Global behaviour of radially symmetric solutions stable at infinity for gradient systems

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This paper is concerned with radially symmetric solutions of systems of the form
\[ u_t = -\nabla V(u) + \Delta_x u \]
where space variable \( x \) and state-parameter \( u \) are multidimensional, and the potential \( V \) is coercive at infinity. For such systems, under generic assumptions on the potential, the asymptotic behaviour of solutions stable at infinity, that is approaching a spatially homogeneous equilibrium when \( |x| \) approaches \(+\infty\), is investigated. It is proved that every such solutions approaches a stacked family of radially symmetric bistable fronts travelling to infinity. This behaviour is similar to the one of bistable solutions for gradient systems in one unbounded spatial dimension, described in a companion paper. It is expected (but unfortunately not proved at this stage) that behind these travelling fronts the solution again behaves as in the one-dimensional case (that is, the time derivative approaches zero and the solution approaches a pattern of stationary solutions).

1 Introduction

This paper deals with the global dynamics of radially symmetric solutions of nonlinear parabolic systems of the form
\[ u_t = -\nabla V(u) + \Delta_x u \]

[http://math.univ-lyon1.fr/~erisler/]
where time variable $t$ is real, space variable $x$ lies in the spatial domain $\mathbb{R}^d$ with $d$ a positive integer, the the function $(x,t) \mapsto u(x,t)$ takes its values in $\mathbb{R}^n$ with $n$ a positive integer, and the nonlinearity is the gradient of a scalar potential function $V : \mathbb{R}^n \to \mathbb{R}$, which is assumed to be regular (of class at least $C^2$) and coercive at infinity (see hypothesis (H_{coerc}) in subsection 2.1 on page 5).

If $(v,w)$ is a pair of vectors of $\mathbb{R}^n$ or $\mathbb{R}^d$, let $v \cdot w$ and $|v| = \sqrt{v \cdot v}$ denote the usual Euclidean scalar product and the usual Euclidean norm, respectively, and let us simply write $v^2$ for $|v|^2$.

Radially symmetric solutions of system (1) are functions of the form

$$u(x,t) = \tilde{u}(r(t)) \quad \text{where} \quad r = |x|.$$  

For such functions, system (1) takes the following form:

$$\tilde{u}_t = -\nabla V(\tilde{u}) + \frac{d-1}{r} \tilde{u}_r + \tilde{u}_{rr} \quad \text{with the boundary condition} \quad \partial_r \tilde{u}(0,t) = 0,$$

and it this last system that we are going to consider in this paper.

A fundamental feature of system (1) is that it can be recast, at least formally, as the gradient flow of an energy functional. If $(x,t) \mapsto u(x,t)$ is a solution of system (1), the energy (or Lagrangian or action) functional of the solution reads:

$$E(u(\cdot,t)) = \int_{\mathbb{R}^d} \left( \frac{|\nabla_x u(x,t)|^2}{2} + V(u(x,t)) \right) dx,$$

where

$$|\nabla_x u(x,t)|^2 = \sum_{i=1}^{d} \sum_{j=1}^{n} \frac{\partial_x u_j(x,t)^2}{2}.$$  

Its time derivative reads, at least formally,

$$\frac{d}{dt} E(u(\cdot,t)) = -\int_{\mathbb{R}^d} u_t(x,t)^2 \, dx \leq 0, \quad u_t(x,t)^2 = \sum_{j=1}^{n} \partial_t u_j(x,t)^2,$$

and system (1) can formally be rewritten as:

$$u_t(\cdot,t) = -\frac{\delta}{\delta u} E(u(\cdot,t)).$$

Obviously, the same assertions hold for the (reduced) system (2), with the following expression of the (formal) energy:

$$\mathcal{E}[	ilde{u}(\cdot,t)] = \mathcal{E}[r \mapsto \tilde{u}(r,t)] = \int_{[0,\infty)} r^{d-1} \left( \frac{\tilde{u}_r(r,t)^2}{2} + V(\tilde{u}(r,t)) \right) dr;$$

its time derivative reads, at least formally,

$$\frac{d}{dt} \mathcal{E}[	ilde{u}(\cdot,t)] = -\int_{[0,\infty)} r^{d-1} u_t(r,t)^2 \, dr \leq 0,$$

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and again system (2) can formally be rewritten as:

$$\tilde{u}_t(\cdot, t) = -\frac{\delta}{\delta \tilde{u}} \tilde{E}[\tilde{u}(\cdot, t)].$$

A perhaps more surprising feature of system (1) is that a formal gradient structure exists not only in the laboratory frame, but also in every frame travelling at constant speed. For every $c$ in $\mathbb{R}^d$, if we consider two functions $(x, t) \mapsto u(x, t)$ and $(y, t) \mapsto v(y, t)$ related by

$$u(x, t) = v(y, t) \quad \text{as soon as} \quad x = ct + y$$

then $u$ is a solution of (1) if and only if $v$ is a solution of

$$v_t - c \cdot \nabla_y v = -\nabla V(v) + \Delta_y v.$$ 

Now, if we consider the energy

$$\mathcal{E}_c[v(\cdot, t)] = \int_{\mathbb{R}^d} e^{c \cdot y} \left( \frac{|\nabla_y v(y, t)|^2}{2} + V(v(y, t)) \right) dy,$$

then, at least formally,

$$\frac{d}{dt} \mathcal{E}_c[v(\cdot, t)] = - \int_{\mathbb{R}} e^{c \cdot y} v_t(y, t)^2 dy,$$

and system (5) can formally be rewritten as:

$$v_t(\cdot, t) = -e^{-c \cdot y} \frac{\delta}{\delta v} \mathcal{E}_c[v(\cdot, t)].$$

This gradient structure has been known for a long time, but it was not until recently that it received a more detailed attention from several authors (among them C. B. Muratov, Th. Gallay, R. Joly, and the author [7, 8, 12, 20]), and that it was shown that this structure is sufficient (in itself, that is without the use of the maximum principle) to prove results of global convergence towards travelling fronts. These ideas have been applied since in different contexts, for instance by G. Chapuisat [2], Muratov and M. Novaga [13–15], N. D. Alikakos and N. I. Katzourakis [1], C. Luo [11].

Even more recently, the same ideas enabled the author ([21–23]) to push one step further (that is, extend to systems) the program initiated by P. C. Fife and J. MacLeod in the late seventies with the aim of describing the global asymptotic behaviour (when space is one-dimensional) of every bistable solution, that is every solution close to stable homogeneous equilibria at both ends of space ([4–6]). Under generic assumptions on the potential $V$, these solutions approach a stacked (possibly empty) family of bistable travelling fronts at both ends of space, and approach in between a pattern of stationary solutions going slowly away from one another. These stacked families of fronts are called terraces by some authors (see [3, 17, 19], where new results of the same flavour were recently obtained in the scalar case $n = 1$).
The aim of this paper is to extend to the case of radially symmetric solutions in higher space dimensions the results (description of the global asymptotic behaviour) obtained in [21] for bistable solutions when spatial domain is one-dimensional. Thus we are going to consider solutions of system (1) (in higher spatial dimension $d$) that are altogether radially symmetric and stable at infinity (in space). Or equivalently solutions of system (2) that approach a stable homogeneous equilibrium when the radius $r$ approaches $+\infty$. Our aim is to prove that, under generic assumptions on the potential, these solutions approach a stacked family of (radially symmetric) bistable front going to infinity (a “propagating terrace”), and behind these travelling fronts a pattern of stationary solutions going slowly away from one another (a “standing terrace”). Unfortunately, the results that we were able to reach at this stage only achieve the first step of this program (Theorem 1 on page 10), and few information will be given concerning the behaviour of the solution behind the propagating terrace (Proposition 1 on page 11).

In the early eighties, the global behaviour of radially symmetric solutions of reaction-diffusion equations (in the scalar case $n = 1$) has been studied by several authors, in particular C. R. C. T. Jones and K. Uchiyama, [10, 24]. They extended a number of results that had been established for similar equations on the real line, concerning global convergence towards monostable or bistable fronts, and threshold phenomena (always with the use of the maximum principle). An extensive study of threshold phenomena for radially symmetric solutions was recently provided by Muratov and Zhong, [16]. The present paper can be viewed as a first attempt to extend some of those results (bistable setting only) to the more general case of systems (and thus without the maximum principle).

The path of the proof is very similar to the one used in the spatial dimension one case, [21]. It is based on a careful study of the relaxation properties of energy or $L^2$ functionals (localized in space by adequate weight functions), both in the laboratory frame and in frames travelling at various speeds. The differences are mainly of technical nature, they are related to the two following specific features of the (reduced) system (2):

- the “curvature” term $(d - 1)u_r/r$ (which fortunately approaches zero when the radius $r$ approaches plus infinity);
- the fact that space is reduced to the half-line $[0, +\infty)$ (thus is in this sense less “homogeneous” than the full real line).

Observe that, due to these two features, the gradient structure in every travelling frame does not persist, strictly speaking, for the reduced system (2): the additional curvature term is not of gradient type, and in a travelling frame the space domain itself depends on time. Only for $r$ approaching plus infinity do we recover (asymptotically) the gradient structures in travelling frames.

The whole paper is thus nothing but an attempt to show that the proof set up in [21] can be adapted to the case of system (2), in other words that the two technical difficulties above can be overcome.
2 Assumptions, notation, and statement of the results

This section presents strong similarities with sections 2 of [21] and 2 of [22], where more details and comments can be found.

For the remaining of the paper it will be assumed than the space dimension \( d \) is not smaller than 2. Indeed the case \( d = 1 \) was already treated in [21], and several definitions, estimates, and statements will turn out to be irrelevant without this assumption (see for instance the definition of the weight function \( T_{\rho \psi_0} \) in subsection 4.4 on page 19).

2.1 Semi-flow in uniformly local Sobolev space and coercivity hypothesis

Let us consider the two following Banach spaces of continuous and uniformly bounded functions equipped with the uniform norm:

\[
X = \left( C_0^1(\mathbb{R}^d, \mathbb{R}^n), \| \ldots \|_{L^\infty(\mathbb{R}^d, \mathbb{R}^n)} \right) \quad \text{and} \quad Y = \left( C_0^0(\mathbb{R}^+, \mathbb{R}^n), \| \ldots \|_{L^\infty(\mathbb{R}^+, \mathbb{R}^n)} \right).
\]

System (1) defines a local semi-flow in \( X \) (see for instance D. B. Henry’s book [9]).

As in [21, 22], let us assume that the potential function \( V : \mathbb{R}^n \to \mathbb{R} \) is of class \( C^k \) where \( k \) is an integer not smaller than 2, and that this potential function is strictly coercive at infinity in the following sense:

\[
(H_{\text{coerc}}) \quad \liminf_{R \to +\infty} \inf_{|u| \geq R} \frac{u \cdot \nabla V(u)}{|u|^2} > 0
\]

(or in other words there exists a positive quantity \( \varepsilon \) such that the quantity \( u \cdot \nabla V(u) \) is larger than \( \varepsilon |u|^2 \) as soon as \( |u| \) is sufficiently large).

According to this hypothesis \((H_{\text{coerc}})\), the semi-flow of system (1) on \( X \) is actually global (see Lemma 1 on page 11). As a consequence, considering the restriction of this semi-flow to radially symmetric functions, it follows that system (2) defines a global semi-flow on \( Y \). Let us denote by \((S_t)_{t \geq 0}\) this last semi-flow on \( Y \).

2.2 First generic hypothesis on the potential: critical points are nondegenerate

The results of this paper require several generic hypotheses on the potential \( V \). The simplest of those hypotheses is:

\[
(H_{\text{non-deg}}) \quad \text{Every critical point of } V \text{ is nondegenerate.}
\]

Notation. Let \( \mathcal{M} \) denote the set of (nondegenerate, local or global) minimum points of \( V \):

\[
\mathcal{M} = \{ u \in \mathbb{R}^n : \nabla V(u) = 0 \quad \text{and} \quad D^2V(u) \text{ is positive definite} \}.
\]
2.3 Solutions stable at infinity

**Definition.** A solution \((r,t) \mapsto u(r,t)\) of system (2) is said to be **stable at infinity** if there exists a (local or global) minimum point \(m\) of \(V\) such that:

\[
\lim_{r \to +\infty} \sup |u(r,t) - m| \to 0 \quad \text{when} \quad t \to +\infty.
\]

More precisely, such a solution is said to be **stable and close to** \(m\) **at infinity**.

2.4 Stationary solutions and travelling fronts: definition and notation

Let \(c\) be a real quantity. A function \(\phi : \mathbb{R} \to \mathbb{R}^n, \xi \mapsto \phi(\xi)\) is the profile of a wave travelling at speed \(c\) (or is a stationary solution if \(c\) vanishes) for system (1) if the function \((x,t) \mapsto \phi(x_1 - ct)\) is a solution of this system, that is if \(\phi\) is a solution of the differential system

\[
\phi'' = -c\phi' + \nabla V(\phi).
\]

**Notation.** If \(u_-\) and \(u_+\) are critical points of \(V\) (and \(c\) is a real quantity), let \(\Phi_c(u_-, u_+)\) denote the set of **nonconstant** solutions of system (9) connecting \(u_-\) to \(u_+\). With symbols,

\[
\Phi_c(u_-, u_+) = \{ \phi : \mathbb{R} \to \mathbb{R}^n : \phi \text{ is a nonconstant solution of system (9)} \}
\]

and \(\phi(\xi) \xrightarrow{\xi \to -\infty} u_-\) and \(\phi(\xi) \xrightarrow{\xi \to +\infty} u_+\).

2.5 Breakup of space translation invariance for stationary solutions and travelling fronts

Let \(\lambda_{\text{min}}\) (\(\lambda_{\text{max}}\)) denote the minimum (respectively, maximum) of all eigenvalues of the Hessian matrices of the potential \(V\) at (local) minimum points. In other words, if \(\sigma(D^2V(u))\) denotes the spectrum of the Hessian matrix of \(V\) at a point \(u\) in \(\mathbb{R}^n\),

\[
\lambda_{\text{min}} = \min_{m \in \mathcal{M}} \min \left(\sigma(D^2V(m))\right) \quad \text{and} \quad \lambda_{\text{max}} = \max_{m \in \mathcal{M}} \max \left(\sigma(D^2V(m))\right)
\]

(according to the coercivity hypothesis \((H_{\text{coerc}})\) the set \(\mathcal{M}\) is finite). Obviously,

\[
0 < \lambda_{\text{min}} \leq \lambda_{\text{max}} < +\infty.
\]

**Notation.** For the remaining of this paper, let us fix a positive quantity \(d_{\text{Esc}}\), sufficiently small so that, for every (local) minimum point \(m\) of \(V\) and for all \(u\) in \(\mathbb{R}^n\) satisfying \(|u - m| \leq d_{\text{Esc}}\), every eigenvalue \(\lambda\) of \(D^2V(u)\) satisfies:

\[
\frac{\lambda_{\text{min}}}{2} \leq \lambda \leq 2\lambda_{\text{max}}.
\]

Obviously this notation \(d_{\text{Esc}}\) refers to the word “distance” (and “Escape”) and should not be mingled with the space dimension \(d\).
It is well known (see for instance [20, 22] for a proof of this elementary result) that very nonconstant stationary solution of system (9), connecting two points of $\mathcal{M}$, “escapes” at least at distance $d_{\text{Esc}}$ from each of these two points (whatever the value of the speed $c$ and even if these two points are equal) at some position of space (see figure 1). Thus, for every quantity $c$ in $\mathbb{R}$ and every pair $(m_-, m_+)$ in $\mathcal{M}^2$, we may consider the set of normalized bistable fronts (if $c$ is nonzero) or stationary solutions (if $c$ equals 0) connecting $m_-$ to $m_+$ (see figure 2):

$$\Phi_{c,\text{norm}}(m_-, m_+) = \{ \phi \in \Phi_c(m_-, m_+) : |\phi(0) - m_+| = d_{\text{Esc}} \text{ and } |\phi(\xi) - m_+| < d_{\text{Esc}} \text{ for every positive quantity } \xi \}.$$  

2.6 Additional generic hypotheses on the potential

The result below requires additional generic hypotheses on the potential $V$, that will now be stated. A formal proof of the genericity of this hypothesis is scheduled (work in progress by Romain Joly and the author).

$$(H_{\text{bist}})$$ Every front travelling at a nonzero speed and invading a stable equilibrium (a minimum point of $V$) is bistable.
In other words, for every minimum point \( m_+ \) in \( M \), every critical point \( u_- \) of \( V \), and every positive quantity \( c \), if the set \( \Phi_c(u_-, m_+) \) is nonempty, then \( u_- \) must belong to \( M \). As a consequence of this hypothesis, only bistable travelling fronts will be involved in the asymptotic behaviour of bistable solutions.

The statement of the two remaining hypotheses requires the following notation.

Notation. If \( m_+ \) is a point in \( M \) and \( c \) is a positive quantity, let \( \Phi_c(m_+) \) denote the set of fronts travelling at speed \( c \) and “invading” the equilibrium \( m_+ \) (note that according to hypothesis (H\(_{\text{bist}}\)) all these fronts are bistable), and let us define similarly \( \Phi_{c,\text{norm}}(m_+) \).

With symbols,
\[
\Phi_c(m_+)=\bigcup_{m_-\in M}\Phi_c(m_-, m_+) \quad \text{and} \quad \Phi_{c,\text{norm}}(m_+)=\bigcup_{m_-\in M}\Phi_{c,\text{norm}}(m_-, m_+).
\]

The two additional generic hypotheses that will be made on \( V \) are the following.

\((H_{\text{disc}-c})\) For every \( m_+ \) in \( M \), the set:
\[
\{ c \text{ in } [0, +\infty) : \Phi_c(m_+) \neq \emptyset \}
\]
has an empty interior.

\((H_{\text{disc}-\Phi})\) For every minimum point \( m_+ \) in \( M \) and every positive quantity \( c \), the set
\[
\{ (\phi(0),\phi'(0)) : \phi \in \Phi_{c,\text{norm}}(m_+) \}
\]
is totally discontinuous — if not empty — in \( \mathbb{R}^{2n} \). That is, its connected components are singletons. Equivalently, the set \( \Phi_{c,\text{norm}}(m_+) \) is totally disconnected for the topology of compact convergence (uniform convergence on compact subsets of \( \mathbb{R} \)).

In these two last definitions, the subscript “disc” refers to the concept of “discontinuity” or “discreteness”.

Finally, let us define the following “group of generic hypotheses”:

\((G)\) \( (H_{\text{non-deg}}) \) and \( (H_{\text{bist}}) \) and \( (H_{\text{disc}-c}) \) and \( (H_{\text{disc}-\Phi}) \).

2.7 Propagating terrace of bistable solutions

This subsection is devoted to the next definition. Its purpose is to enable a compact formulation of the main result of this paper (Theorem 1 below). Some comments on the terminology and related references are given after this definition.

Definition (propagating terrace of bistable fronts, figure 3). Let \( m_{\text{loc}} \) and \( m_{\text{far}} \) be two minimum points of \( V \) (satisfying \( V(m_{\text{loc}}) \leq V(m_{\text{far}}) \)). A function
\[
T : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n, \quad (r, t) \mapsto T(r, t)
\]
is called a propagating terrace of bistable fronts (travelling to the right), connecting \( m_{\text{loc}} \) to \( m_{\text{far}} \), if there exists a nonnegative integer \( q \) such that:
1. if $q$ equals 0, then $m_{\text{loc}} = m_{\text{far}}$ and, for every nonnegative quantity $r$ and every nonnegative time $t$,

$$T(r,t) = m_{\text{loc}} = m_{\text{far}};$$

2. if $q$ equals 1, then there exist
   - a positive quantity $c_1$
   - and a function $\phi_1$ in $\Phi_c(m_{\text{loc}}, m_{\text{far}})$ (that is, the profile of a bistable front travelling at speed $c_1$ and connecting $m_{\text{loc}}$ to $m_{\text{far}}$)
   - and a $C^1$-function $\mathbb{R}_+ \to \mathbb{R}_+, t \mapsto r_1(t)$, satisfying $r_1'(t) \to c_1$ when $t$ approaches $+\infty$

such that, for every nonnegative quantity $r$ and every nonnegative time $t$,

$$T(r,t) = \phi_1(r - r_1(t));$$

3. if $q$ is not smaller than 2, then there exists $q - 1$ minimum points $m_1, \ldots, m_{q-1}$ of $V$, satisfying (if we denote $m_{\text{far}}$ by $m_0$ and $m_{\text{loc}}$ by $m_q$)

$$V(m_0) > V(m_1) > \cdots > V(m_q),$$

and there exist $q$ positive quantities $c_1, \ldots, c_q$ satisfying:

$$c_1 \geq \cdots \geq c_q,$$

and for each integer $i$ in $\{1, \ldots, q\}$, there exist:
   - a function $\phi_i$ in $\Phi_{c_i}(m_i, m_{i-1})$ (that is, the profile of a bistable front travelling at speed $c_i$ and connecting $m_i$ to $m_{i-1}$)
   - and a $C^1$-function $\mathbb{R}_+ \to \mathbb{R}_+, t \mapsto r_i(t)$, satisfying $r_i'(t) \to c_i$ when $t$ approaches $+\infty$
such that, for every integer $i$ in $\{1, \ldots, q - 1\}$,

$$r_{i+1}(t) - r_i(t) \to +\infty \quad \text{when} \quad t \to +\infty,$$

and such that, for every nonnegative quantity $r$ and every nonnegative time $t$,

$$T(r, t) = m_0 + \sum_{i=1}^{q} \left[ \phi_i(r - r_i(t)) - m_{i-1} \right].$$

Obviously, item 2 may have been omitted in this definition, since it fits with item 3 with $q$ equals 1.

The terminology “propagating terrace” was introduced by A. Ducrot, T. Giletti, and H. Matano in [3] (and subsequently used by P. Poláčik, [17–19]) to denote a stacked family (a layer) of travelling fronts in a (scalar) reaction-diffusion equation. This led the author to keep the same terminology in the present context. This terminology is convenient to denote objects that would otherwise require a long description. It is also used in the companion papers [21, 23]. We refer to [21] for additional comments on this terminological choice.

To finish, observe that in the present context terraces are only made of bistable fronts, by contrast with the propagating terraces introduced and used by the authors cited above; that (still in the present context) terraces are approached by solutions but are (in general) not solutions themselves; and that a propagating terrace may be nothing but a single stable homogeneous equilibrium (when $q$ equals 0) or may involve a single travelling front (when $q$ equals 1).

### 2.8 Main result: convergence towards a propagating terrace of bistable solutions

The following theorem is the main result of this paper.

**Theorem 1** (convergence towards a propagating terrace). Assume that the potential $V$ satisfies the coercivity hypothesis ($H_{\text{coerc}}$) and the generic hypotheses (G). Then, for every solution stable at infinity $(r, t) \mapsto u(r, t)$ of system (2), there exists a propagating terrace $T$ of bistable fronts (travelling to the right) such that, for every sufficiently small positive quantity $\varepsilon$,

$$\sup_{r \in [\varepsilon t, +\infty)} |u(r, t) - T(r, t)| \to 0$$

when $t$ approaches $+\infty$.

By contrast with the main result of [21], this result does not provide any information concerning the behaviour of the solution “close to the origin”, behind the propagating terrace. A limited extension is provided by the next proposition.
2.9 Residual asymptotic energy

The following proposition provides a limited extension to the conclusions of Theorem 1 concerning the behaviour of the solution behind the propagating terrace of bistable fronts.

**Proposition 1** (residual asymptotic energy). Assume that all the hypotheses of Theorem 1 hold and take the same notation as in this theorem. The following additional conclusion holds.

There exists a quantity \( \mathcal{E} \) ("residual asymptotic energy") in \( \{-\infty\} \cup \mathbb{R} \) such that, if we denote by \( m_{\text{far}} \) the local minimum of \( V \) such that the solution is close to \( m_{\text{far}} \) at infinity, and if we denote by \( m_{\text{loc}} \) the the local minimum of \( V \) such that the propagating terrace \( T \) connects \( m_{\text{loc}} \) to \( m_{\text{far}} \), then for every sufficiently small positive quantity \( \varepsilon \),

\[
\int_0^t r^{d-1} \left( \frac{u_r(r,t)^2}{2} + V(u(r,t)) - V(m_{\text{loc}}) \right) dr \to \mathcal{E}
\]

when \( t \) approaches \( +\infty \).

The next step would be to prove that the residual asymptotic energy \( \mathcal{E} \) is not equal to minus infinity (and maybe even nonnegative, as is the case in space dimension one, [21, 23]). For this purpose, an obvious strategy is to try to extend the “non-invasion implies relaxation” argument set up in [21] to the case of radially symmetric solutions considered here. But doing so requires to deal with a difficulty: the non-variational “curvature terms” occurring in a frame travelling at a nonzero speed, especially when considered close to the origin. It is not clear at this stage — at least in the author’s mind — how this difficulty could be overcome.

2.10 Organization of the paper

The organization of this paper closely follows that of [21].

- The next section 3 is devoted to some preliminaries (existence of solutions, preliminary computations on spatially localized functionals, notation).
- Section 4 on page 16 is devoted to the proof of Proposition 2 on page 18 “invasion implies convergence”. This proposition almost proves Theorem 1.
- Section 5 on page 43 is devoted to the proof of Proposition 7 on page 44 which is almost identical to Proposition 1.
- Finally, the proof of Theorem 1 and Proposition 1 are completed in the short section 6 on page 50.

3 Preliminaries

3.1 Global existence of solutions and attracting ball for the semi-flow

**Lemma 1** (global existence of solutions and attracting ball). For every function \( u_0 \) in \( Y \), system (2) has a unique globally defined solution \( t \mapsto S_t u_0 \) in \( \mathcal{C}^0([0, +\infty), Y) \) with
initial data $u_0$. In addition, there exist a positive quantity $R_{\text{att,}\infty}$ (radius of attracting ball for the $L^\infty$-norm), depending only on $V$, such that, for every sufficiently large time $t$,

$$\sup_{x \in \mathbb{R}} |(S_t u_0)(x)| \leq R_{\text{att,}\infty}.$$ 

Proof. It is sufficient to prove the same result for the semi-flow defined by system (1) on the Banach space $X$ (that is without radial symmetry) defined in subsection 2.1 on page 5 and this is what we are going to do.

As mentioned in subsection 2.1, system (1) is locally well-posed in $X$. Let $u_0$ denote a function in $X$, and let $u : \mathbb{R}^d \times [0, T_{\text{max}}) \to u(x, t)$

denote the (maximal) solution of system (1) with initial data $u_0$, where $T_{\text{max}}$ in $(0, +\infty]$ denotes the upper bound of the maximal time interval where this solution is defined. For all $(x, t)$ in $\mathbb{R}^d \times [0, T_{\text{max}})$, let

$$v(x, t) = \frac{1}{2}|u(x, t)|^2.$$ 

A immediate calculation shows that this function is a solution of the system:

$$\partial_t v = -u \cdot \nabla V(u) + \Delta_x v - |\nabla_x u|^2. \quad (12)$$ 

Besides, according to the coercivity hypothesis (H$\text{coerc}$), there exist positive quantities $q_{\text{coerc}}$ and $K_{\text{coerc}}$ such that, for all $w$ in $\mathbb{R}^n$,

$$w \cdot \nabla V(w) \geq q_{\text{coerc}} w^2 - K_{\text{coerc}}. \quad (13)$$

It follows from (12) and (13) that $v(\cdot, t)$ satisfies the partial differential inequality:

$$\partial_t v \leq K_{\text{coerc}} - 2q_{\text{coerc}} v + \Delta_x v,$$

and as a consequence, if we consider the solution $t \mapsto m(t)$ of the differential equation (and initial condition):

$$\dot{m} = K_{\text{coerc}} - 2q_{\text{coerc}} m, \quad m(0) = \sup_{x \in \mathbb{R}^d} v(x, 0) = \frac{1}{2} \|u(\cdot, 0)\|^2_X,$$

then it follows from the maximum principle that, for all $(x, t)$ in $\mathbb{R}^d \times [0, T_{\text{max}})$,

$$v(x, t) \leq m(t). \quad (15)$$

As a consequence blow-up cannot occur, the solution $u(x, t)$ must be defined up to $+\infty$ in time, and if we consider the quantity

$$R_{\text{att,}\infty} = \sqrt{\frac{K_{\text{coerc}}}{q_{\text{coerc}}} + 1},$$
then it follows from (14) and (15) that, for every sufficiently large time $t$,

$$\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{att, \infty}.$$ 

Restricting the semi-flow of system (1) to radially symmetric functions, the conclusions of Lemma 1 follow. Lemma 1 is proved. \hfill \square

In addition, system (2) has smoothing properties (Henry [9]). Due to these properties, since $V$ is of class $C^k$ (with $k$ not smaller than 2), every solution $t \mapsto S_t u_0$ in $C^0([0, +\infty), Y)$ actually belongs to

$$C^0((0, +\infty), C_0^{k+1}([0, +\infty), \mathbb{R}^n)) \cap C^1((0, +\infty), C_0^{k-1}([0, +\infty), \mathbb{R}^n)),$$

and, for every positive quantity $\varepsilon$, the following quantities (16)

$$\sup_{t \geq \varepsilon} \|S_t u_0\|_{C_0^{k+1}([0, +\infty), \mathbb{R}^n)} \quad \text{and} \quad \sup_{t \geq \varepsilon} \left\| \frac{d}{dt}S_t u_0 \right\|_{C_0^{k-1}([0, +\infty), \mathbb{R}^n)}$$

are finite.

### 3.2 Time derivative of (localized) energy and $L^2$-norm of a solution in a standing or travelling frame

Let $u_0$ be a function in $Y$ and, for every pair $(r, t)$ of nonnegative quantities, let $u(r, t) = (S_t u_0)(r)$ denote the corresponding solution of system (2). In the next calculations we assume that time $t$ is positive, so that according to (16) the regularities of $u$ and $u_t$ ensure that all integrals converge.

#### 3.2.1 Standing frame

Let $r \mapsto \psi(r)$ denote a function in the space $W^{1,1}(\mathbb{R}_+, \mathbb{R})$ (that is a function belonging to $L^1(\mathbb{R}_+)$ together with its first derivative), and let us consider the energy (Lagrangian) and the $L^2$-norm of the solution, localized by the weight function $\psi$:

$$\int_0^{+\infty} \psi(r) \left( \frac{u(r, t)^2}{2} + V(u(r, t)) \right) dr \quad \text{and} \quad \int_0^{+\infty} \psi(r) \frac{u(r, t)^2}{2} dr.$$

Let us assume in addition that $\psi(0) = 0$. The time derivatives of these two quantities read:

(17) \hspace{1cm} \frac{d}{dt} \int_0^{+\infty} \psi \left( \frac{u^2}{2} + V(u) \right) dr = \int_0^{+\infty} \left[ -\psi u_t^2 + \left( \frac{d-1}{r} \psi - \psi' \right) u_t \cdot u_r \right] dr

and

(18) \hspace{1cm} \frac{d}{dt} \int_0^{+\infty} \psi \frac{u^2}{2} dr = \int_0^{+\infty} \left[ \psi (-u \cdot \nabla V(u) - u_r^2) + \left( \frac{d-1}{r} \psi - \psi' \right) u \cdot u_r \right] dr.
In both expressions, the border term at \( r \) equals 0 coming from the integration by parts vanishes since \( \psi(0) = 0 \). In both expressions again, the last term disappears on every domain where \( \psi(r) \) is proportional to \( r^{d-1} \) (this corresponds to a uniform weight for the Lebesgue measure on \( \mathbb{R}^d \)).

We refer to [22] for more comments on these expressions. The sole difference with the one-dimensional space case treated in [22] is the “\((d - 1)/r\)” curvature terms on the right-hand side of these expressions. Fortunately, this additional term will not induce many changes with respect to the arguments developed in [22], since:

- close to the origin \( r = 0 \), the weight function \( \psi \) can be chosen proportional to \( r^{d-1} \),
- far away from the origin \( r = 0 \), this curvature term is just small.

### 3.2.2 Travelling frame

Now let us consider nonnegative quantities \( c \) and \( t_{\text{init}} \) and \( r_{\text{init}} \) (the speed, origin of time, and initial origin of space for the travelling frame respectively, see figure 12 on page 27). For every nonnegative quantity \( s \), let us consider the interval:

\[
I(s) = [-r_{\text{init}} - cs, +\infty),
\]

and, for every \( \rho \) in \( I(s) \), let

\[
v(\rho, s) = u(r, t) \quad \text{where} \quad r = r_{\text{init}} + cs + \rho \quad \text{and} \quad t = t_{\text{init}} + s
\]

denote the same solution viewed in a referential travelling at speed \( c \). This function \((\rho, s) \mapsto v(\rho, s)\) is a solution of the system:

\[
v_s - cv_\rho = -\nabla V(v) + \frac{d - 1}{r_{\text{init}} + cs + \rho} v_\rho + v_{\rho\rho}.
\]

This time, let us assume that the weight function \( \psi \) is a function of the two variables \( \rho \) and \( s \), defined on the domain

\[
\{(\rho, s) \in \mathbb{R} \times [0, +\infty) : \rho \in I(s)\}
\]

and such that, for all \( s \) in \([0, +\infty)\), the function \( \rho \mapsto \psi(\rho, s) \) belongs to \( W^{2,1}(I(s), \mathbb{R}) \) and the time derivative \( \rho \mapsto \psi_\rho(\rho, s) \) is defined and belongs to \( L^1(I(s), \mathbb{R}) \). Again, let us consider the energy (Lagrangian) and the \( L^2 \)-norm of the solution, localized by the weight function \( \psi \):

\[
\int_{I(s)} \psi(\rho, s) \left( \frac{v_\rho(\rho, s)^2}{2} + V(v(\rho, s)) \right) d\rho \quad \text{and} \quad \int_{I(s)} \psi(\rho, s) \frac{v(\rho, s)^2}{2} d\rho.
\]

Let us assume in addition that, for all \( s \) in \([-t_{\text{init}}, +\infty)\), the functions \( \rho \mapsto \psi(\rho, s) \) and \( \rho \mapsto \psi_\rho(\rho, s) \) vanish at \( \rho = -r_{\text{init}} - cs \) (at the left end of its domain of definition). Then
the time derivatives of these two quantities read:

\[
\frac{d}{ds} \int_{I(s)} \psi \left( \frac{v^2}{2} + V(v) \right) \, d\rho = \int_{I(s)} \left[ -\psi v_s^2 + \psi_s \left( \frac{v^2}{2} + V(v) \right) \right. \\
+ \left. \left( \frac{d - 1}{r_{\text{init}} + cs + \rho} \psi + c\psi - \psi_\rho \right) v_s \cdot v_\rho \right] \, d\rho.
\]

and

\[
\frac{d}{ds} \int_{I(s)} \psi \frac{v^2}{2} \, d\rho = \int_{I(s)} \left[ \psi \left( -v \cdot \nabla V(v) - v_\rho^2 \right) + \psi_s \frac{v^2}{2} \right. \\
+ \left. \left( \frac{d - 1}{r_{\text{init}} + cs + \rho} \psi + c\psi - \psi_\rho \right) v \cdot v_\rho \right] \, d\rho \\
= \int_{I(s)} \left[ \psi \left( -v \cdot \nabla V(v) - v_\rho^2 \right) + \psi_s \frac{v^2}{2} \right. \\
+ \left. \left( \psi_{\rho\rho} - c\psi_\rho \right) \frac{v^2}{2} + \frac{d - 1}{r_{\text{init}} + cs + \rho} \psi v \cdot v_\rho \right] \, d\rho.
\]

In these expressions again, the integration by part border terms at \( \rho = -r_{\text{init}} - cs \) vanish, and some terms simplify where the quantity

\[
\frac{d - 1}{r_{\text{init}} + cs + \rho} \psi + c\psi - \psi_\rho
\]

vanishes, that is where \( \psi \) is proportional to the expression

\[
(r_{\text{init}} + cs + \rho)^{d-1} \exp(c\rho)
\]

(combining the Lebesgue measure and the exponential weight \( \exp(c\rho) \)). For the time derivative of the \( L^2 \)-functional, a second expression (after integrating by parts the factor \( c\psi - \psi_\rho \)) is given (it is actually this second expression that will turn out to be the most appropriate for the calculations and estimates to come).

We refer to [21] for more comments on these expressions. As in the laboratory frame case, the sole difference with the one-dimensional space case treated in [21] is the \("(d - 1)/(r_{\text{init}} + cs + \rho)"\) curvature terms on the right-hand side of these expressions. Fortunately, this additional term will not induce many changes with respect to the arguments developed in [21], since:

- close to the “origin” \( \rho = -r_{\text{init}} - cs \), the weight function will be chosen in such a way that the quantity \([21]\) (involving this curvature term) vanishes or remains small,
- far away from the origin, this curvature term is just small.

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3.3 Miscellanea

3.3.1 Estimates derived from the definition of the “escape distance”

For every minimum point $m$ in $\mathcal{M}$ and every vector $v$ in $\mathbb{R}^n$ satisfying $|v - m| \leq d_{\text{Esc}}$, it follows from inequalities (10) on page 6 derived from the definition of $d_{\text{Esc}}$ that

\[
\begin{align*}
\frac{\lambda_{\min}}{4} (u - m)^2 & \leq V(u) \leq \lambda_{\max} (u - m)^2, \\
\frac{\lambda_{\min}}{2} (u - m)^2 & \leq (u - m) \cdot \nabla V(u) \leq 2\lambda_{\max} (u - m)^2.
\end{align*}
\]

3.3.2 Minimum of the convexities of the lower quadratic hulls of the potential at local minimum points

As in [21, 22], let

\[
q_{\text{low-hull}} = \min_{m \in \mathcal{M}} \min_{u \in \mathbb{R}^n \setminus \{m\}} \frac{V(u) - V(m)}{(u - m)^2}
\]

(see figure 4) and let

\[
V(u)
\]

\[
\begin{array}{c}
\text{m} \\
\text{q}_{\text{low-hull}} (u-m)^2
\end{array}
\]

\[
\text{u}
\]

Figure 4: Lower quadratic hull of the potential at a minimum point (definition of the quantity $q_{\text{low-hull}}$).

\[
w_{\text{en,0}} = \frac{1}{\max(1, -4 q_{\text{low-hull}})}.
\]

It follows from this definition that, for every $m$ in the set $\mathcal{M}$ and for all $u$ in $\mathbb{R}^n$,

\[
w_{\text{en,0}} V(u) + \frac{(u - m)^2}{4} \geq 0.
\]

4 Invasion implies convergence

4.1 Definitions and hypotheses

Let us assume that $V$ satisfies the coercivity hypothesis (H$_{\text{coerc}}$) and the generic hypotheses (G) (see subsection 2.6 on page 7). Let us consider a minimum point $m$ in $\mathcal{M}$, a function (initial data) $u_0$ in $Y$, and the corresponding solution $r \mapsto u(r, t) = (S_t u_0)(r)$ defined on $[0, +\infty)^2$. 

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We are not going to assume that this solution is stable at infinity, but instead we assume (as stated by the next hypothesis \( \text{H}^{\text{hom}} \)) that there exists a growing interval, travelling at a positive speed, where the solution is close to \( m \) (the subscript “hom” in the definitions below refers to this “homogeneous” area), see figure 5.

\[ (\text{H}^{\text{hom}}) \quad \text{There exists a positive quantity } c^{\text{hom}} \text{ and a } C^1\text{-function} \]

\[ r^{\text{hom}} : [0, +\infty) \to \mathbb{R}, \text{ satisfying } r'^{\text{hom}}(t) \to c^{\text{hom}} \text{ when } t \to +\infty, \]

such that, for every positive quantity \( L \),

\[ \| r \mapsto u(r^{\text{hom}}(t) + r, t) - m \|_{H^1([-L,L])} \to 0 \text{ when } t \to +\infty. \]

For every \( t \) in \([0, +\infty)\), let us denote by \( r^{\text{Esc}}(t) \) the supremum of the set:

\[ \left\{ r \in [0, r^{\text{hom}}(t)] : |u(r, t) - m| = d^{\text{Esc}} \right\}, \]

with the convention that \( r^{\text{Esc}}(t) \) equals \(-\infty\) if this set is empty. In other words, \( r^{\text{Esc}}(t) \) is the first point at the left of \( r^{\text{hom}}(t) \) where the solution “Escapes” at the distance \( r^{\text{Esc}} \) from the stable homogeneous equilibrium \( m \). We will refer to this point as the “Escape point” (there will also be an “escape point”, with a small “e” and a slightly different definition later). Let us consider the upper limit of the mean speeds between 0 and \( t \) of this Escape point:

\[ c^{\text{Esc}} = \limsup_{t \to +\infty} \frac{r^{\text{Esc}}(t)}{t}, \]

and let us make the following hypothesis, stating that the area around \( r^{\text{hom}}(t) \) where the solution is close to \( m \) is “invaded” from the left at a nonzero (mean) speed.

\[ (\text{H}^{\text{inv}}) \quad \text{The quantity } c^{\text{Esc}} \text{ is positive.} \]
4.2 Statement

The aim of section 4 is to prove the following proposition, which is the main step in the proof of Theorem 1. The proposition is illustrated by figure 6.

**Proposition 2** (invasion implies convergence). Assume that $V$ satisfies the coercivity hypothesis $(H_{coerc})$ and the generic hypotheses $(G)$, and, keeping the definitions and notation above, let us assume that for the solution under consideration hypotheses $(H_{hom})$ and $(H_{inv})$ hold. Then the following conclusions hold.

- For $t$ sufficiently large, the function $t \mapsto r_{Esc}(t)$ is of class $C^1$ and $r'_{Esc}(t) \to c_{Esc}$ when $t \to +\infty$.

- There exist:
  - a minimum point $m_{next}$ in $M$ satisfying $V(m_{next}) < V(m)$,
  - a profile of travelling front $\phi$ in $\Phi_{c_{Esc},\text{norm}}(m_{next}, m)$,
  - a $C^1$-function $[0, +\infty) \to \mathbb{R}$, $t \mapsto r_{hom-next}(t)$,

such that, when $t$ approaches $+\infty$, the following limits hold:

$$r_{Esc}(t) - r_{hom-next}(t) \to +\infty \quad \text{and} \quad r'_{hom-next}(t) \to c_{Esc}$$

and

$$\sup_{r \in [r_{hom-next}(t), r_{hom}(t)]} |u(r, t) - \phi(r - r_{Esc}(t))| \to 0$$

and, for every positive quantity $L$,

$$\|r \mapsto u(r_{hom-next}(t) + r, t) - m_{next}\|_{H^1([-L,L],\mathbb{R}^n)} \to 0.$$

4.3 Settings of the proof, 1: normalization and choice of origin of times

Let us keep the notation and assumptions of subsection 4.1 and let us assume that the hypotheses $(H_{coerc})$ and $(G)$ and $(H_{hom})$ and $(H_{inv})$ of Proposition 2 hold.

Before doing anything else, let us clean up the place.
• For notational convenience, let us assume without loss of generality that $m = 0_{\mathbb{R}^n}$ and $V(0_{\mathbb{R}^n}) = 0$.

• According to Lemma 1 on page 11 ("global existence of solutions and attracting ball"), we may assume (without loss of generality, up to changing the origin of time) that, for all $t$ in $[0, +\infty)$,

$$ \sup_{r \in [0, +\infty)} |u(r, t)| \leq R_{\text{att}, \infty}. $$

• According to the a priori bounds (16) on page 13, we may assume (without loss of generality, up to changing the origin of time) that

$$ \sup_{t \geq 0} \| r \mapsto u(r, t) \|_{C^{k+1}([0, \infty), \mathbb{R}^n)} < +\infty \quad \text{and} \quad \sup_{t \geq 0} \| r \mapsto u_t(r, t) \|_{C^{k-1}([0, \infty), \mathbb{R}^n)} < +\infty. $$

• According to (H\text{hom}), we may assume (without loss of generality, up to changing the origin of time) that, for all $t$ in $[0, +\infty)$,

$$ r_{\text{hom}}'(t) \geq 0. $$

4.4 Firewall functional in the laboratory frame

Let $\kappa_0$ and $r_{s-c}$ denote two positive quantities, with $\kappa_0$ sufficiently small and $r_{s-c}$ sufficiently large so that

$$ \frac{w_{\text{en}, 0}}{4} \left( \frac{d-1}{r_{s-c}} + \kappa_0 \right)^2 \leq \frac{1}{4} \quad \text{and} \quad \frac{1}{4} \left( \frac{d-1}{r_{s-c}} + \kappa_0 \right) \leq \frac{1}{4} \quad \text{and} \quad \frac{d-1}{r_{s-c}} + \kappa_0 \leq \frac{\lambda_{\min}}{4} $$

(those properties will be used to prove inequality (34) below), namely (since according to its definition (23) on page 16 the quantity $w_{\text{en}, 0}$ is not larger than 1):

$$ \kappa_0 = \min \left( \frac{1}{2}, \frac{\lambda_{\min}}{8} \right) $$

and

$$ r_{s-c} = \max \left( 2(d-1), \frac{8(d-1)}{\lambda_{\min}} \right). $$

Let us consider the weight function $\psi_0$ defined by

$$ \psi_0(r) = \exp(-\kappa_0|r|), $$

and, for every quantity $\rho$ not smaller than $r_{s-c}$, let $T_\rho \psi_0$ denote the function $[0, +\infty) \to \mathbb{R}$ defined by:

$$ T_\rho \psi_0(r) = \begin{cases} 
\psi_0(r - \rho) \left( \frac{r}{r_{s-c}} \right)^{d-1} & \text{if } 0 \leq r \leq r_{s-c}, \\
\psi_0(r - \rho) & \text{if } r \geq r_{s-c}, 
\end{cases} $$

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Figure 7: Graph of the weight function $r \mapsto T_\rho \psi_0(r)$ used to define the firewall function $F_0(\rho, t)$. The quantity $r_{s-c}$ is large, and, according to the definition of $\kappa_0$, the slope is small.

As the following computations will show, for $r$ larger than this quantity $r_{s-c}$, the “curvature terms” that take place in the time derivatives of energy and $L^2$ functionals (see expressions (17) and (18) on page 13) will be sufficiently small for the desired estimates to hold. The subscript “s-c” thus refers to “small curvature” (or equivalently, “large radius”).

Thus, the function $T_\rho \psi_0$ defined above is:

- a translate of the function $\psi_0$ far from the origin (for $r$ larger than $r_{s-c}$),
- the same translate multiplied by a factor proportional to the “Lebesgue measure” weight $r^{d-1}$ close to the origin (for $r$ smaller than $r_{s-c}$), this factor being equal to 1 at $r = r_{s-c}$ to ensure the continuity of the function.

One purpose of this definition is to control the last terms in the expressions (17) and (18) for the time derivatives of the energy and $L^2$ functionals. For every pair $(\rho, r)$ of nonnegative quantities with $\rho$ not smaller than $r_{s-c}$,

$$d - 1 \frac{1}{r} T_\rho \psi_0(r) - T_\rho \psi_0'(r) = \begin{cases} - \kappa_0 T_\rho \psi_0(r) & \text{if } r < r_{s-c}, \\ (d - 1 \frac{1}{r} - \kappa_0) T_\rho \psi_0(r) & \text{if } r_{s-c} < r < \rho, \\ (d - 1 \frac{1}{r} + \kappa_0) T_\rho \psi_0(r) & \text{if } \rho < r, \end{cases}$$

thus, in all three cases,

$$\left| \frac{d - 1}{r} T_\rho \psi_0(r) - T_\rho \psi_0'(r) \right| \leq \left( \frac{d - 1}{r_{s-c}} + \kappa_0 \right) T_\rho \psi_0(r).$$

For every pair $(\rho, t)$ of nonnegative quantities with $\rho$ not smaller than $r_{s-c}$, let us consider the “firewall” function

$$F_0(\rho, t) = \int_0^{+\infty} T_\rho \psi_0(r) \left[ w_{en,0} \left( \frac{u_r(r,t)^2}{2} + V(u(r,t)) \right) + \frac{u(r,t)^2}{2} \right] dr.$$
For every nonnegative time \( t \), let us consider the following set (the set of radii where the solution “Escapes” at a certain distance from \( 0 \)):

\[
\Sigma_{\text{Esc},0}(t) = \{ r \in [0, +\infty) : |u(r, t)| > d_{\text{Esc}} \},
\]

(32)

**Lemma 2** (firewall decrease up to pollution term). There exist positive quantities \( \nu_{F_0} \) and \( K_{F_0} \), depending only on \( V \), such that, for every pair \((\rho, t)\) of nonnegative quantities,

\[
\partial_t F_0(\rho, t) \leq -\nu_{F_0} F_0(\rho, t) + K_{F_0} \int_{\Sigma_{\text{Esc},0}(t)} T_{\rho} \psi_0(r) \, dr.
\]

(33)

**Proof.** It follows from expressions (17) and (18) on page 13 for the time derivatives of localized energy and \( L^2 \)-functionals that

\[
\partial_t F_0(\rho, t) = \int_0^{+\infty} T_{\rho} \psi_0 \left( -w_{\text{en},0} u_t^2 - u \cdot \nabla V(u) - u_r^2 \right)
\]

\[
+ \left( \frac{d-1}{r} - \frac{T_{\rho} \psi_0 - T_{\rho} \psi_0'} \right) \left( w_{\text{en},0} u_t \cdot u_r + u \cdot u_r \right) \, dr.
\]

Thus, according to the upper bound (30),

\[
\partial_t F_0(\rho, t) \leq \int_0^{+\infty} T_{\rho} \psi_0 \left( -w_{\text{en},0} u_t^2 - u \cdot \nabla V(u) - u_r^2 \right)
\]

\[
+ \left( \frac{d-1}{r_{\text{s-c}}} + \kappa_0 \right) \left( w_{\text{en},0} |u_t \cdot u_r| + |u \cdot u_r| \right) \, dr,
\]

thus, polarizing the scalar products \( u_t \cdot u_r \) and \( u \cdot u_r \),

\[
\partial_t F_0(\rho, t) \leq \int_0^{+\infty} T_{\rho} \psi_0 \left[ \left( \frac{w_{\text{en},0}}{4} \left( \frac{d-1}{r_{\text{s-c}}} + \kappa_0 \right)^2 + \frac{1}{4} \left( \frac{d-1}{r_{\text{s-c}}} + \kappa_0 \right) - 1 \right) u_r^2 \right.
\]

\[
- u \cdot \nabla V(u) + \left( \frac{d-1}{r_{\text{s-c}}} + \kappa_0 \right) u_r^2 \] 

\[
\left. \right) \, dr,
\]

and according to inequalities (28) satisfied by the quantities \( \kappa_0 \) and \( r_{\text{s-c}} \),

(34)

\[
\partial_t F_0(\rho, t) \leq \int_0^{+\infty} T_{\rho} \psi_0 \left( -\frac{u_r^2}{2} - u \cdot \nabla V(u) + \frac{\lambda_{\text{min}}}{4} u_r^2 \right) \, dr.
\]

Let \( \nu_{F_0} \) be a positive quantity sufficiently small so that

\[
\nu_{F_0} w_{\text{en},0} \leq 1 \quad \text{and} \quad \nu_{F_0} \left( w_{\text{en},0} \lambda_{\text{max}} + \frac{1}{2} \right) \leq \frac{\lambda_{\text{min}}}{4}
\]

(35)

(these two properties will be used below), namely

\[
\nu_{F_0} = \min \left( \frac{1}{w_{\text{en},0}}, \frac{\lambda_{\text{min}}}{4(w_{\text{en},0} \lambda_{\text{max}} + 1/2)} \right).
\]
Adding and subtracting the same quantity to the right-hand side of inequality (34) yields

\[
\partial_t F_0(\rho, t) \leq \int_0^{+\infty} T_\rho \psi_0 \left[ -\frac{u_r^2}{2} - \nu F_0 \left( w_{en,0} V(u) + \frac{u^2}{2} \right) \right] \, dr \\
+ \int_0^{+\infty} T_\rho \psi_0 \left[ \nu F_0 \left( w_{en,0} V(u) + \frac{u^2}{2} \right) - u \cdot \nabla V(u) + \frac{\lambda_{\min}}{4} u^2 \right] \, dr.
\]

(36)

Observe that:

- according to the first property in (35) the first term of the right-hand side of this inequality (36) is bounded from above by \(-\nu F_0 F_0(\rho, t)\);

- according to the second property in (35) and to the estimates (22) related to the definition of \(d_{\text{Esc}}\), the integrand of the second integral of the right-hand side of this inequality (36) is nonpositive as soon as \(\rho\) is not in the set \(\Sigma_{\text{Esc},0}(t)\), therefore inequality (36) remains true if the integration domain of this second integral is restricted to \(\Sigma_{\text{Esc},0}(t)\).

The quantity:

\[
K_{F_0} = \max_{|v| \leq R_{\text{att}} \infty} \left( \nu F_0 \left( w_{en,0} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{4} v^2 \right) + 1
\]

(this quantity is positive and depends only on \(V\) — the “+1” is just here to ensure its positivity to fit with the conclusions of Lemma 2), then inequality (33) follows from inequality (36). Lemma 2 is proved.

4.5 Upper bound on the invasion speed

Let

\[
d_{\text{esc}} = d_{\text{Esc}} \sqrt{\min \left( \frac{w_{en,0}}{2}, \frac{1}{3} \right)}.
\]

(37)

As the quantity \(d_{\text{Esc}}\) defined in subsection 2.5 on page 6 this quantity \(d_{\text{esc}}\) will provide a way to measure the vicinity of the solution to the minimum point \(0_Rn\), this time in terms of the firewall function \(F_0\). The value chosen for \(d_{\text{esc}}\) depends only on \(V\) and ensures the validity of the following lemma.

Lemma 3 (escape/Escape). For all \((\rho, t)\) in \([r_{\text{esc}}, +\infty) \times [0, +\infty)\), the following assertion holds:

\[
F_0(\rho, t) \leq d_{\text{esc}}^2 \implies |u(\rho, t)| \leq d_{\text{esc}}.
\]

(38)
Proof. Let \( v \) be a function \( Y \), and assume in addition that \( v \) is of class \( C^1 \) and that its derivative is uniformly bounded on \( \mathbb{R}_+ \). Then, for all \( \rho \) in \( [r_{sc}, +\infty) \),
\[
v(\rho)^2 = T_\rho \psi_0(\rho)v(\rho)^2
\]
\[
\leq \int_\rho^{+\infty} \left| \frac{d}{dr}(T_\rho \psi_0(r)v(r)^2) \right| dr
\]
\[
\leq \int_\rho^{+\infty} \left( |T_\rho \psi'_0(r)| v(r)^2 + 2T_\rho \psi_0(r) v(r) \cdot v'(r) \right) dr
\]
\[
\leq \int_\rho^{+\infty} T_\rho \psi_0(r) ((\kappa_0 + 1) v(r)^2 + v'(r)^2) dr
\]
\[
\leq (\kappa_0 + 1) \int_\rho^{+\infty} T_\rho \psi_0(r) (v(r)^2 + v'(r)^2) dr.
\]
Thus it follows from inequality (31) on the coercivity of \( F_0(\cdot, \cdot) \) that, for all \( \rho \) in \( [r_{sc}, +\infty) \) and \( t \) in \( [0, +\infty) \),
\[
u(\rho, t)^2 \leq \frac{\kappa_0 + 1}{\min \left( \frac{w_{max, 0}}{2}, \frac{1}{2} \right)} F_0(\rho, t),
\]
and this ensures the validity of implication (38) with the value of \( d_{esc} \) chosen in definition (37).

Let \( L \) be a positive quantity, sufficiently large so that
\[
2K_{F_0} \exp\left(-\frac{\kappa_0 L}{\kappa_0} \right) \leq \nu_{F_0} \frac{d_{esc}^2}{8}, \text{ namely } L = \frac{1}{\kappa_0} \log \left( \frac{16 K_{F_0}}{\nu_{F_0} d_{esc}^2 \kappa_0} \right)
\]
(this quantity depends only on \( V \)), let \( \eta_{no-esc} : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) (“no-escape hull”) be the function defined by (see figure 8):
\[
\eta_{no-esc}(x) = \begin{cases} 
+\infty & \text{for } x < 0, \\
\frac{d_{esc}^2}{2} \left( 1 - \frac{x}{2L} \right) & \text{for } 0 \leq x \leq L, \\
\frac{d_{esc}^2}{4} & \text{for } x \geq L,
\end{cases}
\]
and let \( c_{no-esc} \) (“no-escape speed”) be a positive quantity, sufficiently large so that
\[
c_{no-esc} \frac{d_{esc}^2}{4L} \geq 2 \frac{K_{F_0}}{\kappa_0}, \text{ namely } c_{no-esc} = \frac{8 K_{F_0} L}{\kappa_0 d_{esc}^2}
\]
(this quantity depends only on \( V \). The following lemma, illustrated by figure 9 is a variant of lemma 4 of [22]; its proof is identical.

Lemma 4 (bound on invasion speed). For every pair \( (r_{left}, r_{right}) \) of points in the interval \( [r_{sc}, +\infty) \) and every \( t_0 \) in \( [0, +\infty) \), if
\[
F(r, t_0) \leq \max(\eta_{no-esc}(r - r_{left}), \eta_{no-esc}(r_{right} - r)) \text{ for all } r \in [r_{sc}, +\infty),
\]

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Figure 8: Graph of the hull function $\eta_{no-esc}$.

Figure 9: Illustration of Lemma 4: if the firewall function is below the maximum of two mirror hulls at a certain time $t_0$ and if these two hulls travel at opposite speeds $\pm c_{no-esc}$, then the firewall will remain below the maximum of those travelling hulls in the future (note that after they cross this maximum equals $+\infty$ thus the assertion of being “below” is empty).
then, for every date \( t \) not smaller than \( t_0 \) and all \( r \) in \([r_{s-c}, +\infty)\),

\[
\mathcal{F}(r, t) \leq \max \left( \eta_{\text{no-esc}}(r_{\text{left}} - c_{\text{no-esc}}(t - t_0)), \eta_{\text{no-esc}}(r_{\text{right}} + c_{\text{no-esc}}(t - t_0) - r) \right).
\]

4.6 Settings of the proof, 2: escape point and associated speeds

Let us pursue the settings for the proof of Proposition 2 “invasion implies convergence”.

According to hypothesis (H\(_{\text{hom}}\)), we may assume, up to changing the origin of time, that, for all \( t \) in \([0, +\infty)\) and for all \( r \) in \([r_{s-c}, +\infty)\),

\[
(40) \quad r_{s-c} \leq r_{\text{hom}}(t) - 1 \quad \text{and} \quad \mathcal{F}_0(r, t) \leq \max \left( \eta_{\text{no-esc}}\left(r - (r_{\text{hom}}(t) - 1)\right), \eta_{\text{no-esc}}(r_{\text{hom}}(t) - r) \right).
\]

As a consequence, for all \( t \) in \([0, +\infty)\), the set

\[
I_{\text{Hom}}(t) = \left\{ r_{\ell} \in [r_{s-c}, r_{\text{hom}}(t)] : \text{for all } r \in [r_{s-c}, +\infty), \right. \\
\left. \mathcal{F}_0(r, t) \leq \max \left( \eta_{\text{no-esc}}(r - r_{\ell}), \eta_{\text{no-esc}}(r_{\text{hom}}(t) - r) \right) \right\}
\]

is a nonempty interval (containing \([r_{\text{hom}}(t) - 1, r_{\text{hom}}(t)]\)), see figure 10. For all \( t \) in

\[
\gamma_{\text{Esc}}(t) = \inf \left( I_{\text{Hom}}(t) \right) \quad (\text{thus } \gamma_{\text{Esc}}(t) \in [r_{s-c}, r_{\text{hom}}(t) - 1]).
\]

Somehow like \( r_{\text{Esc}}(t) \), this point represents the first point at the left of \( r_{\text{hom}}(t) \) where the solution “escapes” (in a sense defined by the firewall function \( \mathcal{F}_0 \) and the no-escape hull \( \eta_{\text{no-esc}} \)) at a certain distance from \( 0_{\mathbb{R}^n} \) (except if \( I_{\text{Hom}}(t) \) is the whole interval \([r_{s-c}, r_{\text{hom}}(t)]\), in this case this “escape” does not occur). In the following, this point \( r_{\text{esc}}(t) \) will be called the “escape point” (by contrast with the “Escape point” \( r_{\text{Esc}}(t) \) defined before). According to the “hull inequality” \((40)\) and Lemma 3 (“escape/Escape”), for all \( t \) in \([0, +\infty)\),

\[
(42) \quad r_{\text{Esc}}(t) \leq r_{\text{esc}}(t) \leq r_{\text{hom}}(t) - 1 \quad \text{and} \quad \Sigma_{\text{Esc}, 0}(t) \cap [r_{\text{Esc}}(t), r_{\text{hom}}(t)] = \emptyset,
\]

and, according to hypothesis (H\(_{\text{hom}}\)),

\[
(43) \quad r_{\text{hom}}(t) - r_{\text{esc}}(t) \to +\infty \quad \text{when} \quad t \to +\infty.
\]
The big advantage of $r_{\text{esc}}(\cdot)$ with respect to $r_{\text{Esc}}(\cdot)$ is that, according to Lemma 4 ("bound on invasion speed"), the growth of $r_{\text{esc}}(\cdot)$ is more under control. More precisely, according to this lemma, for every pair $(t, s)$ of points of $[0, +\infty)$,

$$r_{\text{esc}}(t + s) \leq r_{\text{esc}}(t) + c_{\text{no-esc}} \cdot s.$$  

For every $s$ in $[0, +\infty)$, let us consider the “upper and lower bounds of the variations of $r_{\text{esc}}(\cdot)$ over all time intervals of length $s$” (see figure 11):

$$\bar{r}_{\text{esc}}(s) = \sup_{t \in [0, +\infty)} r_{\text{esc}}(t + s) - r_{\text{esc}}(t) \quad \text{and} \quad \underline{r}_{\text{esc}}(s) = \inf_{t \in [0, +\infty)} r_{\text{esc}}(t + s) - r_{\text{esc}}(t).$$

According to these definitions and to inequality (44) above, for all $t$ and $s$ in $[0, +\infty)$,

$$-\infty \leq \underline{r}_{\text{esc}}(s) \leq r_{\text{esc}}(t + s) - r_{\text{esc}}(t) \leq \bar{r}_{\text{esc}}(s) \leq c_{\text{no-esc}} \cdot s.$$  

Let us consider the four limit mean speeds:

$$c_{\text{esc-inf}} = \lim_{t \to +\infty} \inf r_{\text{esc}}(t) \quad \text{and} \quad c_{\text{esc-sup}} = \lim_{t \to +\infty} \sup r_{\text{esc}}(t) \quad \text{and} \quad c_{\text{esc-inf}} = \lim_{s \to +\infty} \inf \frac{r_{\text{esc}}(s)}{s} \quad \text{and} \quad c_{\text{esc-sup}} = \lim_{s \to +\infty} \sup \frac{r_{\text{esc}}(s)}{s}.$$  

The following inequalities follow readily from these definitions and from hypothesis (H_{inv}):

$$-\infty \leq c_{\text{esc-inf}} \leq c_{\text{esc-inf}} \leq c_{\text{esc-sup}} \leq \bar{c}_{\text{esc-sup}} \leq c_{\text{no-esc}} \quad \text{and} \quad 0 < c_{\text{Esc}} \leq c_{\text{esc-sup}}.$$  

We are going to prove that the four limit mean speeds defined just above are equal. The proof is based of the “relaxation scheme” set up in the next subsection.
4.7 Relaxation scheme in a travelling frame

The aim of this subsection is to set up an appropriate relaxation scheme in a travelling frame. This means defining an appropriate localized energy and controlling the “flux” terms occurring in the time derivative of this localized energy. The considerations made in subsection 3.2 on page 13 will be put in practice.

4.7.1 Preliminary definitions

Let us introduce the following real quantities that will play the role of “parameters” for the relaxation scheme below (see figure 12):

- the “initial time” $t_{\text{init}}$ of the time interval of the relaxation;
- the position $r_{\text{init}}$ of the origin of the travelling frame at initial time $t = t_{\text{init}}$ (in practice it will be chosen equal to $r_{\text{esc}}(t_{\text{init}})$);
- the speed $c$ of the travelling frame;
- a quantity $\rho_{\text{cut-init}}$ that will be the position of the maximum point of the weight function $\rho \mapsto \chi(\rho, t_{\text{init}})$ localizing energy at initial time $t = t_{\text{init}}$ (this weight function is defined below); the subscript “cut” refers to the fact that this weight function displays a kind of “cut-off” on the interval between this maximum point and $+\infty$. Thus the maximum point is in some sense the point “where the cut-off begins”.

Let us make on these parameters the following hypotheses:

$$0 \leq t_{\text{init}} \quad \text{and} \quad 0 < c \leq c_{\text{no-esc}} \quad \text{and} \quad 0 \leq \rho_{\text{cut-init}} \quad \text{and} \quad r_{\text{init}} \geq r_{s-c}. $$

Figure 12: Space coordinate $\rho$ and time coordinate $s$ in the travelling frame, and parameters $t_{\text{init}}$ and $r_{\text{init}}$ and $c$ and $\rho_{\text{cut-init}}$. 

- the position $r_{\text{init}}$ of the origin of the travelling frame at initial time $t = t_{\text{init}}$ (in practice it will be chosen equal to $r_{\text{esc}}(t_{\text{init}})$);
For all $\rho$ in $[-r_{\text{init}} - cs, +\infty)$ and $s$ in $[0, +\infty)$, let

$$v(\rho, s) = u(r, t) \quad \text{where} \quad r = r_{\text{init}} + cs + \rho \quad \text{and} \quad t = t_{\text{init}} + s.$$ 

This function satisfies the differential system

$$(47) \quad v_s - cv_{\rho} = -\nabla V(v) + \frac{d-1}{r_{\text{init}} + cs + \rho} v_\rho + v_{\rho\rho}.$$ 

Let $\kappa$ (rate of decrease of the weight functions), $c_{\text{cut}}$ (speed of the cutoff point in the travelling frame), and $w_{\text{en}}$ (weighting of energy in the “firewall” function) be three positive quantities, sufficiently small so that

$$(48) \quad \frac{c_{\text{cut}}(c + \kappa)w_{\text{en}}}{2} \leq \frac{1}{8} \quad \text{and} \quad \frac{w_{\text{en}}(c + \kappa + 1/2)^2}{4} \leq \frac{1}{8}$$

and $w_{\text{en}}c_{\text{cut}}(c + \kappa) \leq \frac{\lambda_{\min}}{8\lambda_{\max}}$ and $\frac{(c_{\text{cut}} + \kappa)(c + \kappa)}{2} \leq \frac{\lambda_{\min}}{16}$

(these conditions will be used to prove inequality on page 37), and so that

$$w_{\text{en}} \leq w_{\text{en},0};$$

thus we may choose the quantities $\kappa$ and $c_{\text{cut}}$ and $w_{\text{en}}$ as follows:

$$\kappa = \min\left(\frac{1}{4}, \frac{\lambda_{\min}}{16(c_{\text{no-esc}} + 1)}\right),$$

$$c_{\text{cut}} = \min\left(\frac{\lambda_{\min}}{8\lambda_{\max}}, \frac{\lambda_{\min}}{8(c_{\text{no-esc}} + 1)}\right),$$

$$w_{\text{en}} = \min\left(w_{\text{en},0}, \frac{1}{(c_{\text{no-esc}} + 1)^2}\right)$$

(these quantities depend only on $V$).

4.7.2 Localized energy

For every nonnegative quantity $s$, let us consider the intervals:

$$I_{\text{left}}(s) = [-r_{\text{init}} - cs, -r_{\text{init}} - cs + r_{s-c}],$$

$$I_{\text{main}}(s) = [-r_{\text{init}} - cs + r_{s-c}, \rho_{\text{cut-init}} + c_{\text{cut}}s],$$

$$I_{\text{right}}(s) = [\rho_{\text{cut-init}} + c_{\text{cut}}s, +\infty),$$

$$I_{\text{tot}}(s) = [-r_{\text{init}} - cs, +\infty) = I_{\text{left}}(s) \cup I_{\text{main}}(s) \cup I_{\text{right}}(s)$$

(see figure 13). Observe that, since according to hypotheses the quantity $r_{\text{init}}$ is not smaller than $r_{s-c}$, the interval $I_{\text{main}}(s)$ is nonempty. Let us consider the function $\chi(\rho, s)$ (weight function for the localized energy) defined as follows:

$$\chi(\rho, s) = \begin{cases} 
\exp(c\rho) \left(\frac{r_{\text{init}} + cs + \rho}{r_{s-c}}\right)^{d-1} & \text{if } \rho \in I_{\text{left}}(s), \\
\exp(c\rho) & \text{if } \rho \in I_{\text{main}}(s), \\
\exp((c + \kappa)(\rho_{\text{cut-init}} + c_{\text{cut}}s) - \kappa\rho) & \text{if } \rho \in I_{\text{right}}(s).
\end{cases}$$
Figure 13: Intervals $I_{\text{left}}(s)$ and $I_{\text{main}}(s)$ and $I_{\text{right}}(s)$ and graphs of the weight functions $\chi(y, s)$ and $\psi(y, s)$. 
For all $s$ in $[0, +\infty)$, let us define the “energy function” $E(s)$ by:

$$E(s) = \int_{I_{\text{tot}}(s)} \chi(\rho, s) \left( \frac{v_\rho(\rho, s)^2}{2} + V(v(\rho, s)) \right) d\rho.$$ 

### 4.7.3 Time derivative of the localized energy

For every nonnegative quantity $s$, let

$$D(s) = \int_{I_{\text{tot}}(s)} \chi(\rho, s) v_s(\rho, s)^2 d\rho.$$ 

It follows from expression (19) on page 15 for the derivative of a localized energy that

$$E'(s) = -D(s) + \int_{I_{\text{tot}}(s)} \chi s \left( \frac{v_\rho^2}{2} + V(v) \right) d\rho$$

$$+ \int_{I_{\text{tot}}(s)} \left( \frac{d-1}{r_{\text{init}} + cs + \rho} \chi + c\chi - \chi_\rho \right) v_s \cdot v_\rho d\rho.$$ 

It follows from the definition of $\chi$ that, for every real quantity $\rho$,

$$\chi_s(\rho, s) = \begin{cases} \frac{c(d-1)}{r_{\text{init}} + cs + \rho} \chi(\rho, s) & \text{if } \rho \in I_{\text{left}}(s), \\ 0 & \text{if } \rho \in I_{\text{main}}(s), \\ c_{\text{cut}}(c + \kappa) \chi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s), \end{cases}$$

and

$$\chi_\rho(\rho, s) = \begin{cases} c\chi(\rho, s) + \frac{d-1}{r_{\text{init}} + cs + \rho} \chi(\rho, s) & \text{if } \rho \in I_{\text{left}}(s), \\ c\chi(\rho, s) & \text{if } \rho \in I_{\text{main}}(s), \\ -\kappa \chi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s). \end{cases}$$

thus

$$\frac{d-1}{r_{\text{init}} + cs + \rho} \chi(\rho, s) + c\chi(\rho, s) - \chi_\rho(\rho, s) =$$

$$\begin{cases} 0 & \text{if } \rho \in I_{\text{left}}(s), \\ \frac{d-1}{r_{\text{init}} + cs + \rho} \chi(\rho, s) & \text{if } \rho \in I_{\text{main}}(s), \\ \left(c + \kappa + \frac{d-1}{r_{\text{init}} + cs + \rho}\right) \chi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s). \end{cases}$$
As a consequence, 

\[ E'(s) = -D(s) \]

\[ + \int_{I_{\text{left}}(s)} \chi \frac{c(d-1)}{r_{\text{init}} + cs + \rho} \left( \frac{v_{\rho}^2}{2} + V(v) \right) d\rho \]

\[ + \int_{I_{\text{main}}(s)} \chi \frac{d-1}{r_{\text{init}} + cs + \rho} v_s \cdot v_\rho d\rho \]

\[ + \int_{I_{\text{right}}(s)} \chi \left( c_{\text{cut}} (c + \kappa) \left( \frac{v_{\rho}^2}{2} + V(v) \right) + \left( \frac{d-1}{r_{\text{init}} + cs + \rho} + c + \kappa \right) v_s \cdot v_\rho \right) d\rho . \]

Polarizing the scalar products \( v_s \cdot v_\rho \), it follows that

\[ E'(s) \leq -\frac{1}{2} D(s) \]

\[ + \int_{I_{\text{left}}(s)} \chi \frac{c(d-1)}{r_{\text{init}} + cs + \rho} \left( \frac{v_{\rho}^2}{2} + V(v) \right) d\rho \]

\[ + \int_{I_{\text{main}}(s)} \chi \left( \frac{d-1}{r_{\text{init}} + cs + \rho} \right)^2 \frac{v_{\rho}^2}{2} d\rho \]

\[ + \int_{I_{\text{right}}(s)} \chi \left( c_{\text{cut}} (c + \kappa) \left( \frac{v_{\rho}^2}{2} + V(v) \right) + \left( \frac{d-1}{r_{\text{init}} + cs + \rho} + c + \kappa \right)^2 \frac{v_{\rho}^2}{2} \right) d\rho . \]

Let us make a brief comment on this inequality, in comparison with the (simpler) case \( d = 1 \) (see subsubsection 4.7.3 of [21]).

Observe that the last term of this inequality (the integral over \( I_{\text{right}}(s) \)) is very similar to the \( d = 1 \) case. As in the \( d = 1 \) case, its control will require the definition of a “firewall function” that will be defined in the next sub-subsection [4.7.4]. Thus the main novelty with respect to the \( d = 1 \) case is the existence of the two other integrals over \( I_{\text{left}}(s) \) and \( I_{\text{main}}(s) \) (according to the calculations above, the integral over \( I_{\text{left}}(s) \) follows from the fact that \( \chi_s(\rho,s) \) is positive when \( \rho \) belongs to this interval, and the integral over \( I_{\text{main}}(s) \) comes from the curvature term in system (47)).

Unfortunately, the firewall function that will be defined in the next sub-subsection will turn out to be of no help to control these two terms, since the weight function \( \psi(\rho,s) \) involved in its definition will have to be chosen much smaller than \( \chi(\rho,s) \) on both intervals \( I_{\text{left}}(s) \) and \( I_{\text{main}}(s) \). As a consequence, these two terms need to be treated separately. The aim of the two following lemmas is to do this job, that is to provide appropriate upper bounds for these two terms. The sole required feature of these bounds is that they should be small if the quantity \( r_{\text{init}} \) is large.

**Lemma 5** (upper bound for curvature term on \( I_{\text{left}}(s) \)). There exists a positive quantity \( K_{E_{\text{left}}} \), depending only on \( V \) and \( d \), such that, for every nonnegative quantity \( s \), the following estimate holds:

\[ \int_{I_{\text{left}}(s)} \chi \frac{c(d-1)}{r_{\text{init}} + cs + \rho} \left( \frac{v_{\rho}^2}{2} + V(v) \right) d\rho \leq K_{E_{\text{left}}} \exp\left(-cr_{\text{init}}\right) . \]
Proof. For every nonnegative quantity $s$ and every $\rho$ in $I_{\text{left}}(s)$,

$$\chi(\rho, s) \frac{c(d - 1)}{r_{\text{init}} + cs + \rho} = \frac{c(d - 1) \exp(cp)}{r_{s-c}} \left( \frac{r_{\text{init}} + cs + \rho}{r_{s-c}} \right)^{d-2} \leq \frac{c(d - 1) \exp(cp)}{r_{s-c}}$$

(this inequality still holds if $d = 1$, however recall that for clarity we decided to exclude the $d = 1$ case thus $d$ is assumed to be not smaller than 2). Thus,

$$\int_{I_{\text{left}}(s)} \chi \left( \frac{c(d - 1)}{r_{\text{init}} + cs + \rho} \right) \rho \leq \frac{d - 1}{r_{s-c}} \exp \left( c(-r_{\text{init}} - cs + r_{s-c}) \right) \leq \left( \frac{d - 1}{r_{s-c}} \exp(cr_{s-c}) \right) \exp(-cr_{\text{init}}),$$

thus inequality (52) follows from the bound (46) on the speed $c$ and the a priori bounds (25) on page 19 for the solution. Lemma 5 is proved.

Let us make the following additional hypothesis on the parameter $r_{\text{init}}$:

$$r_{\text{init}} \geq 2r_{s-c}. \tag{53}$$

Lemma 6 (upper bound for curvature term on $I_{\text{main}}(s)$). There exists a positive quantity $K_{E,\text{main}}$, depending only on $V$ and $d$, such that, for every nonnegative quantity $s$, the following estimate holds:

$$\int_{I_{\text{main}}(s)} \chi \left( \frac{d - 1}{r_{\text{init}} + cs + \rho} \right)^{2} d\rho \leq K_{E,\text{main}} \left( \exp \left( \frac{-cr_{\text{init}}}{2} \right) + \frac{2}{r_{\text{init}}} \exp \left( c(r_{\text{cut-init}} + c_{\text{cut}}s) \right) \right). \tag{54}$$

Proof. Let us consider the integral:

$$J = \int_{I_{\text{main}}(s)} \chi \left( \frac{d - 1}{r_{\text{init}} + cs + \rho} \right)^{2} d\rho$$

$$= \int_{-r_{\text{init}} - cs + r_{s-c}}^{r_{\text{init}} + cs + r_{s-c}} \exp(c \rho) \left( \frac{r_{\text{init}} + cs + \rho}{r_{s-c}} \right)^{2} d\rho$$

$$= \exp(-cr_{\text{init}} - c^{2}s) \int_{r_{s-c}}^{r_{\text{init}} + cr_{\text{cut-init}} + (c + c_{cut})s} \frac{\exp(\frac{cr}{r^{2}})}{r^{2}} dr.$$

To bound from above this expression, we may cut the integral into two pieces, namely:

$$J = \exp(-cr_{\text{init}} - c^{2}s) \left( \int_{r_{s-c}}^{r_{\text{init}}/2} \frac{\exp(\frac{cr}{r^{2}})}{r^{2}} dr + \int_{r_{\text{init}}/2}^{r_{\text{init}} + cr_{\text{cut-init}} + (c + c_{cut})s} \frac{\exp(\frac{cr}{r^{2}})}{r^{2}} dr \right)$$
(observe that according to hypothesis (53) the quantity \( r_{\text{init}}/2 \) is not smaller than \( r_{s-c} \)). Thus, bounding from above the two quantities \( \exp(cr) \) in this expression (by replacing the quantity \( r \) by the upper bound of the respective integration domain), it follows that

\[
J \leq \frac{\exp\left(-cr_{\text{init}}/2 - c^2 s\right)}{r_{s-c}} + \frac{2}{r_{\text{init}}} \exp\left(c(r_{\text{cut-init}} + c_{\text{cut}})\right),
\]

and since according to its definition (29) on page 19 the quantity \( r_{s-c} \) is not smaller than 1, it follows that

\[
J \leq \exp\left(-\frac{cr_{\text{init}}}{2}\right) + \frac{2}{r_{\text{init}}} \exp\left(c(r_{\text{cut-init}} + c_{\text{cut}})\right).
\]

Thus inequality (54) follows from the a priori bounds (25) on page 19 for the solution. Lemma 6 is proved.

Observe that, according to the definition (29) on page 19 of \( r_{s-c} \), the quantity \( (d-1)/(r_{\text{init}} + cs + \rho) \) is not larger than \( 1/2 \) as soon as \( \rho \) is \( I_{\text{right}}(s) \) (and even by the way in \( I_{\text{main}}(s) \)). Thus it follows from inequality (51) and from Lemmas 5 and 6 that, for every nonnegative quantity \( s \),

\[
E'(s) \leq -\frac{1}{2}D(s) + K_{E,\text{left}} \exp(-cr_{\text{init}})
+ K_{E,\text{main}} \left( \exp\left(-\frac{cr_{\text{init}}}{2}\right) + \frac{2}{r_{\text{init}}} \exp\left(c(r_{\text{cut-init}} + c_{\text{cut}})\right) \right)
+ \int_{I_{\text{right}}(s)} \chi\left(c_{\text{cut}}(c + \kappa)\left(\frac{v^2}{2} + V(v)\right) + \left(1/2 + c + \kappa\right)\frac{v^2}{2}\right) d\rho.
\]

4.7.4 Definition of the “firewall” function and bound on the time derivative of energy

A second function (the “firewall”) will now be defined, to get some control over the last term of the right-hand side of inequality (51). Let us consider the function \( \psi(y, s) \) (weight function for the firewall function) defined as follows (for every nonnegative quantity \( s \) and every quantity \( \rho \) in \( I_{\text{tot}}(s) \)):

\[
\psi(\rho, s) = \begin{cases} 
\exp\left((c + \kappa)\rho - \kappa(r_{\text{cut-init}} + c_{\text{cut}})\right)\left(\frac{r_{\text{init}} + cs + \rho}{r_{s-c}}\right)^{d-1} & \text{if } \rho \in I_{\text{left}}(s), \\
\exp\left((c + \kappa)\rho - \kappa(r_{\text{cut-init}} + c_{\text{cut}})\right) & \text{if } \rho \in I_{\text{main}}(s), \\
\chi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s)
\end{cases}
\]

(see figure 13) and, for every nonnegative quantity \( s \), let us define the “firewall” function by:

\[
F(s) = \int_{I_{\text{tot}}(s)} \psi(\rho, s) \left(w_{\text{en}}\left(\frac{v^2}{2}\right) + V(v(\rho, s)) + \frac{v^2}{2}\right) d\rho.
\]
In view of the property (24) on page 16 concerning \( w_{en,0} \) and since \( w_{en} \) is not larger than \( w_{en,0} \), this function is coercive in the sense that, for all \( s \) in \([0, +\infty)\),

\[
\mathcal{F}(s) \geq \min\left(\frac{w_{en}}{2}, \frac{1}{4}\right) \int_{I_{tot}(s)} \psi(\rho, s)(v_\rho(\rho, s)^2 + v(\rho, s)^2) \, d\rho.
\]

(56)

Let us consider the following positive quantity (depending only on \( V \)):

\[
K_{E,\text{right}} = \frac{c_{\text{cut}}(c_{\text{no-esc}} + \kappa) + (1/2 + c_{\text{no-esc}} + \kappa)^2}{w_{en}}.
\]

It follows from the upper bound (55) on \( E'(\cdot) \) that, for all \( s \) in \([0, +\infty)\), the following inequality holds (see subsubsection 4.7.4 of [21] for a detailed justification):

\[
E'(s) \leq -\frac{1}{2} D(s) + K_{E,\text{left}} \exp(-cr_\text{init}) \\
+ K_{E,\text{main}} \left( \exp\left(-\frac{cr_\text{init}}{2}\right) + \frac{2}{r_\text{init}} \exp\left(c(r_\text{cut-init} + c_{\text{cut}}s)\right) \right) + K_{E,\text{right}} \mathcal{F}(s).
\]

Let \( s_{\text{fin}} \) be a nonnegative quantity (denoting the length of the time interval on which the relaxation scheme will be applied), and let us consider the expression:

\[
K_{E,\text{curv}}(r, s, c) = K_{E,\text{left}} s \exp(-cr) \\
+ K_{E,\text{main}} s \left( \exp\left(-\frac{cr}{2}\right) + \frac{2}{r_\text{init}} \exp\left(c(r_\text{cut-init} + c_{\text{cut}}s)\right) \right).
\]

It follows from the previous inequality that

\[
1 \int_0^{s_{\text{fin}}} D(s) \, ds \leq E(0) - E(s_{\text{fin}}) + K_{E,\text{curv}}(r_\text{init}, s_{\text{fin}}, c) + K_{E,\text{right}} \int_0^{s_{\text{fin}}} \mathcal{F}(s) \, ds.
\]

This “relaxation scheme inequality” is the core of the arguments carried out through this section to prove Proposition 2. The crucial property of the “curvature term” \( K_{E,\text{curv}}(r, s, c) \) is that is approaches 0 when \( r \) approaches +\( \infty \), uniformly with respect to \( s \) bounded and \( c \) bounded away from 0 and +\( \infty \). Our next goal is to gain some control over the firewall function (and as a consequence over the last term of this inequality).

### 4.7.5 Time derivative of the firewall function

For every nonnegative quantity \( s \), let us consider the set (the domain of space where the solution “Escapes” at distance \( d_{\text{Esc}} \) from \( 0_{\mathbb{R}^n} \)):

\[
\Sigma_{\text{Esc}}(s) = \left\{ \rho \in I_{\text{main}}(s) \cup I_{\text{right}}(s) : |v(\rho, s)| > d_{\text{Esc}} \right\}.
\]

To make the connection with definition (32) on page 21 of the related set \( \Sigma_{\text{Esc},0}(t) \), observe that, for every quantity \( \rho \) in \( I_{tot}(s) \),

\[
\rho \in \Sigma_{\text{Esc}}(s) \iff r_\text{init} + cs + \rho \in \Sigma_{\text{Esc},0}(t_\text{init} + s).
\]

Our next goal is to prove the following lemma (observe the strong similarity with Lemma 2 on page 21).
Lemma 7 (firewall decrease up to pollution term). There exist positive quantities $\nu_F$ and $K_{F,\text{left}}$, depending only on $V$, and a positive quantity $K_{F,\text{left}}$, depending only on $V$ and $d$, such that, for every nonnegative quantity $s$,

\begin{equation}
F'(s) \leq -\nu_F F(s) + K_F \int_{\Sigma_{\text{Esc}}(s)} \psi(\rho, s) \, d\rho + K_{F,\text{left}} \exp(-c r_{\text{init}}).
\end{equation}

Proof. According to expressions (19) and (20) on page 15 for the time derivatives of a localized energy and a localized $L^2$ functional, for all $s$ in $[0, +\infty)$,

\begin{equation}
F'(s) = \int_{I_{\text{tot}}(s)} \left[ \psi(-w_{\text{en}} v_s^2 - v \cdot \nabla V(v) v^2) + \psi_s \left( w_{\text{en}} \frac{v_{\rho}^2}{2} + V(v) + \frac{v^2}{2} \right) \right]
\end{equation}

(at this stage we used the “first” version of the time derivative of the $L^2$-functional written in (20), without the additional integration by parts of $c\psi - \psi_{\rho}$). The aim of the next calculations is to “control” the two last terms below this integral.

It follows from the definition of $\psi$ that, for every nonnegative quantity $s$,

\begin{equation}
\psi_s(\rho, s) = \begin{cases} 
-\kappa \psi(\rho, s) & \text{if } \rho \in I_{\text{left}}(s), \\
\kappa \psi(\rho, s) & \text{if } \rho \in I_{\text{main}}(s), \\
c_{\text{cut}} (c + \kappa) \psi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s),
\end{cases}
\end{equation}

and

\begin{equation}
\psi_{\rho}(\rho, s) = \begin{cases} 
\frac{d - 1}{r_{\text{init}} + cs + \rho} \psi(\rho, s) & \text{if } \rho \in I_{\text{left}}(s), \\
(c + \kappa) \psi(\rho, s) & \text{if } \rho \in I_{\text{main}}(s), \\
-\kappa \psi(\rho, s) & \text{if } \rho \in I_{\text{right}}(s),
\end{cases}
\end{equation}

\begin{equation}
d - 1 \frac{d - 1}{r_{\text{init}} + cs + \rho} \psi(\rho, s) + c\psi(\rho, s) - \psi_{\rho}(\rho, s) =
\end{equation}

As in the case $d = 1$ (see [21]), the sole problematic term in the right-hand side of expression (59) (with respect to the conclusions of Lemma 7) is the product $(c\psi - \psi_{\rho}) v \cdot v_{\rho}$.
on the interval \( I_{\text{right}}(s) \). As in \(^2\), we are going to make an (additional) integration by parts on this term to take advantage of the smallness of \( \psi_{\text{pp}} - c\psi_{\rho} \) on \( I_{\text{right}}(s) \). There are several ways to proceed, since the integration by parts may be performed either only on \( I_{\text{main}}(s) \cup I_{\text{right}}(s) \) or on the whole interval \( I_{\text{tot}}(s) \). Since the first option would create a border term at the left of \( I_{\text{main}}(s) \) let us go on with the second option. Doing so, it follows from \(^5\) that

\[
F'(s) = \int_{I_{\text{tot}}(s)} \left[ \psi\left(-w_{\text{en}}v_s^2 - v \cdot \nabla V(v) - v_{\rho}^2\right) + \psi_s\left(w_{\text{en}}\left(\frac{v_{\rho}^2}{2} + V(v)\right) + \frac{v^2}{2}\right)\right.
+ \left.w_{\text{en}}\left(\frac{d - 1}{r_{\text{init}} + cs + \rho} \psi + c\psi - \psi_{\rho}\right)v_s \cdot v_{\rho} + \frac{d - 1}{r_{\text{init}} + cs + \rho} \psi v \cdot v_{\rho}\right]
+ (\psi_{\text{pp}} - c\psi_{\rho})\frac{v^2}{2} \, d\rho.
\]

(62)

It follows from the expression of \( \psi_{\rho} \) above that, for every nonnegative quantity \( s \),

\[
\psi_{\text{pp}}(\rho, s) - c\psi_{\rho}(\rho, s) \leq \theta(\rho, s) \quad \text{for all} \quad \rho \in I_{\text{tot}}(s)
\]

where

\[
\theta(\rho, s) = \begin{cases} 
\kappa(c + \kappa) + \frac{(c + 2\kappa)(d - 1)}{r_{\text{init}} + cs + \rho} + \frac{(d - 1)(d - 2)}{(r_{\text{init}} + cs + \rho)^2} & \text{if} \quad \rho \in I_{\text{left}}(s), \\
\kappa(c + \kappa)\psi(\rho, s) & \text{if} \quad \rho \in I_{\text{main}}(s), \\
\kappa(c + \kappa)\psi(\rho, s) & \text{if} \quad \rho \in I_{\text{right}}(s). 
\end{cases}
\]

Indeed, \( \psi_{\text{pp}} - c\psi_{\rho} \) equals \( \theta \) plus two Dirac masses of negative weight (one at the junction between \( I_{\text{left}}(s) \) and \( I_{\text{main}}(s) \), and one at the junction between \( I_{\text{main}}(s) \) and \( I_{\text{right}}(s) \)).

Observe that for every \( \rho \) in the interval \( I_{\text{main}}(s) \cup I_{\text{right}}(s) \), the quantity \( r_{\text{init}} + cs + \rho \) is not smaller than \( r_{s,c} \). As a consequence, it follows from equality \(^6\) that, for every nonnegative quantity \( s \),

\[
F'(s) \leq \int_{I_{\text{tot}}(s)} \left[ -w_{\text{en}}v_s^2 - v \cdot \nabla V(v) - v_{\rho}^2 + c_{\text{cut}}(c + \kappa)\left(w_{\text{en}}\left(\frac{v_{\rho}^2}{2} + V(v)\right) + \frac{v^2}{2}\right)\right.
+ \left.w_{\text{en}}\left(\frac{d - 1}{r_{s,c} + c + \kappa} |v_s \cdot v_{\rho}| + \frac{d - 1}{r_{s,c}} |v \cdot v_{\rho}| + \kappa(c + \kappa)\frac{v^2}{2}\right) \, d\rho
+ F_{\text{residue, left}}(s)
\]

where

\[
F_{\text{residue, left}}(s) = \int_{I_{\text{left}}(s)} \left[ \frac{c(d - 1)}{r_{\text{init}} + cs + \rho} \left(w_{\text{en}}\left(\frac{v_{\rho}^2}{2} + V(v)\right) + \frac{v^2}{2}\right) + \frac{d - 1}{r_{\text{init}} + cs + \rho} v \cdot v_{\rho}\right]
+ \left(\frac{(c + 2\kappa)(d - 1)}{r_{\text{init}} + cs + \rho} + \frac{(d - 1)(d - 2)}{(r_{\text{init}} + cs + \rho)^2}\right)\frac{v^2}{2} \, d\rho.
\]

The following lemma deals with the “residual” term \( F_{\text{residue, left}}(s) \).
Lemma 8 (control on the residual integral over $I_{\text{left}}(s)$). There exists a positive quantity $K_{\mathcal{F},\text{left}}$, depending only on $V$ and $d$, such that, for every nonnegative quantity $s$, the following estimate holds:

$$\mathcal{F}_{\text{residue, left}}(s) \leq K_{\mathcal{F},\text{left}} \exp(-cr_{\text{init}}).$$

Proof. Since $\psi$ is smaller than $\chi$ on the interval $I_{\text{left}}(s)$, the proof is identical to that of Lemma 5 on page 31 (observe the vanishing term in $\mathcal{F}_{\text{residue, left}}(s)$ if $d = 2$).

Thus it follows from inequality (64) (polarizing the scalar products $v_s \cdot v_\rho$ and $v \cdot v_\rho$) and from Lemma 8 that, for every nonnegative quantity $s$,

$$\mathcal{F}'(s) \leq \int_{I_{\text{tot}}(s)} \psi \left[ \left( -1 + \frac{c_{\text{cut}}(c + \kappa)w_{\text{en}}}{2} + w_{\text{en}} \frac{(c + \kappa + \frac{d-1}{2r_{s-c}})^2}{4} + \frac{d-1}{2r_{s-c}} \right) \frac{v_\rho^2}{2} \right. $$

$$\left. - v \cdot \nabla V(v) + c_{\text{cut}}(c + \kappa)w_{\text{en}}|V(v)| + \left( \frac{c_{\text{cut}}(c + \kappa)}{2} + \frac{d-1}{2r_{s-c}} + \frac{\kappa(c + \kappa)}{2} \right) v^2 \right] \, d\rho$$

$$+ K_{\mathcal{F},\text{left}} \exp(-cr_{\text{init}}).$$

Since according to the definition (29) on page 19 for $r_{s-c}$ the quantity $(d - 1)/r_{s-c}$ is smaller than $1/2$ and than $\lambda_{\min}/8$, it follows that

$$\mathcal{F}'(s) \leq \int_{I_{\text{tot}}(s)} \psi \left[ \left( -1 + \frac{c_{\text{cut}}(c + \kappa)w_{\text{en}}}{2} + w_{\text{en}} \frac{(c + \kappa + \frac{1}{2})^2}{4} + \frac{1}{4} \right) \frac{v_\rho^2}{2} \right. $$

$$\left. - v \cdot \nabla V(v) + c_{\text{cut}}(c + \kappa)w_{\text{en}}|V(v)| + \left( \frac{c_{\text{cut}}(c + \kappa)}{2} + \frac{\lambda_{\min}}{16} + \frac{\kappa(c + \kappa)}{2} \right) v^2 \right] \, d\rho$$

$$+ K_{\mathcal{F},\text{left}} \exp(-cr_{\text{init}}),$$

and according to the properties (48) on page 28 satisfied by the quantities $\kappa$ and $c_{\text{cut}}$ and $w_{\text{en}}$, it follows that

$$\mathcal{F}'(s) \leq \int_{I_{\text{tot}}(s)} \psi \left( -\frac{v_\rho^2}{2} - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{8\lambda_{\max}} |V(v)| + \frac{\lambda_{\min}}{8} v^2 \right) \, d\rho$$

$$+ K_{\mathcal{F},\text{left}} \exp(-cr_{\text{init}}).$$

Let $\nu_{\mathcal{F}}$ be a positive quantity, sufficiently small so that

$$\nu_{\mathcal{F}} \leq 1 \quad \text{and} \quad \nu_{\mathcal{F}} \left( w_{\text{en}} \lambda_{\max} + \frac{1}{2} \right) \leq \frac{\lambda_{\min}}{4}$$

(these two properties will be used below), namely

$$\nu_{\mathcal{F}} = \min \left( \frac{1}{w_{\text{en}}}, \frac{\lambda_{\min}}{4(w_{\text{en}} \lambda_{\max} + 1/2)} \right).$$

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Let us add and subtract to the right-hand side of inequality (65) the same quantity (with
the purpose of making appear a term proportional to $-\mathcal{F}(t)$), as follows:

$$
\mathcal{F}'(s) \leq \int_{\Omega_{\text{tot}}(s)} \psi \left[ -v^2 \rho - \nu \left( w_{\text{en}} V(v) + \frac{v^2}{2} \right) \right] d\rho \\
+ \int_{\Omega_{\text{tot}}(s)} \psi \left[ \nu \left( w_{\text{en}} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{8\lambda_{\max}} |V(v)| + \frac{\lambda_{\min}}{8} v^2 \right] d\rho \\
+ K_{\mathcal{F}, \text{left}} \exp(-cr_{\text{init}}).
$$

Observe that:

- according to the first property in (66), the first integral of the right-hand side of
  this inequality is not larger than $-\nu \mathcal{F}(s)$;

- according to the second property in (66) and to estimates (22) derived from the
  definition of $d_{\text{Esc}}$, the integrand of the second integral is nonpositive as soon as
  $\rho$ is not in $\Sigma_{\text{Esc}}(s)$. As a consequence, this inequality still holds if the domain of
  integration of this integral is restricted to $\Sigma_{\text{Esc}}(s)$.

Thus, if we consider the quantity

$$
K_{\mathcal{F}} = \max_{|v| \leq R_{\text{att}, \infty}} \left( \nu \left( w_{\text{en}} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{8\lambda_{\max}} |V(v)| + \frac{\lambda_{\min}}{8} v^2 \right) + 1
$$

(this quantity is positive and depends only on $V$, the “$+1$” is only here to ensure its
positivity according to the conclusions of Lemma 7, then inequality (58) follows from
inequality (67). Lemma 7 is proved.

For every nonnegative quantity $s$, let

$$
\mathcal{G}(s) = \int_{\Sigma_{\text{Esc}}(s)} \psi(\rho, s) d\rho.
$$

Integrating inequality (58) between 0 and a nonnegative quantity $s_{\text{fin}}$ yields, since $\mathcal{F}(s_{\text{fin}})$
is nonnegative,

$$
\int_0^{s_{\text{fin}}} \mathcal{F}(s) ds \leq \frac{1}{\nu \mathcal{F}} \left( \mathcal{F}(0) + K_{\mathcal{F}} \int_0^{s_{\text{fin}}} \mathcal{G}(s) ds + K_{\mathcal{F}, \text{left}} s_{\text{fin}} \exp(-cr_{\text{init}}) \right).
$$

Thus, if we consider the expression:

$$
\tilde{K}_{\mathcal{E}, \text{curv}}(r, s, c) = K_{\mathcal{E}, \text{curv}}(r, s, c) + \frac{K_{\mathcal{E}, \text{right}} K_{\mathcal{F}, \text{left}} s_{\text{fin}}}{\nu \mathcal{F}} \exp(-cr_{\text{init}}),
$$

then the “relaxation scheme” inequality (57) on page 34 becomes:

$$
\frac{1}{2} \int_0^{s_{\text{fin}}} \mathcal{D}(s) ds \leq \mathcal{E}(0) - \mathcal{E}(s_{\text{fin}}) + \tilde{K}_{\mathcal{E}, \text{curv}}(r_{\text{init}}, s_{\text{fin}}, c)
$$

$$
+ \frac{K_{\mathcal{E}, \text{right}}}{\nu \mathcal{F}} \left( \mathcal{F}(0) + K_{\mathcal{F}} \int_0^{s_{\text{fin}}} \mathcal{G}(s) ds \right).
$$

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Observe that, as was the case for \( K_{\text{curv}}(r, s, c) \), the “curvature term” \( \tilde{K}_{\text{curv}}(r, s, c) \) (still) approaches 0 when \( r \) approaches \(+\infty\), uniformly with respect to \( s \) bounded and \( c \) bounded away from 0 and \(+\infty\). Our next goal is to gain some control over the quantity \( G(s) \).

### 4.7.6 Control over the flux term in the time derivative of the firewall function

For every nonnegative quantity \( s \) let

\[
\rho_{\text{hom}}(s) = r_{\text{hom}}(t_{\text{init}} + s) - r_{\text{init}} - cs ,
\]

and

\[
\rho_{\text{esc}}(s) = r_{\text{esc}}(t_{\text{init}} + s) - r_{\text{init}} - cs .
\]

According to properties (42) on page 25 for the set \( \Sigma_{\text{Esc},0}(t) \),

\[
\Sigma_{\text{Esc}}(s) \subset (-\infty, \rho_{\text{esc}}(s)] \cup [\rho_{\text{hom}}(s), +\infty) ,
\]

thus if we consider the quantities

\[
G_{\text{back}}(s) = \int_{-r_{\text{init}} - cs}^{\rho_{\text{esc}}(s)} \psi(\rho, s) \, d\rho \quad \text{and} \quad G_{\text{front}}(s) = \int_{\rho_{\text{hom}}(s)}^{+\infty} \psi(\rho, s) \, d\rho .
\]

(observe that, by definition — see (41) on page 25 — the quantity \( \rho_{\text{esc}}(s) \) is not smaller than \( r_{\text{esc}} - r_{\text{init}} - cs \), and is therefore larger than \(-r_{\text{init}} - cs\)). Then

\[
G(s) \leq G_{\text{back}}(s) + G_{\text{front}}(s) .
\]

Let us make the following hypothesis (required for the next lemma to hold):

\[
(c + \kappa)(\bar{c}_{\text{esc-sup}} - c) \leq \frac{\kappa c_{\text{cut}}}{4} .
\]

(this hypothesis is satisfied as soon as \( c \) is close enough to \( \bar{c}_{\text{esc-sup}} \)).

**Lemma 9** (upper bounds on flux terms in the derivative of the firewall). There exists a positive quantity \( K[u_0] \), depending only on \( V \) and on the initial data \( u_0 \) (but not on the parameters \( t_{\text{init}} \) and \( r_{\text{init}} \) and \( c \) and \( \rho_{\text{cut-init}} \) of the relaxation scheme) such that, for every nonnegative quantity \( s \),

\[
G_{\text{back}}(s) \leq K[u_0] \exp(-\kappa \rho_{\text{cut-init}}) \exp \left( -\frac{\kappa c_{\text{cut}}}{2} s \right) ,
\]

\[
G_{\text{front}}(s) \leq \frac{1}{\kappa} \exp((c_{\text{no-esc}} + 1) \rho_{\text{cut-init}}) \exp((c_{\text{no-esc}} + \kappa)(c_{\text{cut}} + \kappa)s) \exp(-\kappa \rho_{\text{hom}}(0)) .
\]

**Proof.** The proof is identical to that of lemma 6 of [21].
4.7.7 Final form of the “relaxation scheme” inequality

Let us consider the quantity

$$K_{G,\text{back}}[u_0] = \frac{2K_{E,\text{right}}K_FK[u_0]}{\nu_F\kappa_{\text{cut}}},$$

and, for every nonnegative quantity $s$, the quantity

$$K_{G,\text{front}}(s) = \frac{K_{E,\text{right}}K_F}{\nu_F\kappa} \exp\left((c_{\text{no-esc}} + 1)(c_{\text{cut}} + 1)s\right).$$

Then, for every nonnegative quantity $s_{\text{fin}}$, according to inequalities (70), the “relaxation scheme” inequality (68) can be rewritten as follows:

$$\frac{1}{2} \int_0^{s_{\text{fin}}} \mathcal{D}(s) \, ds \leq \mathcal{E}(0) - \mathcal{E}(s_{\text{fin}}) + \frac{K_{E,\text{right}}}{\nu_F} \mathcal{F}(0) + K_{G,\text{back}}[u_0] \exp(-\kappa \rho_{\text{cut-init}})$$

$$+ K_{G,\text{front}}(s_{\text{fin}}) \exp\left((c_{\text{no-esc}} + 1) \rho_{\text{cut-init}}\right) \exp(-\kappa \rho_{\text{hom}}(0))$$

$$+ K_{E,\text{curv}}(r_{\text{init}},s_{\text{fin}},c).$$

Recall that the “curvature term” $K_{E,\text{curv}}(r,s,c)$ approaches 0 when $r$ approaches $+\infty$, uniformly with respect to $s$ bounded and $c$ bounded away from 0 and $+\infty$. Recall by the way that this last inequality requires the additional hypothesis (53) on page 32 made on the quantity $r_{\text{init}}$ (namely, $r_{\text{init}}$ should not be smaller than $2r_{s-c}$).

4.8 Compactness

The end of the proof of Proposition 2 “invasion implies convergence” will make use several times of the following compactness argument.

**Lemma 10** (compactness). Let $(r_p, t_p)_{p \in \mathbb{N}}$ denote a sequence in $[0, +\infty) \times [0, +\infty)$ such that $r_p$ approaches $+\infty$ when $p$ approaches $+\infty$, and for every integer $p$ let us consider the functions $r \mapsto u_p(r)$ and $r \mapsto \tilde{u}_p(r)$ defined by:

$$u_p(r) = u(r_p + r, t_p) \quad \text{and} \quad \tilde{u}_p(r) = u_t(r_p + r, t_p).$$

Then, up to replacing the sequence $(r_p, t_p)_{p \in \mathbb{N}}$ by a subsequence, there exist functions $u_\infty$ in $C_b^1(\mathbb{R}, \mathbb{R}^n)$ and $\tilde{u}_\infty$ in $C_b^{k-2}(\mathbb{R}, \mathbb{R}^n)$ such that, for every positive quantity $L$,

$$\|u_p(\cdot) - u_\infty(\cdot)\|_{C_b([-L,L], \mathbb{R}^n)} \to 0 \quad \text{and} \quad \|\tilde{u}_p(\cdot) - \tilde{u}_\infty(\cdot)\|_{C_b^{k-2}([-L,L], \mathbb{R}^n)} \to 0$$

when $p \to +\infty$, and such that, for all $r$ in $\mathbb{R}$,

$$\tilde{u}_\infty(r) = -\nabla V(u_\infty(r)) + u_\infty''(r).$$

**Proof.** According to the a priori bounds (28) on page 19 for the derivatives of the solutions of system (2), by compactness and a diagonal extraction procedure, there exist functions $u_\infty$ and $\tilde{u}_\infty$ such that, up to extracting a subsequence,

$$u_p(\cdot) \to u_\infty(\cdot) \quad \text{and} \quad \tilde{u}_p(\cdot) \to \tilde{u}_\infty \quad \text{when} \quad p \to +\infty,$$

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uniformly on every compact subset of $\mathbb{R}$. The limits $u_\infty$ and $\tilde{u}_\infty$ belong respectively to $C^k_b(\mathbb{R}, \mathbb{R}^n)$ and $C^{k-2}_b(\mathbb{R}, \mathbb{R}^n)$ and the convergences hold in $C^k([-L,L], \mathbb{R}^n)$ and in $C^{k-2}([-L,L], \mathbb{R}^n)$ respectively, for every positive quantity $L$.

Passing to the limit in system (2) the last conclusion follows.

### 4.9 Convergence of the mean invasion speed

The aim of this subsection is to prove the following proposition.

**Proposition 3 (mean invasion speed).** The following equalities hold:

$$c_{\text{esc-inf}} = c_{\text{esc-sup}} = \bar{c}_{\text{esc-sup}}.$$ 

**Proof.** Let us proceed by contradiction and assume that

$$c_{\text{esc-inf}} < \bar{c}_{\text{esc-sup}}.$$ 

Then, let us take and fix a positive quantity $c$ satisfying the following conditions:

$$(72) \quad c_{\text{esc-inf}} < c < \bar{c}_{\text{esc-sup}} \leq \bar{c}_{\text{esc-sup}} + \kappa c_{\text{cut}} 4(c_{\text{no-esc}} + \kappa)$$ and $\Phi_c(0 \mathbb{R}^n) = \emptyset$.

The first condition is satisfied as soon as $c$ is smaller than and sufficiently close to $\bar{c}_{\text{esc-sup}}$, thus existence of a quantity $c$ satisfying the two conditions follows from hypothesis (H_{disc-c}).

The contradiction will follow from the relaxation scheme set up in subsection 4.7. The main ingredient is: since the set $\Phi_c(0 \mathbb{R}^n)$ is empty, some dissipation must occur permanently around the escape point in a referential travelling at speed $c$. This is stated by the following lemma.

**Lemma 11 (nonzero dissipation in the absence of travelling front).** There exist positive quantities $L$ and $\varepsilon_{\text{dissip}}$ such that, for every $t$ in $[0, +\infty)$, if the quantity $r_{\text{esc}}(t)$ is not smaller than $L$, then the following inequality holds:

$$\|\rho \mapsto u_t(r_{\text{esc}}(t) + \rho, t) + cu_r(r_{\text{esc}}(t) + \rho, t)\|_{L^2([-L,L], \mathbb{R}^n)} \geq \varepsilon_{\text{dissip}}.$$ 

**Proof of Lemma 11** Let us proceed by contradiction and assume that the converse is true. Then, for every nonzero integer $p$, there exists $t_p$ in $[0, +\infty)$ such that the quantity $r_{\text{esc}}(t_p)$ is not smaller than $L$ and such that

$$(73) \quad \|\rho \mapsto u_t(r_{\text{esc}}(t_p) + \rho, t_p) + cu_r(r_{\text{esc}}(t_p) + \rho, t_p)\|_{L^2([-p,p], \mathbb{R}^n)} \leq \frac{1}{p}.$$ 

By compactness (Lemma 10), up to replacing the sequence $(t_p)_{p \in \mathbb{N}}$ by a subsequence, there exist $u_\infty$ in $C^k_b(\mathbb{R}, \mathbb{R}^n)$ and $\tilde{u}_\infty$ in $C^{k-2}_b(\mathbb{R}, \mathbb{R}^n)$ such that, for every positive quantity $L$, $\|u_p(\cdot) - u_\infty(\cdot)\|_{C^k([-L,L], \mathbb{R}^n)} \to 0$ and $\|\tilde{u}_p(\cdot) - \tilde{u}_\infty(\cdot)\|_{C^{k-2}([-L,L], \mathbb{R}^n)} \to 0$.

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when \( p \to +\infty \) and such that, for all \( r \) in \( \mathbb{R} \),

\[
\tilde{u}_\infty(r) = -\nabla V(u_\infty(r)) + u_\infty''(r).
\]

According to hypothesis (73), the function \( u_\infty + c\tilde{u}_\infty \) vanishes identically, so that \( u_\infty \) is a solution of

\[
u''_\infty + cu'_\infty - \nabla V(u_\infty) = 0.
\]

According to the properties of the escape point (42) and (43) on page 25,

\[
\sup_{\rho \in [0, +\infty)} |u_\infty(\rho)| \leq d_{\text{Esc}},
\]

thus it follows from Lemma 42 of \([21]\) that \( u_\infty(\rho) \) approaches \( 0_{\mathbb{R}^n} \) when \( \rho \) approaches \( +\infty \). On the other hand, according to the a priori bounds on the solution, \( |u_\infty(\cdot)| \) is bounded (by \( R_{\text{att}, \infty} \)), and since \( \Phi_c(0_{\mathbb{R}^n}) \) is empty, it follows from Lemma 42 of \([21]\) that \( u_\infty(\cdot) \) vanishes identically, a contradiction with the definition of \( r_{\text{esc}}(\cdot) \).

The remaining of the proof of Proposition 3 is identical to that of Proposition 3 of \([21]\), therefore it will not be reproduced here.

According to Proposition 3, the three quantities \( c_{\text{esc-inf}} \) and \( c_{\text{esc-sup}} \) and \( \bar{c}_{\text{esc-sup}} \) are equal; let

\[ c_{\text{esc}} \]

denote their common value.

### 4.10 Further control on the escape point

**Proposition 4** (mean invasion speed, further control). The following equality holds:

\[
c_{\text{esc-inf}} = c_{\text{esc}}.
\]

**Proof.** The proof is identical to that of Proposition 4 of \([21]\).

### 4.11 Dissipation approaches zero at regularly spaced times

For every \( t \) in \([1, +\infty)\), the following set

\[
\left\{ \varepsilon \in (0, +\infty) : \int_{-1}^{1} \left( \int_{-1/\varepsilon}^{1/\varepsilon} \left( u_t(r_{\text{esc}}(t) + \rho, t) + c_{\text{esc}} u_r(r_{\text{esc}}(t) + \rho, t) \right)^2 d\rho \right) dt \leq \varepsilon \right\}
\]

is (according to the a priori bounds \([25]\) on page 19 for the solution) a nonempty interval (which by the way is unbounded from above). Let

\[ \delta_{\text{dissip}}(t) \]

denote the infimum of this interval. This quantity measures to what extent the solution is, at time \( t \) and around the escape point \( r_{\text{esc}}(t) \), close to be stationary in a frame travelling at speed \( c_{\text{esc}} \).
This definition of the quantity \( \delta_{\text{dissip}}(t) \) is slightly different from the one used in [21]. The reason is that I have not been able to recover, in the case of radially symmetric solutions considered here, an estimate on the derivative of the dissipation \( D(s) \) similar to the one obtained in Lemma 7 of [21]. However this slight change in the definition of \( \delta_{\text{dissip}}(t) \) induces no significant change in the remaining arguments.

Our goal is to prove that

\[
\delta_{\text{dissip}}(t) \to 0 \quad \text{when} \quad t \to +\infty.
\]

Proposition 5 below can be viewed as a first step towards this goal.

**Proposition 5** (regular occurrence of small dissipation). For every positive quantity \( \varepsilon \), there exists a positive quantity \( T(\varepsilon) \) such that, for every \( t \) in \([0, +\infty)\),

\[
\inf_{t' \in [t, t + T(\varepsilon)]} \delta_{\text{dissip}}(t') \leq \varepsilon.
\]

**Proof.** The proof is identical to that of Proposition 5 of [21]. \( \square \)

### 4.12 Relaxation

**Proposition 6** (relaxation). The following assertion holds:

\[
\delta_{\text{dissip}}(t) \to 0 \quad \text{when} \quad t \to +\infty.
\]

**Proof.** The proof is identical to that of Proposition 6 of [21]. \( \square \)

### 4.13 Convergence

The end of the proof of Proposition 2 on page 18 (“invasion implies convergence”) is a straightforward consequence of Proposition 6, and is identical to the case of space dimension one treated in [21] (see subsections 4.13 and 4.14 of that reference). As mentioned above, the definition of the quantity \( \delta_{\text{dissip}}(t) \) is slightly different from that of [21], however since this quantity approaches 0 when \( t \) approaches +\( \infty \), limits of the profiles of the solution around the escape point \( r_{\text{esc}}(t) \) must (still with this new definition of \( \delta_{\text{dissip}}(t) \)) necessarily be solutions of system (9) satisfied by the profiles of travelling fronts. Thus all arguments remain the same, and details will not be reproduced here. Proposition 2 is proved.

## 5 Residual asymptotic energy

The aim of this section is to prove Proposition 7 below (this will prove Proposition 1 on page 11 in introduction). The generic hypotheses (G) are not required for this statement, thus let us just assume that \( V \) satisfies hypothesis \( (H_{\text{coerc}}) \).
5.1 Definitions and hypotheses

As in section 4 on page [16], let us consider a minimum point \( m \) in \( M \), a function (initial data) \( u_0 \) in \( Y \), and the corresponding solution \( (r,t) \mapsto u(r,t) = (S_t u_0)(r) \) defined on \([0, +\infty)^2\), and let us make the following hypothesis (which is identical to the one made in subsection 4.1):

\[ \text{(H\text{hom})} \quad \text{There exists a positive quantity } c_{\text{hom}} \text{ and a } C^1\text{-function } r_{\text{hom}} : [0, +\infty) \to \mathbb{R}, \text{ satisfying } r_{\text{hom}}'(t) \to c_{\text{hom}} \text{ when } t \to +\infty, \]

such that, for every positive quantity \( L \),

\[ \| r \mapsto u(r_{\text{hom}}(t) + r,t) - m \|_{H^1([-L,L])} \to 0 \text{ when } t \to +\infty. \]

Let us define the function \( t \mapsto r_{\text{Esc}}(t) \) and the quantity \( c_{\text{Esc}} \) exactly as in subsection 4.1. However, by contrast with subsection 4.1, we are not going to assume hypothesis \( \text{(H\text{inv})} \), but we are going to assume the following hypothesis instead.

\[ \text{(H\text{no-inv})} \quad \text{The quantity } c_{\text{Esc}} \text{ is nonpositive.} \]

5.2 Statement

**Proposition 7 (Residual asymptotic energy).** Assume that \( V \) satisfies hypothesis \( \text{(H\text{coerc})} \) (only) and that the solution \( (x,t) \mapsto u(x,t) \) under consideration satisfies hypotheses \( \text{(H\text{hom})} \) and \( \text{(H\text{no-inv})} \). Then, there exists a quantity \( \mathcal{E}_\infty \) ("residual asymptotic energy") in \( \{-\infty\} \cup \mathbb{R} \) such that, for every quantity \( c \) satisfying \( 0 < c < c_{\text{hom}} \),

\[ \left(74\right) \quad \int_0^t r^{d-1} \left( \frac{u(r,t)^2}{2} + V(u(r,t)) - V(m) \right) dr \to \mathcal{E}_\infty \text{ when } t \to +\infty. \]

5.3 Settings of the proof

Let us keep the notation and assumptions of subsection 5.1, and let us assume that hypotheses \( \text{(H\text{coerc})} \) and \( \text{(H\text{hom})} \) and \( \text{(H\text{no-inv})} \) of Proposition 7 hold. For notational convenience, let us assume, without loss of generality, that

\[ m = 0_{\mathbb{R}^n} \text{ and } V(0_{\mathbb{R}^n}) = 0. \]

According to Lemma 1 on page 11, we may assume (without loss of generality, up to changing the origin of times) that, for all \( t \) in \([0, +\infty)\),

\[ \left(75\right) \quad \sup_{r \in [0, +\infty)} |u(r,t)| \leq R_{\text{att,}\infty}. \]
5.4 Relaxation scheme in standing frame

As in subsection 4.7 on page 27, we are going to define a localized energy and a localized firewall function (this time in the standing frame). The steps are very similar to those of subsection 4.7, although the computations are much simpler. We will use a similar (sometimes identical) notation to denote objects analogous to those defined in subsection 4.7.

5.4.1 Localized energy

Let \( \tilde{\kappa} \) (rate of decrease of the weight functions) and \( \tilde{c}_{\text{cut}} \) (speed of the cutoff point in the standing frame) be two positive quantities, sufficiently small so that

\[
\tilde{\kappa}\tilde{c}_{\text{cut}} + \tilde{\kappa} \leq \frac{\lambda_{\text{min}}}{4\lambda_{\text{max}}} \quad \text{and} \quad \tilde{\kappa} + \frac{1}{2} \leq \frac{\lambda_{\text{min}}}{2(1 + c_{\text{hom}})}
\]

(those properties will be applied to prove inequality (83) below) and so that \( \tilde{c}_{\text{cut}} < c_{\text{hom}} \).

We may choose the quantity \( \tilde{\kappa} \) as follows:

\[
\tilde{\kappa} = \min\left(1, \frac{\lambda_{\text{min}}}{8\lambda_{\text{max}} c_{\text{hom}}}, \frac{\lambda_{\text{min}}}{4(1 + c_{\text{hom}})} \right)
\]

(then the properties (76) are satisfied for every value of \( \tilde{c}_{\text{cut}} \) in \((0, c_{\text{hom}})\)). For every positive time \( t \), let us consider the intervals:

\[
I_{\text{main}, \chi}(t) = [0, \tilde{c}_{\text{cut}} t], \quad I_{\text{right}, \chi}(t) = [\tilde{c}_{\text{cut}} t, +\infty),
\]

and let us consider the function \( \chi(r, t) \) (weight function for the localized energy) defined by:

\[
\chi(r, t) = \begin{cases} 
  r^{d-1} & \text{if } r \in I_{\text{main}, \chi}(t), \\
  r^{d-1} \exp(-\tilde{\kappa}(r - \tilde{\kappa} t)) & \text{if } r \in I_{\text{right}, \chi}(t)
\end{cases}
\]

(see figure 14) and let us define the “energy” function \( \mathcal{E}(t) \) by:

\[
\mathcal{E}(t) = \int_{0}^{+\infty} \chi(r, t) \left( \frac{u_t(r, t)^2}{2} + V(u(r, t)) \right) dr.
\]

5.4.2 Time derivative of the localized energy

For every positive time \( t \), let

\[
D(t) = \int_{0}^{+\infty} \chi(r, t) u_t(r, t)^2 dr.
\]

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Figure 14: Graphs of the weight functions $\chi(y, s)$ and $\psi(y, s)$.

It follows from expression (17) on page 13 for the derivative of a localized energy that

$$E'(t) = -D(t) + \int_0^{+\infty} \left[ \chi_t \left( \frac{u_r^2}{2} + V(u) \right) + \left( \frac{d-1}{r} \chi - \chi_r \right) u_t \cdot u_r \right] dr.$$  

It follows from the definition of $\chi$ that, for every positive time $t$,

$$\chi_t(r, t) = \begin{cases} 0 & \text{if } r \in I_{\text{main}}(t), \\ \tilde{\kappa} \tilde{c}_{\text{cut}}(r, t) & \text{if } r \in I_{\text{right}}(t), \end{cases}$$

and

$$\frac{d-1}{r} \chi(r, t) - \chi_r(r, t) = \begin{cases} 0 & \text{if } r \in I_{\text{main}}(t), \\ \tilde{\kappa} \chi(r, t) & \text{if } r \in I_{\text{right}}(t). \end{cases}$$

Thus it follows from expression (17) on page 13 for the derivative of a localized energy that

$$E'(t) = -D(t) + \int_{I_{\text{right}, \chi}(t)} \chi \left[ \tilde{\kappa} \tilde{c}_{\text{cut}} \left( \frac{u_r^2}{2} + V(u) \right) + \tilde{\kappa} u_t \cdot u_r \right] dr$$

$$\leq -\frac{1}{2} D(t) + \int_{I_{\text{right}, \chi}(t)} \chi \left[ \tilde{\kappa} \tilde{c}_{\text{cut}} \left( \frac{w_{en,0}^2}{2} + V(u) \right) + \frac{u_r^2}{2} \right] dr$$

$$+ \frac{\tilde{\kappa}^2}{2} \frac{u_r^2}{2} \right] dr$$

$$\leq -\frac{1}{2} D(t) + \frac{\tilde{\kappa} \tilde{c}_{\text{cut}} + \tilde{\kappa}}{w_{en,0}} \int_{I_{\text{right}, \chi}(t)} \chi \left[ w_{en,0} \left( \frac{u_r^2}{2} + V(u) \right) + \frac{u_r^2}{2} \right] dr.$$  

(78)
5.4.3 Definition of the “firewall” function and bound on the time derivative of energy

As in sub-subsection 4.7.4 on page 33 we are going to define a firewall function to control the “pollution” term in the derivative of energy above. Let \( \tilde{c}_{\text{cut, left}} \) and \( \tilde{c}_{\text{cut, right}} \) denote two quantities satisfying:

\[
0 < \tilde{c}_{\text{cut, right}} \leq \tilde{c}_{\text{cut}} \leq \tilde{c}_{\text{cut, left}}
\]

and, for every positive time \( t \), let us consider the intervals:

\[
I_{\text{main,} \psi}(t) = [0, \tilde{c}_{\text{cut, right}} t],
I_{\text{main,} \psi}(t) = [\tilde{c}_{\text{cut, right}} t, \tilde{c}_{\text{cut, left}} t],
I_{\text{right,} \psi}(t) = [\tilde{c}_{\text{cut, left}} t, +\infty).
\]

Let us consider the function \( \psi(r,t) \), defined on \( \mathbb{R}^2_+ \), as follows:

\[
\psi(r,t) = \begin{cases} 
    r^{d-1} \exp(-\tilde{\kappa}(\tilde{c}_{\text{cut, right}} - r)) & \text{if} \quad r \in I_{\text{main,} \psi}(t), \\
    r^{d-1} \exp(-\tilde{\kappa}(r - \tilde{c}_{\text{cut, left}} t)) & \text{if} \quad \rho \in I_{\text{right,} \psi}(t)
\end{cases}
\]

(see figure 14) and, for every positive time \( t \), let us define the “firewall” function \( F(t) \) by

\[
F(t) = \int_0^{+\infty} \psi \left[ w_{\text{en},0} \left( \frac{u_r^2}{2} + V(u) \right) + \frac{u^2}{2} \right] dr.
\]

According to inequality (24) on page 16 satisfied by \( w_{\text{en},0} \), the quantity \( F(t) \) is coercive in the following sense:

\[
F(t) \geq \min \left( \frac{w_{\text{en},0}}{2}, \frac{1}{4} \right) \int_0^{+\infty} \psi(r,t)(u_r(r,t))^2 + u(r,t)^2) \right) dr.
\]

Besides, if we consider the following quantity:

\[
\tilde{K}_{E,\text{right}} = \frac{\tilde{\kappa}(\tilde{c}_{\text{cut}} + \tilde{\kappa})}{w_{\text{en},0}},
\]

then it follows from inequality (78) that, for every positive time \( t \),

\[
\mathcal{E}'(t) \leq -\frac{1}{2} D(t) + \tilde{K}_{E,\text{right}} F(t).
\]

5.4.4 Time derivative of the firewall function

For every nonnegative time \( t \), let us consider the set \( \Sigma_{\text{Esc},0}(t) \) defined in (32) on page 21 and let us consider the same quantity \( \nu_{\mathcal{F}_0} \) as in Lemma 2 on page 21.
Lemma 12 (firewall decrease up to pollution term). There exists a nonnegative quantity $\tilde{K}_F$, depending only on $V$, such that, for every positive time $t$,

(82) \[ F'(t) \leq -\nu_F F(t) + \tilde{K}_F \int_{\Sigma_{\text{Esc},0}(t)} \psi(r, t) \, dr. \]

Proof. It follows from expressions (19) and (20) on page 15 that, for every positive time $t$,

\[ F'(t) = \int_0^{+\infty} \left[ \psi \left( -w_{en,0}u_t^2 - u \cdot \nabla V(u) - u_r^2 \right) + \psi_t \left( w_{en,0} \left( \frac{u_t^2}{2} + V(u) \right) + \frac{u_r^2}{2} \right) \\ + \left( \frac{d-1}{r} \psi - \psi_r \right) (w_{en,0} u_t \cdot u_r + u \cdot u_r) \right] \, dr. \]

It follows from the definition of $\psi$ that, for every positive time $t$,

\[ \psi_t(r, t) = \begin{cases} -\tilde{\kappa} \tilde{c}_{\text{cut, right}} \psi(r, t) & \text{if } r \in I_{\text{main}, \psi}(t), \\ 0 & \text{if } r \in I_{\text{main}, \psi}(t), \\ \tilde{\kappa} \tilde{c}_{\text{cut, left}} \psi(r, t) & \text{if } r \in I_{\text{right}}(t), \end{cases} \]

and

\[ \frac{d-1}{r} \psi(r, t) - \psi_r(r, t) = \begin{cases} -\tilde{\kappa} \psi(r, t) & \text{if } r \in I_{\text{main}, \psi}(t), \\ 0 & \text{if } r \in I_{\text{main}, \psi}(t), \\ \tilde{\kappa} \psi(r, t) & \text{if } r \in I_{\text{right}}(t). \end{cases} \]

Thus, for every positive time $t$,

\[ F'(t) \leq \int_0^{+\infty} \psi \left[ -w_{en,0}u_t^2 - u \cdot \nabla V(u) - u_r^2 + \tilde{\kappa} \tilde{c}_{\text{cut}} \left( w_{en,0} \left( \frac{u_t^2}{2} + V(u) \right) + \frac{u_r^2}{2} \right) \\ + \tilde{\kappa} (w_{en,0} |u_t| + |u_r|) \right] \, dr. \]

Thus

\[ F'(t) \leq \int_0^{+\infty} \psi \left[ -1 + \frac{\tilde{\kappa} \tilde{c}_{\text{cut}} w_{en,0}}{2} + \frac{w_{en,0} \tilde{\kappa}^2}{2} + \frac{\tilde{\kappa}}{2} \right] u_r^2 - u \cdot \nabla V(u) + \tilde{\kappa} \tilde{c}_{\text{cut}} w_{en,0} V(u) + \left( \frac{\tilde{\kappa} \tilde{c}_{\text{cut}}}{2} + \frac{\tilde{\kappa}}{2} \right) u_r^2 \, dr. \]

It follows from the properties (76) satisfied by $\tilde{c}_{\text{cut}}$ and $\tilde{\kappa}$ that

(83) \[ F'(t) \leq \int_0^{+\infty} \psi \left[ -\frac{u_r^2}{2} - u \cdot \nabla V(u) + \frac{\lambda_{\min}}{4 \lambda_{\max}} |V(u)| + \frac{\lambda_{\min}}{8} u_r^2 \right] \, dr. \]

Let us consider the quantity

\[ \tilde{K}_F = \max_{|v| \leq R_{\text{cut}, \infty}} \left( \nu_F \left( w_{en,0} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{8 \lambda_{\max}} |V(v)| + \frac{\lambda_{\min}}{8} v^2 \right) \]

(this quantity is nonnegative and depends only on $V$). Proceeding as in the proof of Lemma 7 or Lemma 8, inequality (82) follows from (83). Lemma 12 is proved. \( \square \)
5.4.5 Convergence of the localized energy

For every positive time $t$, let

$$ G(t) = \int_{\Omega_{\text{esc},0}(t)} \psi(r, t) \, dr $$

**Lemma 13** (exponential decrease of pollution terms). There exist positive quantities $K_G$ and $\varepsilon$ such that, for every positive time $t$, the following estimate holds:

$$ G(t) \leq K_G \exp(-\varepsilon t). \tag{84} $$

**Proof.** For every positive time $t$, let

$$ G_{\text{left}}(t) = \int_{0}^{r_{\text{esc}}(t)} \psi(r, t) \, dr \quad \text{and} \quad G_{\text{right}}(t) = \int_{r_{\text{hom}}(t)}^{+\infty} \psi(r, t) \, dr. $$

Observe that according to the definitions of $r_{\text{esc}}(t)$

$$ G(t) \leq G_{\text{left}}(t) + G_{\text{right}}(t). $$

On the one hand, according to the definition of $\psi(r, t)$,

$$ G_{\text{left}}(t) \leq \int_{0}^{r_{\text{esc}}(t)} r^{d-1} \exp(-\kappa (\tilde{c}_{\text{cut, right}} t - r)) \, dr \leq \exp(-\kappa \tilde{c}_{\text{cut, right}} t) r_{\text{esc}}(t)^d \exp(\kappa r_{\text{esc}}(t)). $$

According to hypothesis (H$_{\text{no-inv}}$), for every sufficiently large time $t$,

$$ r_{\text{esc}}(t) \leq \frac{\tilde{c}_{\text{cut, right}} t}{4} $$

thus there exists a positive quantity $K_{G, \text{left}}$ such that, for every sufficiently large time $t$,

$$ G_{\text{left}}(t) \leq K_{G, \text{left}} \exp\left(-\frac{\kappa \tilde{c}_{\text{cut, right}} t}{2}\right). \tag{85} $$

On the other hand, again according to the definition of $\psi(r, t)$,

$$ G_{\text{right}}(t) \leq \int_{r_{\text{hom}}(t)}^{+\infty} r^{d-1} \exp(-\kappa r - \tilde{c}_{\text{cut, left}} t) \, dr \leq \exp(\kappa \tilde{c}_{\text{cut, left}}) \int_{r_{\text{hom}}(t)}^{+\infty} r^{d-1} \exp(-\kappa r) \, dr. $$

Since the quantity $\tilde{c}_{\text{cut}}$ was chosen smaller than $c_{\text{hom}}$ and since $r_{\text{hom}}(t) \equiv c_{\text{hom}} t$ for large positive time $t$, it follows that there exists a positive quantity $K_{G, \text{right}}$ such that, for every sufficiently large time $t$,

$$ G_{\text{right}}(t) \leq K_{G, \text{right}} \exp\left(-\frac{\kappa (c_{\text{hom}} - \tilde{c}_{\text{cut}}) t}{2}\right). \tag{86} $$
It follows from (85) and (86) that there exist positive quantities $K_G$ and $\varepsilon$ such that inequality (84) holds, at least for sufficiently large time $t$. In view of the definition of $\psi(\cdot, \cdot)$, the quantity $G(t)$ is uniformly bounded on every bounded time interval, thus up to increasing the quantity $K_G$ inequality (84) actually holds for every positive time $t$. Lemma 13 is proved.

It follows from inequalities (82) and (84) that $F(t)$ approaches zero at an exponential rate when time approaches plus infinity. Thus if we consider the quantity

$$E_\infty = \liminf_{t \to +\infty} E(t) \in \{-\infty\} \cup \mathbb{R},$$

then it follows from inequality (81) that

$$E(t) \to E_\infty \quad \text{when} \quad t \to +\infty.$$

The fact that $F(t)$ approaches 0 when time approaches plus infinity for every choice of the quantities $\tilde{c}_{\text{cut, left}}$ and $\tilde{c}_{\text{cut, right}}$ satisfying (79) shows that the quantity $E_\infty$ does not depend on the choice of the quantity $\tilde{c}_{\text{cut}}$, and that the same convergence holds for the integral in (74) instead of $F(t)$. This finishes the proof of Proposition 7.

6 Proof of Theorem 1 and Proposition 1

Convergence to the propagating terrace of bistable travelling fronts follows from Proposition 2. The proof is identical to that in Theorem 1 of [21], see section 6 of that reference. Thus details will not be reproduced here. Proposition 1 (residual asymptotic energy) follows readily from Proposition 7. Proof of Theorem 1 and Proposition 1 is complete.

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References


