
\sup_{x \in [-\varepsilon t, \varepsilon t]} \int_{x}^{x+1} u_t(z,t)^2 \, dz \to 0
\]

when \( t \) approaches \(+\infty\).

In this statement the convergence towards the two propagating terraces is expressed with a uniform norm, but it follows from the proof that the same limits holds for the uniformly local \( H^1_{ul} \times L^2_{ul} \)-norm on the intervals \((-\infty, -\varepsilon t]\) and \([\varepsilon t, +\infty)\).

This statement is quite similar to the main result (Theorem 1) of \([14]\), at least with respect to the convergence towards the two propagating terraces of bistable fronts traveling to the left and to the right. However, by contrast, it is much less precise concerning the behaviour of the solution between these two propagating terraces: as illustrated by figure \([5]\) the approach to a “standing terrace” in the center area behind these two terraces is missing, and it is solely stated that the time derivative of the solution goes to zero in this area. This is quite unsatisfactory and due to a technical obstacle that I have not been able to overcome, as will be explained in section \([7]\) on page \([72]\).

2.9 Nonnegative residual asymptotic energy

The next proposition provides an extension (still unsatisfactory however) to the conclusions of Theorem \([1]\) concerning the behaviour of the solution in the center area (the
notation is illustrated on figure 5).

**Proposition 1** (nonnegative residual asymptotic energy). Let us assume that all the hypotheses of Theorem 1 hold. Then, with the same notation, if we denote by \( m_{\text{left}} \) and \( m_{\text{right}} \) the two local minimum points of \( V \) such that the solution \( (x,t) \mapsto u(x,t) \) connects \( m_{\text{left}} \) to \( m_{\text{right}} \), and if we denote by \( m_{\text{center-left}} \) and \( m_{\text{center-right}} \) the two local minimum points of \( V \) such that the terrace \( T_{\text{left}} \) connects \( m_{\text{left}} \) to \( m_{\text{center-left}} \) and the terrace \( T_{\text{right}} \) connects \( m_{\text{center-right}} \) to \( m_{\text{right}} \), then

\[
V(m_{\text{center-left}}) = V(m_{\text{center-right}}),
\]

and there exists a nonnegative quantity \( \mathcal{E} \) ("residual asymptotic energy") such that, if we denote by \( h \) the quantity \( V(m_{\text{center-left}}) = V(m_{\text{center-right}}) \), then for every sufficiently small positive quantity \( \varepsilon \) the following limit holds:

\[
\int_{-\varepsilon t}^{\varepsilon t} \left( \frac{u_x(x,t)^2}{2} + V(u(x,t)) - h \right) \, dx \to \mathcal{E}
\]

when \( t \) approaches \( +\infty \).

### 2.10 Additional questions

The main question about the weakness of the conclusions concerning the behaviour of the solution in the "center" area will be addressed in section 7 on page 72. Besides, let us briefly mention some other questions that are naturally raised by this result; analogous questions were already discussed in [14, 15], where additional comments can be found.

- Does the correspondence between a solution and its asymptotic pattern display some form of regularity? (we refer to [14] for known results and comments on this question, in the parabolic case).
- Does Theorem 1 hold without hypothesis (H\( \text{disc-c} \))?
- Provide quantitative estimates on the rate of convergence of a solution towards its asymptotic pattern.

### 2.11 Organization of the paper

The organization of this paper closely follows that of the companion paper [14] where the parabolic case is treated.

- The next section 3 is devoted to some preliminaries (existence of solutions, asymptotic compactness, preliminary computations on spatially localized functionals, notation).
- Proof of Theorem 1 is mainly based on two propositions: Propositions 4 and 9. Proposition 4 "invasion implies convergence" is the main step and is proved in section 4 on page 72. Proposition 9 "non-invasion implies relaxation" is proved in section 5 on page 58.
• These two propositions are combined together in section 6 on page 71, where the proof of Theorem 1 is completed.

• Section 7 on page 72 is devoted to some explanations for the lack of precise description of the asymptotic behaviour of the solution in the “center” area, behind (between) the terraces of fronts travelling to the right and to the left.

• Finally, elementary properties of the profiles of travelling fronts are recalled in section 8 on page 73.

3 Preliminaries

3.1 Global existence of solutions and attracting ball for the flow

Let us consider the functional space (uniformly local energy space)

\[ X = H^1_{ul}(\mathbb{R}, \mathbb{R}^n) \times L^2_{ul}(\mathbb{R}, \mathbb{R}^n). \]

The following proposition is stated and proved in [5] in the case \( n = 1 \). The proof is identical in the case of systems \( n > 1 \). In the statement of this proposition, existence of an attracting ball for the \( L^\infty \)-norm is redundant; the reason for this redundancy is that the radius \( R_{att, \infty} \) of an attracting ball for the \( L^\infty \)-norm will be explicitly used in several estimates.

**Proposition 2** (global existence of solutions and attracting ball). For all initial data \((u_0, u_1)\) in \( X \), system (1) has a unique solution global solution \( u \) in the space

\[ C^0([0, +\infty), H^1_{ul}(\mathbb{R}, \mathbb{R}^n)) \cap C^1([0, +\infty), L^2_{ul}(\mathbb{R}, \mathbb{R}^n)) \]

satisfying \( u(0) = u_0 \) and \( u_t(0) = u_1 \). In addition, there exist positive quantities \( R_{att, X} \) and \( R_{att, \infty} \) depending only on \( V \) and \( \alpha \) (radius of attracting balls for the \( X \)-norm and the \( L^\infty \)-norm, respectively), such that, for every sufficiently large quantity \( t \),

\[
\sqrt{\|u(\cdot, t)\|_{H^1_{ul}(\mathbb{R}, \mathbb{R}^n)}^2 + \|u_t(\cdot, t)\|_{L^2_{ul}(\mathbb{R}, \mathbb{R}^n)}^2} \leq R_{att, X}
\]

and

\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}, \mathbb{R}^n)} \leq R_{att, \infty}.
\]

3.2 Asymptotic compactness of the solutions

The following proposition reproduces Proposition 2.3 of [5].

**Proposition 3** (asymptotic compactness). For every solution \( u \in C^0([0, +\infty), H^1_{ul}(\mathbb{R}, \mathbb{R}^n)) \cap C^1([0, +\infty), L^2_{ul}(\mathbb{R}, \mathbb{R}^n)) \) of system (1) and for every sequence \((x_p, t_p)\) in \( \mathbb{R} \times [0, +\infty) \) such that \( t_p \) approaches \(+\infty\) when \( p \) approaches \(+\infty\), there exists a subsequence (still denoted by \((x_p, t_p)\)) and there exists a solution

\[ \bar{u} \in C^0(\mathbb{R}, H^1_{ul}(\mathbb{R}, \mathbb{R}^n)) \cap C^1(\mathbb{R}, L^2_{ul}(\mathbb{R}, \mathbb{R}^n)). \]
of the same system \((1)\) such that, for all positive quantities \(L\) and \(T\), both quantities
\[
\sup_{t \in [-T,T]} \| u(x_p + \cdot, t_p + t) - \bar{u}(\cdot, t) \|_{H^1([-L,L], \mathbb{R}^n)}
\]
and
\[
\sup_{t \in [-T,T]} \| u_t(x_p + \cdot, t_p + t) - \bar{u}_t(\cdot, t) \|_{L^2([-L,L], \mathbb{R}^n)}
\]
approach 0 when \(p\) approaches \(+\infty\).

3.3 Time derivative of (localized) energy and \(L^2\)-norm of a solution in a standing or travelling frame

Take \((u_0, u_1)\) in \(X\) and consider the solution \((x, t) \mapsto u(x, t)\) of system \((1)\) with initial data \((u_0, u_1)\).

3.3.1 Standing frame

As in [5], we are going to make an extensive use of two functionals, obtained by considering the scalar product of system \((1)\) either with \(u_t\) or with \(u\), and integrating with respect to time. This leads to the “energy” (Lagrangian):
\[
\int_{\mathbb{R}} \left( \frac{\alpha u_t(x,t)^2}{2} + \frac{u_x(x,t)^2}{2} + V(u(x,t)) \right) dx,
\]
and the following variant of the \(L^2\)-norm:
\[
\int_{\mathbb{R}} \left( \alpha u(x,t) \cdot u_t(x,t) + \frac{u(x,t)^2}{2} \right) dx
\]
(it would be the \(L^2\)-norm if we were considering the parabolic case \(\alpha\) equals zero, see [14, 15]).

In order to ensure the convergence of such integrals, it is necessary to localize the integrands. Let \(x \mapsto \psi(x)\) denote a function in the space \(W^{2,1}(\mathbb{R}, \mathbb{R})\) (that is a function belonging to \(L^1(\mathbb{R})\), together with its first and second derivatives). Then, the time derivatives of these two functionals — localized by \(\psi(x)\) — read:

\[
\frac{d}{dt} \int_{\mathbb{R}} \psi \left( \frac{\alpha u_t^2}{2} + \frac{u_x^2}{2} + V(u) \right) dx = -\int_{\mathbb{R}} \psi u_t^2 dx - \int_{\mathbb{R}} \psi u_t u_x dx.
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}} \psi \left( \alpha u \cdot u_t + \frac{u^2}{2} \right) dx = \int_{\mathbb{R}} \psi \left( -u \cdot \nabla V(u) - u_x^2 + \alpha u_t^2 \right) dx + \int_{\mathbb{R}} \psi'' u_t^2 dx.
\]

These two functionals can be appropriately combined in order to prove, say, the local stability of a homogeneous solution at a local minimum of the potential (assumed to be equal to 0 in \(\mathbb{R}^n\) and where the value of \(V\) is assumed to be zero). For this purpose, the combination must display two features: coercivity and decrease. If the weight of the
second functional is 1, then in order to ensure decrease (dissipation term), the weight of the first functional must be at least $\alpha$; assume it is $\alpha + \beta$, where $\beta$ is a nonnegative quantity to be chosen appropriately. Then, neglecting the terms involving the derivatives of $\psi$,

- with respect to the local coercivity, the combination is (roughly) bounded from below by the integral of an integrand made of $\psi$ times the following expression

$$\frac{\beta \alpha}{2} u_t^2 + \frac{\alpha + \beta}{2} u_x^2 + (\alpha + \beta)V(u),$$

- and with respect to the decrease, its derivative is (roughly) bounded from above by the integral of an integrand made of $\psi$ times the following expression

$$-\beta u_t^2 - u \cdot \nabla V(u) - u_x^2.$$

Thus a reasonable choice is to choose $\beta = \alpha$, or in other words to consider the following combined functional:

$$\int_{\mathbb{R}} \psi \left( \frac{\alpha^2 u_t^2}{2} + \alpha u_x^2 + 2\alpha V(u) + \alpha u \cdot u_t + \frac{u^2}{2} \right) \, dx.$$

### 3.3.2 Travelling frame

Let $c$ and $t_{\text{init}}$ and $x_{\text{init}}$ denote three real quantities (the “parabolic” speed, origin of time, and initial origin of space for the travelling frame, see figure 10 on page 30), with $t_{\text{init}}$ nonnegative. Usually, besides the parabolic speed $c$ in $(0, +\infty)$, it is convenient to define the physical speed $\sigma$ in $(0, 1/\sqrt{\alpha})$, these two speeds being related by:

$$\sigma = \frac{c}{\sqrt{1 + \alpha c^2}} \iff c = \frac{\sigma}{\sqrt{1 - \alpha \sigma^2}}.$$

Let us consider the function $(y, s) \mapsto v(y, s)$ (for every real quantity $y$ and nonnegative quantity $s$) defined by:

$$v(y, s) = u(x, t)$$

where $(y, s)$ and $(x, t)$ are related by:

$$t = t_{\text{init}} + s \quad \text{and} \quad x = x_{\text{init}} + \sigma s + \frac{y}{\sqrt{1 + \alpha c^2}} \iff y = \sqrt{1 + \alpha c^2} (x - x_{\text{init}}) - cs.$$

The evolution system for the function $(y, s) \mapsto v(y, s)$ reads:

$$\alpha v_{ss} + v_s - 2\alpha cv_{ys} = -\nabla V(v) + cv + v_{yy}.$$

Let us consider a function $(y, s) \mapsto \psi(y, s)$ such that, for every nonnegative quantity $s$, the function $y \mapsto \psi(y, s)$ belongs to $W^{2,1}(\mathbb{R}, \mathbb{R})$. As in [5], we are going to consider the natural analogues for the travelling frame of the two functionals considered above in a standing frame (again, they are obtained by considering the scalar product of system (11))
either with \( v_s \) or with \( v \) and integrating with respect to time). Their time derivatives read

\[
\frac{d}{ds} \int_{\mathbb{R}} \psi \left( \frac{\alpha v^2}{2} + \frac{v_y^2}{2} + V(v) \right) dy = \int_{\mathbb{R}} \psi_s \left( \frac{\alpha v^2}{2} + \frac{v_y^2}{2} + V(v) \right) dy
\]

\[
- \int_{\mathbb{R}} (\psi + \alpha c \psi_y) v_s^2 dy + \int_{\mathbb{R}} (c \psi - \psi_y) v_y \cdot v_s dy ,
\]

and

\[
\frac{d}{ds} \int_{\mathbb{R}} \psi \left( \alpha v \cdot v_s + \frac{v^2}{2} - 2 \alpha c v \cdot v_y \right) dy = \int_{\mathbb{R}} \psi_s \left( \alpha v \cdot v_s + \frac{v^2}{2} - 2 \alpha c v \cdot v_y \right)
\]

\[
+ \psi \left( - v \cdot \nabla V(v) - v_y^2 + \alpha v_s^2 - 2 \alpha c v \cdot v_s \right) + \frac{1}{2} (\psi_{yy} - c \psi_y v^2) \right] dy.
\]

If we replace \( \psi(y, s) \) by \( e^{cy} \) in equality (12), we obtain the following (formal) decrease of energy (be aware that integrals are not necessarily convergent):

\[
\frac{d}{ds} \int_{\mathbb{R}} e^{cy} \left( \frac{\alpha v^2}{2} + \frac{v_y^2}{2} + V(v) \right) dy = -(1 + \alpha c) \int_{\mathbb{R}} e^{cy} v_s^2 dy.
\]

Let us assume that the function \( \psi_{yy} - c \psi_y \) is small, that \( \psi_y \) is either small or close to \( cv \), and that \( \psi \) varies slowly with time, and let us again wonder what would be an appropriate combination of these two functionals, to recover altogether decrease and coercivity (provided that \( 0_{\mathbb{R}^n} \) is a local minimum of \( V \) where the value of \( V \) is zero). Assume that the weight of the second functional equals 1. Then the weight of the first functional must be at least equal to \( \alpha \) (to ensure decrease due to dissipation); assume it is \( \alpha + \beta \), where \( \beta \) is a nonnegative quantity to be chosen appropriately. Then, assuming that \( \psi \) varies slowly with time and neglecting terms that are small according to the assumptions on \( \psi \), it follows that:

- with respect to the coercivity, the combination is (roughly) bounded from below by the integral of an integrand made of \( \psi \) times the following expression

\[
\frac{\alpha \beta}{2} v_s^2 + \frac{\alpha + \beta}{2} v_y^2 + (\alpha + \beta) V(v) + \alpha c \frac{\psi_y}{\psi} v^2
\]

(the last term being either positive or close to 0),

- with respect to the decrease, the time derivative of the combination is bounded from above by the integral of an integrand made of \( \psi \) times the following expression

\[
(-\beta - (\alpha + \beta) \alpha c \frac{\psi_y}{\psi}) v_s^2 + \left( c(\beta - \alpha) - (\alpha + \beta) \frac{\psi_y}{\psi} \right) v_y \cdot v_s - v \cdot \nabla V(v) - v_y^2 .
\]

As in the case of a standing frame, it thus turns out that \( \beta = \alpha \) is a reasonable choice, and even that this choice is specially relevant here since it fires one of the terms in the
derivative (the term with the factor $\beta - \alpha$). The corresponding combined functional thus reads:

$$\int \! \mathcal{R} \psi \left( \alpha^2 v_s^2 + \alpha v_y^2 + 2\alpha V(v) + \alpha v \cdot v_s + \frac{v^2}{2} - 2\alpha v \cdot v_y \right) \, dy$$

(16) \quad = \int \! \mathcal{R} \left[ \psi \left( \alpha^2 v_s^2 + \alpha v_y^2 + 2\alpha V(v) + \alpha v \cdot v_s + \frac{v^2}{2} \right) + \alpha \psi_v v^2 \right] \, dy$$

and expression (15) simplifies into

$$\left( -\alpha - 2\alpha^2 c^2 \frac{\psi_y}{\psi} \right) v_s^2 - 2\alpha \frac{\psi_y}{\psi} v_y \cdot v_s - v \cdot \nabla V(v) - v_y^2.$$

If $\psi_y/\psi$ is close to zero, this last quantity is roughly equal to

$$-\alpha v_s^2 - v \cdot \nabla V(v) - v_y^2,$$

and if $\psi_y/\psi$ is close to $c$, it is roughly equal to

$$\left( -\alpha - 2\alpha^2 c^2 \right) v_s^2 - 2\alpha c v_y \cdot v_s - v \cdot \nabla V(v) - v_y^2 \leq -\alpha v_s^2 - v \cdot \nabla V(v) - \frac{v_y^2}{2};$$

in both cases this provides the desired decrease (at least on the domain of space where $v$ is close to $0_{\mathbb{R}^n}$).

3.4 Miscellanea

3.4.1 Estimates derived from the definition of the “escape distance”

For every minimum point $m$ in $\mathcal{M}$ and every vector $v$ in $\mathbb{R}^n$ satisfying $|v - m|_D \leq d_{\text{Esc}}$, it follows from inequalities (6) on page 5 that

$$\frac{\lambda_{\text{max}}}{4} (u - m)^2 \leq V(u) - V(m) \leq \lambda_{\text{max}} (u - m)^2,$$

(17)

$$\frac{\lambda_{\text{min}}}{2} (u - m)^2 \leq (u - m) \cdot \nabla V(u) \leq 2\lambda_{\text{max}} (u - m)^2.$$

3.4.2 Maximum of the convexities of the lower quadratic hulls of the potential at local minimum points

For the computations carried in the next section, it will be convenient to introduce the quantity $q_{\text{low-hull}}$ defined as the minimum of the convexities of the positive quadratic hulls of $V$ at the points of $\mathcal{M}$. With symbols:

$$q_{\text{upp-hull}, V} = \max_{m \in \mathcal{M}} \max_{u \in \mathbb{R}^n, |u| \leq R_{\text{att,oc}}} \frac{V(u) - V(m)}{(u - m)^2}.$$
3.4.3 Maximum distance between the values of $V$ the potential at local minimum points

The following quantity $\Delta V$ will be used to define the a priori bound on the speed of propagation of the travelling waves (see (35) on page 28):

$$\Delta V = \max \{ V(m_1) - V(m_2) : (m_1, m_2) \in \mathcal{M}^2 \}.$$

4 Invasion implies convergence

4.1 Definitions and hypotheses

Let us assume that $V$ satisfies the coercivity hypothesis ($H_{\text{coerc}}$) and the generic hypotheses ($G$) (see subsection 2.6 on page 7). Let us consider a minimum point $m$ in $\mathcal{M}$, a pair (initial data) $(u_0, u_1)$ in $X$, and, for all quantities $x$ in $\mathbb{R}$ and $t$ in $[0, +\infty)$, the corresponding solution $(x, t) \mapsto u(x, t)$. Let us make the following hypothesis, illustrated by figure 6.

**($H_{\text{hom-right}}$)** There exists a positive quantity $\sigma_{\text{hom}}$ and a $C^1$-function

$$x_{\text{hom}} : [0, +\infty) \to \mathbb{R}, \text{ satisfying } x'_{\text{hom}}(t) \to \sigma_{\text{hom}} \text{ when } t \to +\infty,$$

such that, for every positive quantity $L$, the quantity

$$\left\| y \mapsto \left( u(x_{\text{hom}}(t) + y, t) - m, u(t)(x_{\text{hom}}(t) + y, t) \right) \right\|_{H^1([-L,L],\mathbb{R}^n) \times L^2([-L,L],\mathbb{R}^n)}$$

approaches 0 when $t$ approaches $+\infty$.

For every $t$ in $[0, +\infty)$, let us denote by $x_{\text{Esc}}(t)$ the supremum of the set:

$$\left\{ x \in (-\infty, x_{\text{hom}}(t)) : |u(x, t) - m| = d_{\text{Esc}} \right\}.$$
with the convention that \( x_{\text{Esc}}(t) \) equals \(-\infty\) if this set is empty. In other words, \( x_{\text{Esc}}(t) \) is the first point at the left of \( x_{\text{hom}}(t) \) where the solution “Escapes” at the distance \( d_{\text{Esc}} \) from the stable homogeneous equilibrium \( m \). We will refer to this point as the “Escape point” (there will also be an “escape point”, with a small “e” and a slightly different definition later). Observe that, if \( x_{\text{Esc}}(t) > -\infty \), then

\[
\left| u\left( x_{\text{Esc}}(t), t \right) \right| = d_{\text{Esc}} \quad \text{and} \quad |u(x, t)| < d_{\text{Esc}} \quad \text{for all} \quad x \quad \text{in} \quad (x_{\text{Esc}}(t), x_{\text{hom}}(t)).
\]

Let us consider the upper limit of the mean speeds between 0 and \( t \) of this Escape point:

\[
\sigma_{\text{Esc}} = \limsup_{t \to +\infty} \frac{x_{\text{Esc}}(t)}{t},
\]

and let us make the following hypothesis, stating that the area around \( x_{\text{hom}}(t) \) where the solution is close to \( m \) is “invaded” from the left at a nonzero (mean) speed.

\((H_{\text{inv}})\) The quantity \( \sigma_{\text{Esc}} \) is positive.

4.2 Statement

The aim of section 4 is to prove the following proposition, which is the main step in the proof of Theorem 1. The proposition is illustrated by figure 7. The first assertion of

\[u(x,t)\]

\[\sigma_{\text{Esc}}, \sigma_{\text{hom}}\]

\[x_{\text{hom,next}}(t), x_{\text{Esc}}(t), x_{\text{hom}}(t)\]

Figure 7: Illustration of Proposition 4

this proposition is that the mean “physical” speed \( \sigma_{\text{Esc}} \) is smaller than \( 1/\sqrt{\alpha} \); thus it is legitimate to use the following notation for the “parabolic” counterpart of that speed:

\[
\sigma_{\text{Esc}} = \frac{\sigma_{\text{Esc}}}{\sqrt{1 - \alpha \sigma^2_{\text{Esc}}}}.
\]

**Proposition 4** (invasion implies convergence). Assume that \( V \) satisfies the coercivity hypothesis \((H_{\text{coerc}})\) and the generic hypotheses \((G)\), and, keeping the definitions and notation above, let us assume that for the solution under consideration hypotheses \((H_{\text{hom-right}})\) and \((H_{\text{inv}})\) hold. Then the following conclusions hold.

- The mean speed \( \sigma_{\text{Esc}} \) is smaller than \( 1/\sqrt{\alpha} \).
• There exist:
  
  – a minimum point \( m_{\text{next}} \) in \( M \) satisfying \( V(m_{\text{next}}) < V(m) \),
  
  – a profile of travelling front \( \phi \) in \( \Phi_{c,\text{norm}}(m_{\text{next}}, m) \),
  
  – \( C^1 \)-functions \( t \mapsto x_{\text{hom}-\text{next}}(t) \) and \( t \mapsto \tilde{x}_{\text{Esc}}(t) \) defined on \([0, +\infty)\) and with values in \( \mathbb{R} \),

such that, when \( t \) approaches \(+\infty\), the following limits hold:

\[
\tilde{x}_{\text{Esc}}(t) - x_{\text{Esc}}(t) \to 0 \quad \text{and} \quad \tilde{x}'_{\text{Esc}}(t) \to \sigma_{\text{Esc}}
\]

and

\[
x_{\text{Esc}}(t) - x_{\text{hom}-\text{next}}(t) \to +\infty \quad \text{and} \quad x'_{\text{hom}-\text{next}}(t) \to \sigma_{\text{Esc}}
\]

and

\[
\sup_{x \in [x_{\text{hom}-\text{next}}(t), x_{\text{hom}}(t)]} \left| u(x, t) - \phi \left( \sqrt{1 + \alpha^2_{\text{Esc}}(x - x_{\text{Esc}}(t))} \right) \right| \to 0
\]

and, for every positive quantity \( L \), the norm in \( H^1([-L, L], \mathbb{R}^n) \times L^2([-L, L], \mathbb{R}^n) \) of the function

\[
y \mapsto \left( u(x_{\text{hom}-\text{next}}(t) + y, t) - m_{\text{next}}, u_t(x_{\text{hom}-\text{next}}(t) + y, t) \right)
\]

approaches \( 0 \).

In this statement, the very last conclusion is partly redundant with the previous one. The reason why this last conclusion is stated this way is that it emphasizes the fact that a property similar to \((H_{\text{hom-right}})\) is recovered “behind” the travelling front. As can be expected this will be used to prove Theorem 1 by re-applying Proposition 4 as many times as required (to the left and to the right), as long as “invasion of the equilibria behind the last front” occurs.

### 4.3 Settings of the proof, 1: normalization and choice of origin of times

Let us keep the notation and assumptions of subsection 4.1, and let us assume that the hypotheses \((H_{\text{coerc}})\) and \((G)\) and \((H_{\text{hom-right}})\) and \((H_{\text{inv}})\) of Proposition 4 hold.

Before doing anything else, let us clean up the place.

• For notational convenience, let us assume without loss of generality that \( m = 0_{\mathbb{R}^n} \) and \( V(0_{\mathbb{R}^n}) = 0 \).

• According to Proposition 2 on page 12, we may assume (without loss of generality, up to changing the origin of time) that, for all \( t \) in \([0, +\infty)\),

\[
\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att,}\infty} \quad \text{and} \quad \|x \mapsto (u(x, t), u_t(x, t))\|_X \leq R_{\text{att,}X}.
\]

• According to \((H_{\text{hom-right}})\), we may assume (without loss of generality, up to changing the origin of time) that, for all \( t \) in \([0, +\infty)\),
Unfortunately, the Escape point $x_{\text{Esc}}(t)$ presents a significant drawback: there is no a priori reason why it should display any form of continuity (it may jump back and forth while time increases). This lack of control is problematic with respect to our purpose, which is to write down a dissipation argument precisely around the position in space where the solution escapes from $0_{\mathbb{R}^n}$.

The answer to this difficulty will be to define another “escape point” (this one will be denoted by “$x_{\text{esc}}(t)$” — with a small “e” — instead of $x_{\text{Esc}}(t)$). This second definition is a bit more involved than that of $x_{\text{Esc}}(t)$, but the resulting escape point will have the significant advantage of growing at a finite (and even bounded) rate (Lemma 7 on page 25). The material required to define this escape point is introduced in the next subsection.

4.4 Upper bound on the invasion speed

4.4.1 Firewall function and its time derivative

Let $\kappa_0$ denote a positive quantity satisfying

\[ \kappa_0 \leq \frac{1}{2} \quad \text{and} \quad \alpha \kappa_0 \leq \frac{1}{2} \quad \text{and} \quad \frac{\kappa_0^2}{2} \leq \frac{\lambda_{\text{min}}}{4} \]

namely

\[ \kappa_0 = \min\left(\sqrt{\frac{\lambda_{\text{min}}}{2}}, \frac{1}{2}, \frac{1}{2\alpha}\right). \]

Let us consider the weight function $\psi_0$ defined by

\[ \psi_0(x) = \exp(-\kappa_0|x|). \]

For $\xi$ in $\mathbb{R}$, let $T_\xi \psi_0$ denote the translate of $\psi_0$ by $\xi$, that is the function defined by:

\[ T_\xi \psi_0(x) = \psi_0(x - \xi). \]

For every real quantity $x$ and nonnegative quantity $t$, following expression (10) on page 14, let

\[ F_0(x,t) = \alpha^2 u_t^2 + \alpha u_x^2 + 2\alpha V(u) + \alpha u \cdot u_t + \frac{u^2}{2}, \]

(the argument of $u$ and $u_x$ and $u_t$ on the right-hand side being $(x,t)$), and, for every real quantity $\xi$ let us consider the “firewall” and “quadratic” functions

\[ \mathcal{F}_0(\xi,t) = \int_{\mathbb{R}} T_\xi \psi_0(x) F_0(x,t) \, dx, \]

\[ \mathcal{Q}_0(\xi,t) = \int_{\mathbb{R}} T_\xi \psi_0(x) \left(\alpha u_t(x,t)^2 + u_x(x,t)^2 + u(x,t)^2\right) \, dx. \]
In this definition of $Q_0(\xi, t)$, the reason for the factor $\alpha$ in front of the term $u_t(x,t)^2$ is that it simplifies the expression of the time derivative of $Q_0$, and that will be convenient to bound this time derivative in Lemma 4 on page 24. Let

(23) \[ \Sigma_{\text{Esc}, 0}(t) = \{ x \in \mathbb{R} : |u(x, t)| > d_{\text{Esc}} \} . \]

The desired coercivity and decrease properties of the firewall function $F_0(\cdot, \cdot)$ are stated in the following two lemmas.

**Lemma 1** (firewall coercivity up to pollution term). There exist a positive quantity $\varepsilon_{F_0, \text{coerc}}$ and a nonnegative quantity $K_{F_0, \text{coerc}}$, both depending only on $\alpha$ and $V$, such that for every real quantity $\xi$ and every nonnegative quantity $t$,

(24) \[ F_0(\xi, t) \geq \varepsilon_{F_0, \text{coerc}} Q_0(\xi, t) - K_{F_0, \text{coerc}} \int_{\Sigma_{\text{Esc}, 0}(t)} T_\xi \psi_0(x) \, dx . \]

**Lemma 2** (firewall decrease up to pollution term). There exist a positive quantity $\varepsilon_{F_0, \text{decr}}$ and a nonnegative quantity $K_{F_0, \text{decr}}$, both depending only on $\alpha$ and $V$, such that for every real quantity $\xi$ and nonnegative quantity $t$,

(25) \[ \partial_t F_0(\xi, t) \leq -\varepsilon_{F_0, \text{decr}} F_0(\xi, t) + K_{F_0, \text{decr}} \int_{\Sigma_{\text{Esc}, 0}(t)} T_\xi \psi_0(x) \, dx . \]

**Proof of Lemma 1.** For every real quantity $\xi$ and nonnegative quantity $t$ (adding and subtracting the same quantity),

\[ F_0(\xi, t) \geq \int_{\mathbb{R}} T_\xi \psi_0 \left( \frac{\alpha}{2} u_t^2 + \alpha u_x^2 + \frac{\alpha \lambda_{\text{min}}}{2} u^2 + 2\alpha \left( V(u) - \frac{\lambda_{\text{min}}}{4} u^2 \right) \right) \, dx . \]

thus, since according to properties (17) on page 16 derived from the definition of $d_{\text{Esc}}$ the last term of the integrand is nonnegative when $x$ is not in the set $\Sigma_{\text{Esc}, 0}(t)$,

\[ F_0(\xi, t) \geq \int_{\mathbb{R}} T_\xi \psi_0 \left( \frac{\alpha}{2} u_t^2 + \alpha u_x^2 + \frac{\alpha \lambda_{\text{min}}}{2} u^2 \right) \, dx \]

\[ + \int_{\Sigma_{\text{Esc}, 0}(t)} T_\xi \psi_0 \left( 2\alpha \left( V(u) - \frac{\lambda_{\text{min}}}{4} u^2 \right) \right) \, dx . \]

Let

$$\varepsilon_{F_0, \text{coerc}} = \min \left( \frac{1}{2}, \frac{\alpha \lambda_{\text{min}}}{2} \right) \text{ and } K_{F_0, \text{coerc}} = -\min_{v \in \mathbb{R}^n, |v| \leq R_{\text{att}, \infty}} 2\alpha \left( V(v) - \frac{\lambda_{\text{min}}}{4} v^2 \right)$$

(observe that this quantity $K_{F_0, \text{coerc}}$ is nonnegative); with this notation, inequality (24) follows from the last inequality above. **Lemma 1** is proved.

**Proof of Lemma 2.** According to expressions (8) and (9) on page 13, for every real quantity $\xi$ and nonnegative quantity $t$,

\[ \partial_t F_0(\xi, t) = \int_{\mathbb{R}} T_\xi \psi_0' \left( -\alpha u_t^2 - u_x^2 - u \cdot \nabla V(u) - 2\alpha \frac{T_\xi \psi_0'}{T_\xi \psi_0} u_x \cdot u_t + \frac{T_\xi \psi_0 u^2}{T_\xi \psi_0} \right) \, dx \]

\[ \leq \int_{\mathbb{R}} T_\xi \psi_0 \left( \alpha (-1 + \kappa_0) u_t^2 + (-1 + \alpha \kappa_0) u_x^2 + \frac{\kappa_0^2}{2} u^2 - u \cdot \nabla V(u) \right) \, dx \]

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(we used the fact that $T_\xi \psi''_0 / T_\xi \psi_0$ equals $\kappa_0$ and $T_\xi \psi''_0 / T_\xi \psi_0$ equals $\kappa_0^2$ plus a Dirac mass of negative weight). Thus, according to the two first conditions of (22) on page 20 satisfied by $\kappa_0$, it follows that (adding and subtracting the same quantity)

$$\partial_t F_0(\xi, t) \leq - \int_\mathbb{R} T_\xi \psi_0 \left( \frac{\alpha}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda_{\text{min}}}{4} u^2 \right) dx + \int_\mathbb{R} T_\xi \psi_0 \left( \frac{\lambda_{\text{min}}}{4} + \frac{\kappa_0^2}{2} \right) u^2 - u \cdot \nabla V(u) \right) dx.$$ 

Thus according to the last condition of (22) satisfied by $\kappa_0$ and to the property (17) on page 16 derived from the definition of $d_{\text{Esc}}$, this inequality still holds if we restrict the second integral of the right-hand side to the set $\Sigma_{\text{Esc},0}(t)$. Namely,

$$\partial_t F_0(\xi, t) \leq - \int_\mathbb{R} T_\xi \psi_0 \left( \frac{\alpha}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda_{\text{min}}}{4} u^2 \right) dx + \int_{\Sigma_{\text{Esc},0}(t)} T_\xi \psi_0 \left( \frac{\lambda_{\text{min}}}{2} u^2 - u \cdot \nabla V(u) \right) dx.$$

Let

$$\varepsilon_{\mathcal{F}_0} = \min \left( \frac{1}{2}, \frac{\lambda_{\text{min}}}{4} \right) \quad \text{and} \quad K_{\mathcal{F}_0,\text{decr}} = \max_{v \in \mathbb{R}^n, |v| \leq R_{\text{att},\infty}} \frac{\lambda_{\text{min}}}{2} v^2 - u \cdot \nabla V(v)$$

(observe that this quantity $K_{\mathcal{F}_0}$ is nonnegative); with this notation, it follows from the last inequality that

$$\partial_t F_0(\xi, t) \leq - \varepsilon_{\mathcal{F}_0} Q_0(\xi, t) + K_{\mathcal{F}_0,\text{decr}} \int_{\Sigma_{\text{Esc},0}(t)} T_\xi \psi_0(x) dx.$$ 

Finally, let

$$C_{\mathcal{F}_0} = \max \left( \frac{3\alpha}{2}, \alpha, 1 + 2\alpha q_{\text{upp-hull},V} \right) \quad \text{and} \quad \varepsilon_{\mathcal{F}_0,\text{decr}} = \frac{\varepsilon_{\mathcal{F}_0}}{C_{\mathcal{F}_0}};$$

with this notation, according to the definition (18) on page 16 of the quantity $q_{\text{upp-hull},V}$,

$$F_0(\xi, t) \leq C_{\mathcal{F}_0} Q_0(\xi, t),$$

thus inequality (25) follows from inequality (26). Lemma 2 is proved.

4.4.2 Elementary inequalities involving $u(\cdot, \cdot)$ and $Q_0(\cdot, \cdot)$ and $F_0(\cdot, \cdot)$ and $\partial_t F_0(\cdot, \cdot)$ and $\partial_t Q_0(\cdot, \cdot)$

The aim of the following definitions and statements is to prove Lemma 7 below, providing a bound on the speed at which a spatial domain where the solution is close to 0 can be “invaded”. This lemma involves the two “hull functions” $\eta_{\text{no-esc}, Q_0}$ and $\eta_{\text{no-esc}, F_0}$ controlling $F_0(\cdot, \cdot)$ and $Q_0(\cdot, \cdot)$ respectively. The definition of these two hull functions is based on the three quantities $d_{\text{esc},Q_0}$ and $d_{\text{esc},F_0}$ and $L$ that will be defined now with Lemma 7 as a purpose. Let

$$d_{\text{esc},Q_0} = d_{\text{Esc}}.$$
Lemma 3 ($Q_0$ controls $u$). For every real quantity $\xi$ and every nonnegative quantity $t$, the following assertion holds

$$Q_0(\xi, t) \leq d^2_{\text{esc}, Q_0} \implies |u(\xi, t)| \leq d_{\text{Esc}}.$$ 

Proof. Let $v$ denote a function in $H^1_0(\mathbb{R}, \mathbb{R}^n)$. Then,

$$v(0)^2 = \psi_0(0)v(0)^2$$

$$\leq \frac{1}{2} \int_\mathbb{R} \frac{d}{dx} (\psi_0(x)v(x)^2) \, dx$$

$$\leq \frac{1}{2} \int_\mathbb{R} (|\psi_0'(x)v(x)|^2 + 2\psi_0(x)v(x) \cdot v'(x)) \, dx$$

$$\leq \frac{1}{2} \int_\mathbb{R} \psi_0(x)((1 + \kappa_0)v(x)^2 + v'(x)^2) \, dx$$

$$\leq \int_\mathbb{R} \psi_0(x)(v(x)^2 + v'(x)^2) \, dx$$

(indeed according to hypotheses (22) on page 20 the quantity $\kappa_0$ is not larger than 1), and the conclusion follows from the definitions of $d_{\text{esc}, Q_0}$ and $Q_0(\cdot, \cdot)$. 

Let

$$d_{\text{esc}, F_0} = \sqrt{\frac{\varepsilon_{F_0, \text{coerc}}}{8} d_{\text{esc}, Q_0}},$$

and let $L$ be a positive quantity satisfying the following properties (that will be used below)

$$K_{F_0, \text{coerc}} \frac{2}{\varepsilon_{F_0, \text{coerc}} \kappa_0} \exp(-\kappa_0 L) \leq \frac{1}{8} d^2_{\text{esc}, Q_0} \quad (27)$$

and

$$K_{F_0, \text{decr}} \frac{2}{\kappa_0} \exp(-\kappa_0 L) \leq \frac{\varepsilon_{F_0, \text{decr}} d^2_{\text{esc}, F_0}}{4} \quad (28)$$

namely

$$L = \frac{1}{\kappa_0} \log\left( \max\left( \frac{16 K_{F_0, \text{coerc}}}{\kappa_0 \varepsilon_{F_0, \text{coerc}} d^2_{\text{esc}, Q_0}}, \frac{8 K_{F_0, \text{decr}}}{\kappa_0 \varepsilon_{F_0, \text{decr}} d^2_{\text{esc}, F_0}} \right) \right).$$

Those requirements on $L$ are related to the fact that

$$\int_{(-\infty,-L] \cup [L,\infty)} \psi_0(x) \, dx = \frac{2}{\kappa_0} \exp(-\kappa_0 L).$$

Lemma 4 ($F_0$ controls $Q_0$). For every real quantity $\xi$ and every nonnegative quantity $t$,

$$F_0(\xi, t) \leq d^2_{\text{esc}, F_0} \implies Q_0(\xi, t) \leq \frac{1}{4} d^2_{\text{esc}, Q_0}. \quad (23)$$
Proof. This assertion is an immediate consequence of the coercivity property \( (24) \) for \( F \), the definition of the quantity \( d_{\text{esc},F_0} \) above, and the first property \( (27) \) satisfied by the quantity \( L \).

Lemma 5 (**\( F_0 \) remains small far from \( \Sigma_{\text{Esc},0}(t) \)**). For every real quantity \( \xi \) and every nonnegative quantity \( t \),

\[
F_0(\xi,t) \geq \frac{1}{2}d_{\text{esc},F_0}^2 \implies \partial_t F_0(\xi,t) < 0.
\]

and, for every \( \xi' \) in \([\xi - L, \xi + L]\), \(|u(\xi',t)| \leq d_{\text{Esc}}\).

Proof. This assertion is an immediate consequence of the decrease property \( (25) \) and the second property \( (28) \) satisfied by the quantity \( L \).

Lemma 6 (bound on growth of \( Q_0 \)). There exists a positive quantity \( K_{Q_0,\text{growth}} \), depending only on \( \alpha \) and \( V \), such that, for every real quantity \( \xi \) and every nonnegative quantity \( t \),

\[
\partial_t Q_0(\xi,t) \leq K_{Q_0,\text{growth}}.
\]

Proof. For every real quantity \( \xi \) and every nonnegative quantity \( t \),

\[
\partial_t Q_0(\xi,t) = 2 \int_{\mathbb{R}} \left( T_{\xi}\psi_0(u_t \cdot (-u_t - \nabla V(u)) + u \cdot u_t) - T_{\xi}\psi_0' u_x \cdot u \right) dx.
\]

thus the conclusion follows from the a priori bounds on the solution (Proposition 2 on page 12).

4.4.3 No-escape hulls and upper bound on the invasion speed

Let us consider the two following “no-escape hull” functions

\[
\eta_{\text{no-esc},Q_0} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad \eta_{\text{no-esc},F_0} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}
\]

defined by (see figure 8).

Figure 8: Graphs of the hull functions \( \eta_{\text{no-esc},Q_0} \) and \( \eta_{\text{no-esc},F_0} \).
\[ \eta_{\text{no-esc,} Q_0}(x) = \begin{cases} +\infty & \text{for } x < 0, \\ \frac{d^2_{\text{esc,} Q_0}}{2} \left(1 - \frac{x}{2L}\right) & \text{for } 0 \leq x \leq L, \\ \frac{d^2_{\text{esc,} Q_0}}{4} & \text{for } x \geq L, \end{cases} \]

and

\[ \eta_{\text{no-esc,} F_0}(x) = \begin{cases} +\infty & \text{for } x < L, \\ d^2_{\text{esc,} F_0} & \text{for } x \geq L, \end{cases} \]

and let us consider the positive quantity \( \sigma_{\text{no-esc}} \) ("no-escape speed") defined by:

\[ \sigma_{\text{no-esc}} = \frac{4LK_{Q_0, \text{growth}}}{d^2_{\text{esc,} Q_0}}. \]

The following lemma is a variant of lemma 4 of \[15\].

**Lemma 7** (bound on invasion speed). For every pair \((x_{\text{left}}, x_{\text{right}})\) of points of \(\mathbb{R}\) and every \(t_0\) in \([0, +\infty)\), if the following properties holds for all \(x\) in \(\mathbb{R}\),

\[ Q_0(x, t_0) \leq \max(\eta_{\text{no-esc,} Q_0}(x - x_{\text{left}}), \eta_{\text{no-esc,} Q_0}(x_{\text{right}} - x)) \]

and

\[ F_0(x, t_0) \leq \max(\eta_{\text{no-esc,} F_0}(x - x_{\text{left}}), \eta_{\text{no-esc,} F_0}(x_{\text{right}} - x)) , \]

then, for every date \(t\) not smaller than \(t_0\) and for all \(x\) in \(\mathbb{R}\), the two following inequalities hold

\[ Q_0(x, t) \leq \max(\eta_{\text{no-esc,} Q_0}(x_{\text{left}} - \sigma_{\text{no-esc}} (t - t_0)), \eta_{\text{no-esc,} Q_0}(x_{\text{right}} + \sigma_{\text{no-esc}} (t - t_0) - x)) \]

and

\[ F_0(x, t) \leq \max(\eta_{\text{no-esc,} F_0}(x_{\text{left}} - \sigma_{\text{no-esc}} (t - t_0)), \eta_{\text{no-esc,} F_0}(x_{\text{right}} + \sigma_{\text{no-esc}} (t - t_0) - x)) \]

**Proof.** The proof follows from Lemmas 3 to 6. It is almost identical to the proof of Lemma 4 of \[15\] (see also lemma 4 and figure 10 of \[14\]). We leave the details to the reader. \(\square\)

4.5 Settings of the proof, 2: escape point and associated speeds

With the notation and results of the previous subsection 4.4 in pocket, let us pursue the settings for the proof of Proposition 4 "invasion implies convergence".

According to \(H_{\text{hom-right}}\), we may assume, up to changing the origin of time, that, for all \(t\) in \([0, +\infty)\) and for all \(x\) in \(\mathbb{R}\),

\[ Q_0(x, t) \leq \max\left(\eta_{\text{no-esc,} Q_0}(x - (x_{\text{hom}}(t) - 1)), \eta_{\text{no-esc,} Q_0}(x_{\text{hom}}(t) - x)\right) \]

(29)

and

\[ F_0(x, t) \leq \max\left(\eta_{\text{no-esc,} F_0}(x - (x_{\text{hom}}(t) - 1)), \eta_{\text{no-esc,} F_0}(x_{\text{hom}}(t) - x)\right). \]
As a consequence, for all $t$ in $[0, +\infty)$, the set
\[
I_{\text{Hom}}(t) = \left\{ x_t \leq x_{\text{hom}}(t) : \text{for all } x \in \mathbb{R}, \right. \\
Q_0(x, t) \leq \max \left\{ \eta_{\text{no-esc}, Q_0}(x - x_t), \eta_{\text{no-esc}, Q_0}(x_{\text{hom}}(t) - x) \right\} \quad \text{and} \\
F_0(x, t) \leq \max \left\{ \eta_{\text{no-esc}, F_0}(x - x_t), \eta_{\text{no-esc}, F_0}(x_{\text{hom}}(t) - x) \right\}
\]
is a nonempty interval (containing $[x_{\text{hom}}(t) - 1, x_{\text{hom}}(t)]$) that must be bounded from below. Indeed, if at a certain time it was not bounded from below — in other words if it was equal to $(-\infty, x_{\text{hom}}(t)]$ — then according to Lemma 7 this would remain unchanged in the future, thus according to Lemma 3 the point $x_{\text{Esc}}(t)$ would remain equal to $-\infty$ in the future, a contradiction with hypothesis (H$_{\text{inv}}$).

For all $t$ in $[0, +\infty)$, let
\[
x_{\text{esc}}(t) = \inf(I_{\text{Hom}}(t)) \quad (\text{thus } x_{\text{esc}}(t) > -\infty).
\]

Somehow like $x_{\text{Esc}}(t)$, this point represents the first point at the left of $x_{\text{hom}}(t)$ where the solution “escapes” (in a sense defined by the functions $Q_0$ and $F_0$ and the no-escape hulls $\eta_{\text{no-esc}, Q_0}$ and $\eta_{\text{no-esc}, F_0}$) at a certain distance from $0_{\mathbb{R}^n}$. In the following, this point $x_{\text{esc}}(t)$ will be called the “escape point” (by contrast with the “Escape point” $x_{\text{Esc}}(t)$ defined before). According to the first of the “hull inequalities” (29) above, for all $t$ in $[0, +\infty)$, let
\[
x_{\text{esc}}(t) = \inf(I_{\text{Hom}}(t)) \quad (\text{thus } x_{\text{esc}}(t) > -\infty).
\]

For every $s$ in $[0, +\infty)$, let us consider the “upper and lower bounds of the variations of $x_{\text{esc}}(\cdot)$ over all time intervals of length $s$” (see figure 9):
\[
\bar{x}_{\text{esc}}(s) = \sup_{t \in [0, +\infty)} x_{\text{esc}}(t + s) - x_{\text{esc}}(t) \quad \text{and} \quad \underline{x}_{\text{esc}}(s) = \inf_{t \in [0, +\infty)} x_{\text{esc}}(t + s) - x_{\text{esc}}(t).
\]

According to these definitions and to inequality (33) above, for all $t$ and $s$ in $[0, +\infty)$,
\[
-\infty \leq \underline{x}_{\text{esc}}(s) \leq x_{\text{esc}}(t + s) - x_{\text{esc}}(t) \leq \bar{x}_{\text{esc}}(s) \leq \sigma_{\text{no-esc}} \cdot s.
\]

Let us consider the four limit mean speeds:
\[
\sigma_{\text{esc-inf}} = \lim_{t \to +\infty} \inf \frac{x_{\text{esc}}(t)}{t} \quad \text{and} \quad \sigma_{\text{esc-sup}} = \lim_{t \to +\infty} \sup \frac{x_{\text{esc}}(t)}{t}.
\]

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The following inequalities follow readily from these definitions and from hypothesis (H_{inv}):

\[ -\infty \leq \sigma_{\text{esc-inf}} \leq \sigma_{\text{esc-inf}} \leq \sigma_{\text{esc-sup}} \leq \sigma_{\text{no-esc}} \quad \text{and} \quad 0 < \sigma_{\text{Esc}} \leq \sigma_{\text{esc-sup}}. \]

We are going to prove that the four limit mean speeds defined just above are equal. The proof is based on the “relaxation scheme” that will be set up in subsection 4.7 below. To set up this relaxation scheme, an additional a priori estimates on these speeds (namely, the fact that they are strictly smaller than the maximum speed of propagation $1/\sqrt{\alpha}$) is required. This is the purpose of the next subsection.

### 4.6 Further (subsonic) bound on invasion speed, definition

The next subsection will be devoted to the relaxation scheme in a travelling frame that is the core of the proof of Theorem 1. This relaxation scheme will require a bound on the parabolic speed of the travelling frame, in other words it will require that the physical speed of the travelling frame be (strictly) subsonic (without this requirement all estimates would literally blow up). The aim of this subsection is to define the value of this bound (namely the quantity $c_{\text{max}}$ defined below). Using the relaxation scheme set up in the next subsection, it will be proved later (Lemma 14 in sub-subsection 4.7.7) that the (upper) limit mean speed $\bar{\sigma}_{\text{esc-sup}}$ is not larger than this (subsonic) bound $c_{\text{max}}$.

These observations and statements are very similar to (and much inspired by) those made by Gallay and Joly in [5]. To define the subsonic bound on invasion speed, these authors used a Poincaré inequality in the weighted Sobolev spaces $H^1_c(\mathbb{R}, \mathbb{R}^n)$ (see subsection 4.2 of [5]). Although based on the same idea, the definition of $c_{\text{max}}$ below is slightly
different and more convenient for the situation we are going to deal with (convergence towards a stacked family of travelling fronts).

Let us recall the quantity $\Delta V$ defined in sub-subsection 3.4.3 on page 14 and let us consider the (positive) quantities

\begin{equation}
(35) \quad c_{\text{max}} = 1 + \frac{4\Delta V}{\min(1/2, \lambda_{\min}/4)d_{\text{Esc}}^{2}} \quad \text{and} \quad E_{\text{Esc}} = \frac{\min(1/2, \lambda_{\min}/4)d_{\text{Esc}}^{2}}{4}.
\end{equation}

The following lemma provides a justification for this value of $c_{\text{max}}$ and will be used in sub-subsection 4.7.7 to prove Lemma 14 stating that the (upper) limit mean speed $\bar{\sigma}_{\text{esc-sup}}$ is not larger than $c_{\text{max}}$. Note that the “$1+$” in the definition of $c_{\text{max}}$ is only to ensure that $c_{\text{max}}$ is nonzero (and actually not smaller than 1), since the quantity $\Delta V$ may be equal to 0.

**Lemma 8** (positive energy at Escape point when travelling frame speed is large). For every function $w$ in $H_{ul}^{1}(\mathbb{R}, \mathbb{R}^{n})$ and every quantities $y_{0}$ and $c$ satisfying the following conditions:

\[|w(y_{0})| = d_{\text{Esc}} \quad \text{and} \quad |w(y)| \leq d_{\text{Esc}} \text{ for all } y \text{ in } [y_{0}, y_{0}+1] \quad \text{and} \quad c \geq c_{\text{max}},\]

the following estimate holds:

\begin{equation}
(36) \quad \int_{-\infty}^{y_{0}+1} e^{cy} \left( \frac{w'(y)^{2}}{2} + V(w(y)) \right) dy \geq e^{cy_{0}} E_{\text{Esc}}.
\end{equation}

**Proof.** Let us consider a function $w$ in $H_{ul}^{1}(\mathbb{R}, \mathbb{R}^{n})$ and quantities $y_{0}$ and $c$ satisfying the hypotheses above. Then, according to properties (17) on page 16 derived from the definition $d_{\text{Esc}}$,

\begin{align*}
\int_{-\infty}^{y_{0}+1} e^{cy} \left( \frac{w'(y)^{2}}{2} + V(w(y)) \right) dy & \geq \int_{-\infty}^{y_{0}} e^{cy} (-\Delta V) dy + \int_{y_{0}}^{y_{0}+1} e^{cy} \left( \frac{w'(y)^{2}}{2} + \frac{\lambda_{\min}}{4}w(y)^{2} \right) dy \\
& \geq e^{cy_{0}} \left( -\frac{\Delta V}{c} + \min \left( \frac{1}{2}, \frac{\lambda_{\min}}{4} \right) \int_{y_{0}}^{y_{0}+1} (w'(y)^{2} + w(y)^{2}) dy \right).
\end{align*}

Let us denote by $\theta$ the affine function taking the value 1 at $y_{0}$ and 0 at $y_{0}+1$, namely defined by: $\theta(y) = y_{0} + 1 - y$. Then,

\begin{align*}
d_{\text{Esc}}^{2} = w(y_{0})^{2} = \theta(y_{0})w(y_{0})^{2} &= -\int_{y_{0}}^{y_{0}+1} \frac{d}{dy} (\theta(y)w(y)^{2}) dy \\
& = -\int_{y_{0}}^{y_{0}+1} \left( \theta'(y)w(y)^{2} + 2\theta(y)w(y)w'(y) \right) dy \\
& \leq 2\int_{y_{0}}^{y_{0}+1} (w(y)^{2} + w'(y)^{2}) dy.
\end{align*}

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It follows from these two inequalities that
\[
\int_{-\infty}^{y_0+1} e^{cy} \left( \frac{w'(y)^2}{2} + V(w(y)) \right) dy \geq e^{cy_0} \left( -\frac{\Delta V}{c} + \frac{1}{2} \min\left( \frac{1}{2}, \lambda_{\text{min}} \right) d_{\text{Esc}}^2 \right),
\]
and in view of the definitions of \( c_{\text{max}} \) and \( E_{\text{Esc}} \), inequality (36) follows. Lemma 8 is proved.

4.7 Relaxation scheme in a travelling frame

The aim of this subsection is to set up an appropriate relaxation scheme in a travelling frame. This means defining an appropriate localized energy and controlling the "flux" terms occurring in the time derivative of this localized energy. The considerations made in subsection 3.3 on page 13 will be put in practice.

4.7.1 Preliminary definitions

Let us keep the notation and hypotheses introduced above (since the beginning of subsection 4.3), and let us introduce the following real quantities that will play the role of "parameters" for the relaxation scheme below:

- the "initial time" \( t_{\text{init}} \) of the time interval of the relaxation;
- the initial position \( x_{\text{init}} \) of the origin of the travelling frame;
- the "parabolic" speed \( c \) of the travelling frame and its "physical" speed \( \sigma \), related by:
  \[ \sigma = \frac{c}{\sqrt{1 + \alpha c^2}} \iff c = \frac{\sigma}{\sqrt{1 - \alpha \sigma^2}}; \]
- a quantity \( y_{\text{cut-init}} \) that will be the the position of the maximum point of the weight function \( y \mapsto \chi(y, t_{\text{init}}) \) localizing energy at initial time \( t = t_{\text{init}} \) (this weight function is defined below).

Let us recall the (positive) quantity \( c_{\text{max}} \) defined in the previous sub-subsection and let us make on these parameters the following hypotheses:

\[
0 \leq t_{\text{init}} \quad \text{and} \quad 0 < c \leq c_{\text{max}} \quad \text{and} \quad 0 \leq y_{\text{cut-init}}.
\]

The relaxation scheme will be applied several time in the next pages, for various choices of this set of parameters.

For every real quantity \( y \) and every nonnegative quantity \( s \), let
\[
v(y, s) = u(x, t)
\]
where \((y, s)\) and \((x, t)\) are related by:
\[
t = t_{\text{init}} + s \quad \text{and} \quad x = x_{\text{init}} + \sigma s + \frac{y}{\sqrt{1 + \alpha c^2}} \iff y = \sqrt{1 + \alpha c^2}(x - x_{\text{init}}) - cs
\]
Figure 10: Space coordinate \( y \) and time coordinate \( s \) in the travelling frame, and parameters \( t_{\text{init}} \) and \( x_{\text{init}} \) and \( c \) and \( y_{\text{cut-init}} \).

(see figure 10). The system satisfied by \( v(\cdot, \cdot) \) reads

\[
\alpha v_{ss} + v_{s} - 2\alpha cv_{ys} = -\nabla V(v) + cv_{y} + v_{yy}.
\]

Let \( \kappa \) (rate of decrease of the weight functions) and \( c_{\text{cut}} \) (speed of the cutoff point in the travelling frame) be two positive quantities, sufficiently small so that the following conditions be satisfied (all those conditions will be used during forthcoming calculations):

\[
\frac{\kappa c_{\text{max}}}{2} \leq \frac{\lambda_{\text{min}}}{8} \quad \text{and} \quad 2\alpha c_{\text{cut}}(c_{\text{max}} + \kappa) \leq \frac{1}{4} \quad \text{and} \quad \frac{\kappa}{2}(c_{\text{max}} + \kappa) \leq \frac{\lambda_{\text{min}}}{8}
\]

and

\[
\left(c_{\text{cut}}\left(\alpha + \frac{1}{2}\right)(c_{\text{max}} + \kappa)\right) \leq \frac{1}{4} \quad \text{and} \quad \alpha c_{\text{cut}}(c_{\text{max}} + \kappa)(c_{\text{max}} + 1) \leq \frac{1}{4}
\]

\[
\text{and} \quad (c_{\text{max}} + \kappa)c_{\text{cut}}\left(\frac{1}{2} + \alpha(1/2 + c_{\text{max}} + 2\lambda_{\text{max}})\right) \leq \frac{\lambda_{\text{min}}}{8}.
\]

We may, for instance choose these two quantities as follows:

\[
\kappa = \min\left(1, \frac{1}{8\alpha(c_{\text{max}} + 1)} : \frac{\lambda_{\text{min}}}{4(c_{\text{max}} + 1)}\right)
\]

and

\[
c_{\text{cut}} = \min\left(\frac{1}{4(\alpha + 1/2)(c_{\text{max}} + 1)^2} : \frac{\lambda_{\text{min}}}{8(c_{\text{max}} + 1)(1/2 + \alpha(1/2 + c_{\text{max}} + 2\lambda_{\text{max}}))}\right).
\]

4.7.2 Localized energy

For every real quantity \( s \), let us consider the two intervals:

\[ I_{\text{main}}(s) = (-\infty, y_{\text{cut-init}} + c_{\text{cut}}s] \quad \text{and} \quad I_{\text{right}}(s) = [y_{\text{cut-init}} + c_{\text{cut}}s, +\infty) , \]
and let us consider the function \( \chi(y, s) \) (weight function for the localized energy) defined by:

\[
\chi(y, s) = \begin{cases} 
\exp(cy) & \text{if } y \in I_{\text{main}}(s), \\
\exp((c + \kappa)(y_{\text{cut-init}} + c_{\text{cut}}s) - \kappa y) & \text{if } y \in I_{\text{right}}(s),
\end{cases}
\]

(see figure [11]), and, for all \( s \) in \([0, +\infty)\), let us define the “energy function” \( E(s) \) by:

\[
E(s) = \int_{\mathbb{R}} \chi(y, s) \left( \frac{\alpha v_y(y, s)^2}{2} + \frac{v_y(y, s)^2}{2} + V(v(y, s)) \right) dy.
\]

**4.7.3 Time derivative of the localized energy**

For every nonnegative quantity \( s \), let

\[
\Sigma_{\text{Esc}}(s) = \{ y \in \mathbb{R} : |v(y, s)| > d_{\text{Esc}} \}.
\]

**Lemma 9** (approximate decrease of energy, 1). There exists a positive quantity \( K_{E, Q} \) and a nonnegative quantity \( K_{E, \text{Esc}, 1} \) such that, for every nonnegative quantity \( s \),

\[
E'(s) \leq - (1 + \alpha c^2) D(s) + K_{E, Q} \int_{I_{\text{right}}(s)} \chi(y, s) (v_y^2 + v_y^2 + v_y^2) dy + K_{E, \text{Esc}, 1} \int_{I_{\text{right}}(s) \cap \Sigma_{\text{Esc}}(s)} \chi(y, s) dy.
\]

**Proof.** According to expression [12] on page 15 for the derivative of a localized energy, the quantities \( \chi_s \) and \( \chi + \alpha c \chi_y \) and \( c \chi - \chi_y \) are involved in the derivative of \( E(s) \); it follows from the definition of \( \chi \) that:

\[
\chi_s(y, s) = \begin{cases} 
0 & \text{if } y \in I_{\text{main}}(s), \\
c_{\text{cut}}(c + \kappa) \chi(y, s) & \text{if } y \in I_{\text{right}}(s),
\end{cases}
\]

and

\[
(\chi + \alpha c \chi_y)(y, s) = \begin{cases} 
(1 + \alpha c^2) \chi(y, s) & \text{if } y \in I_{\text{main}}(s), \\
(1 - \alpha c \kappa) \chi(y, s) & \text{if } y \in I_{\text{right}}(s),
\end{cases}
\]

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Thus, if for all \( s \) in \([0, +\infty)\) we define the “dissipation” function by
\[
(41) \quad D(s) = \int_{\mathbb{R}} (c_{\chi} - \chi)(y, s) v_s(y, s)^2 \, dy,
\]
then, for all \( s \) in \([0, +\infty)\), it follows from expression (12) on page 15 for the derivative of a localized energy that
\[
E'(s) = -(1 + \alpha c^2) D(s)
+ \int_{I_{\text{right}}(s)} (c + \kappa) \chi \left( \frac{c_{\text{cut}}(c + \kappa)}{2} \frac{v_s^2}{2} + \frac{v_y^2}{2} + V(v) \right) + \alpha(c + \kappa) c_{\text{cut}}^2 (c + \kappa) v_s \cdot v_s \, dy
\leq -(1 + \alpha c^2) D(s)
+ \int_{I_{\text{right}}(s)} (c + \kappa) \chi \left( \frac{c_{\text{cut}}(c + \kappa)}{2} \frac{v_s^2}{2} + \frac{1}{2} v_s^2 + \left( \frac{c_{\text{cut}}}{2} + \frac{1}{2} \right) v_y^2 + c_{\text{cut}} \lambda_{\text{max}} v_s^2 \right) \, dy.
\]
Adding and subtracting the same quantity to the right-hand side of this inequality, it follows that
\[
E'(s) \leq -(1 + \alpha c^2) D(s)
+ \int_{I_{\text{right}}(s)} (c + \kappa) \chi \left( \frac{c_{\text{cut}}(c + \kappa)}{2} \frac{v_s^2}{2} + \frac{1}{2} v_s^2 + \left( \frac{c_{\text{cut}}}{2} + \frac{1}{2} \right) v_y^2 + c_{\text{cut}} \lambda_{\text{max}} v_s^2 \right) \, dy
+ \int_{I_{\text{right}}(s)} (c + \kappa) \chi c_{\text{cut}} \left( V(v) - \lambda_{\text{max}} v_s^2 \right) \, dy.
\]
Let
\[
K_{\xi, Q} = (c_{\max} + \kappa) \max \left( \frac{c_{\text{cut}}}{2} + \alpha c_{\max} + \frac{1}{2} \frac{c_{\text{cut}}}{2} + \frac{1}{2} \frac{c_{\text{cut}}}{2} c_{\text{cut}} \lambda_{\text{max}} \right)
\]
and
\[
K_{\xi, \text{Esc}, 1} = \max_{u \in \mathbb{R}^n, |u| \leq R_{\text{att}}, \infty} (c_{\max} + \kappa) c_{\text{cut}} \left( V(u) - \lambda_{\text{max}} u^2 \right)
\]
(observe that this quantity \( K_{\xi, \text{Esc}, 1} \) is nonnegative). According to properties (17) on page 16 derived from the definition of \( d_{\text{Esc}} \), the quantity \( V(v) - \lambda_{\text{max}} v_s^2 \) is nonpositive for \( y \) in \( \mathbb{R} \setminus \Sigma_{\text{Esc}}(s) \), thus the requested inequality (40) follows from the last inequality above. Lemma 9 is proved.

4.7.4 Definition of the “firewall” function and bound on the time derivative of energy

A second function (the “firewall”) will now be defined, to get some control over the second term of the right-hand side of inequality (40). Let us consider the function \( \psi(y, s) \) (weight
function for the firewall function) defined by:

\[
\psi(y, s) = \begin{cases} 
\exp((c + \kappa)y - \kappa(y_{\text{cut-init}} + c_{\text{cut}}s)) & \text{if } y \in I_{\text{main}}(s), \\
\chi(y, s) & \text{if } y \in I_{\text{right}}(s).
\end{cases}
\]

(see figure 11). For every real quantity \(y\) and every nonnegative quantity \(s\), following expression (16) on page 16, let

\[
F(y, s) = \alpha v_x^2 + \alpha v_y^2 + 2\alpha v \cdot v_s + \left(\frac{1}{2} + \alpha \frac{\psi}{\psi_y}\right) v^2
\]

(the argument of every function on the right-hand side being \((y, s)\)), and let

\[
\mathcal{F}(s) = \int_{\mathbb{R}} \psi(y, s) F(y, s) \, dy.
\]

Besides, let

\[
\mathcal{Q}(s) = \int_{\mathbb{R}} \psi(y, s)(v_x(y, s)^2 + v_y(y, s)^2 + v(y, s)^2) \, dy.
\]

Observe that, by contrast with the definition of \(Q_0(\cdot, \cdot)\) in sub-subsection 4.4.1 on page 20, this definition does not include any factor \(\alpha\) in front of the term \(v_x(y, s)^2\) (indeed no benefit would follow from such a factor in the next computations). The following lemma is the “travelling frame” analogue of Lemma 1 on page 21.

**Lemma 10** (firewall coercivity up to pollution term). There exist a positive quantity \(\varepsilon_{\mathcal{F},\text{coerc}}\) and a nonnegative quantity \(K_{\mathcal{F},\text{coerc}}\), depending only on \(\alpha\) and \(V\), such that for every nonnegative quantity \(s\),

\[
\mathcal{F}(s) \geq \varepsilon_{\mathcal{F},\text{coerc}} \mathcal{Q}(s) - K_{\mathcal{F},\text{coerc}} \int_{\Sigma_{\text{Esc}}(s)} \psi(y, s) \, dy.
\]

**Proof.** For every real quantity \(y\) and every nonnegative quantity \(s\),

\[
F(y, s) \geq \frac{\alpha^2}{2} v_x^2 + \alpha v_y^2 + \alpha \left(2V(v) - \kappa cv^2\right)
\]

\[
\geq \frac{\alpha^2}{2} v_x^2 + \alpha v_y^2 + \frac{\alpha \lambda_{\min}}{4} v^2 + \alpha \left(2V(v) - \kappa cv^2 - \frac{\lambda_{\min}}{4} v^2\right).
\]

According to properties (17) on page 16 derived from the definition of \(d_{\text{Esc}}\) and the properties (38) on page 16 satisfied by \(\kappa\), the quantity

\[
2V(v) - \kappa cv^2 - \frac{\lambda_{\min}}{4} v^2
\]

is nonnegative for \(y\) in \(\mathbb{R} \setminus \Sigma_{\text{Esc}}(s)\), thus

\[
\mathcal{F}(s) \geq \int_{\mathbb{R}} \psi \left(\frac{\alpha^2}{2} v_x^2 + \alpha v_y^2 + \frac{\alpha \lambda_{\min}}{4} v^2\right) \, dy + \int_{\Sigma_{\text{Esc}}(s)} \psi \alpha \left(2V(v) - \kappa cv^2 - \frac{\lambda_{\min}}{4} v^2\right) \, dy.
\]

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Let
\[\varepsilon_{F,\text{coerc}} = \min\left(\alpha^2, \alpha, \frac{\alpha \lambda_{\min}}{4}\right)\]
and
\[K_{F,\text{coerc}} = -\min_{u \in \mathbb{R}^n, |u| \leq R_{\text{att},\infty}} \alpha \left(2V(u) - \kappa c_{\text{max}} u^2 - \frac{\lambda_{\min}}{4} u^2\right)\]
(observe that this quantity \(K_{F,\text{coerc}}\) is nonnegative); with this notation inequality (42) follows from the last inequality above. Lemma 10 is proved.

Lemma 11 (approximate decrease of energy, 2). There exist nonnegative quantities \(K_{\varepsilon,F}\) and \(K_{\varepsilon,\text{Esc}}\), depending only on \(\alpha\) and \(V\), such that for every nonnegative quantity \(s\),
\[
(43) \quad \mathcal{E}'(s) \leq -(1 + \alpha c^2) D(s) + K_{\varepsilon,F} \mathcal{F}(s) + K_{\varepsilon,\text{Esc}} \int_{\Sigma_{\text{Esc}}(s)} \psi(y,s) \, dy.
\]

Proof. For every nonnegative quantity \(s\), since \(\chi(y,s) = \psi(y,s)\) for all \(y\) in \(I_{\text{right}}(s)\), it follows from inequality (40) (replacing \(\chi\) by \(\psi\))
\[
\mathcal{E}'(s) \leq -(1 + \alpha c^2) D(s) + K_{\varepsilon,Q} \int_{I_{\text{right}}(s)} \psi(y,s)(v^2_x + v^2_y + v^2) \, dy
\]
\[
+ K_{\varepsilon,\text{Esc},1} \int_{I_{\text{right}}(s) \cap \Sigma_{\text{Esc}}(s)} \psi(y,s) \, dy
\]
\[
\leq -(1 + \alpha c^2) D(s) + K_{\varepsilon,Q} \mathcal{Q}(s) + K_{\varepsilon,\text{Esc},1} \int_{\Sigma_{\text{Esc}}(s)} \psi(y,s) \, dy.
\]
Let
\[K_{\varepsilon,F} = \frac{K_{\varepsilon,Q}}{\varepsilon_{F,\text{coerc}}} \quad \text{and} \quad K_{\varepsilon,\text{Esc}} = K_{\varepsilon,\text{Esc},1} + \frac{K_{\varepsilon,Q} K_{F,\text{coerc}}}{\varepsilon_{F,\text{coerc}}},\]
with this notation, inequality (43) follows readily from inequality (42) of Lemma 10. Lemma 11 is proved.

For every nonnegative quantity \(s\), let
\[G(s) = \int_{\Sigma_{\text{Esc}}(s)} \psi(y,s) \, dy.
\]
Let \(s_{\text{fin}}\) be a nonnegative quantity (denoting the length of the time interval on which the relaxation scheme will be applied). It follows from Lemma 11 that
\[
(44) \quad (1 + \alpha c^2) \int_0^{s_{\text{fin}}} D(s) \, ds \leq \mathcal{E}(0) - \mathcal{E}(s_{\text{fin}}) + K_{\varepsilon,F} \int_0^{s_{\text{fin}}} \mathcal{F}(s) \, ds + K_{\varepsilon,\text{Esc}} \int_0^{s_{\text{fin}}} G(s) \, ds.
\]
This is the first version of the relaxation scheme inequality that is the key argument to prove Proposition 3 (invasion implies convergence). The aim of the two next sub-subsection is to gain some control over the quantities \(\mathcal{F}(s)\) and \(G(s)\).
4.7.5 Time derivative of the firewall function

The following lemma is the “travelling frame” analogue of Lemma 2.

**Lemma 12** (firewall decrease up to pollution term). There exist a positive quantity $\varepsilon_{\mathcal{F}, \text{decr}}$ and a nonnegative quantity $K_{\mathcal{F}, \text{decr}}$, depending only on $\alpha$ and $V$, such that for every nonnegative quantity $s$,

$$\mathcal{F}'(s) \leq -\varepsilon_{\mathcal{F}, \text{decr}} \mathcal{F}(s) + K_{\mathcal{F}, \text{decr}} \mathcal{G}(s).$$

**Proof.** According to expressions (12) and (13) on page 15 for the time derivatives of the functionals in a travelling frame, for every nonnegative quantity $s$,

$$\mathcal{F}'(s) = \int \left[ \psi_s \left( \alpha^2 v_s^2 + \alpha v_y^2 + 2\alpha V(v) \right) - \left( 2\alpha \psi + 2\alpha^2 c \psi_y \right) v_s^2 - 2\alpha (c \psi - \psi_y) v_y \cdot v_s 
+ \psi_s \left( \alpha v \cdot v_s + \frac{v_y^2}{2} - 2\alpha c v \cdot v_y \right) + \psi \left( -v \cdot \nabla V(v) - v_y^2 + \alpha v_s^2 - 2\alpha c v_y \cdot v_s \right) 
+ \frac{1}{2} (\psi_{yy} - c \psi_y) v^2 \right] dy.$$

Simplifying the terms involving $\psi v_s^2$ and those involving $\psi v_y \cdot v_s$, and rearranging terms, it follows that

$$\mathcal{F}'(s) = \int \left[ v_y^2 \left( -\alpha \psi - 2\alpha^2 c \psi_y + \alpha^2 \psi_s \right) + v_s^2 \left( -\psi + \alpha \psi_s \right) - v \cdot \nabla V(v) 
- 2\alpha \psi_s v \cdot v_s + \frac{v^2}{2} (\psi_s + \psi_{yy} - c \psi_y) + \psi_s \left( 2\alpha V(v) + \alpha v \cdot v_s - 2\alpha c v \cdot v_y \right) \right] dy.$$

According to the definition of $\psi(\cdot, \cdot)$, the following inequalities hold for all values of its arguments:

$$|\psi_s| \leq c_{\text{cut}} (c + \kappa) \psi \quad \text{and} \quad \psi_{yy} - c \psi_y \leq \kappa (c + \kappa) \psi$$

(indeed, $\psi_{yy} - c \psi_y$ equals $\kappa (c + \kappa) \psi$ plus a Dirac mass of negative weight at $y = y_{\text{cut-init}} + c_{\text{cut}} s$). Thus, for every nonnegative quantity $s$,

$$\mathcal{F}'(s) \leq \int \left[ v_y^2 \left( -\alpha - 2\alpha^2 c \frac{\psi_y}{\psi} + \alpha^2 c_{\text{cut}} (c + \kappa) \right) + v_s^2 \left( -1 + \alpha c_{\text{cut}} (c + \kappa) \right) - v \cdot \nabla V(v) 
- 2\alpha \frac{\psi_y}{\psi} v_y \cdot v_s + v^2 \frac{c_{\text{cut}} + \kappa (c + \kappa)}{2} 
+ c_{\text{cut}} (c + \kappa) \left( 2\alpha |V(v)| + \alpha |v \cdot v_s| + 2\alpha c |v \cdot v_y| \right) \right] dy.$$
Thus, polarizing the scalar products $v_y \cdot v_s$ and $v \cdot v_s$ and $v \cdot v_y$,
\[
\mathcal{F}'(s) \leq \int_R \psi \left[ v_s^2 \left( -\alpha - 2\alpha^2 \frac{v_y}{\psi} + \alpha^2 \text{cut}(c + \kappa) + 2\alpha^2 \frac{\psi^2}{\psi^2} + \frac{\alpha \text{cut}(c + \kappa)}{2} \right) 
+ v_y^2 \left( -\frac{1}{2} + \alpha \text{cut}(c + \kappa) + \alpha \text{cut}(c + \kappa) \right) \right. 
+ \left. \frac{\alpha \text{cut}(c + \kappa)}{2} + \frac{\alpha \text{cut}(c + \kappa)}{2} + \alpha \text{cut}(c + \kappa) \right] 
+ 2\alpha \text{cut}(c + \kappa) \psi dy.
\]

Observe that the following equality holds, be the argument $y$ in $I_{\text{main}}(s)$ or in $I_{\text{right}}(s)$:
\[
-2\alpha^2 \frac{v_y}{\psi} + 2\alpha^2 \frac{\psi^2}{\psi^2} = 2\alpha^2 \kappa(c + \kappa).
\]

Thus, adding and subtracting the same quantities,
\[
\mathcal{F}'(s) \leq \int_R \psi \left[ v_s^2 \left( -\alpha + \alpha^2 \text{cut}(c + \kappa) + 2\alpha^2 \kappa(c + \kappa) + \frac{\alpha \text{cut}(c + \kappa)}{2} \right) 
+ v_y^2 \left( -\frac{1}{2} + \alpha \text{cut}(c + \kappa) + \alpha \text{cut}(c + \kappa) \right) + v^2 \left( -\frac{\lambda_{\text{min}}}{2} + \frac{(c + \kappa)(c + \kappa)}{2} \right) 
+ \frac{\alpha \text{cut}(c + \kappa)}{2} + \alpha \text{cut}(c + \kappa) + 2\alpha \text{cut}(c + \kappa) \lambda_{\text{max}} \right] 
+ \lambda_{\text{min}} v^2 - v \cdot \nabla V(v) + 2\alpha \text{cut}(c + \kappa) (|V(v)| - \lambda_{\text{max}} v^2) \right] dy.
\]

Observe that, according to the properties [17] on page 16 derived from the definition of $d_{\text{Esc}}$, the terms displayed on the last line are nonpositive if $y$ is not in $\Sigma_{\text{Esc}}(s)$. Therefore, the inequality still holds if for these terms the integration domain is restricted to $\Sigma_{\text{Esc}}(s)$.

Thus, factorizing the other lines of the right-hand side of this inequality, it follows that
\[
\mathcal{F}'(s) \leq \int_R \psi \left[ \alpha v_s^2 \left( -1 + (c + \kappa)(c_{\text{cut}}(\alpha + 1/2) + 2\alpha \kappa) \right) \right. 
+ \left. \frac{1}{2} + \alpha \text{cut}(c + \kappa)(1 + c) \right] 
+ \frac{\lambda_{\text{min}}}{2} v^2 - v \cdot \nabla V(v) + 2\alpha \text{cut}(c + \kappa) (|V(v)| - \lambda_{\text{max}} v^2) \right] dy.
\]

Let us consider the positive quantity
\[
\varepsilon_F = \min \left( \alpha, \frac{1}{2}, \lambda_{\text{min}} \frac{1}{4} \right)
\]
and the (nonnegative) quantity

\[ K_{F,\text{decr}} = \max_{u \in \mathbb{R}^n, |u| \leq R_{\text{att},\infty}} \frac{\lambda_{\min}^2}{2} u^2 - u \cdot \nabla V(u) + 2\alpha c_{\text{cut}}(c_{\text{max}} + \kappa)(|V(u)| - \lambda_{\max} u^2). \]

According to the properties of \( \kappa \) and \( c_{\text{cut}} \) (38) and (39) on page 30, it follows that

\[ \mathcal{F}'(s) \leq -\varepsilon F_Q(s) + K_F G(s). \]

Finally, let

\[ C_F = \max \left( \frac{3\alpha^2}{2}, \alpha(1 + c_{\text{max}}), 2\alpha q_{\text{up-hull},V} + 1 + \alpha c_{\text{max}} \right) \quad \text{and} \quad \varepsilon_{F,\text{decr}} = \frac{\varepsilon F}{C_F}; \]

It follows from the definition of \( F(\cdot) \) that, for every nonnegative quantity \( s \),

\[ F(s) \leq C_F Q(s). \]

Thus inequality (45) follows from inequality (46). Lemma 12 is proved. \( \Box \)

For every nonnegative quantity \( s_{\text{fin}} \), inequality (45) yiedls

\[ \int_0^{s_{\text{fin}}} F(s) \, ds \leq \frac{1}{\varepsilon_{F,\text{decr}}} \left( F(0) - F(s_{\text{fin}}) + K_{F,\text{decr}} \int_0^{s_{\text{fin}}} G(s) \, ds \right), \]

and in view of inequality (12) of Lemma 10 (firewall coercivity up to pollution term),

\[ -F(s_{\text{fin}}) \leq K_{F,\text{coerc}} G(s_{\text{fin}}). \]

Thus the “relaxation scheme” inequality (44) becomes

\[ (1 + \alpha c^2) \int_0^{s_{\text{fin}}} D(s) \, ds \leq \mathcal{E}(0) - \mathcal{E}(s_{\text{fin}}) + \frac{K_{\mathcal{E},F}}{\varepsilon_{F,\text{decr}}} F(0) + \frac{K_{\mathcal{E},F} K_{F,\text{coerc}}}{\varepsilon_{F,\text{decr}}} G(s_{\text{fin}}) \]

\[ + \left( \frac{K_{\mathcal{E},F} K_{F,\text{decr}}}{\varepsilon_{F,\text{decr}}} + K_{\mathcal{E},\text{Esc}} \right) \int_0^{s_{\text{fin}}} G(s) \, ds. \]

This is the second version of the relaxation scheme inequality. The aim of the next sub-subsection is to gain some control over the quantity \( G(s) \).

### 4.7.6 Control over the additional flux terms

For every nonnegative quantity \( s \), let

\[ y_{\text{hom}}(s) = \sqrt{1 + \alpha c^2} (x_{\text{hom}}(t_{\text{init}} + s) - x_{\text{init}} - \sigma s) \]

\[ y_{\text{esc}}(s) = \sqrt{1 + \alpha c^2} (x_{\text{esc}}(t_{\text{init}} + s) - x_{\text{init}} - \sigma s) \]

\[ y_{\text{Esc}}(s) = \sqrt{1 + \alpha c^2} (x_{\text{Esc}}(t_{\text{init}} + s) - x_{\text{init}} - \sigma s) \]

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(see figures 6 and 7 on page 17 and on page 18). According to properties (31) on page 26 for the set \( \Sigma_{Esc,0}(t) \), for all \( s \) in \( [0, +\infty) \),

\[
\Sigma_{Esc}(s) \subset (-\infty, y_{esc}(s)] \cup [y_{hom}(s), +\infty),
\]

thus if we consider the quantities

\[
G_{\text{back}}(s) = \int_{-\infty}^{y_{esc}(s)} \psi(y, s) \, dy \quad \text{and} \quad G_{\text{front}}(s) = \int_{y_{hom}(s)}^{+\infty} \psi(y, s) \, dy,
\]

then, for all \( s \) in \( [0, +\infty) \),

\[
G(s) \leq G_{\text{back}}(s) + G_{\text{front}}(s).
\]

The aim of this sub-subsection is to prove the bounds on \( G_{\text{back}}(s) \) and \( G_{\text{front}}(s) \) provided by the next lemma.

**Lemma 13** (upper bounds on \( G_{\text{back}}(s) \) and \( G_{\text{front}}(s) \)). For every nonnegative quantity \( s \), the following estimates hold:

\[
\begin{align*}
G_{\text{back}}(s) &\leq \frac{1}{c + \kappa} \exp\left( (c + \kappa) y_{esc}(s) - \kappa y_{\text{cut-init}} - \kappa c_{\text{cut}} s \right), \\
G_{\text{front}}(s) &\leq \frac{1}{\kappa} \exp\left[ (c + \kappa) y_{\text{cut-init}} + (c + \kappa) c_{\text{cut}} + \kappa c_{\text{cut}} s - \kappa y_{\text{hom}}(0) \right].
\end{align*}
\]

**Proof.** We are going to bound the integrand \( \psi(y, s) \) in the expression of \( G_{\text{back}}(s) \) and \( G_{\text{front}}(s) \) by:

\[
\exp\left[ (c + \kappa) y - \kappa (y_{\text{cut-init}} + c_{\text{cut}} s) \right] \quad \text{for} \quad G_{\text{back}}(s),
\]

\[
\exp\left[ (c + \kappa) (y_{\text{cut-init}} + c_{\text{cut}} s) - \kappa y \right] \quad \text{for} \quad G_{\text{front}}(s).
\]

By explicit calculation,

\[
G_{\text{back}}(s) \leq \frac{1}{c + \kappa} \exp\left[ (c + \kappa) y_{esc}(s) - \kappa y_{\text{cut-init}} - \kappa c_{\text{cut}} s \right]
\]

and inequality (49) follows.

Concerning \( G_{\text{front}}(s) \), since \( x'_{\text{hom}}(\cdot) \) is nonnegative (inequality (21) on page 20), for all \( s \) in \( [0, +\infty) \),

\[
y'_{\text{hom}}(s) \geq -c \quad \text{thus} \quad y_{\text{hom}}(s) \geq y_{\text{hom}}(0) - cs.
\]

By explicit calculation, it follows that

\[
G_{\text{front}}(s) \leq \frac{1}{\kappa} \exp\left[ (c + \kappa) y_{\text{cut-init}} + (c + \kappa) c_{\text{cut}} + \kappa c_{\text{cut}} s - \kappa y_{\text{hom}}(0) \right]
\]

and inequality (50) follows. Lemma 13 is proved. □
4.7.7 Further (subsonic) bound on invasion speed, statement and proof

Up to now, the quantity $c_{\text{max}}$ has only been used to make the hypothesis (37) that the parabolic speed of the travelling frame under consideration does not exceed this quantity. Now, we are going to use the relaxation scheme set up above to prove that this quantity $c_{\text{max}}$ is indeed an upper bound on the speed of invasion. The aim of this sub-subsection is to prove the following lemma.

**Lemma 14** (invasion speed is subsonic). *The following inequality holds

$$\bar{\sigma}_{\text{esc-sup}} \leq \frac{c_{\text{max}}}{\sqrt{1 + \alpha c_{\text{max}}^2}}.$$*

In particular the mean speed $\bar{\sigma}_{\text{esc-sup}}$ is smaller than $1/\sqrt{\alpha}$, therefore if we denote by $\sigma_{\text{max}}$ the “physical” counterpart of $c_{\text{max}}$ and by $\bar{\sigma}_{\text{esc-sup}}$ the “parabolic” counterpart of $\bar{\sigma}_{\text{esc-sup}}$, defined by

$$\sigma_{\text{max}} = \frac{c_{\text{max}}}{\sqrt{1 + \alpha c_{\text{max}}^2}} \quad \text{and} \quad \bar{\sigma}_{\text{esc-sup}} = \frac{\bar{\sigma}_{\text{esc-sup}}}{\sqrt{1 - \alpha \bar{\sigma}_{\text{esc-sup}}^2}},$$

then the conclusion of Lemma 14 may be stated under the form of the following two equivalent inequalities:

$$\bar{\sigma}_{\text{esc-sup}} \leq \sigma_{\text{max}} \iff \bar{\sigma}_{\text{esc-sup}} \leq c_{\text{max}}.$$*

The idea of the proof of Lemma 14 provided below is due to Gallay and Joly see Lemma 5.2 of [5]). The principle is that, if we consider the previous relaxation scheme in a travelling frame with a parabolic speed $c$ not smaller than $c_{\text{max}}$, then, according to Lemma 8 on page 28 the following lower bound holds:

$$\int_{-\infty}^{y_{\text{Esc}}(s)+1} e^{cy} \left( \alpha v_{y}(y, s)^2 + \frac{v_{y}(y, s)^2}{2} + V(v(y, s)) \right) dy \geq E_{\text{Esc}} \exp(y_{\text{Esc}}(s)),$$

and as a consequence the same kind of lower bound holds for the localized energy $E(s)$ defined in sub-subsection 4.7.2. On the other hand, the relaxation scheme inequality (47) provides an upper bound for this localized energy, and under appropriate conditions this will enable us to prove that this localized energy remains bounded from above. Finally, it will follow from these bounds that the Escape point $y_{\text{Esc}}(s)$ must itself be bounded from above. We are going to see that this is contradictory with arbitrarily large values of the escape point $y_{\text{esc}}(s)$, and in turn contradictory with a mean speed $\bar{\sigma}_{\text{esc-sup}}$ exceeding $c_{\text{max}}$.

**Proof of Lemma 14**. Let us proceed by contradiction and assume that the converse assertion holds:

$$\sigma_{\text{max}} < \bar{\sigma}_{\text{esc-sup}} \iff c_{\text{max}} < \bar{\sigma}_{\text{esc-sup}}.$$*

Let $\varepsilon$ denote a positive quantity, sufficiently small so that

$$\sigma_{\text{max}} < \bar{\sigma}_{\text{esc-sup}} - \varepsilon,$$

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and let us make in addition the following technical hypothesis (see the comment below after the statement of Lemma 15):

\[ \varepsilon < \frac{1}{\sqrt{1 + \alpha c_{\max}^2}} \frac{\kappa c_{\text{cut}}}{2(c_{\max} + \kappa)}. \]

The following lemma provides appropriate time intervals where the relaxation scheme will be applied. Here are the features of these time intervals:

- the mean speed of the escape point is almost maximal on them;
- their length is arbitrarily large;
- for a given length they occur at arbitrarily large times.

**Lemma 15** (time intervals with controlled length and large initial times where mean speed of escape point is almost maximal). For every positive integer \( p \), there exists a sequence \( (t_{p,q})_{q \in \mathbb{N}} \) of positive quantities approaching \(+\infty\) when \( q \) approaches \(+\infty\), and such that, for every integer \( q \),

\[ x_{\text{esc}}(t_{p,q} + p) - x_{\text{esc}}(t_{p,q}) \geq (\bar{\sigma}_{\text{esc-sup}} - \varepsilon)p. \]

The technical hypothesis (51) above will be used in the proof of Lemma 17 on page 42 stating that the escape point ends “far to the right” at the end of the relaxation scheme we are going to consider.

**Proof of Lemma 15.** If the converse was true, then there would exist a positive integer \( p \) and a positive quantity \( t_0 \) such that, for every \( t \) not smaller than \( t_0 \),

\[ \frac{x_{\text{esc}}(t + p) - x_{\text{esc}}(t)}{p} \leq \bar{\sigma}_{\text{esc-sup}} - \varepsilon \]

and this would imply:

\[ \limsup_{s \to +\infty} \sup_{t \in [0, +\infty)} \frac{x_{\text{esc}}(t + s) - x_{\text{esc}}(t_1)}{s} \leq \bar{\sigma}_{\text{esc-sup}} - \varepsilon, \]

a contradiction with the definition of \( \bar{\sigma}_{\text{esc-sup}} \). \( \square \)

For every positive integer \( p \), let us consider a sequence \( (t_{p,q})_{q \in \mathbb{N}} \) satisfying the conclusions of Lemma 15 above, and let \( q(p) \) and \( x_{\text{init}}^{(p)} \) denote an integer and a real quantity to be chosen below. Finally, let us take the following notation:

\[ t_{\text{init}}^{(p)} = t_{p,q(p)}. \]

We are going to apply the relaxation scheme set up in the previous sub-subsection for the following set of parameters:

\[ t_{\text{init}} = t_{\text{init}}^{(p)} \quad \text{and} \quad x_{\text{init}} = x_{\text{init}}^{(p)} \quad \text{and} \quad c = c_{\max} \quad \text{and} \quad y_{\text{cut-init}} = 0. \]
Let us denote by

\[ y_{\text{esc}}^{(p)}(\cdot) \quad \text{and} \quad y_{\text{hom}}^{(p)}(\cdot) \quad \text{and} \quad \chi^{(p)}(\cdot, \cdot) \quad \text{and} \quad \mathcal{E}^{(p)}(\cdot) \quad \text{and} \quad \mathcal{F}^{(p)}(\cdot) \]

and \[ y_{\text{esc}}^{(p)}(\cdot) \quad \text{and} \quad y_{\text{hom}}^{(p)}(\cdot) \quad \text{and} \quad \mathcal{E}_{\text{back}}^{(p)}(\cdot) \quad \text{and} \quad \mathcal{G}_{\text{front}}^{(p)}(\cdot) \]

the objects defined in the previous sub-subsections (with the same notation except the “\( (p) \)” superscripts to emphasize the fact that these objects depend on \( p \)). We are going to consider the relaxation scheme on a time interval of length \( s_{\text{fin}} = p \), that is between the times \( t_{\text{init}}^{(p)} \) and \( t_{\text{init}}^{(p)} + p \). Observe that, according to the conclusion (52) of Lemma 15, whatever the choice of \( q(p) \) and \( x_{\text{init}}^{(p)} \),

\[
\frac{y_{\text{esc}}^{(p)}(p) - y_{\text{esc}}^{(p)}(0)}{p} \geq \sqrt{1 + \frac{c_{\text{max}}^2}{\sigma_{\text{esc-sup}} - \varepsilon - \sigma_{\text{max}}}} > 0
\]

(see figure [12]).

Figure 12: Definition of the quantity \( x_{\text{init}}^{(p)}(p) \). An increase of \( x_{\text{init}}^{(p)} \) translates the graph of \( x \mapsto y_{\text{esc}}^{(p)}(s) \) downwards. The value chosen for \( x_{\text{init}}^{(p)} \) is the smallest one so that this graph remains below the slope starting from the origin on the interval \([0, p]\). The figure aims at displaying the assertion of Lemma 17, that is the fact that \( y_{\text{esc}}^{(p)}(p) \) approaches \( +\infty \) when \( p \) approaches \( +\infty \).

To set up this relaxation scheme there still remains to define the two quantities \( q(p) \) and \( x_{\text{init}}^{(p)} \). Our purpose is to make this choice in such a way that the following two conditions be fulfilled:

- the quantity \( \mathcal{E}^{(p)}(p) \) (the localized energy in travelling frame at the end of the relaxation time interval) remains bounded when \( p \) approaches \( +\infty \);
- the quantity \( y^{(p)}(p) \) (the escape point in travelling frame at the end of the relaxation time interval) approaches \( +\infty \) when \( p \) approaches \( +\infty \).
Guided by expression inequality (49) on \( G_{\text{back}}(\cdot) \), let us choose the quantity \( x_{\text{init}}^{(p)} \) as the smallest possible quantity such that, for every \( s \) in the interval \([0, p]\), the following condition be fulfilled:

\[
(c_{\text{max}} + \kappa)y_{\text{esc}}^{(p)}(s) \leq \frac{\kappa \text{cut}}{2} s
\]

(see figure 12).

According to definition (48)

\[
y_{\text{esc}}^{(p)}(s) = \sqrt{1 + \alpha c_{\text{max}}^2 (x_{\text{esc}}^{(p)}(t_{\text{init}}^{(p)} + s) - x_{\text{init}}^{(p)} - \sigma_{\text{max}} s)},
\]

thus in other words, let us choose the quantity \( x_{\text{init}}^{(p)} \) as follows:

\[
x_{\text{init}}^{(p)} = \sup_{s \in [0, p]} x_{\text{esc}}^{(p)}(t_{\text{init}}^{(p)} + s) - \left( \sigma_{\text{max}} + \frac{\kappa \text{cut}}{2 \sqrt{1 + \alpha c_{\text{max}}^2}} (c_{\text{max}} + \kappa) \right)s
\]

(according to inequality (33) on page 26 controlling the increase of \( x_{\text{esc}}(\cdot) \), this supremum is finite). Condition (54) will ensure that the terms involving \( G_{\text{back}}(\cdot) \) in the relaxation scheme inequality (47) remain bounded.

The relevance of this definition for the quantity \( x_{\text{init}}^{(p)} \) is justified by the two following lemmas.

**Lemma 16** (boundedness of energy at the end of the time intervals). For every positive integer \( p \), if the integer \( q(p) \) is chosen sufficiently large, then the “final” energy \( E^{(p)}(p) \) is bounded from above by a quantity that does not depend on \( p \).

**Lemma 17** (escape point ends up far to the right in travelling frame). The following convergence holds:

\[
y_{\text{esc}}^{(p)}(p) \to +\infty \quad \text{when} \quad p \to +\infty.
\]

**Proof of Lemma 17** The proof is based of the relaxation scheme inequality (47). Thus, let us consider the various terms involved in this inequality.

First, let us observe that since the quantity \( y_{\text{cut-init}} \) is equal to 0, the quantities \( E^{(p)}(0) \) and \( F^{(p)}(0) \) are bounded from above by quantities depending only on \( \alpha \) and \( V \) (this follows from the a priori bounds (20) on page 19 for the solution).

Now, according to inequalities (49) and (54), for every \( s \) in \([0, p]\),

\[
G_{\text{back}}^{(p)}(s) \leq \frac{1}{\kappa} \exp(-\kappa \text{cut} s/2),
\]

and this ensures that the terms involving \( G_{\text{back}}^{(p)}(\cdot) \) in inequality (47) are bounded from above by quantities that do not depend on \( p \).

Finally, let us deal with the function \( G_{\text{front}}^{(p)}(\cdot) \). According to inequality (50), for every nonnegative quantity \( s \),

\[
G_{\text{front}}^{(p)}(s) \leq \frac{1}{\kappa} \exp((c_{\text{max}} + \kappa)(c_{\text{cut}} + \kappa) s - \kappa y_{\text{hom}}^{(p)}(0))
\]
and according to definition (48),
\[ y_{\text{hom}}^{(p)}(0) = \sqrt{1 + \alpha c_{\max}^2(x_{\text{hom}}(t_{\text{init}}^{(p)}) - x_{\text{init}}^{(p)})}. \]

On the other hand, according to the definition of \( x_{\text{init}}^{(p)} \) and to inequality (33) on page 26 controlling the growth of \( x_{\text{esc}}^{(\cdot)} \),

\[ x_{\text{init}}^{(p)} \leq x_{\text{esc}}(t_{\text{init}}^{(p)}) + \sigma_{\text{no-esc}} p, \]

thus
\[ y_{\text{hom}}^{(p)}(0) \geq \sqrt{1 + \alpha c_{\max}^2(x_{\text{hom}}(t_{\text{init}}^{(p)}) - x_{\text{esc}}(t_{\text{init}}^{(p)}) - \sigma_{\text{no-esc}} p)} \]

and this shows that the quantity \( y_{\text{hom}}^{(p)}(0) \) is arbitrarily large provided that the integer \( q(p) \) is chosen large enough (depending on \( p \)). As a consequence, if the integer \( q(p) \) is chosen large enough (depending on \( p \)), then the terms involving \( G_{\text{front}}^{(p)}(\cdot) \) in inequality (47) are bounded from above by quantities that do not depend on \( p \). Lemma [13] is proved. \( \square \)

**Proof of Lemma [7]** According to inequality (53) and to definition (48) on page 37
\[ y_{\text{esc}}^{(p)}(p) \geq \sqrt{1 + \alpha c_{\max}^2(\tilde{\sigma}_{\text{esc-sup}} - \varepsilon - \sigma_{\max})p + y_{\text{esc}}^{(p)}(0)} \]
\[ \geq \sqrt{1 + \alpha c_{\max}^2((\tilde{\sigma}_{\text{esc-sup}} - \varepsilon - \sigma_{\max})p + x_{\text{esc}}(t_{\text{init}}^{(p)}) - x_{\text{init}}^{(p)})}. \]

Now, according to the definition (55) of \( x_{\text{init}}^{(p)} \), there exists a quantity \( s_p \) in \([0, p]\) such that
\[ x_{\text{init}}^{(p)} \leq 1 + x_{\text{esc}}(t_{\text{init}}^{(p)} + s_p) - (\sigma_{\max} + \frac{\kappa c_{\text{cut}}}{2\sqrt{1 + \alpha c_{\max}^2(c_{\max} + \kappa)}})s_p. \]

It follows from the two previous inequalities that
\[ x_{\text{esc}}(t_{\text{init}}^{(p)} + s_p) - x_{\text{esc}}(t_{\text{init}}^{(p)}) \geq \]
\[ (\tilde{\sigma}_{\text{esc-sup}} - \varepsilon - \sigma_{\max})p + \left(\sigma_{\max} + \frac{\kappa c_{\text{cut}}}{2\sqrt{1 + \alpha c_{\max}^2(c_{\max} + \kappa)}}\right)s_p - 1 - \frac{y_{\text{esc}}^{(p)}(p)}{\sqrt{1 + \alpha c_{\max}^2}}, \]
thus, provided that \( s_p \) is nonzero,
\[ \frac{x_{\text{esc}}(t_{\text{init}}^{(p)} + s_p) - x_{\text{esc}}(t_{\text{init}}^{(p)})}{s_p} \geq \]
\[ \tilde{\sigma}_{\text{esc-sup}} - \varepsilon + \frac{\kappa c_{\text{cut}}}{2\sqrt{1 + \alpha c_{\max}^2(c_{\max} + \kappa)}} - \frac{1}{s_p} - \frac{y_{\text{esc}}^{(p)}(p)}{s_p \sqrt{1 + \alpha c_{\max}^2}}. \]

Let us proceed by contradiction and assume that there exists a quantity \( C \) such that, for arbitrarily large values of \( p \), the quantity \( y_{\text{esc}}^{(p)}(p) \) is not larger than \( C \). Then, according to inequality (53), for such values of \( p \) the quantity \( y_{\text{esc}}^{(p)}(0) \) is large negative, and according to inequality (33) controlling the growth of \( x_{\text{esc}}^{(\cdot)} \), the quantity \( s_p \) must be large positive.
According to the technical hypothesis (51), it follows that, for such sufficiently large values of $p$, 
\[
\frac{x_{\text{esc}}(t_{\text{init}}^\cdot + s_p) - x_{\text{esc}}(t_{\text{init}}^\cdot)}{s_p} > \bar{\sigma}_{\text{esc-sup}},
\]
a contradiction with the definition of $\bar{\sigma}_{\text{esc-sup}}$. Lemma 17 is proved.

The following lemma is a slight variant of Lemma 16 above.

**Lemma 18** (boundedness of energy at the end of the time intervals, variant). For every positive integer $p$, if the integer $q(p)$ is chosen sufficiently large, then the quantity
\[
\int_{-\infty}^{c_{\text{cut}}p} e^{cy} \left( \alpha \frac{v_s(p)(y,p)^2}{2} + \frac{v_y(p)(y,p)^2}{2} + V(v(p)(y,p)) \right) dy
\]
is bounded from above by a quantity that does not depend on $p$.

**Proof.** According to the definition ((48)) of $y_{\text{hom}}(\cdot)$,
\[
y_{\text{hom}}(p) = \sqrt{1 + \alpha \max^2 \left( x_{\text{hom}}(t_{\text{init}}^\cdot + p) - x_{\text{esc}}(t_{\text{init}}^\cdot) - \bar{\sigma}_{\text{esc-sup}} \right)},
\]
thus, according to inequality ((56)),
\[
y_{\text{hom}}(p) \geq \sqrt{1 + \alpha \max^2 \left( x_{\text{hom}}(t_{\text{init}}^\cdot + p) - x_{\text{esc}}(t_{\text{init}}^\cdot) - (\bar{\sigma}_{\text{esc-sup}} + \bar{\sigma}_{\text{max}}) \right)}.
\]
Thus, for every positive quantity $p$, if the integer $q(p)$ is chosen sufficiently large, then the quantity $y_{\text{hom}}(p)$ is arbitrarily large, and in particular larger than the point $c_{\text{cut}}p$.

In this case, according to the definition of the localized energy $E(\cdot)$ and of the weight function $\chi(\cdot, \cdot)$, since $\chi(p)(y,p)$ equals $e^{cy}$ for every $y$ in the interval $(-\infty, c_{\text{cut}}p]$, the following inequality holds:
\[
E(p)(p) \geq \int_{-\infty}^{c_{\text{cut}}p} e^{cy} \left( \alpha \frac{v_s(p)(y,p)^2}{2} + \frac{v_y(p)(y,p)^2}{2} + V(v(p)(y,p)) \right) dy
\]
\[+ \int_{y_{\text{hom}}(p)}^{+\infty} \chi(p)(y,p)V(v(p)(y,p)) dy.
\]
According to the definition of the weight function $\chi(\cdot, \cdot)$, the second integral of the right-hand side of this inequality is arbitrarily close to 0 is the quantity $y_{\text{hom}}(p)$ is sufficiently large, or in other words if the integer $q(p)$ is chosen sufficiently large. In view of Lemma 16 this finishes the proof of Lemma 18.

Let us assume from now on that for every positive integer $p$, the integer $q(p)$ is chosen large enough so that the conclusions of Lemmas 16 to 18 be satisfied, and so that (as assumed in the proof of Lemma 18),
\[
(57) \quad c_{\text{cut}}p \leq y_{\text{hom}}(p).
\]

Last not least, the definition of the quantity $c_{\max}$ in subsection 4.6 on page 27 (and the fact that the speed of the travelling frame under consideration is as large as $c_{\max}$) will now finally be used to prove the following lemma.
Lemma 19 (upper bound of Escape point in travelling frame). The quantity $y_{\text{Esc}}^{(p)}(p)$ remains bounded from above when $p$ approaches $+\infty$.

Proof. According to inequalities (31) and (54) on page 26 and on page 42 for every positive integer $p$,

$$ y_{\text{Esc}}^{(p)}(p) + 1 \leq y_{\text{esc}}^{(p)}(p) + 1 \leq \frac{c_{\text{cut}}}{2} p + 1, $$

thus as soon as $p$ is sufficiently large,

$$ y_{\text{Esc}}^{(p)}(p) + 1 \leq c_{\text{cut}} p, $$

and it follows from Lemma 18 and from inequality (57) that the quantity

$$ \int_{-\infty}^{y_{\text{Esc}}^{(p)}(p)+1} e^y \left( \frac{v_y^{(p)}(y,p)^2}{2} + V(v^{(p)}(y,p)) \right) \, dy $$

is bounded from above by a quantity that does not depend on $p$. On the other hand, according to Lemma 8 on page 28 (involving the positive quantity $E_{\text{Esc}}$),

$$ \int_{-\infty}^{y_{\text{Esc}}^{(p)}(p)+1} e^y \left( \frac{v_y^{(p)}(y,p)^2}{2} + V(v^{(p)}(y,p)) \right) \, dy \geq \exp(c_{\text{cut}}^{(p)} E_{\text{Esc}}), $$

and the conclusion follows.

The final step is provided by the following lemma that will turn out to be contradictory to the definition of the escape point $x_{\text{esc}}(\cdot)$.

Lemma 20 (approach to zero around escape point). For every positive quantity $L$, the integral

$$ \int_{y_{\text{Esc}}^{(p)}(p)-L}^{y_{\text{Esc}}^{(p)}(p)+L} \left( v_x^{(p)}(y,p)^2 + v_y^{(p)}(y,p)^2 + v^{(p)}(y,p)^2 \right) \, dy $$

approaches 0 when $p$ approaches $+\infty$.

Proof. Let $L$ denote a positive quantity. According to Lemmas 17 and 19 and to inequalities (57) and (58), for every sufficiently large integer $p$, the following inequalities hold:

$$ y_{\text{Esc}}^{(p)}(p) \leq y_{\text{esc}}^{(p)}(p) - L \leq y_{\text{esc}}^{(p)}(p) \leq y_{\text{Esc}}^{(p)}(p) + L \leq c_{\text{cut}} p \leq y_{\text{hom}}^{(p)}(p). $$
Then, it follows from these inequalities that

\[
\int_{-\infty}^{c_{\text{cut}}} e^{cy} \left( \frac{v_x^2(y, p)}{2} + \frac{v_y^2(y, p)}{2} + V(v(y, p)) \right) dy \\
\geq \int_{-\infty}^{y_{\text{esc}}(p)} e^{cy} V(v(y, p)) dy + \\
\int_{y_{\text{esc}}(p)}^{c_{\text{cut}}} e^{cy} \left( \frac{v_x^2(y, p)}{2} + \frac{v_y^2(y, p)}{2} + \frac{\lambda_{\min}}{4} v(y, p)^2 \right) dy \\
\geq -\frac{\Delta v}{c} \exp(y_{\text{esc}}(p)) + \\
\min \left( \frac{\alpha}{2}, \frac{1}{2}, \frac{\lambda_{\min}}{4} \right) \exp(y_{\text{esc}}(p) - L) \int_{y_{\text{esc}}(p) - L}^{y_{\text{esc}}(p) + L} \left( v_x^2(y, p) + v_y^2(y, p) + v(y, p)^2 \right) dy.
\]

In view of Lemmas 17 to 19, the conclusion follows. Lemma 20 is proved.

For every positive integer \(p\), let us denote by \(t'_p\) the time \(t(\cdot)_{\text{init}} + p\). It follows from Lemma 20 that, for every positive quantity \(L\), the quantity

\[
\int_{x_{\text{esc}}(t'_p) - L}^{x_{\text{esc}}(t'_p) + L} \left( u_x(x, t'_p) + u_x(x, t'_p) + u(x, t'_p)^2 \right) dx
\]

approaches 0 when \(p\) approaches \(+\infty\). In view of the definitions of the functions \(F(\cdot, \cdot)\) and \(Q(\cdot, \cdot)\) in sub-subsection 4.4.1 on page 20 and according to the a priori bounds (20) on page 19 for the solution, it follows that, for every positive quantity \(L\), both quantities

\[
\sup \{ |F(\xi, t'_p)| : \xi \in [x_{\text{esc}}(t'_p) - L, x_{\text{esc}}(t'_p) + L] \}
\]

and \(\sup \{ Q(\xi, t'_p) : \xi \in [x_{\text{esc}}(t'_p) - L, x_{\text{esc}}(t'_p) + L] \}\)

approach 0 when \(p\) approaches \(+\infty\), a contradiction with the definition of the “escape” point \(x_{\text{esc}}(\cdot)\) in subsection 4.5 on page 25. Lemma 14 on page 39 is proved.

\[\square\]

### 4.7.8 Final form of the “relaxation scheme” inequality

From now on the relaxation scheme will always be applied with the following choice for \(x_{\text{init}}\):

\[x_{\text{init}} = x_{\text{esc}}(t_{\text{init}})\]

The aim of this sub-subsection is to take advantage of this additional hypothesis and of the estimates of sub-subsection 4.7.6 and of Lemma 14 on page 39 to provide a more explicit version of the relaxation scheme inequality (47) on page 37.

The following additional technical hypothesis will be required to prove the next lemma providing another expression for the upper bound on \(G_{\text{back}}(s)\)

\[
\bar{\sigma}_{\text{esc-sup}} - \frac{\kappa_{\text{cut}}}{4(c_{\max} + \kappa)\sqrt{1 + \alpha_{\max}^2}} \leq \sigma.
\]
This hypothesis is satisfied as soon as the physical speed \( \sigma \) is sufficiently close to \( \bar{\sigma}_{\text{esc-sup}} \) (or equivalently as soon as the parabolic speed \( c \) is sufficiently close to \( \bar{c}_{\text{esc-sup}} \)). It ensures that the escape point \( y_{\text{esc}}(s) \) remains “more and more far away to the left” with respect to the position \( y_{\text{cut-init}} + c_s s \) of the cut-off, when \( s \) increases.

**Lemma 21** (new upper bound on \( G_{\text{back}}(s) \)). *There exists a positive quantity \( K[(u_0, u_1)] \), depending only on \( V \) and on the initial condition \((u_0, u_1)\) under consideration, such that for every nonnegative quantity \( s \) the following estimates hold:*

\[
G_{\text{back}}(s) \leq K[(u_0, u_1)] \exp(-\kappa y_{\text{cut-init}}) \exp\left(-\frac{\kappa c_s}{2} s \right).
\]

**Proof.** According to inequality (49) on page 38,

\[
G_{\text{back}}(s) \leq \frac{1}{\kappa} \exp(-\kappa y_{\text{cut-init}}) \exp\left((c + \kappa) y_{\text{esc}}(s) - \frac{\kappa c_s}{2} s \right) \exp\left(-\frac{\kappa c_s}{2} s \right).
\]

Let us us denote by \( \beta(s) \) the argument of the second exponential of the right-hand side of this last inequality:

\[
\beta(s) = (c + \kappa) y_{\text{esc}}(s) - \frac{\kappa c_s}{2} s
\]

\[
= (c + \kappa) \left( \sqrt{1 + \alpha^2} (x_{\text{esc}}(t_{\text{init}} + s) - x_{\text{esc}}(t_{\text{init}})) - c s \right) - \frac{\kappa c_s}{2} s
\]

\[
\leq (c + \kappa) \left( \sqrt{1 + \alpha^2} x_{\text{esc}}(s) - c s \right) - \frac{\kappa c_s}{2} s
\]

\[
\leq (c + \kappa) \sqrt{1 + \alpha^2} x_{\text{esc}}(s) - \bar{\sigma}_{\text{esc-sup}} s
\]

\[
+ \left( (c + \kappa) \sqrt{1 + \alpha^2 \bar{\sigma}_{\text{esc-sup}} - c} - \frac{\kappa c_s}{2} s \right).
\]

Besides, according to the condition [59] on the “physical” speed \( \sigma \), the following inequality holds:

\[
(c + \kappa) \sqrt{1 + \alpha^2 \bar{\sigma}_{\text{esc-sup}} - c} \leq \frac{\kappa c_s}{4},
\]

thus, for every nonnegative quantity \( s \),

\[
\beta(s) \leq (c + \kappa) \sqrt{1 + \alpha^2 \bar{\sigma}_{\text{esc-sup}} - c} - \frac{\kappa c_s}{4} s,
\]

and according to the definition of \( \bar{\sigma}_{\text{esc-sup}} \) this quantity approaches \(-\infty\) when \( s \) approaches \(+\infty\). The following (nonnegative) quantity:

\[
\tilde{\beta}[(u_0, u_1)] = \sup_{s \geq 0} (c_{\text{max}} + \kappa) \sqrt{1 + \alpha^2} (x_{\text{esc}}(s) - \bar{\sigma}_{\text{esc-sup}} s) - \frac{\kappa c_s}{4} s
\]

is an upper bound for all the values of \( \beta(s) \), for all \( s \) in \([0, +\infty)\). This quantity depends on \( V \) and on the function \( x \mapsto x_{\text{esc}}(s) \), in other words on the initial condition \((u_0, u_1)\), but not on the parameters \( t_{\text{init}} \) and \( c \) and \( y_{\text{cut-init}} \) of the relaxation scheme. Let

\[
K[(u_0, u_1)] = \frac{1}{\kappa} \exp(\tilde{\beta}[(u_0, u_1)]);
\]

with this notation, the upper bound [60] on \( G_{\text{back}}(s) \) follows from inequality (61). \( \square \)
Let us consider the quantities
\[ K_1 = \frac{K_{\mathcal{E},F} K_{\mathcal{F},\text{coerc}}}{\varepsilon_{\mathcal{F},\text{decr}}} \quad \text{and} \quad K_2 = \frac{K_{\mathcal{E},F} K_{\mathcal{F},\text{decr}}}{\varepsilon_{\mathcal{F},\text{decr}}} + K_{\mathcal{E},\text{Esc}} \]
and
\[ K_{\mathcal{G},\text{back}}[(u_0,u_1)] = K[(u_0,u_1)] \left( K_1 + \frac{2}{\kappa_{\text{cut}}} K_2 \right), \]
and, for every nonnegative quantity \( s \), the quantity
\[ K_{\mathcal{G},\text{front}}(s) = \left( K_1 + \frac{K_2}{c_{\text{max}} + \kappa}(c_{\text{cut}} + \kappa) \right) \exp \left( (c_{\text{max}} + \kappa)(c_{\text{cut}} + \kappa)s \right). \]

Then, for every nonnegative quantity \( s_{\text{fin}} \), according to inequalities (50) on \( \mathcal{G}_{\text{front}}(s) \) and (60) on \( \mathcal{G}_{\text{back}}(s) \), the relaxation scheme inequality (47) on page 37 can be rewritten as follows:
\[ (62) \]
\[ (1 + \alpha c^2) \int_0^{s_{\text{fin}}} D(s) \, ds \leq E(0) - E(s_{\text{fin}}) + \frac{K_{\mathcal{E},F}(0) + K_{\mathcal{G},\text{back}}[(u_0,u_1)]}{\varepsilon_{\mathcal{F},\text{decr}}} \exp(-\kappa y_{\text{cut-init}}) \]
\[ + K_{\mathcal{G},\text{front}}(s_{\text{fin}}) \exp \left( (c_{\text{max}} + \kappa) y_{\text{cut-init}} \right) \exp(-\kappa y_{\text{hom}}(0)). \]

This is the last version of the relaxation scheme inequality. The nice feature is that it has exactly the same form as in the parabolic case treated in [14] (actually, the sole difference is the value of the factor in front of the integral of the left-hand side, but this detail plays absolutely no role in the arguments carried out in [14]).

4.8 Convergence of the mean invasion speed

The aim of this subsection is to prove the following proposition.

**Proposition 5** (mean invasion speed). The following equalities hold:
\[ \sigma_{\text{esc-inf}} = \sigma_{\text{esc-sup}} = \bar{\sigma}_{\text{esc-sup}}. \]

**Proof.** Let us proceed by contradiction and assume that
\[ \sigma_{\text{esc-inf}} < \bar{\sigma}_{\text{esc-sup}}. \]

Then, let us take and fix a positive quantity \( \sigma \) ("physical speed") such that, if we denote by \( c \) the corresponding "parabolic speed":
\[ c = \frac{\sigma}{\sqrt{1 - \alpha c^2}} \iff \sigma = \frac{c}{\sqrt{1 + \alpha c^2}}, \]
then the following conditions are satisfied:
\[ \sigma_{\text{esc-inf}} < \sigma < \bar{\sigma}_{\text{esc-sup}} \leq \sigma + \frac{\kappa c_{\text{cut}}}{4(c_{\text{max}} + \kappa)\sqrt{1 + \alpha c_{\text{max}}^2}} \quad \text{and} \quad \Phi_c(0_{\mathbb{R}^n}) = \emptyset. \]

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The first condition is satisfied as soon as $c$ is smaller than and sufficiently close to $\bar{c}_{\text{esc-sup}}$, thus existence of a quantity $c$ satisfying the two conditions follows from hypothesis $\text{(H}_{\text{disc-c}})$. The contradiction will follow from the relaxation scheme set up in subsection 4.7.

The main ingredient is: since the set $\Phi_c(0_{\mathbb{R}^n})$ is empty, some dissipation must occur permanently around the escape point in a referential travelling at physical speed $\sigma$. This is stated by the following lemma.

**Lemma 22** (nonzero dissipation in the absence of travelling front). *There exist positive quantities $L$ and $\varepsilon_{\text{dissip}}$ such that*

$$\liminf_{t \to +\infty} \int_{t-1}^{t+1} \int_{-L}^{L} \left( u_t \left( x_{\text{esc}}(t) + x, t \right) + \sigma u_x \left( x_{\text{esc}}(t) + x, t \right) \right)^2 \, dx \, dt \geq \varepsilon_{\text{dissip}}.$$  

*Proof of Lemma 22.* Let us proceed by contradiction and assume that the converse is true. Then, there exists a sequence $(t_p)_{p \in \mathbb{N}}$ in $[1, +\infty)$ approaching $+\infty$ such that, for every nonzero integer $p$,

$$\int_{t_p-1}^{t_p+1} \int_{-L}^{L} \left( u_t \left( x_{\text{esc}}(t_p) + x, t_p \right) + \sigma u_x \left( x_{\text{esc}}(t_p) + x, t_p \right) \right)^2 \, dx \, dt \leq \frac{1}{p}.$$  

By compactness (Proposition 3 on page 12), up to replacing the sequence $(t_p)_{p \in \mathbb{N}}$ by a subsequence, we may assume that there exists a solution $\bar{u} \in \mathcal{C}^0([\mathbb{R}, H^1_{ul}(\mathbb{R}, \mathbb{R}^n)]) \cap \mathcal{C}^1([\mathbb{R}, L^2_{ul}(\mathbb{R}, \mathbb{R}^n)])$ of system (1) such that, for every positive quantity $L$, both quantities

$$\sup_{t \in [-1, 1]} \left\| x \mapsto u \left( x_{\text{esc}}(t_p) + x, t_p + t \right) - \bar{u}(x, t) \right\|_{H^1([-L,L], \mathbb{R}^n)}$$  

and

$$\sup_{t \in [-1, 1]} \left\| x \mapsto u_t \left( x_{\text{esc}}(t_p) + x, t_p + t \right) - \bar{u}_t(x, t) \right\|_{L^2([-L,L], \mathbb{R}^n)}$$

approach 0 when $p$ approaches $+\infty$. For every $y$ in $\mathbb{R}$ and $t$ in $[-1, 1]$, let

$$\tilde{v}(y, t) = \bar{u} \left( \frac{y}{\sqrt{1 + \alpha c^2}} + \sigma t, t \right).$$

It follows from inequality (63) that the function $t \mapsto \tilde{v}_t(\cdot, t)$ vanishes in

$$\mathcal{C}^0([-1, 1], L^2(\mathbb{R}, \mathbb{R}^n))$$

and as a consequence the function $\bar{w}$ defined by $\bar{w}(y) = \tilde{v}(y, 0)$ is a solution of the differential equation:

$$\bar{w}'' + c\bar{w}' - \nabla V(\bar{w}) = 0.$$  

According to the properties of the escape point (31) and (32) on page 26

$$\sup_{y \in [0, +\infty)} |\bar{w}(y)| \leq d_{\text{esc}},$$

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thus it follows from Lemma 41 on page 73 that \( \bar{w}(y) \) approaches 0 in \( \mathbb{R}^n \) when \( y \) approaches \( +\infty \). On the other hand, according to the a priori bounds on the solution, \( |\bar{w}(\cdot)| \) is bounded (by \( R_{\text{att,}\infty} \)), and since \( \Phi_s(0_{\mathbb{R}^n}) \) is empty, it follows from Lemma 40 on page 73 that \( \bar{w}(\cdot) \) vanishes identically, a contradiction with the definition of \( x_{\text{esc}}(\cdot) \).

The remaining of the proof of Proposition 5 is almost identical to the parabolic case treated in [14], where more explanations and details can be found. The next step is the choice of the time interval and the travelling frame (at physical speed \( \sigma \)) where the relaxation scheme will be applied. Here is a first attempt.

**Lemma 23** (large excursions to the right and returns for escape point in travelling frame). There exist sequences \( (t_p)_{p \in \mathbb{N}} \) and \( (s_p)_{p \in \mathbb{N}} \) and \( (\bar{s}_p)_{p \in \mathbb{N}} \) of real quantities such that the following properties hold.

- For every integer \( p \), the following inequalities hold: \( 0 \leq t_p \) and \( 0 \leq s_p \leq \bar{s}_p \);
- \( x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - \sigma s_p \rightarrow +\infty \) when \( p \rightarrow +\infty \);
- For every integer \( p \), the following inequality holds: \( x_{\text{esc}}(t_p + \bar{s}_p) - x_{\text{esc}}(t_p) - \sigma \bar{s}_p \leq 0 \).

**Proof of Lemma 23** The proof is identical to that of Lemma 10 of [14].

Let \( \tau \) denote a (large) positive quantity, to be chosen below. The following lemma provides appropriate time intervals to apply the relaxation scheme.

**Lemma 24** (escape point remains to the right and ends up to the left in travelling frame, controlled duration). There exist sequences \( (t'_p)_{p \in \mathbb{N}} \) and \( (s'_p)_{p \in \mathbb{N}} \) such that, for every integer \( p \) the following properties hold:

- \( 0 \leq t'_p \) and \( \tau \leq s'_p \leq 2\tau \),
- for all \( s \) in \( [0, \tau] \), the following inequality holds: \( x_{\text{esc}}(t'_p + s) - x_{\text{esc}}(t'_p) - \sigma s \geq 0 \),
- \( x_{\text{esc}}(t'_p + s'_p) - x_{\text{esc}}(t'_p) - \sigma s'_p \leq 1 \),

and such that \( t'_p \rightarrow +\infty \) when \( p \rightarrow +\infty \).

**Proof of Lemma 24** The proof is identical to that of Lemma 11 of [14].

For every integer \( p \) we are going to apply the relaxation scheme for the following parameters:

\[ t_{\text{init}} = t'_p \quad \text{and} \quad x_{\text{init}} = x_{\text{esc}}(t_{\text{init}}) \quad \text{and} \quad \sigma \quad \text{as chosen above, and} \quad y_{\text{cut-init}} = 0 \]

(the relaxation scheme thus depends on \( p \)). Let us denote by

\( e^{(p)}(\cdot, \cdot) \) and \( \mathcal{E}^{(p)}(\cdot) \) and \( D^{(p)}(\cdot) \) and \( \mathcal{F}^{(p)}(\cdot) \) and \( y_{\text{esc}}^{(p)}(\cdot) \) and \( y_{\text{hom}}^{(p)}(\cdot) \)
the objects defined in subsection 4.7 (with the same notation except the “(p)” superscript that is here to remind that all these objects depend on the integer $p$). By definition the quantity $y_{\text{esc}}^{(p)}(0)$ equals zero, and according to the conclusions of Lemma 24,

$$y_{\text{esc}}^{(p)}(s) \geq 0 \text{ for all } s \in [0, \tau] \text{ and } y_{\text{esc}}^{(p)}(s') \leq \sqrt{1 + \alpha c^2}.$$  

The two following lemmas will be shown to be in contradiction with the relaxation scheme final inequality (62) on page 48.

**Lemma 25** (bounds on energy and firewall at the ends of relaxation scheme). The quantities $\mathcal{E}^{(p)}(0)$ and $\mathcal{F}^{(p)}(0)$ are bounded from above and the quantity $\mathcal{E}^{(p)}(s'_p)$ is bounded from below, and these bounds are uniform with respect to $\tau$ and $p$.

*Proof of Lemma 25.* The proof is identical to that of Lemma 12 of [14].

**Lemma 26** (large dissipation integral). The quantity

$$\int_0^{s'_p} D^{(p)}(s) \, ds$$

approaches $+\infty$ when $\tau$ approaches $+\infty$, uniformly with respect to $p$.

*Proof of Lemma 26.* The proof is identical to that of Lemma 13 of [14].

According to Lemma 25 and since $y_{\text{hom}}^{(p)}(0)$ approaches $+\infty$ when $p$ approaches $+\infty$, the right-hand side of inequality (62) on page 48 is bounded, uniformly with respect to $\tau$, provided that $p$ (depending on $\tau$) is sufficiently large. This is contradictory to Lemma 26 and completes the proof of Proposition 5 on page 48.

According to Proposition 5 the three quantities $\sigma_{\text{esc-inf}}$ and $\sigma_{\text{esc-sup}}$ and $\bar{\sigma}_{\text{esc-sup}}$ are equal; let $\sigma_{\text{esc}}$ denote their common value.

### 4.9 Further control on the escape point

**Proposition 6** (mean invasion speed, further control). The following equality holds:

$$\sigma_{\text{esc-inf}} = \sigma_{\text{esc}}.$$ 

*Proof.* The proof is identical to that of Proposition 4 of [14].
4.10 Dissipation approaches zero at regularly spaced times

For every $t$ in $[1, +\infty)$, the following set

$$\left\{ \varepsilon \in (0, +\infty) : \int_{-1}^{1} \left( \int_{-1/\varepsilon}^{1/\varepsilon} \left( u_t(x_{\text{esc}}(t) + x, t) + \sigma_{\text{esc}} u_x(x_{\text{esc}}(t) + x, t) \right)^2 dx \right) dt \leq \varepsilon \right\}$$

is (according to the a priori bounds (20) on page 19 for the solution) a nonempty interval (which by the way is unbounded from above). Let

$$\delta_{\text{dissip}}(t)$$

denote the infimum of this interval. This quantity measures to what extent the solution is, at time $t$ and around the escape point $x_{\text{esc}}(t)$, close to be stationary in a frame travelling at physical speed $\sigma_{\text{esc}}$. Our goal is to to prove that

$$\delta_{\text{dissip}}(t) \to 0 \quad \text{when} \quad t \to +\infty.$$  

Proposition 7 below can be viewed as a first step towards this goal.

**Proposition 7** (regular occurrence of small dissipation). For every positive quantity $\varepsilon$, there exists a positive quantity $T(\varepsilon)$ such that, for every $t$ in $[0, +\infty)$,

$$\inf_{t' \in [t, t + T(\varepsilon)]} \delta_{\text{dissip}}(t') \leq \varepsilon.$$  

**Proof.** The proof is identical to that of Proposition 5 of [14]. \qed

4.11 Relaxation

**Proposition 8** (relaxation). The following assertion holds:

$$\delta_{\text{dissip}}(t) \to 0 \quad \text{when} \quad t \to +\infty.$$  

**Proof.** The proof is identical to that of Proposition 6 of [14]. \qed

4.12 Convergence

The end of the proof of Proposition 4 on page 18 (“invasion implies convergence”) is a straightforward consequence of Proposition 8. We will make use of the notation $x_{\text{Esc}}(t)$ and $x_{\text{esc}}(t)$ and $x_{\text{hom}}(t)$ introduced in subsections 4.1 and 4.5. Recall that, according to properties (31) on page 26 and to the hypotheses of Proposition 4 for every nonnegative time $t$,

$$-\infty \leq x_{\text{Esc}}(t) \leq x_{\text{esc}}(t) \leq x_{\text{hom}}(t) < +\infty.$$  

However, by contrast with the parabolic case treated in [14], we cannot use the point $x_{\text{Esc}}(t)$ to “track” the position of the travelling front approached by the solution around this point, since the solution lacks the required regularity in order the map $t \mapsto x_{\text{Esc}}(t)$ to be of class $C^1$. A convenient way to get around this difficulty is to use the decomposition
of the solution into two parts, one regular, and one approaching zero when time goes to infinity, as stated by the following lemma (reproduced from [5]).

Recall the notation $X$ of subsection 3.1 on page 12 and let $Y = H^2_{ul}(\mathbb{R}, \mathbb{R}^n) \times H^1_{ul}(\mathbb{R}, \mathbb{R}^n)$, and, for every nonnegative time $t$, let $U(t) = (u(\cdot, t), u_t(\cdot, t))$ denote the “position / impulsion” form of the solution. We know from Proposition 2 on page 12 that $U \in C^0([0, +\infty), X)$.

Lemma 27 (“smooth plus small” decomposition, [5]). There exists $U_{\text{small}} \in C^0([0, +\infty), X) \quad \text{and} \quad U_{\text{smooth}} \in C^1([0, +\infty), X) \cap C^0([0, +\infty), Y)$

such that: $U = U_{\text{small}} + U_{\text{smooth}}$ and

$$
\|U_{\text{small}}(t)\|_X \to 0 \quad \text{when} \quad t \to +\infty
$$

and

$$
\sup_{t \geq 0} \|U_{\text{smooth}}\|_Y < +\infty.
$$

Proof. Let

$$
A = \frac{1}{\alpha} \begin{pmatrix} 0 & \alpha \\ \partial^2_x - 1 & -1 \end{pmatrix} \quad \text{when} \quad F(u, u_t) = \frac{1}{\alpha} \begin{pmatrix} 0 \\ u - \nabla V(u) \end{pmatrix}.
$$

and let $U_0 = U(0) = (u_0, u_1)$ denote the initial condition for the solution under consideration. Then, for every nonnegative time $t$, the following representation holds for the solution at time $t$:

$$
U(t) = e^{tA}U_0 + \int_0^t e^{(t-s)A}F(U(s)) \, ds
$$

thus we may choose $U_{\text{small}}(t)$ and $U_{\text{smooth}}(t)$ as the first and the second term of the right-hand side of this equality, respectively. For more details see [5, p. 113]. Observe by the way that this decomposition is not unique. \qed

For every $t$ in $[0, +\infty)$, let us write

$$
U_{\text{smooth}}(t) = (u_{\text{smooth}}(t), \partial_t u_{\text{smooth}}(t))
$$

and let us denote by $x_{\text{Esc}-\text{smooth}}(t)$ the supremum of the set

$$
\{ x \in (-\infty, x_{\text{hom}}(t)) : |u_{\text{smooth}}(t)| = d_{\text{Esc}} \}
$$

with the convention that $x_{\text{Esc}-\text{smooth}}(t)$ equals $-\infty$ if this set is empty.
Lemma 28 (distance between $x_{\text{Esc-smooth}}(t)$ and $x_{\text{esc}}(t)$ remains bounded). The following limit holds:

$$\limsup_{t \to +\infty} x_{\text{esc}}(t) - x_{\text{Esc-smooth}}(t) < +\infty.$$ 

Proof. Let us proceed by contradiction and assume that the converse holds. Then there exists a sequence $(t_p)_{p \in \mathbb{N}}$ of nonnegative times approaching $+\infty$ such that

$$x_{\text{esc}}(t_p) - x_{\text{Esc-smooth}}(t_p) \to +\infty \quad \text{when} \quad p \to +\infty.$$ 

We proceed as in the proof of Lemma 22 on page 49. By compactness (Proposition 3 on page 12), up to replacing the sequence $(t_p)_{p \in \mathbb{N}}$ by a subsequence, we may assume that there exists a solution

$$\bar{u} \in C^0(\mathbb{R}, H^1(\mathbb{R}, \mathbb{R}^n)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{R}^n))$$

of system (1) such that, for every positive quantity $L$, both quantities

$$\sup_{t \in [-1,1]} \| x \mapsto u(x_{\text{esc}}(t_p) + x, t_p + t) - \bar{u}(x, t) \|_{H^1([-L,L], \mathbb{R}^n)}$$

and

$$\sup_{t \in [-1,1]} \| x \mapsto u_t(x_{\text{esc}}(t_p) + x, t_p + t) - \bar{u}_t(x, t) \|_{L^2([-L,L], \mathbb{R}^n)}$$

approach 0 when $p$ approaches $+\infty$. For every $y$ in $\mathbb{R}$ and $t$ in $[-1,1]$, let

$$\bar{v}(y, t) = \bar{u} \left( \frac{y}{\sqrt{1 + \alpha c_{\text{esc}}^2} + \sigma_{\text{esc}} t, t} \right).$$

It follows from Proposition 8 on page 52 that the function $t \mapsto \bar{v}(\cdot, t)$ vanishes in $C^0([-1,1], L^2(\mathbb{R}, \mathbb{R}^n))$, and as a consequence the function $\phi$ defined by $\phi(y) = \bar{v}(y, 0)$ is a solution of the differential equation:

$$\phi'' + c\phi' - \nabla V(\phi) = 0.$$

According to the properties of the escape point (31) and (32) on page 26

$$\sup_{y \in [0, +\infty]} |\phi(y)| \leq d_{\text{Esc}},$$

thus it follows from Lemma 41 on page 73 that $\phi(y)$ approaches $0_{\mathbb{R}^n}$ when $y$ approaches $+\infty$. In addition, according to the a priori bounds on the solution, $|\cdot|$ is bounded (by $R_{\text{att,}\infty}$). In addition again, according to the definition of $x_{\text{esc}}(\cdot)$, the function $\phi$ cannot be identically equal to $0_{\mathbb{R}^n}$. In short, the function $\phi$ belongs to the set $\Phi_{\text{esc}}(0_{\mathbb{R}^n})$ of profiles of front travelling at the parabolic speed $c_{\text{esc}}$.

On the other hand, it follows from hypothesis (66), from the definition of $x_{\text{Esc-smooth}}(\cdot)$, and from the asymptotics (64) for $U_{\text{small}}(\cdot)$, that

$$\sup_{y \in \mathbb{R}} |\phi(y)| \leq d_{\text{Esc}},$$

a contradiction with Lemma 11 on page 73. Lemma 28 is proved. \qed
Lemma 29 (vicinity of Escape points and transversality). The following limit hold: \( x_{\text{Esc-smooth}}(t) - x_{\text{Esc}}(t) \to 0 \) when \( t \to +\infty \) and
\[
\limsup_{t \to +\infty} u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) \cdot \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) < 0.
\]

Proof. Let us proceed by contradiction (for the two assertions simultaneously) and assume that the converse holds. Then there exists a sequence \((t_p)_{p \in \mathbb{N}}\) of nonnegative times approaching \(+\infty\) such that:
1. either
   \[
   \limsup_{p \to +\infty} |x_{\text{Esc-smooth}}(t_p) - x_{\text{Esc}}(t_p)| > 0,
   \]
2. or for every nonzero integer \(p\)
   \[
   u_{\text{smooth}}(x_{\text{Esc-smooth}}(t_p), t_p) \cdot \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t_p), t_p) \geq -\frac{1}{p}.
   \]

Proceeding as in the proof of Lemma 28 above, and according to this lemma, we may assume, up to replacing the sequence \((t_p)_{p \in \mathbb{N}}\) by a subsequence, that there exists a function \(\phi\) in the set \(\Phi_{\text{esc}}(0 \mathbb{R}^n)\) of profiles of front travelling at the parabolic speed \(c_{\text{esc}}\) such that, for every positive quantity \(L\),
\[
\|x \mapsto u(x_{\text{Esc-smooth}}(t_p) + x, t_p) - \phi(\sqrt{1 + \alpha c_{\text{esc}}^2 x})\|_{H^1([-L,L],\mathbb{R}^n)} \to 0
\]
when \(p\) approaches \(+\infty\). It follows from this assertion, from the definition of the quantity \(x_{\text{Esc-smooth}}(\cdot)\), and from the asymptotics (64) for \(U_{\text{small}}(\cdot)\), that
\[
|\phi(0)| = d_{\text{Esc}} \quad \text{and} \quad |\phi(y)| \leq d_{\text{Esc}} \quad \text{for every positive quantity}\ y.
\]
Thus, it follows from Lemma 41 on page 73 that
\[
|\phi(y)| < d_{\text{Esc}} \quad \text{for every positive quantity}\ y.
\]
in other words \(\phi\) actually belongs to the set \(\Phi_{\text{esc-norm}}(0 \mathbb{R}^n)\) of normalized profiles of front travelling at the parabolic speed \(c_{\text{esc}}\). It follows in addition from Lemma 41 that
\[
\phi(y) \cdot \phi'(y) < 0 \quad \text{for every } y \in [0, +\infty)
\]
and this shows that
\[
\lim_{p \to +\infty} |x_{\text{Esc-smooth}}(t_p) - x_{\text{Esc}}(t_p)| = 0.
\]
Thus case 1 above cannot hold.

On the other hand, since both \(\phi(\cdot)\) and \(u_{\text{smooth}}(\cdot;\cdot)\) are of class \(C^1\), it follows from the limit (67) and from the asymptotics (64) for \(U_{\text{small}}(\cdot)\) that
\[
u_{\text{smooth}}(x_{\text{Esc-smooth}}(t_p), t_p) \cdot \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t_p), t_p) \to \phi(0) \cdot \phi'(0)
\]
when \(p\) approaches \(+\infty\), and since this limit is a negative quantity, this shows that case 2 above cannot hold, a contradiction. Lemma 29 is proved. \(\Box\)
Lemma 30 (smoothness and asymptotic speed of $x_{\text{Esc-smooth}}(\cdot)$). The map $t \mapsto x_{\text{Esc-smooth}}(t)$ is of class $C^1$ on a neighbourhood of $+\infty$ and $x'_{\text{Esc-smooth}}(t) \to \sigma_{\text{esc}}$ when $t \to +\infty$.

Proof. Let us consider the function

$$f : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}, \quad (x, t) \mapsto \frac{1}{2} \left( u_{\text{smooth}}(x, t)^2 - d_{\text{Esc}}^2 \right).$$

According to the regularity of $u_{\text{smooth}}(\cdot, \cdot)$ (Lemma 27 on page 53), this function is of class at least $C^1$, and, for every sufficiently large time $t$, the quantity $f(x_{\text{Esc-smooth}}(t), t)$ is equal to zero, and

$$\partial_x f(x_{\text{Esc-smooth}}(t), t) = u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) \cdot \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) < 0.$$ 

Thus it follows from the implicit function theorem that the function $x \mapsto x_{\text{Esc-smooth}}(t)$ is of class (at least) a neighbourhood of $+\infty$, and that, for every sufficiently large time $t$,

$$x'_{\text{Esc-smooth}}(t) = - \frac{\partial_t f(x_{\text{Esc-smooth}}(t), t)}{\partial_x f(x_{\text{Esc-smooth}}(t), t)} = - \frac{u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) \cdot \partial_t u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t)}{u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t) \cdot \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t), t)}. \quad (68)$$

According to Lemma 29 above, the denominator of this expression remains bounded away from zero when $t$ approaches $+\infty$. On the other hand, according to Lemma 28 and to Proposition 8 on page 52 and to the asymptotics (64) for $U_{\text{small}}(\cdot)$ and to the a priori bounds (65) on $U_{\text{smooth}}(\cdot)$,

$$\partial_t u_{\text{smooth}}(x_{\text{Esc-smooth}}(t) + y, t) + \sigma_{\text{esc}} \partial_x u_{\text{smooth}}(x_{\text{Esc-smooth}}(t) + y, t) \to 0 \quad \text{when} \quad t \to +\infty.$$ 

Thus it follows from expression (68) above that $x'_{\text{Esc-smooth}}(t)$ approaches $\sigma_{\text{esc}}$ when time approaches $+\infty$. Lemma 30 is proved.

The next lemma is the only place throughout the proof of Proposition 4 where hypothesis (H_{\text{disc-}\Phi}) — which is part of the generic hypotheses (G) — is required.

Lemma 31 (convergence around Escape point). There exists a function $\phi$ in the set $\Phi_{\text{esc}, \text{norm}}(0_{\mathbb{R}^n})$ of (normalized) profiles of fronts travelling at speed $c_{\text{esc}}$ and invading the equilibrium $0_{\mathbb{R}^n}$ such that, for every positive quantity $L$,

$$\sup_{x \in [x_{\text{Esc}}(t) - L, x_{\text{Esc}}(t) + L]} \left| u(x, t) - \phi \left( (1 + \alpha_{\text{esc}}^2)(x - x_{\text{Esc-smooth}}(t)) \right) \right| \to 0$$

when $t$ approaches $+\infty$. In particular, the set $\Phi_{\text{esc}, \text{norm}}(0_{\mathbb{R}^n})$ is nonempty.
Proof. Take a sequence \((t_p)_{p \in \mathbb{N}}\) of positive times approaching \(+\infty\) when \(p\) approaches \(+\infty\). Proceeding as in the proof of Lemma 28 above, and according to this lemma, we may assume, up to replacing the sequence \((t_p)_{p \in \mathbb{N}}\) by a subsequence, that there exists a function \(\phi\) in the set \(\Phi_{c_{\text{esc}}}(0_{\mathbb{R}^n})\) of profiles of fronts travelling at speed \(c_{\text{esc}}\) and “invading” the local minimum \(0_{\mathbb{R}^n}\) such that, for every positive quantity \(L\),

\[
\left\| x \mapsto u(x_{\text{Esc-smooth}}(t_p) + x, t_p) - \phi(\sqrt{1 + \alpha c_{\text{esc}}^2 x}) \right\|_{H^1([4-L,L],\mathbb{R}^n)} \to 0 \quad \text{when} \quad p \to +\infty.
\]

According to the definition of \(x_{\text{Esc-smooth}}(\cdot)\) and to the asymptotics \((64)\) for \(U_{\text{small}}(\cdot)\), it follows that

\[
|\phi(0)| = d_{\text{Esc}} \quad \text{and} \quad |\phi(y)| \leq d_{\text{Esc}} \quad \text{for all} \quad y \in [0, +\infty),
\]

thus according to Lemma 41 on page 73, it follows that \(\phi\) actually belongs to the set \(\Phi_{c_{\text{esc}},\text{norm}}(0_{\mathbb{R}^n})\) of “normalized” profiles of fronts.

Let \(L\) denote the set of all possible limits (in the sense of uniform convergence on compact subsets of \(\mathbb{R}\)) of sequences of maps

\[
x \mapsto u(x_{\text{Esc-smooth}}(t'_p) + x, t'_p)
\]

for all possible sequences \((t'_p)_{p \in \mathbb{N}}\) such that \(t'_p\) approaches \(+\infty\) when \(p\) approaches \(+\infty\). This set \(L\) is included in the set \(\Phi_{c_{\text{esc}},\text{norm}}(0_{\mathbb{R}^n})\), and, because the semi-flow of system \((1)\) is continuous on \(X\), this set \(L\) is a continuum (a compact connected subset) of \(H^1_{ul}(\mathbb{R}, \mathbb{R}^n)\).

Since on the other hand — according to hypothesis \((H_{\text{disc-\Phi}})\) — the set \(\Phi_{c_{\text{esc}},\text{norm}}(0_{\mathbb{R}^n})\) is totally disconnected in \(H^1_{ul}(\mathbb{R}, \mathbb{R}^n)\), this set \(L\) must actually be reduced to the singleton \(\{\phi\}\). Lemma 31 is proved. \(\square\)

Lemma 32 (convergence up to \(x_{\text{hom}}(t)\)). For every positive quantity \(L\),

\[
\sup_{x \in [x_{\text{Esc}}(t) - L, x_{\text{hom}}(t)]} \left| u(x, t) - \phi\left(1 + \alpha c_{\text{esc}}^2 (x - x_{\text{Esc-smooth}}(t))\right) \right| \to 0 \quad \text{when} \quad t \to +\infty.
\]

Proof. Blabla... \(\square\)

4.13 Homogeneous point behind the travelling front

According to Lemma 40 on page 73 and hypothesis \((H_{\text{bist}})\), the limit

\[
\lim_{x \to -\infty} \phi(x)
\]

exists and belongs to \(\mathcal{M}\); let us denote by \(m_{\text{next}}\) this limit. According to the same Lemma 40

\[
V(m_{\text{next}}) < 0.
\]

The following lemma completes the proof of Proposition 4 (“invasion implies convergence”).
Lemma 33 ("next" homogeneous point behind the front). There exists a $\mathbb{R}$-valued function $x_{\text{hom-next}}$, defined and of class $C^1$ on a neighbourhood of $+\infty$, such that the following limits hold when $t$ approaches $+\infty$:

$$x_{\text{Esc}}(t) - x_{\text{hom-next}}(t) \to +\infty \quad \text{and} \quad x'_{\text{hom-next}}(t) \to \sigma_{\text{esc}}$$

and

$$\sup_{x \in [x_{\text{hom-next}}(t), x_{\text{hom}}(t)]} |u(x, t) - \phi((1 + \alpha^2_{\text{esc}})(x - x_{\text{Esc-smooth}}(t)))| \to 0,$$

and, for every positive quantity $L$,

$$\|x \mapsto (u(x_{\text{hom-next}}(t) + x, t) - m_{\text{next}}, u_t(x_{\text{hom-next}}(t) + x, t)\|_{H^1([-L,L], \mathbb{R}^n) \times L^2([-L,L], \mathbb{R}^n)}$$

approaches zero.

Proof. The proof is identical to the proof of lemma 35 of [14].

The proof of Proposition 4 is complete.

5 Non invasion implies relaxation

The aim of this section is to prove Proposition 9 below. For this purpose the generic hypotheses (G) are not required, thus let us assume that $V$ satisfies hypothesis (H$_{\text{coerc}}$) only. All the arguments of this section are very similar to those of section 5 of [14], where more details and comments can be found.

5.1 Definitions and hypotheses

Let us consider two minimum points $m_-$ and $m_+$ in $M$, a pair of functions (initial data) $(u_0, u_1)$ in $X$, and, for all $x$ in $\mathbb{R}$ and $t$ in $[0, +\infty)$, let $u(x, t)$ denote the corresponding solution.

Without assuming that this solution is bistable, let us make the following hypothesis (H$_{\text{hom}}$), which is similar to hypothesis (H$_{\text{hom-right}}$) made in section 4 ("invasion implies convergence"), but this time both to the right and to the left in space (see figure 13).

![Figure 13: Illustration of hypothesis (H$_{\text{hom}}$) and of Proposition 9](image)

(H$_{\text{hom}}$) There exist a positive quantity $\sigma_{\text{hom,+}}$ and a negative quantity $\sigma_{\text{hom,-}}$ and $C^1$-functions

$$x_{\text{hom,+}} : [0, +\infty) \to \mathbb{R} \quad \text{satisfying} \quad x'_{\text{hom,+}}(t) \to \sigma_{\text{hom,+}} \quad \text{when} \quad t \to +\infty$$

and

$$x_{\text{hom,-}} : [0, +\infty) \to \mathbb{R} \quad \text{satisfying} \quad x'_{\text{hom,-}}(t) \to \sigma_{\text{hom,-}} \quad \text{when} \quad t \to +\infty.$$
The aim of section 5 is to prove the following proposition.

5.2 Statement

Proposition 9 (non-invasion implies relaxation). Assume that $V$ satisfies hypothesis $(H_{coerc})$ (only) and that the solution $(x, t) \mapsto u(x, t)$ under consideration satisfies hypotheses $(H_{hom})$ and $(H_{no-inv})$. Then $V(m_-) = V(m_+)$ and there exists a nonnegative quantity $\mathcal{E}_\infty$ such that the following limits hold:

\[
\sup_{x \in [x_{hom, -}(t), x_{hom, +}(t)]} \int_{x-1}^{x+1} u_t(z, t)^2 \, dz \to 0
\]

and

\[
\int_{x_{hom, -}(t)}^{x_{hom, +}(t)} \left[ \frac{u_x(x, t)^2}{2} + V(u(x, t)) - V(m_\pm) \right] \, dx \to \mathcal{E}_\infty
\]

when $t$ approaches $+\infty$.
5.3 Settings of the proof

Let us keep the notation and assumptions of subsection 5.1, and let us assume that hypotheses \((H_{\text{coerc}})\) and \((H_{\text{hom}})\) and \((H_{\text{no-inv}})\) of Proposition 9 hold. Before doing anything else, let us clean up the place.

- For notational convenience, let us assume, without loss of generality, that

\[
\max(V(m_-), V(m_+)) = 0.
\]

- According to Proposition 2 on page 12, we may assume (without loss of generality, up to changing the origin of times) that, for all \(t\) in \([0, +\infty)\),

\[
\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att}, \infty}.
\]

5.4 Relaxation scheme in a standing or almost standing frame

The aim of this subsection is to set up an appropriate relaxation scheme in a standing or almost standing frame. This means defining an appropriate localized energy and controlling the “flux” terms occurring in the time derivative of that localized energy. The argument will be quite similar to that of subsection 4.7 on page 29 (the relaxation scheme in the travelling frame), the main difference being that speed of the travelling frame will now be either equal or close to zero, and as a consequence the weight function for the localized energy will be defined with a cut-off on the right and another on the left, instead of a single one; accordingly firewall functions will be introduced to control the fluxes along each of these cuts-off.

5.4.1 Preliminary definitions

Let us keep the notation and hypotheses introduced above (since the beginning of subsection 5.3), and, as in subsection 4.7 on page 29, let us introduce as parameters the “parabolic” speed \(c\) of the travelling frame and its “physical” speed \(\sigma\) related by

\[
\sigma = \frac{c}{\sqrt{1 + \alpha c^2}} \Leftrightarrow c = \frac{\sigma}{\sqrt{1 - \alpha \sigma^2}}.
\]

By contrast with subsection 4.7, the other parameters — namely \(t_{\text{init}}\) and \(x_{\text{init}}\) and \(y_{\text{cut-init}}\) — are not be required here. The relaxation scheme will be applied in the next subsection for a speed \(c\) very close or equal to zero.

Let us consider the function \((y, t) \mapsto v(y, t)\), defined for every real quantity \(y\) and every nonnegative time \(t\) by

\[v(y, t) = u(x, t)\]

where \(x\) and \(y\) are related by:

\[x = \sigma t + \frac{y}{\sqrt{1 + \alpha c^2}} \Leftrightarrow y = \sqrt{1 + \alpha c^2}x - ct.\]
The evolution equation for the function \((y, t) \mapsto v(y, t)\) reads
\[
\alpha v_{tt} + v_t - 2\alpha c v_y = -\nabla V(v) + cv_y + v_{yy}.
\]

We are going to define a localized energy and two firewall functions associated with this solution. As in sub-subsection 4.7.1 on page 29, we are going to use the quantities \(\kappa\) and \(c_{\text{cut}}\), with exactly the same definitions as the ones following conditions (38) and (39) on page 30. Let
\[
c_{\text{cut},0} = \min\left( c_{\text{cut}}, \frac{\sigma_{\text{hom}},+}{2}, \frac{|\sigma_{\text{hom}},-|}{2} \right),
\]
and let us make on the parameter \(c\) the following hypotheses:
\[
|c| \leq \frac{\kappa}{6} \quad \text{and} \quad |c| \leq c_{\text{max}} \quad \text{and} \quad |c| \leq \frac{1}{\sqrt{\alpha}} \quad \text{and} \quad |c| \leq \frac{c_{\text{cut},0}}{6}.
\]

According to \((H_{\text{hom}})\) and \((H_{\text{no-inv}})\) and to the choice of \(c_{\text{cut},0}\) above, we may assume, up to changing the origin of time, that, for all \(t\) in \([0, +\infty)\),
\[
\begin{align*}
x_{\text{hom}},-(t) &\leq -\frac{11}{6}c_{\text{cut},0}t \quad \text{and} \quad -\frac{1}{6\sqrt{2}}c_{\text{cut},0}t \leq x_{\text{Esc},-}(t) \\
\text{and} \quad x_{\text{Esc},+}(t) &\leq \frac{1}{6\sqrt{2}}c_{\text{cut},0}t \quad \text{and} \quad \frac{11}{6}c_{\text{cut},0}t \leq x_{\text{hom},+}(t).
\end{align*}
\]

For every nonnegative time \(t\), let
\[
\begin{align*}
y_{\text{hom},+}(t) &= \sqrt{1 + \alpha c^2}x_{\text{hom},+}(t) - ct \quad \text{and} \quad y_{\text{hom},-}(t) = \sqrt{1 + \alpha c^2}x_{\text{hom},-}(t) - ct \\
\text{and} \quad y_{\text{Esc},+}(t) &= \sqrt{1 + \alpha c^2}x_{\text{Esc},+}(t) - ct \quad \text{and} \quad y_{\text{Esc},-}(t) = \sqrt{1 + \alpha c^2}x_{\text{Esc},-}(t) - ct.
\end{align*}
\]

Observe that according to (71) the quantity \(\sqrt{1 + \alpha c^2}\) is not larger than \(\sqrt{2}\). As a consequence it follows from hypotheses (72) and from the two last hypotheses of (71) that, for every nonnegative time \(t\),
\[
\begin{align*}
y_{\text{hom},-}(t) &\leq -\frac{5}{3}c_{\text{cut},0}t \quad \text{and} \quad -\frac{1}{3}c_{\text{cut},0}t \leq y_{\text{Esc},-}(t) \\
\text{and} \quad y_{\text{Esc},+}(t) &\leq \frac{1}{3}c_{\text{cut},0}t \quad \text{and} \quad \frac{5}{3}c_{\text{cut},0}t \leq y_{\text{hom},+}(t)
\end{align*}
\]
(see figure 14).

5.4.2 Localized energy

For every nonnegative time \(t\), let us consider the three intervals:
\[
\begin{align*}
I_{\text{left}}(t) &= (-\infty, -c_{\text{cut},0}t], \\
I_{\text{main}}(t) &= [-c_{\text{cut},0}t, c_{\text{cut},0}t], \\
I_{\text{right}}(t) &= [c_{\text{cut},0}t, +\infty),
\end{align*}
\]
Figure 14: Illustration of setting assumptions for the proof of Proposition 9.

and let us consider the function $\chi(y, t)$ (weight function for the localized energy) defined by:

$$
\chi(y, t) = \begin{cases} 
\exp(-c\text{cut,0}t + \kappa(y + \text{cut,0}t)) & \text{if } y \in I_{\text{left}}(t), \\
\exp(cy) & \text{if } y \in I_{\text{main}}(t), \\
\exp(c\text{cut,0}t - \kappa(y - \text{cut,0}t)) & \text{if } y \in I_{\text{right}}(t),
\end{cases}
$$

(see figure 15), and, for all $t$ in $[0, +\infty)$, let us define the “energy function” by:

$$
\mathcal{E}(t) = \int_{\mathbb{R}} \chi(y, t) \left( \alpha \frac{v_t(y, t)^2}{2} + \frac{v_y(y, t)^2}{2} + V(v(y, t)) \right) dy.
$$

**5.4.3 Time derivative of the localized energy**

It follows from the definition of $\chi$ that:

$$
\chi_t(y, t) = \begin{cases} 
cut,0(-c + \kappa)\chi(y, t) & \text{if } y \in I_{\text{left}}(t), \\
0 & \text{if } y \in I_{\text{main}}(t), \\
cut,0(c + \kappa)\chi(y, t) & \text{if } y \in I_{\text{right}}(t),
\end{cases}
$$

and

$$
(c \chi - \chi_y)(y, t) = \begin{cases} 
(c - \kappa)\chi(y, t) & \text{if } y \in I_{\text{left}}(t), \\
0 & \text{if } y \in I_{\text{main}}(t), \\
(c + \kappa)\chi(y, t) & \text{if } y \in I_{\text{right}}(t).
\end{cases}
$$
Thus, if for all \( t \) in \([0, +\infty)\) we define the “dissipation” function by

\[
\mathcal{D}(t) = \int_{\mathbb{R}} \chi(y, t) v_t(y, t)^2 \, dy,
\]

then, for all \( t \) in \([0, +\infty)\), it follows from expression (12) on page 15 (time derivative of a localized energy) that

\[
\mathcal{E}'(t) = -(1 + \alpha c^2) \mathcal{D}(t)
+ \int_{I_{\text{left}}(t)} \chi \left( c_{\text{cut}, 0} (-c + \kappa) \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \frac{\alpha c_{\text{cut}}}{2} + \frac{v_t^2}{2} + \frac{v_y^2}{2} + V(v) \right) + \alpha c(c - \kappa)v_t^2 + (c - \kappa)v_y \cdot v_t \right) \, dy
+ \int_{I_{\text{right}}(t)} \chi \left( c_{\text{cut}, 0} (c + \kappa) \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \frac{\alpha c_{\text{cut}}}{2} + \frac{v_t^2}{2} + \frac{v_y^2}{2} + V(v) \right) + \alpha c(c + \kappa)v_t^2 + (c + \kappa)v_y \cdot v_t \right) \, dy.
\]

Thus, since \(|c| < \kappa\) and since \( \max(V(m_-), V(m_+)) = 0 \)

\[
\mathcal{E}'(t) \leq -(1 + \alpha c^2) \mathcal{D}(t)
+ \int_{I_{\text{left}}(t)} (\kappa - c) \chi \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \alpha |c| + \frac{1}{2} v_t^2 + \frac{c_{\text{cut}, 0}}{2} + \frac{1}{2} v_y^2 \right) \, dy
+ c_{\text{cut}, 0} (V(v) - V(m_-)) \, dy
+ \int_{I_{\text{right}}(t)} (\kappa + c) \chi \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \alpha |c| + \frac{1}{2} v_t^2 + \frac{c_{\text{cut}, 0}}{2} + \frac{1}{2} v_y^2 \right) \, dy
+ c_{\text{cut}, 0} (V(v) - V(m_+)) \, dy.
\]

Adding and subtracting the same quantity to the right-hand side of this inequality, it follows that

\[
\mathcal{E}'(t) \leq -(1 + \alpha c^2) \mathcal{D}(t)
+ \int_{I_{\text{left}}(t)} (\kappa - c) \chi c_{\text{cut}, 0} (V(v) - V(m_-) - \lambda_{\text{max}}(v - m_-)^2) \, dy
+ \int_{I_{\text{right}}(t)} (\kappa + c) \chi c_{\text{cut}, 0} (V(v) - V(m_+) - \lambda_{\text{max}}(v - m_+)^2) \, dy
+ \int_{I_{\text{left}}(t)} (\kappa - c) \chi \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \alpha |c| + \frac{1}{2} v_t^2 + \frac{c_{\text{cut}, 0}}{2} + \frac{1}{2} v_y^2 \right) \, dy
+ c_{\text{cut}, 0} \lambda_{\text{max}}(v - m_-)^2 \, dy
+ \int_{I_{\text{right}}(t)} (\kappa + c) \chi \left( \frac{\alpha c_{\text{cut}, 0}}{2} + \alpha |c| + \frac{1}{2} v_t^2 + \frac{c_{\text{cut}, 0}}{2} + \frac{1}{2} v_y^2 \right) \, dy
+ c_{\text{cut}, 0} \lambda_{\text{max}}(v - m_+)^2 \, dy.
\]
Let
\[ K_{\epsilon,\text{Esc},1} = \max_{m \in M, u \in \mathbb{R}^n, |u| \leq R_{\text{att},\infty}} (c_{\text{max}} + \kappa)c_{\text{cut},0}(V(u) - V(m) - \lambda_{\text{max}}(u - m)^2) \]

(observe that this quantity \( K_{\epsilon,\text{Esc},1} \) is nonnegative), let us define the quantity \( K_{\epsilon,\mathcal{Q}} \) as in sub-subsection 4.7.3 on page 31, and, for every nonnegative time \( t \), let
\[ \Sigma_{\text{Esc},+}(t) = \{ y \in \mathbb{R} : |v(y,t) - m_+| > d_{\text{Esc}} \} \]
and
\[ \Sigma_{\text{Esc},-}(t) = \{ y \in \mathbb{R} : |v(y,t) - m_-| > d_{\text{Esc}} \} . \]

Proceeding as in sub-subsection 4.7.3, it follows that
\[ \mathcal{E}'(t) \leq - (1 + \alpha \epsilon^2) \mathcal{D}(t) \]
\[ + K_{\epsilon,\text{Esc},1} \int_{I_{\text{left}}(t) \cap \Sigma_{\text{Esc},-}(t)} \chi(y,t) \, dy + K_{\epsilon,\mathcal{Q}} \int_{I_{\text{left}}(t)} \chi(y,t) \bigl(v_+^2 + v_-^2 + (v - m_-)^2\bigr) \, dy \]
\[ + K_{\epsilon,\text{Esc},1} \int_{I_{\text{right}}(t) \cap \Sigma_{\text{Esc},+}(t)} \chi(y,t) \, dy + K_{\epsilon,\mathcal{Q}} \int_{I_{\text{right}}(t)} \chi(y,t) \bigl(v_+^2 + v_-^2 + (v - m_+)^2\bigr) \, dy . \]

### 5.4.4 Definition of the “firewall” functions and bound on the time derivative of energy

Proceeding as in sub-subsection 4.7.4 on page 32, we are going to define two firewall functions to control the right-hand side of this inequality. For every real quantity \( y \) and every nonnegative quantity \( t \), let
\[
\psi_+(y,t) = \begin{cases} 
\exp\left(c_{\text{cut},0}t + \kappa(y - c_{\text{cut},0}t)\right) & \text{if } y \in I_{\text{left}}(t) \cup I_{\text{main}}(t), \\
\chi(y,t) & \text{if } y \in I_{\text{right}}(t) \cup I_{\text{main}}(t) \cup I_{\text{right}}(t) 
\end{cases}
\]
\[
\psi_-(y,t) = \begin{cases} 
\chi(y,t) & \text{if } y \in I_{\text{left}}(t), \\
\exp\left(-c_{\text{cut},0}t - \kappa(y + c_{\text{cut},0}t)\right) & \text{if } y \in I_{\text{main}}(t) \cup I_{\text{right}}(t) 
\end{cases}
\]
(see figure 15), and let
\[
F_+(y,t) = \alpha^2 v_+^2 + \alpha v_y^2 + 2\alpha (V(v) - V(m_+)) + \alpha v \cdot v_t + \left(\frac{1}{2} + \alpha c_{\text{cut},0} + \kappa\right) (v - m_+)^2 ,
\]
\[
F_-(y,t) = \alpha^2 v_-^2 + \alpha v_y^2 + 2\alpha (V(v) - V(m_-)) + \alpha v \cdot v_t + \left(\frac{1}{2} + \alpha c_{\text{cut},0} + \kappa\right) (v - m_-)^2 .
\]

(The argument of every function on the right-hand side being \((y,t)\), and finally let
\[
\mathcal{F}_+(t) = \int_{\mathbb{R}} \psi_+(y,t) F_+(y,t) \, dy \quad \text{and} \quad \mathcal{F}_-(t) = \int_{\mathbb{R}} \psi_-(y,t) F_+(y,t) \, dy .
\]

Besides, let
\[
\mathcal{Q}_+(t) = \int_{\mathbb{R}} \psi_+(y,t) \left(v_t(y,t)^2 + v_y(y,t)^2 + (v(y,t) - m_+)^2\right) \, dy ,
\]
\[
\mathcal{Q}_-(t) = \int_{\mathbb{R}} \psi_-(y,t) \left(v_t(y,t)^2 + v_y(y,t)^2 + (v(y,t) - m_-)^2\right) \, dy .
\]

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The following lemma is almost identical to Lemma [10] on page 33 and makes use of
the notation \( \varepsilon_{F,\text{coerc}} \) introduced there.

**Lemma 34** (firewall coercivity up to pollution term). There exist a nonnegative quantity
\( K_{F,\text{coerc}} \), depending only on \( \alpha \) and \( V \), such that, for every nonnegative quantity \( t \),

\[
\mathcal{F}_+(t) \geq \varepsilon_{F,\text{coerc}} \mathcal{Q}_+(t) - K_{F,\text{coerc}} \int_{\Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy
\]

and

\[
\mathcal{F}_-(t) \geq \varepsilon_{F,\text{coerc}} \mathcal{Q}_-(t) - K_{F,\text{coerc}} \int_{\Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy.
\]

**Proof.** We proceed as in the proof of Lemma [10]. For every real quantity \( y \) and every
nonnegative quantity \( t \),

\[
F(y, t) \geq \frac{\alpha}{2} v_t^2 + \alpha v_y^2 + \alpha \left( 2(V(v) - V(m_+)) - \kappa|c|(v - m_+)^2 \right)
\]

\[
\geq \frac{\alpha}{2} v_t^2 + \alpha v_y^2 + \frac{\alpha \lambda_{\text{min}}}{4} v^2 + \alpha \left( 2(V(v) - V(m_+)) - \left( \kappa|c| + \frac{\lambda_{\text{min}}}{4} \right)(v - m_+)^2 \right).
\]

According to properties (17) on page 16 derived from the definition of \( d_{\text{Esc}} \) the quantity

\[
2(V(v) - V(m_+)) - \left( \kappa|c| + \frac{\lambda_{\text{min}}}{4} \right)(v - m_+)^2
\]

is nonpositive for \( y \) in \( \mathbb{R} \setminus \Sigma_{\text{Esc},+}(s) \). Thus, if we consider the (nonnegative) quantity

\[
K_{F,\text{coerc}} = - \min_{m \in \mathcal{M}, u \in \mathbb{R}^n, |u| \leq R_{\text{att},\infty}} \frac{\alpha}{2} \left( 2(V(u) - V(m)) - \left( \kappa|c| + \frac{\lambda_{\text{min}}}{4} \right)(u - m)^2 \right),
\]

the first inequality of (69) follows. The same can be made for the second inequality.

**Lemma 35** (approximate decrease of energy). There exist nonnegative quantities \( K_{\varepsilon,F} \)
and \( K_{\varepsilon,\text{Esc}} \) such that, for every nonnegative time \( t \),

\[
\mathcal{E}'(t) \leq - (1 + \alpha^2) \mathcal{D}(t) + K_{\varepsilon,F} \left( \mathcal{F}_+(t) + \mathcal{F}_-(t) \right)
\]

\[
+ K_{\varepsilon,\text{Esc}} \left( \int_{\Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy + \int_{\Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy \right).
\]

**Proof.** For every nonnegative time \( t \), since \( \chi(y, t) = \psi_+(y, t) \) for all \( y \) in \( I_{\text{right}}(t) \) and
\( \chi(y, t) = \psi_-(y, t) \) for all \( y \) in \( I_{\text{left}}(t) \), it follows from inequality (68) that

\[
\mathcal{E}'(t) \leq - (1 + \alpha^2) \mathcal{D}(t)
\]

\[
+ K_{\varepsilon,\text{Esc},1} \left( \int_{I_{\text{left}}(t) \cap \Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy + \int_{I_{\text{right}}(t) \cap \Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy \right)
\]

\[
+ K_{\varepsilon,\text{Q}} \left( \int_{I_{\text{left}}(t)} \psi_-(y, t) \left( v_s^2 + v_y^2 + (v - m_-)^2 \right) \, dy + \int_{I_{\text{right}}(t)} \psi_+(y, t) \left( v_s^2 + v_y^2 + (v - m_+)^2 \right) \, dy \right),
\]

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thus
\[ E'(t) \leq -(1 + \alpha c^2) D(t) + K E_{\text{Esc},1} \left( \int_{\Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy + \int_{\Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy \right) + K E_{\text{Q}} (Q_-(t) + Q_+(t)). \]

Thus, defining \( K E \) and \( K E_{\text{Esc}} \) as in the proof of Lemma 11 on page 34, inequality (77) follows from (76). Lemma 35 is proved.

5.4.5 Time derivative of the firewall functions

Lemma 36 (firewall decrease up to pollution term). There exist a positive quantity \( \varepsilon_{F, \text{decr}} \) and a nonnegative quantity \( K F_{\text{decr}} \), depending only on \( \alpha \) and \( V \), such that for every nonnegative quantity \( t \),

\begin{align*}
F_+''(t) &\leq -\varepsilon_{F, \text{decr}} F_+(t) + K F_{\text{decr}} \int_{\Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy \\
F_-''(t) &\leq -\varepsilon_{F, \text{decr}} F_-(t) + K F_{\text{decr}} \int_{\Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy.
\end{align*}

(78)

Proof. The proof is identical to that of Lemma 12 on page 35. \( \square \)

For every nonnegative quantity \( t \), let
\[ G_+(t) = \int_{\Sigma_{\text{Esc},+}(t)} \psi_+(y, t) \, dy \quad \text{and} \quad G_-(t) = \int_{\Sigma_{\text{Esc},-}(t)} \psi_-(y, t) \, dy. \]

According to the definition of \( x_{\text{Esc},+}(t) \) and \( x_{\text{Esc},-}(t) \), for all \( t \in [0, +\infty) \),
\[ \Sigma_{\text{Esc},+}(t) \subset (-\infty, y_{\text{Esc},+}(t)] \cup [y_{\text{hom},+}(t), +\infty) \]
\[ \text{and} \quad \Sigma_{\text{Esc},-}(t) \subset (-\infty, y_{\text{hom},-}(t)] \cup [y_{\text{Esc},-}(t), +\infty), \]
thus, if we consider the quantities
\[ G_{\text{front},+}(t) = \int_{y_{\text{hom},+}(t)}^{+\infty} \psi_+(y, t) \, dy \quad \text{and} \quad G_{\text{back},+}(t) = \int_{-\infty}^{y_{\text{Esc},+}(t)} \psi_+(y, t) \, dy, \]
\[ G_{\text{front},-}(t) = \int_{-\infty}^{y_{\text{hom},-}(t)} \psi_-(y, t) \, dy \quad \text{and} \quad G_{\text{back},-}(t) = \int_{y_{\text{hom},-}(t)}^{+\infty} \psi_-(y, t) \, dy, \]
then, for every nonnegative quantity \( t \),
\[ G_+(t) \leq G_{\text{front},+}(t) + G_{\text{back},+}(t) \quad \text{and} \quad G_-(t) \leq G_{\text{front},-}(t) + G_{\text{back},-}(t). \]

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According to the definition of $\psi_+$ and $\psi_-$ and according to hypotheses (71) and inequalities (73) on page 61 it follows from explicit calculations that:

\[
\begin{align*}
G_{\text{front},+}(t) &\leq \frac{1}{\kappa} \exp\left( c_{\text{cut},0}(c + \kappa)t - \kappa y_{\hom,+}(t) \right) \leq \frac{1}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right), \\
G_{\text{back},+}(t) &\leq \frac{1}{\kappa} \exp\left( c_{\text{cut},0}(c - \kappa)t + \kappa y_{\Esc,+}(t) \right) \leq \frac{1}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right), \\
G_{\text{front},-}(t) &\leq \frac{1}{\kappa} \exp\left( c_{\text{cut},0}(-c + \kappa)t + \kappa y_{\hom,-}(t) \right) \leq \frac{1}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right), \\
G_{\text{back},-}(t) &\leq \frac{1}{\kappa} \exp\left( c_{\text{cut},0}(-c - \kappa)t - \kappa y_{\Esc,-}(t) \right) \leq \frac{1}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right).
\end{align*}
\]

As a consequence it follows from inequality (78) that, for all $t$ in $[0, +\infty)$,

\[
F_{\pm}(t) \leq -\varepsilon_{F,\text{decr}} F_{\pm}(0) + \frac{2 K_{F,\text{decr}}}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right).
\]

Let us consider the quantity:

\[
\tilde{\varepsilon}_{F,\text{decr}} = \min\left( \varepsilon_{F,\text{decr}}, \frac{\kappa c_{\text{cut},0}}{4} \right).
\]

It follows from the last inequality that, for all $t$ in $[0, +\infty)$,

\[
(79) \quad F_{\pm}(t) \leq \left( F_{\pm}(0) + \frac{4 K_{F,\text{decr}}}{\kappa^2 c_{\text{cut},0}} \right) \exp(-\tilde{\varepsilon}_{F,\text{decr}} t).
\]

According to Proposition 2 on page 12 (existence of an attracting ball for the semi-flow), there exists a positive quantity $K_{F,\text{init}}$, depending only on $\alpha$ and $V$, such that, up to changing the origin of times, the following estimates hold:

\[
F_{+}(0) \leq K_{F,\text{init}} \quad \text{and} \quad F_{-}(0) \leq K_{F,\text{init}}.
\]

Thus, if we consider the following nonnegative quantity:

\[
K_{\varepsilon,\text{final}} = 2 K_{\varepsilon,F} \left( K_{F,\text{init}} + \frac{4 K_{F,\text{decr}}}{\kappa^2 c_{\text{cut},0}} \right) + \frac{4 K_{\varepsilon,\Esc}}{\kappa},
\]

then it follows from inequalities (77) and (79) that, for every nonnegative quantity $t$,

\[
(80) \quad \varepsilon'(t) \leq -(1 + \alpha c^2) D(t) + K_{\varepsilon,\text{final}} \exp(-\tilde{\varepsilon}_{F,\text{decr}} t).
\]

This last inequality is the key ingredient that will be applied in the next subsection 5.5.

### 5.5 Lower bound on localized energy

We keep the notation adopted and the hypotheses made since the beginning of section 5. For every quantity $c$ sufficiently close to 0 so that hypotheses (71) on page 61 be satisfied, and for every nonnegative quantity $t$ and real quantity $y$, let us denote by $v^{(c)}(\cdot, \cdot)$ and $\chi^{(c)}(\cdot, \cdot)$ and $\varepsilon^{(c)}(\cdot)$ and $D^{(c)}(\cdot)$...
the objects that were defined in subsection 5.4 (with the same notation except the “(c)” superscript that is here to remind that these objects depend on the quantity $c$). For every such $c$, let us consider the quantity $\mathcal{E}^{(c)}(+\infty)$ in $\mathbb{R} \cup \{-\infty\}$ defined by:

$$\mathcal{E}^{(c)}(+\infty) = \liminf_{t \to +\infty} \mathcal{E}^{(c)}(t).$$

According to estimate (80) on the time derivative of the energy, for every such $c$,

$$\mathcal{E}^{(c)}(t) \to \mathcal{E}^{(c)}(+\infty) \quad \text{when} \quad t \to +\infty,$$

and let us call “asymptotic energy at speed $c$” this quantity. The aim of this subsection is to prove the following proposition.

**Proposition 10** (nonnegative asymptotic energy). *The quantity $\mathcal{E}^{(0)}(+\infty)$ (the asymptotic energy at speed zero) is nonnegative.*

The proof proceeds through the following lemmas and corollaries, that are rather direct consequences of the relaxation scheme set up in the previous subsection 5.4, and in particular of the estimate (80) on the time derivative of the energy.

Since according to hypothesis (69) on page 60 the maximum of $V(m_+)$ and $V(m_-)$ is assumed to be equal to zero, we may assume (without loss of generality) that:

$$V(m_+) = 0.$$ 

**Lemma 37** (nonnegative asymptotic energy in frames travelling at small nonzero speed). *For every quantity $c$ sufficiently close to zero so that hypotheses (71) on page 61 be satisfied, if in addition $c$ is positive, then

$$\mathcal{E}^{(c)}(+\infty) \geq 0.$$*

**Proof.** Let $c$ be a positive quantity, sufficiently close to zero so that hypotheses (71) be satisfied. With the notation of subsection 5.4 (for the relaxation scheme in a frame travelling at speed $c$), for all $t$ in $[0, +\infty)$,

$$\mathcal{E}^{(c)}(t) = \int_{\mathbb{R}} \chi^{(c)}(y, t) \left( \alpha \frac{v_x^{(c)}(y, t)^{2}}{2} + \frac{v_y^{(c)}(y, t)^{2}}{2} + V(v^{(c)}(y, t)) \right) dy$$

$$\geq \int_{\mathbb{R}} \chi^{(c)}(y, t) V(v^{(c)}(y, t)) dy$$

$$\geq \int_{\sigma_{E_n, +}(t)} \chi^{(c)}(y, t) V(v^{(c)}(y, t)) dy.$$

Thus, if we consider the global minimum value of $V$:

$$V_{\min} = \min_{u \in \mathbb{R}^n} V(u),$$

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then, for every nonnegative quantity \( t \), according to hypotheses (72) on page 61,

\[
\mathcal{E}^{(c)}(t) \geq V_{\min} \int \chi^{(c)}(y, t) \, dy
\]

\[
\geq V_{\min} \left( \int_{-\infty}^{x_{\text{Esc},+}(t) - ct} \chi^{(c)}(c, y, t) \, dy + \int_{x_{\text{hom},+}(t) - ct}^{+\infty} \chi^{(c)}(y, t) \, dy \right)
\]

\[
\geq V_{\min} \left( \frac{1}{c} \exp \left( c(x_{\text{Esc},+}(t) - ct) \right) + \frac{1}{c} \exp \left( c_{\text{cut},0}(c + \kappa)t - \kappa(x_{\text{hom},+}(t) - ct) \right) \right)
\]

\[
\geq V_{\min} \left( \frac{1}{c} \exp \left( c(x_{\text{Esc},+}(t) - ct) \right) + \frac{1}{c} \exp \left( -\frac{\kappa c_{\text{cut},0}}{2}t \right) \right),
\]

and the conclusion follows.

\[\square\]

**Corollary 1** (almost nonnegative energy in a travelling frame). For every quantity \( c \) sufficiently close to zero so that hypotheses (71) on page 61 be satisfied, if in addition \( c \) is positive, then, for every nonnegative quantity \( t \),

\[
\mathcal{E}^{(c)}(t) \geq -\frac{K_{\varepsilon,\text{final}}}{\varepsilon_{F,\text{decr}}} \exp(-\varepsilon_{F,\text{decr}}t).
\]

**Proof.** The proof follows readily from previous Lemma 37 and inequality (80).

\[\square\]

**Lemma 38** (continuity of energy with respect to the speed at \( c = 0 \)). For every nonnegative quantity \( t \),

\[
\mathcal{E}^{(c)}(t) \to \mathcal{E}^{(0)}(t) \quad \text{when} \quad c \to 0.
\]

**Proof.** For all \( t \) in \((0, +\infty)\),

\[
\mathcal{E}^{(0)}(t) = \int_{\mathbb{R}} \chi^{(0)}(x, t) \left( \alpha \frac{u_t(x, t)^2}{2} + \frac{u_x(x, t)^2}{2} + V(u(x, t)) \right) \, dx,
\]

and, for every quantity \( c \) sufficiently close to zero so that hypotheses (71) on page 61 be satisfied,

\[
\mathcal{E}^{(c)}(t) = \int_{\mathbb{R}} \chi^{(c)}(y, t) \left( \alpha \frac{v_t^{(c)}(y, t)^2}{2} + \frac{v_y^{(c)}(y, t)^2}{2} + V(v^{(c)}(y, t)) \right) \, dy.
\]

Thus, since \( v^{(c)}(\cdot, \cdot) \) is related to \( u(\cdot, \cdot) \) by

\[
u(x, t) = v^{(c)}(y, t) \quad \text{where} \quad y = \sqrt{1 + \alpha c^2} x - ct,
\]

it follows that

\[
\mathcal{E}^{(c)}(t) = \int_{\mathbb{R}} \chi^{(c)}(\sqrt{1 + \alpha c^2} x - ct, t) \left( \frac{\alpha}{2} \left( u_t(x, t) + c\frac{u_x(x, t)}{\sqrt{1 + \alpha c^2}} \right)^2 + \frac{1}{2(1 + \alpha c^2)} u_x(x, t)^2 + V(u(x, t)) \right) \sqrt{1 + \alpha c^2} \, dx.
\]

The result thus follows from the continuity of \( \chi^{(c)}(\cdot, \cdot) \) with respect to \( c \) and from the on the derivatives of \( u(\cdot, \cdot) \) ensured by Proposition 2 on page 12. 

\[\square\]
Corollary 2 (almost nonnegative energy in a standing frame). For every nonnegative quantity \( t \),
\[
\mathcal{E}^{(0)}(t) \geq -\frac{K_{\mathcal{F}_{\text{final}}} \exp(-\tilde{\epsilon}_{\mathcal{F}_{\text{dec}}})}{\tilde{\epsilon}_{\mathcal{F}_{\text{dec}}}}.
\]

Proof. The proof follows readily from Corollary 1 and Lemma 38.

Proposition 10 ("nonnegative asymptotic energy") follows from Corollary 2.

5.6 End of the proof of Proposition 9

According to the estimate (80) on the time derivative of energy, it follows from Proposition 10 that the map
\[
t \mapsto \mathcal{D}^{(0)}(t)
\]
is integrable on \([0, +\infty)\).

Corollary 3 (relaxation — center area). The following limit holds:
\[
\sup_{x \in [-c_{\text{cut}}, 0]} \int_{x-1}^{x+1} u_t(z, t)^2 \, dz \to 0 \quad \text{when} \quad t \to +\infty.
\]

Proof. Let us proceed by contradiction and assume that the converse holds. Then there exists a positive quantity \( \varepsilon \) and a sequence \( (x_p, t_p)_{p \in \mathbb{N}} \) in \( \mathbb{R} \times \mathbb{R_+} \) such that \( t_p \) approaches \( +\infty \) when \( p \) approaches \( +\infty \) and such that, for every integer \( p \), \( x_p \) is in the interval \([-c_{\text{cut}}, 0], c_{\text{cut}}, 0]\] and
\[
\int_{-1}^{1} u_t(x_p + z, t_p)^2 \, dz \geq \varepsilon.
\]

According to Proposition 3 on page 12 ("asymptotic compactness"), up to replacing the sequence \((x_p, t_p)_{p \in \mathbb{N}}\) by a subsequence, we may assume that the sequence of functions \((u, u_t)(x_p + \cdot, t_p + \cdot)\) converges in the space
\[
C^0([-1, 1], H^1([-1, 1], \mathbb{R}^n) \times L^2([-1, 1], \mathbb{R}^n))
\]
to some limit \((\bar{u}, \bar{u}_t)\) that satisfies system (1). Thus
\[
\int_{-1}^{1} \bar{u}_t(z, 0)^2 \, dz \geq \varepsilon.
\]
As a consequence,
\[
\int_{-1}^{1} \left( \int_{-1}^{1} \bar{u}_t(z, 0)^2 \, dz \right) dt > 0,
\]
and as a consequence
\[
\liminf_{p \to +\infty} \int_{-1}^{1} \left( \int_{-1}^{1} u_t(x_p + z, t_p + t)^2 \, dz \right) dt > 0,
\]
a contradiction with the fact that \( t \mapsto \mathcal{D}^{(0)}(t) \) is integrable on \([0, +\infty)\). Corollary 3 is proved.

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Lemma 39 (relaxation — non center area). For every positive quantity \( \varepsilon \), both quantities

\[
\sup_{x \in [x_{\text{hom}}(t), -\varepsilon t]} \int_{x-1}^{x+1} \left( u_t(z,t)^2 + u_x(z,t)^2 + (u(z,t) - m_-)^2 \right) dz \\
\text{and} \sup_{x \in [x_{\text{hom}}(t), \varepsilon t]} \int_{x-1}^{x+1} \left( u_t(z,t)^2 + u_x(z,t)^2 + (u(z,t) - m_+)^2 \right) dz
\]

(82)
approach 0 when \( t \) approaches \( +\infty \).

Proof. Since the distance between the interval \([x_{\text{hom}}(t), -\varepsilon t]\) and the set \( \Sigma_{\text{Esc},-}(t) \) and the distance between the interval \([\varepsilon t, x_{\text{hom}}(t)]\) and the set \( \Sigma_{\text{Esc},+}(t) \) both approach \( +\infty \) when \( t \) approaches \( +\infty \), assertion (82) can be derived (for instance) from inequality (25) of Lemma 2 on page 21 (“firewall decrease up to pollution term” in the laboratory frame) and inequality (24) of Lemma 1 on page 21 (“firewall coercivity up to pollution term” in the laboratory frame).

In view of Proposition 10 and Corollary 3 and Lemma 39, the sole assertion of Proposition 9 that remains to prove is the fact that \( V(m_-) \) is nonnegative. But if \( V(m_-) \) was negative, then, according to Lemma 39 above (and according to the a priori bounds on the solution stated in Proposition 2 on page 12), the following estimate would hold:

\[
\mathcal{E}(0)(t) \sim V(m_-) c_{\text{cut},0} t \quad \text{when} \quad t \to +\infty,
\]
a contradiction with Proposition 10. The proof of Proposition 9 on page 59 (“non-invasion implies relaxation”) is complete.

6 Proof of Theorem 1 and Proposition 1

Let us assume that the coercivity hypothesis \((H_{\text{coerc}})\) and the generic hypotheses \((G)\) hold for the potential \( V \), and let us consider a bistable solution \((x, t) \mapsto u(x, t)\) of system (1). The conclusions of Theorem 1 and Proposition 1 on page 10 and on page 11 can be split into two parts.

1. The approach to the propagating terrace of bistable fronts travelling to the right, and to the one travelling to the left.

2. On the remaining “center” spatial domain, the fact that the time derivative of the solution approaches zero, and the nonnegative “residual asymptotic energy”.

Concerning the first part, it is a rather direct consequence of Proposition 4 on page 18 (“invasion implies convergence”), and the derivation of this first part from this proposition is unchanged with respect to the parabolic case; it is explained in details in section 6 of [14].

Thus we are left with the second part. More precisely, we may assume that between the “last” fronts travelling to the right and to the left, the hypotheses (and thus the
conclusions) of Proposition 9 on page 59 ("non-invasion implies relaxation") hold. Now, in view of Lemma 39 above, the conclusions of Theorem 1 and Proposition 1 concerning the behaviour of the solution in this center area follow readily from the conclusions of Proposition 9. Theorem 1 and Proposition 1 are proved.

7 Asymptotic pattern in the center area

By contrast with Theorem 1 of [14] (and Theorem 2 of [15]), the results obtained in this paper (Theorem 1 and Proposition 1 on page 10 and on page 11) do not provide a complete description (under the form of an asymptotic pattern, a "standing terrace") for the behaviour of the solution in the "center" area between the two stacked families of fronts travelling to the left and to the right. However the same complete description is actually likely to hold, and the reason why these conclusions are not stated here is solely that I have not been able to extend to the hyperbolic case one of the steps of the proof in the parabolic case.

This step is Lemma 10 of [15] ("approach to zero Hamiltonian level set for a sequence of times"). With the same notation as in section 5 on page 58 assume that $V(m_{\pm}) = 0$, and consider the "Hamiltonian" function:

$$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (u, v) \mapsto \frac{v^2}{2} - V(u).$$

For every small positive quantity $\varepsilon$, according to Lemma 39 on page 71

$$\sup_{x \in [x_{\text{hom}} - (t), -\varepsilon t] \cup [\varepsilon t, x_{\text{hom}} + (t)]} |H(u(x, t), u_x(x, t))| \to 0 \quad \text{when} \quad t \to +\infty.$$

Now, in the present context, Lemma 10 of [15] asserts that the quantity

$$\sup_{x \in [-\varepsilon t, \varepsilon t]} |H(u(x, t), u_x(x, t))|$$

approaches 0 at least for a sequence of times going to $+\infty$. The proof proceeds by contradiction and uses the expression (valid in the "parabolic" case):

$$\partial_x \left( H(u(x, t), u_x(x, t)) \right) = u_x \cdot u_t$$

(83)

together with a Hölder inequality, leading to the non-integrability of the function

$$t \mapsto \int_{-\varepsilon t}^{\varepsilon t} u_x^2(x, t) \, dx$$

and to a contradiction with the fact that the residual asymptotic energy $\mathcal{E}_\infty$ is not equal to $-\infty$ (since nonnegative).

Unfortunately, in the hyperbolic case considered here, expression (83) turns (at least formally) into the less tractable expression:

$$\partial_x \left( H(u(x, t), u_x(x, t)) \right) = u_x \cdot (\alpha u_t + u_t)$$

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that I have not been able to use to achieve the conclusions of Lemma 10 of [15]. If this difficulty could be overcome, all the other steps leading to the convergence to the asymptotic pattern in the center area should extend easily to the hyperbolic case considered here.

8 Some properties of the profiles of travelling fronts

Let us assume that $V$ satisfies the coercivity hypothesis (H$_{coerc}$) (subsection 2.1 on page 3) and the non-degeneracy hypothesis (H$_{non-deg}$) (subsection 2.2 on page 3).

8.1 Asymptotic behaviour

Let $c$ denote a positive quantity, and let us consider the differential system governing the profiles of fronts travelling at speed $c$:

$$
\phi'' = -c\phi' + \nabla V(\phi) .
$$

A proof of the two following elementary lemmas can be found in [14].

**Lemma 40** (travelling waves approach critical points). *There exists a quantity $C$, depending only on $V$, such that, for every bounded global solution $x \mapsto \phi(x)$ of the differential system (84),

$$
\sup_{x \in \mathbb{R}} |\phi(x)| \leq C ,
$$

and there are two critical points $m_-$ and $m_+$ of $V$ such that

$$
\phi(x) \xrightarrow{x \to -\infty} m_- \quad \text{and} \quad \phi(x) \xrightarrow{x \to +\infty} m_+ \quad \text{and} \quad V(m_-) < V(m_+).
$$

**Lemma 41** (spatial asymptotics of travelling waves). *Let $m$ be a (local) minimum point of $V$, and let $x \mapsto \phi(x)$ be a global solution of the differential system (84) satisfying

$$
|\phi(x) - m| \leq d_{\text{Esc}} \quad \text{for every} \quad x \in [0, +\infty) \quad \text{and} \quad \phi(\cdot) \not\equiv m .
$$

Then the following conclusions hold.

1. The pair $(\phi(x), \phi'(x))$ approaches $(m, 0)$ when $x$ approaches $+\infty$.
2. The supremum $\sup_{x \in \mathbb{R}} |\phi(x) - m|$ is larger than $d_{\text{Esc}}$.
3. For all $x$ in $[0, +\infty)$, the scalar product $\phi(x) - m \cdot \phi'(x)$ is negative.
4. For all $x$ in $(0, +\infty)$, the distance $|\phi(x) - m|$ is smaller than $d_{\text{Esc}}$.
5. There exists a positive quantity $C$, depending only on $V$ and $c$ (not on $\phi$) such that, for all $x$ in $[0, +\infty)$,

$$
|\phi(x) - m| \leq Ce^{-cx} \quad \text{and} \quad |\phi'(x)| \leq Ce^{-cx} .
$$

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References


