Global behaviour of solutions stable at infinity for gradient systems in higher space dimension: the no invasion case

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This paper is concerned with parabolic gradient systems of the form

$$u_t = -\nabla V(u) + \Delta_x u\,,$$

where the space variable x and the state variable u are multidimensional, and the potential V is coercive at infinity. For such systems, the asymptotic behaviour of solutions *stable at infinity*, that is approaching a stable spatially homogeneous equilibrium as $|x|$ goes to $+\infty$, is investigated. A partial description of the global asymptotic behaviour of such a solution is provided, depending on the mean speed of growth of the spatial domain where the solution is not close to this equilibrium, in relation with the *asymptotic energy* of the solution. If this mean speed is zero, then the asymptotic energy is nonnegative, and the time derivative u_t goes to 0 uniformly in space. If conversely the mean speed is nonzero, then the asymptotic energy equals $-\infty$. This result is called upon in a companion paper where the global behaviour of radially symmetric solutions stable at infinity is described. The proof relies mainly on energy estimates in the laboratory frame and in frames travelling at a small nonzero velocity.

Contents

1 Introduction

This paper deals with the global dynamics of nonlinear parabolic systems of the form

$$(1.1) \qquad u_t = -\nabla V(u) + \Delta_x u$$

where the time variable t is real, the space variable x lies in the spatial domain $\mathbb{R}^{d_{\mathrm{sp}}}$ with d_{sp} an integer not smaller than 2, the function $(x,t) \mapsto u(x,t)$ takes its values in the state domain $\mathbb{R}^{d_{\mathrm{st}}}$ with d_{st} a positive integer, and the nonlinearity is the gradient of a scalar *potential* function $V : \mathbb{R}^{d_{\mathrm{st}}} \to \mathbb{R}$, which is assumed to be regular (of class $\mathcal{C}^2$) and coercive at infinity (see hypothesis ($\mathrm{H_{coerc}}$) in subsection 3.1 on page 7).

A fundamental feature of system (1.1) is that it can be recast, at least formally, as the gradient flow of an energy functional. If (w, w') is a pair of vectors of $\mathbb{R}^{d_{\mathrm{sp}}}$ or $\mathbb{R}^{d_{\mathrm{st}}}$ or $\mathbb{R}^{d_{\mathrm{sp}} \times d_{\mathrm{st}}}$, let $w \cdot w'$ and $|w| = \sqrt{w \cdot w}$ denote the usual Euclidean scalar product and the usual Euclidean norm, respectively, and let us simply write w^2 for $|w|^2$. For every function $v : x \mapsto v(x)$ defined on $\mathbb{R}^{d_{\mathrm{sp}}}$ with values in $\mathbb{R}^{d_{\mathrm{st}}}$ its *energy* (or *Lagrangian* or *action*) is defined, at least formally, as

$$(1.2) \qquad \mathcal{E}[v] = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left(\frac{1}{2} |\nabla_x v(x)|^2 + V(v(x)) \right) dx$$

where

$$(1.3) \qquad |\nabla_x v(x)|^2 = \sum_{i=1}^{d_{\mathrm{sp}}} \sum_{j=1}^{d_{\mathrm{st}}} (\partial_{x_i} v_j(x))^2 \, .$$

Formally, the differential of this functional reads (skipping border terms in the integration by parts):

$$d\mathcal{E}[v] \cdot \delta v = \int_{\mathbb{R}} (\nabla_x v \cdot \nabla_x(\delta v) + \nabla V(v) \cdot \delta v) \, dx$$

$$= \int_{\mathbb{R}} (-\Delta_x v + \nabla V(v)) \cdot \delta v \, dx \, .$$

In other words, the (formal) gradient of this functional with respect to the $L^2(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})$-scalar product reads:

$$\nabla \mathcal{E}[v] = \nabla V(v) - \Delta_x v \, ,$$

and system (1.1) can formally be rewritten under the form:

$$u_t = -\nabla \mathcal{E}[u(\cdot, t)] \, .$$

Accordingly, if $(x,t) \mapsto u(x,t)$ is a solution of this system, then (formally):

$$(1.4) \qquad \frac{d}{dt} \mathcal{E}[u(\cdot, t)] = -\int_{\mathbb{R}^{d_{\mathrm{sp}}}} u_t(x,t)^2 \, dx \leq 0 \, .$$

Here

$$u_t(x,t)^2 = \sum_{j=1}^{d_{\mathrm{st}}} (\partial_t u_j(x,t))^2 \, .$$

1

An additional and related feature of system (1.1) is that a formal gradient structure exists not only in the laboratory frame, but also in every frame travelling at a constant velocity. For every c in $\mathbb{R}^{d_\text{sp}}$, if two functions $(x,t) \mapsto u(x,t)$ and $(\xi,t) \mapsto v(\xi,t)$ are related by

$$u(x,t) = v(\xi,t) \quad \text{for} \quad x = ct + \xi\,,$$

then u is a solution of (1.1) if and only if v is a solution of

$$(1.5) \qquad\qquad v_t - c \cdot \nabla_\xi v = -\nabla V(v) + \Delta_\xi v\,.$$

For every function $w : \xi \mapsto w(\xi)$ defined on $\mathbb{R}^{d_\text{sp}}$ with values in $\mathbb{R}^{d_\text{st}}$ its (formal) energy with respect to system (1.5) is defined as

$$(1.6) \qquad\qquad \mathcal{E}_c[w] = \int_{\mathbb{R}^{d_\text{sp}}} e^{c \cdot \xi} \left(\frac{1}{2} |\nabla_\xi w(\xi)|^2 + V(w(\xi)) \right) d\xi\,.$$

Formally, the differential of this functional reads (skipping border terms in the integration by parts):

$$d\mathcal{E}_c[w] \cdot \delta w = \int_{\mathbb{R}^{d_\text{sp}}} e^{c \cdot \xi} \big(\nabla_\xi w \cdot \nabla_\xi(\delta w) + \nabla V(w) \cdot \delta w \big) d\xi$$
$$= \int_{\mathbb{R}^{d_\text{sp}}} e^{c \cdot \xi} \big(-\Delta_\xi w - c \cdot \nabla_\xi w + \nabla V(w) \big) \cdot \delta w \, d\xi\,.$$

In other words, the (formal) gradient of this functional with respect to the L^2-scalar product with weight $e^{c \cdot \xi}$ on functions $\mathbb{R}^{d_\text{sp}} \to \mathbb{R}^{d_\text{st}}$ reads:

$$\nabla_c \mathcal{E}_c[w] = -\Delta_\xi w - c \cdot \nabla_\xi w + \nabla V(w)\,,$$

system (1.5) can formally be rewritten under the form:

$$(1.7) \qquad\qquad v_t = -\nabla_c \mathcal{E}_c[v(\cdot,t)]\,,$$

and if $(\xi,t) \mapsto v(\xi,t)$ is a solution of system (1.5), then (formally):

$$(1.8) \qquad\qquad \frac{d}{dt} \mathcal{E}_c[v(\cdot,t)] = -\int_{\mathbb{R}^{d_\text{sp}}} e^{c \cdot \xi} v_t(\xi,t)^2 \, d\xi \leq 0\,.$$

This gradient structure has been known for a long time [14], but it is only more recently that it received a more detailed attention from several authors (among which S. Heinze, C. B. Muratov, Th. Gallay, R. Joly, and the author [17, 18, 21, 29, 40]), and that is was shown that this structure is sufficient (in itself, that is without the use of the maximum principle) to prove results of global convergence towards travelling fronts. These ideas have been applied since in different contexts, to prove either global convergence or just existence results, see for instance [1, 2, 4–10, 25, 30–32, 35–37]. More recently, this gradient structure enabled the author to push further the program initiated by P. C. Fife and J. McLeod in the late seventies with the aim of describing the global asymptotic behaviour (when space is one-dimensional) of every *bistable* solution, that is every solution

close to stable homogeneous equilibria at both ends of space ([14–16]): those results were extended to parabolic gradient systems [41, 44], hyperbolic gradient systems [42], and radially symmetric solutions of parabolic gradient systems in higher space dimension [43].

For general solutions (without the radial symmetry hypothesis) stable at infinity of system (1.1), little seems to be known. This is in strike contrast with the scalar case d_{st} equals 1, which is the subject of a large amount of literature: for extinction/invasion (threshold) results in relation with the initial condition and the reaction term see for instance [3, 11, 33, 34, 48], for local convergence and quasi-convergence results see for instance [11, 13, 20, 26, 28, 38, 39], and for further estimates on the location and shape at large positive times of the level sets see for instance [12, 19, 20, 23, 24, 45–47].

The present paper can be viewed as a first attempt to pursue the "variational" strategy for general (not necessarily radially symmetric) solutions stable at infinity of parabolic gradient systems in higher space dimension. The achievement resulting from this attempt is rather modest at this stage since it reduces to providing the "relaxation" conclusion in the "no invasion" case. However, the description of the global behaviour of such solutions in the radially symmetric case [43] actually relies on this achievement: in the proof the main result of [43], a variant (Theorem 2 on page 30) of the main result of this paper (Theorem 1 on page 6) is indeed an essential step.

2 Assumptions, notation, and statement of the results

Throughout all the paper it is be assumed that the integer d_{sp} is not smaller than 2; the case d_{sp} equals 1 is treated in [41, 44] and presents some peculiarities (for instance the domain "far away in space" is not connected if $d_{\mathrm{sp}} = 1$ while it is connected if d_{sp} is not smaller than 2).

2.1 Semi-flow and coercivity hypothesis

Let us consider the Banach space of continuous and uniformly bounded functions equipped with the uniform norm:

$$X = \left(\mathcal{C}_{\mathrm{b}}^0(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}}), \| \dots \|_{L^\infty(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})} \right).$$

System (1.1) defines a local semi-flow in X (see for instance D. B. Henry's book [22]).

As in [41, 43, 44], let us assume that the potential function $V : \mathbb{R}^{d_{\mathrm{st}}} \to \mathbb{R}$ is of class $\mathcal{C}^2$ and that this potential function is strictly coercive at infinity in the following sense:

$$(\mathrm{H}_{\mathrm{coerc}}) \qquad\qquad \lim_{R \to +\infty} \inf_{|u| \geq R} \frac{u \cdot \nabla V(u)}{|u|^2} > 0$$

(in other words there exists a positive quantity ε such that the quantity $u \cdot \nabla V(u)$ is greater than or equal to $\varepsilon |u|^2$ as soon as $|u|$ is large enough).

According to this hypothesis ($\mathrm{H}_{\mathrm{coerc}}$), the semi-flow of system (1.1) on X is actually global (see Proposition 3.1 on page 7). Let us denote by $(S_t)_{t \geq 0}$ this semi-flow.

In the following, a *solution of system* (1.1) will refer to a function

$$\mathbb{R} \times \mathbb{R}^{d_{\mathrm{sp}}} \to \mathbb{R}^{d_{\mathrm{st}}}, \quad (x, t) \mapsto u(x, t),$$

such that the function $u_0 : x \mapsto u(x, t = 0)$ (initial condition) is in X and $u(\cdot, t)$ equals $(S_t u_0)(\cdot)$ for every nonnegative time t.

2.2 Minimum points and solutions stable at infinity

2.2.1 Minimum points

Everywhere in this paper, the term "minimum point" denotes a point where a function — namely the potential V — reaches a local *or* global minimum. Let $\mathcal{M}$ denote the set of *nondegenerate* minimum points:

$$\mathcal{M} = \{u \in \mathbb{R}^{d_{\mathrm{st}}} : \nabla V(u) = 0 \quad \text{and} \quad D^2 V(u) \text{ is positive definite}\}.$$

2.2.2 Solutions stable at infinity

Definition 2.1 (solution stable at infinity)**.** A solution $(x, t) \mapsto u(r, t)$ of system (1.1) is said to be *stable at infinity* if there exists a point m of $\mathcal{M}$ such that

$$\limsup_{r \to +\infty} |(u(r, t) - m| \to 0 \quad \text{as} \quad t \to +\infty.$$

More precisely, such a solution is said to be *stable close to m at infinity*. A function (initial condition) u_0 in X is said to be *stable (close to m) at infinity* if the solution $u(\cdot, t) = (S_t u_0)(\cdot)$ corresponding to this initial condition is stable (close to m) at infinity.

Notation. For every point m of $\mathcal{M}$, let

$$X_{\mathrm{stab\text{-}infty}}(m)$$

denote the subset of X made of initial conditions that are stable close to m at infinity.

By definition, this set is positively invariant under the semi-flow of system (1.1).

2.2.3 Invasion speed of a solution stable at infinity

Definition 2.2 (invasion speed of a solution stable at infinity)**.** For every point m of $\mathcal{M}$ and every solution $u : (x, t) \mapsto u(x, t)$ of system (1.1) which is in $X_{\mathrm{stab\text{-}infty}}(m)$ (that is, stable close to m at infinity), let us call *set of no invasion speeds* of the solution, and let us denote by $S_{\mathrm{no\text{-}inv}}[u]$, the set

$$S_{\mathrm{no\text{-}inv}}[u] = \{c > 0 : \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \ |x| \geq ct} |u(x, t) - m| \to 0 \text{ when } t \to +\infty\};$$

and let us call *invasion speed* of the solution, and let us denote by $c_{\mathrm{inv}}[u]$ the (nonnegative) infimum of this set:

$$c_{\mathrm{inv}}[u] = \inf(S_{\mathrm{no\text{-}inv}}[u]).$$

According to Lemma 2.5 below, the set $S_{\mathrm{no\text{-}inv}}[u]$ is nonempty, so that this invasion speed $c_{\mathrm{inv}}[u]$ is finite.

2.3 Preliminary results

As everywhere in the paper, let us assume that V is of class $\mathcal{C}^2$ and satisfies the coercivity hypothesis (H_{coerc}). Let m denote a point of $\mathcal{M}$.

2.3.1 Sufficient condition for stability at infinity, bound on invasion speed, and exponential decrease beyond invasion

Lemma 2.3 (sufficient condition for stability at infinity)**.** *Let δ denote a positive quantity, small enough so that, for every v in $\mathbb{R}^{d_{\mathrm{st}}}$ such that $|v - m|$ is not larger than δ, the Hessian matrix $D^2 V(v)$ is positive definite. Then, every solution $(x,t) \mapsto u(x,t)$ of system (1.1) satisfying*

$$(2.1) \qquad \lim_{r \to +\infty} \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, |x| \geq r} |u(x,0) - m| \leq \delta$$

is stable close to m at infinity.

Corollary 2.4 (to be stable at infinity is an open condition)**.** *The set $X_{\mathrm{stab\text{-}infty}}(m)$ is open in X.*

Lemma 2.5 (upper bound on the invasion speed of a solution stable at infinity)**.** *For every solution $u : (x,t) \mapsto u(x,t)$ of system (1.1) which is stable close to m at infinity, the quantity $c_{\mathrm{inv}}[u]$ is bounded from above by a quantity depending on V and m, but not on the particular solution u.*

Lemma 2.6 (exponential decrease beyond invasion speed)**.** *For every solution u $(x,t) \mapsto u(x,t)$ of system (1.1) which is stable close to m at infinity, and for every positive quantity c larger than $c_{\mathrm{inv}}[u]$, there exist positive quantities ν and $K[u]$ such that, for every nonnegative time t,*

$$(2.2) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, |x| \geq ct} |(u(x,t) - m| \leq K[u] \exp(-\nu t).$$

The quantity ν depends on V and m and the difference $c - c_{\mathrm{inv}}[u]$ (only), whereas $K[u]$ depends additionally on u.

2.3.2 Asymptotic energy of a solution stable at infinity: definition and upper semi-continuity

Notation. For every nonnegative quantity r, let $B(r)$ denote the open ball of centre $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ and radius r in $\mathbb{R}^{d_{\mathrm{sp}}}$:

$$B(r) = \{v \in \mathbb{R}^{d_{\mathrm{sp}}} : |v| < r\}.$$

Proposition 2.7 (asymptotic energy of a solution stable at infinity)**.** *For every solution $(x,t) \mapsto u(x,t)$ of system (1.1) which is stable close to a point m of $\mathcal{M}$ at infinity, there exists a quantity $\mathcal{E}_{\mathrm{asympt}}[u]$ in $\{-\infty\} \cup \mathbb{R}$ such that, for every positive quantity c larger than $c_{\mathrm{inv}}[u]$, the following limit holds:*

$$\int_{B(ct)} \left(\frac{1}{2} |\nabla_x u(x,t)|^2 + V(u(x,t)) - V(m) \right) dx \to \mathcal{E}_{\mathrm{asympt}}[u] \quad as \quad t \to +\infty.$$

Definition 2.8 (asymptotic energy of a solution stable at infinity). If $u : (x,t) \mapsto u(x,t)$ is a solution stable at infinity of system (1.1), let us call *asymptotic energy of u* the quantity $\mathcal{E}_{\text{asympt}}[u]$ provided by Proposition 2.7.

Similarly, if a function u_0 in X is an initial condition stable at infinity, let us call *asymptotic energy of u_0* the asymptotic energy of the solution of (1.1) corresponding to this initial condition, and let us denote by

$$\mathcal{E}_{\text{asympt}}[u_0]$$

this asymptotic energy.

Thus the *asymptotic energy functional* may be defined, for every point m in $\mathcal{M}$, as

$$(2.3) \qquad \mathcal{E}_{\text{asympt}, m} : X_{\text{stab-infty}}(m) \to \{-\infty\} \sqcup \mathbb{R}, \quad u_0 \mapsto \mathcal{E}_{\text{asympt}}[u_0].$$

As for the (descendent) gradient flow of a regular function on a finite-dimensional manifold, this asymptotic energy is upper semi-continuous with respect to initial condition, as stated by the following proposition. Its statements hold with respect to the topology induced on $X_{\text{stab-infty}}(m)$ by the X-norm and the usual topology on $\{-\infty\} \sqcup \mathbb{R}$.

Proposition 2.9 (upper semi-continuity of the asymptotic energy). *For every m in $\mathcal{M}$, the asymptotic energy functional $\mathcal{E}_{\text{asympt}, m}$ is upper semi-continuous; equivalently, for every real quantity E, the set*

$$\mathcal{E}_{\text{asympt}, m}^{-1}([E, +\infty)) = \left\{ u_0 \in X_{\text{stab-infty}}(m) : \mathcal{E}_{\text{asympt}}[u_0] \geq E \right\}$$

is closed.

According to Theorem 1 below, this asymptotic energy is actually either nonnegative or equal to $-\infty$.

2.4 Main result

Theorem 1 (invasion/no invasion dichotomy). *Let V denote a function in $\mathcal{C}^2(\mathbb{R}^{d_{\text{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis (H_{coerc}). Then, for every m in $\mathcal{M}$ and for every solution $(x,t) \mapsto u(x,t)$ of system (1.1) which is stable close to m at infinity, one of the following two conclusions holds:*

1. *either $c_{\text{inv}}[u] = 0$ ("no invasion"); in this case the asymptotic energy of the solution is nonnegative and the time derivative of the solution goes to 0 as time goes to $+\infty$, uniformly with respect to x in $\mathbb{R}^{d_{\text{sp}}}$:*

   $$(2.4) \qquad \sup_{x \in \mathbb{R}^{d_{\text{sp}}}} |u_t(x,t)| \to 0 \quad as \quad t \to +\infty;$$

2. *or $c_{\text{inv}}[u] > 0$ ("invasion"); in this case the asymptotic energy of the solution equals $-\infty$.*

6

3 Preliminaries

As everywhere else, let us consider a function V in $\mathcal{C}^2(\mathbb{R}^{d_{st}}, \mathbb{R})$ satisfying the coercivity hypothesis (H_{coerc}).

3.1 Global existence of solutions and attracting ball for the semi-flow

Proposition 3.1 (global existence of solutions and attracting ball). *For every function u_0 in X, system (1.1) has a unique globally defined solution $t \mapsto S_t u_0$ in $\mathcal{C}^0([0, +\infty), X)$ with initial condition u_0. In addition, there exist a positive quantity $R_{att,\infty}$ (radius of attracting ball for the L^∞-norm), depending only on V, such that, for every large enough positive time t,*

$$\|x \mapsto (S_t u_0)(x)\|_{L^\infty\left(\mathbb{R}^{d_{sp}}, \mathbb{R}^{d_{st}}\right)} \leq R_{att,\infty} .$$

The following proof was explained to me by Thierry Gallay.

Proof. As mentioned in subsection 2.1 system (1.1) is locally well-posed in X. As a consequence and due to the smoothing properties of the semi-flow it is sufficient to prove that solutions remain bounded in $L^\infty(\mathbb{R}^{d_{sp}}, \mathbb{R}^{d_{st}})$ on their maximal interval of existence to ensure that they are globally defined.

Let u_0 denote a function in X, and let

$$u : \mathbb{R}^{d_{sp}} \times [0, T_{max}), \quad (x, t) \mapsto u(x, t)$$

denote the (maximal) solution of system (1.1) with initial condition u_0, where T_{max} in $(0, +\infty]$ denotes the upper bound of the maximal time interval where this solution is defined. For all (x, t) in $\mathbb{R}^{d_{sp}} \times [0, T_{max})$, let

$$q(x, t) = \frac{1}{2} |u(x, t)|^2 .$$

It follows from this definition that, for (x, t) in $\mathbb{R}^{d_{sp}} \times [0, T_{max})$,

$$q_t = -u \cdot \nabla V(u) + u \cdot \Delta_x u \quad \text{and} \quad \Delta_x q = \sum_{i=1}^{d_{sp}} \partial_{x_i}(u \cdot \partial_{x_i} u) = |\nabla_x u|^2 + u \cdot \Delta_x u$$

(the quantity $|\nabla_x u|^2$ involves a sum over both space and field dimensions, see (1.3) on page 1). Thus the function $q(\cdot, \cdot)$ is a solution of the system:

$$(3.1) \qquad q_t = -u \cdot \nabla V(u) + \Delta_x q - |\nabla_x u|^2 .$$

Besides, according to the coercivity hypothesis (H_{coerc}), there exist positive quantities ε_{coerc} and K_{coerc} such that, for all w in $\mathbb{R}^{d_{st}}$,

$$(3.2) \qquad w \cdot \nabla V(w) \geq \varepsilon_{coerc} w^2 - K_{coerc} .$$

It follows from (3.1) and (3.2) that

$$q_t \leq -\varepsilon_{\mathrm{coerc}} u^2 + K_{\mathrm{coerc}} + \Delta_x q - |\nabla_x u|^2$$
$$\leq -2\varepsilon_{\mathrm{coerc}}\, q + K_{\mathrm{coerc}} + \Delta_x q\,,$$

and as a consequence, introducing the solution $t \mapsto \bar{q}(t)$ of the differential equation (and initial condition):

$$(3.3) \qquad \bar{q}'(t) = -2\varepsilon_{\mathrm{coerc}}\, \bar{q} + K_{\mathrm{coerc}}\,, \quad \bar{q}(0) = \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}} q(x, 0)\,,$$

it follows from the maximum principle that, for all (x, t) in $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, T_{\max})$,

$$(3.4) \qquad\qquad q(x, t) \leq \bar{q}(t)\,.$$

As a consequence blow-up cannot occur, the solution $u(x, t)$ must be defined up to $+\infty$ in time, and introducing the quantity

$$R_{\mathrm{att},\infty} = \sqrt{\frac{K_{\mathrm{coerc}}}{\varepsilon_{\mathrm{coerc}}} + 1}\,,$$

it follows from (3.3) and (3.4) that, for every large enough positive time t,

$$\sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}} |u(x, t)| \leq R_{\mathrm{att},\infty}\,.$$

Proposition 3.1 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In addition, system (1.1) has smoothing properties (Henry [22]). Due to these properties, since V is of class $\mathcal{C}^2$, for every quantity α in $(0, 1)$ every solution $t \mapsto S_t u_0$ in $\mathcal{C}^0([0, +\infty), X)$ actually belongs to

$$\mathcal{C}^0\left((0, +\infty), \mathcal{C}_{\mathrm{b}}^{2,\alpha}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})\right) \cap \mathcal{C}^1\left((0, +\infty), \mathcal{C}_{\mathrm{b}}^{0,\alpha}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})\right),$$

and, for every positive quantity ε, the quantities

$$(3.5) \qquad \sup_{t \geq \varepsilon} \|S_t u_0\|_{\mathcal{C}_{\mathrm{b}}^{2,\alpha}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})} \quad \text{and} \quad \sup_{t \geq \varepsilon} \left\| \frac{d(S_t u_0)}{dt}(t) \right\|_{\mathcal{C}_{\mathrm{b}}^{0,\alpha}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})}$$

are finite. Among these estimates, the one provided by the following corollary will be especially relevant.

Corollary 3.2 (upper bound on the H_{ul}^1-norm). *There exists a positive quantity $R_{\mathrm{att},1,\mathrm{ul}}$ (radius of attracting ball for the H_{ul}^1-norm), depending only on d_{sp} and V, such that, for every function u_0 in X and for every large enough positive time t,*

$$\|x \mapsto (S_t u_0)(x)\|_{H_{\mathrm{ul}}^1(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}})} \leq R_{\mathrm{att},1,\mathrm{ul}}\,.$$

Remark. The method used in the proof of Proposition 3.1 does not seem to extend easily to other settings, for instance if in system (1.1) the Laplace operator $\Delta_x u$ is replaced with $\mathcal{D}\Delta_x u$, with $\mathcal{D}$ a diffusion matrix (a positive definite real symmetric $d_{\mathrm{st}} \times d_{\mathrm{st}}$ matrix) which differs from the identity matrix.

8

3.2 Asymptotic compactness

Lemma 3.3 (asymptotic compactness). *For every solution $(x,t) \mapsto u(x,t)$ of system (1.1), and for every sequence $(x_n, t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty)$ such that $t_n \to +\infty$ as $n \to +\infty$, there exists a entire solution $\bar{u}$ of system (1.1) in*

$$\mathcal{C}^0 \left(\mathbb{R}, \mathcal{C}_{\mathrm{b}}^2 (\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}}) \right) \cap \mathcal{C}^1 \left(\mathbb{R}, \mathcal{C}_{\mathrm{b}}^0 (\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}}) \right) ,$$

such that, up to replacing the sequence $(x_n, t_n)_{n \in \mathbb{N}}$ by a subsequence,

$$(3.6) \qquad D^{2,1} u(x_n + \cdot, t_n + \cdot) \to D^{2,1} \bar{u} \quad as \quad n \to +\infty ,$$

uniformly on every compact subset of $\mathbb{R}^{d_{\mathrm{sp}}+1}$, where the symbol $D^{2,1} v$ stands for $(v, \nabla_x v, \Delta_x v, v_t)$ (for v equal to u or $\bar{u}$).

Proof. See [27, p. 1963] or the proof of [44, Lemma 3.2]. $\qquad\qquad \square$

3.3 Time derivative of localized energy and L^2-norm of a solution

Let m be a point of $\mathcal{M}$ and $u : (x,t) \mapsto u(x,t)$ be a solution of system (1.1). In the calculations below it is assumed that time t is positive, so that according to (3.5) the regularity properties of the solution ensure the existence of all derivatives and integrals.

3.3.1 Standing frame

Let $x \mapsto \psi(x)$ denote a function in $W^{2,1}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R})$ and let us introduce the energy (Lagrangian) and the L^2-distance to m for the solution, localized by the weight function ψ:

$$\int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi(x) \left(\frac{1}{2} |\nabla_x u(x,t)|^2 + V(u(x,t)) - V(m) \right) dx \quad \text{and} \quad \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi(x) \frac{1}{2} \big(u(x,t) - m \big)^2 dx .$$

To simplify the presentation, let us assume here that

$$m = 0_{\mathbb{R}^{d_{\mathrm{st}}}} \quad \text{and} \quad V(m) = V(0_{\mathbb{R}^{d_{\mathrm{st}}}}) = 0 .$$

Then, the time derivatives of these two functionals read:

$$(3.7) \qquad \frac{d}{dt} \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi \left(\frac{1}{2} |\nabla_x u|^2 + V(u) \right) dx = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left(-\psi u_t^2 - \nabla_x \psi \cdot \nabla_x u \cdot u_t \right) dx$$

and

$$(3.8) \qquad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi \frac{1}{2} u^2 \, dx &= \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[\left(\psi \left(-u \cdot \nabla V(u) - |\nabla_x u|^2 \right) - \nabla_x \psi \cdot u \cdot \nabla_x u \right] dx \\ &= \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[\psi \left(-u \cdot \nabla V(u) - |\nabla_x u|^2 \right) + \frac{1}{2} \Delta_x \psi \ u^2 \right] dx . \end{aligned}$$

3.3.2 Travelling frame

Let c denote a vector of $\mathbb{R}^{d_{\mathrm{sp}}}$ (velocity vector) and let us introduce the same solution viewed in a frame travelling at velocity c, that is the function $(\xi, t) \mapsto v(\xi, t)$ defined for every ξ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and nonnegative time t by

$$v(\xi, t) = u(x, t) \quad \text{for} \quad x = ct + \xi \,.$$

This function v is a solution of system (1.5) on page 2. This time, let us introduce a weight function $(\xi, t) \mapsto \psi(\xi, t)$ (depending both on space and time and defined on $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty)$, with values in $\mathbb{R}$), and such that, for every nonnegative time t the function $\xi \mapsto \psi(\xi, t)$ belongs to $W^{2,1}(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R})$ and the function $\xi \mapsto \psi_t(\xi, t)$ is defined and belongs to $L^1(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R})$. Again, let us introduce the energy (Lagrangian) and the L^2-norm of the solution (in travelling frame), localized by the weight function ψ:

$$\int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi(\xi, t) \left(\frac{1}{2} |\nabla_\xi v(\xi, t)|^2 + V(v(\xi, t)) \right) d\xi \quad \text{and} \quad \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi(\xi, t) \frac{1}{2} v(\xi, t)^2 \, d\xi \,.$$

The time derivatives of these two quantities read:

$$\frac{d}{dt} \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi \left(\frac{1}{2} |\nabla_\xi v|^2 + V(v) \right) d\xi =$$

(3.9)

$$\int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[-\psi v_t^2 + \psi_t \left(\frac{1}{2} |\nabla_\xi v|^2 + V(v) \right) + (\psi c - \nabla_\xi \psi) \cdot \nabla_\xi v \cdot v_t \right] d\xi$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi \frac{1}{2} v^2 \, d\xi$$

(3.10)

$$= \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[\psi \big(-v \cdot \nabla V(v) - |\nabla_\xi v|^2 \big) + \frac{1}{2} (\psi_t - c \cdot \nabla_\xi \psi) v^2 - \nabla_\xi \psi \cdot v \cdot \nabla_\xi v \right] d\xi$$

$$= \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[\psi \big(-v \cdot \nabla V(v) - |\nabla_\xi v|^2 \big) + \frac{1}{2} (\psi_t + \Delta_\xi \psi - c \cdot \nabla_\xi \psi) v^2 \right] d\xi \,.$$

3.4 Miscellanea

3.4.1 Escape distance

Notation. For every u in $\mathbb{R}^{d_{\mathrm{st}}}$, let $\sigma\big(D^2 V(u)\big)$ denote the spectrum (the set of eigenvalues) of the Hessian matrix of V at u, and let $\lambda_{\min}(u)$ denote the minimum of this spectrum:

(3.11)
$$\lambda_{\min}(u) = \min\big(\sigma\big(D^2 V(u)\big)\big) \,.$$

Definition 3.4 (Escape distance of a nondegenerate minimum point). For every m in $\mathcal{M}$, let us call *Escape distance of m*, and let us denote by $\delta_{\mathrm{Esc}}(m)$, the supremum of the set

(3.12) $\quad \left\{ \delta \in [0, 1] : \text{ for all } u \text{ in } \mathbb{R}^{d_{\mathrm{st}}} \text{ satisfying } |u - m| \le \delta, \quad \lambda_{\min}(u) \ge \frac{1}{2} \lambda_{\min}(m) \right\} \,.$

Since the quantity $\lambda_{\min}(u)$ varies continuously with u, this Escape distance $\delta_{\mathrm{Esc}}(m)$ is positive (thus in $(0,1]$). In addition, for all u in $\mathbb{R}^{d_{\mathrm{st}}}$ such that $|u - m|$ is not larger than $\delta_{\mathrm{Esc}}(m)$, the following inequality holds:

$$(3.13) \qquad \lambda_{\min}(u) \geq \frac{1}{2} \lambda_{\min}(m).$$

Lemma 3.5 (second order estimates for the potential around a minimum point). *For every minimum point m in $\mathcal{M}$ and every vector u in $\mathbb{R}^{d_{\mathrm{st}}}$ satisfying $|u - m| \leq \delta_{\mathrm{Esc}}(m)$, the following estimates hold:*

$$(3.14) \qquad V u) \geq \frac{\lambda_{\min}(m)}{4} (u - m)^2,$$

$$(3.15) \qquad and \quad (u - m) \cdot \nabla V(u) \geq \frac{\lambda_{\min}(m)}{2} (u - m)^2,$$

$$(3.16) \qquad and \quad (u - m) \cdot \nabla V(u) \geq V(u).$$

Proof. The three inequalities follow from inequality (3.13) and from three variants of Taylor's Theorem with Lagrange remainder applied to the function defined on $[0,1]$ by: $f(\theta) = V(m + \theta(u - m))$ (see [44, Lemma 3.3]). □

3.4.2 Lower quadratic hulls of the potential at minimum points

It will be convenient to introduce the quantity $q_{\text{low-hull}}$ defined as the minimum of the convexities of the lower quadratic hulls of V at the points of $\mathcal{M}$. With symbols:
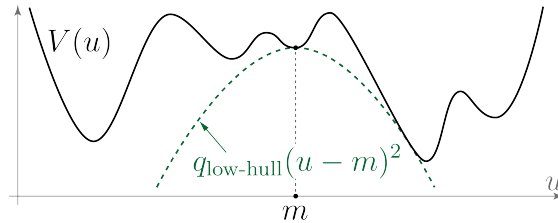
Figure 3.1: Lower quadratic hull of the potential at a minimum point (definition of the quantity $q_{\text{low-hull}}$).

$$q_{\text{low-hull}} = \min_{m \in \mathcal{M}} \ \inf_{u \in \mathbb{R}^{d_{\mathrm{st}}} \setminus \{m\}} \frac{V(u) - V(m)}{(u - m)^2},$$

see figure 3.1. This quantity $q_{\text{low-hull}}$ is negative as soon as m is not a global minimum point of V (and nonnegative otherwise), and according to hypothesis $(\mathrm{H}_{\text{coerc}})$ it is finite (in other words it is not equal to $-\infty$). This definition ensures that, for every m in $\mathcal{M}$ and for all u in $\mathbb{R}^{d_{\mathrm{st}}}$,

$$(3.17) \qquad V(u) - V(m) - q_{\text{low-hull}} (u - m)^2 \geq 0,$$

see figure 3.1. Let us introduce the following quantity:

$$w_{\text{en}} = \frac{1}{\max(1, -4\, q_{\text{low-hull}})}\,.$$

It follows from this definition that w_{en} is in $(0, 1]$ and that, for every m in $\mathcal{M}$ and for all u in $\mathbb{R}^{d_{\text{st}}}$,

$$(3.18) \qquad w_{\text{en}}\big(V(u) - V(m)\big) + \frac{1}{4}(u - m)^2 \geq 0\,.$$

3.4.3 Notation for the values of certain integrals

The following notation and the expression (3.19) below will be useful at several places throughout the paper (to provide expressions for integrals over $\mathbb{R}^{d_{\text{sp}}}$).

Notation. Let $S_{d_{\text{sp}}-1}$ denote the surface area (more precisely the $d_{\text{sp}} - 1$-dimensional volume) of the unit sphere in $\mathbb{R}^{d_{\text{sp}}}$.

For every nonnegative integer n, let e_n denote the *exponential sum function*:

$$e_n(\tau) = \sum_{k=0}^{n} \frac{\tau^k}{k!}\,.$$

For every nonnegative quantity ρ_0 and every nonnegative integer n, it follows from repeated integrations by parts that

$$(3.19) \qquad \int_{\rho_0}^{+\infty} \rho^n e^{-\rho}\, d\rho = n!\, e^{-\rho_0}\, e_n(\rho_0)\,.$$

4 Stability at infinity

4.1 Set-up

As everywhere else, let us consider a function V in $\mathcal{C}^2(\mathbb{R}^{d_{\text{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis (H_{coerc}). Let m be a point of $\mathcal{M}$, and let $(x, t) \mapsto u(x, t)$ be a solution of system (1.1). According to Proposition 3.1, there exists a positive quantity $R_{\text{max},\infty}$ (depending on the solution u under consideration) such that, for every nonnegative time t,

$$(4.1) \qquad \sup_{x \in \mathbb{R}^{d_{\text{sp}}}} |u(x, t)| \leq R_{\text{max},\infty}\,.$$

For notational convenience, let us introduce the "normalized potential" $V^\dagger$ and the "normalized solution" $u^\dagger$ defined as

$$(4.2) \qquad V^\dagger(v) = V(m + v) - V(m) \quad \text{and} \quad u^\dagger(x, t) = u(x, t) - m\,.$$

Thus the origin $0_{\mathbb{R}^{d_{\text{st}}}}$ of $\mathbb{R}^{d_{\text{st}}}$ is to $V^\dagger$ what m is to V, and $u^\dagger$ is a solution of system (1.1) with potential $V^\dagger$ instead of V.

12

4.2 Application of the maximum principle

4.2.1 Local coercivity

Let δ be a positive quantity, small enough so that, for every v in $\mathbb{R}^{d_{\text{st}}}$ such that $|v - m|$ is not larger than δ, the Hessian matrix $D^2 V(v)$ is positive definite. Since the fact that $D^2 V(v)$ is positive definite is an open condition with respect to v, there exist quantities δ' and δ'', satisfying

$$\delta < \delta' < \delta'' ,$$

and such that, for every v in $\mathbb{R}^{d_{\text{st}}}$ such that $|v - m|$ is not larger than δ'', the Hessian matrix $D^2 V(v)$ is positive definite. Thus the quantity λ defined as

$$\lambda = 2 \min_{v \in \mathbb{R}^{d_{\text{st}}}, \, |v-m| \le \delta''} \lambda_{\min}(v)$$

is positive (the factor "2" is here only to ensure homogeneity with the notation of sub-subsection 3.4.1).

Lemma 4.1 (local coercivity)**.** *For every v in $\mathbb{R}^{d_{\text{st}}}$ such that $|v - m|$ is not larger than δ'', the following inequality holds:*

$$(4.3) \qquad\qquad (v - m) \cdot \nabla V(v) \ge \frac{\lambda}{2} v^2 .$$

Proof. Let us introduce the function

$$f : [0, 1] \to \mathbb{R}, \quad v \mapsto V(m + \theta v) .$$

According to Taylor's Theorem with Lagrange remainder, there exists θ in $(0, 1)$ such that

$$f'(1) - f'(0) = f''(\theta) \quad \text{or in other words} \quad v \cdot \nabla V(m + v) - 0 = D^2 V(\theta v) \cdot v \cdot v ,$$

and inequality (4.3) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

With the notation introduced in (4.2), it follows from (4.3) that, for every w in $\mathbb{R}^{d_{\text{st}}}$ such that $|w|$ is not larger than δ'',

$$(4.4) \qquad\qquad\qquad\qquad w \cdot \nabla V^{\dagger}(w) \ge \frac{\lambda}{2} w^2 .$$

4.2.2 Potential $V^{\ddagger}$ quadratic at infinity

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cutoff function satisfying:

$$\chi(x) = 1 \text{ for all } x \text{ in } (-\infty, 0] \text{ and } \chi(x) = 0 \text{ for all } x \text{ in } [1, +\infty),$$

and let us introduce the potential function $V^{\ddagger}$ defined as

$$(4.5) \qquad V^{\ddagger}(v) = \chi(|v| - R_{\max,\infty} - |m|) V^{\dagger}(v) + \left(1 - \chi(|v| - R_{\max,\infty} - |m|)\right) \frac{|v|^2}{2} .$$

13

Thus

$$V^{\ddagger}(v) = V^{\dagger}(v) \text{ for } |v| \leq R_{\max,\infty}+|m| \quad \text{and} \quad V^{\ddagger}(v) = |v|^2 /2 \text{ for } |v| \geq R_{\max,\infty}+|m|+1\,.$$

It follows from (4.1) that $u^{\dagger}$ is still a solution of system (1.1) with potential $V^{\ddagger}$ instead of V:

$$(4.6) \qquad\qquad u_t^{\dagger} = -\nabla V^{\ddagger}(u^{\dagger}) + \Delta_x u^{\dagger}\,.$$

Let us introduce the quantity

$$(4.7) \qquad\qquad R_{\mathrm{coerc}} = R_{\max,\infty} + |m| + 1 + \sqrt{\frac{\lambda(\delta')^2}{2}}\,.$$

Since $v \cdot \nabla V^{\ddagger}(v)$ equals $|v|^2$ for $|v|$ larger than $R_{\max,\infty} + |m| + 1$, the following property holds (this property will be used to prove inequality (4.17) in the proof of Lemma 4.3 below):

$$(4.8) \qquad\qquad |v| \geq R_{\mathrm{coerc}} \implies v \cdot \nabla V^{\ddagger}(v) \geq \frac{\lambda(\delta')^2}{2}\,.$$

4.2.3 Quadratic state function

Let us introduce the "quadratic state" function $q^{\dagger} : \mathbb{R}^{d_{\mathrm{sp}}} \times [0,+\infty) \to \mathbb{R}$ defined as

$$q^{\dagger}(x,t) = \frac{1}{2}\left|u^{\dagger}(x,t)\right|^2\,,$$

and the function

$$(4.9) \qquad \overline{N} : [0,+\infty) \to \mathbb{R}\,, \quad q \mapsto \max\left\{-v \cdot \nabla V^{\ddagger}(v) : \ v \in \mathbb{R}^{d_{\mathrm{st}}} \text{ and } \frac{1}{2}v^2 = q\right\},$$

("upper-bound on the nonlinear part") and the system

$$(4.10) \qquad\qquad q_t = \overline{N}(q) + \Delta_x q\,.$$

Lemma 4.2 (the quadratic state function is a subsolution of system (4.10)). *The function* $(x,t) \mapsto q^{\dagger}(x,t)$ *is a subsolution of system* (4.10).

Proof. The same calculation as in the proof of Proposition 3.1 shows that the function $q^{\dagger}(\cdot,\cdot)$ is a solution of the system:

$$q_t^{\dagger} = -u^{\dagger} \cdot \nabla V^{\ddagger}(u^{\dagger}) + \Delta_x q^{\dagger} - \left|\nabla_x u^{\dagger}\right|^2\,,$$

and as a consequence, for all (x,t) in $\mathbb{R}^{d_{\mathrm{sp}}} \times [0,+\infty)$,

$$q_t^{\dagger} \leq -u^{\dagger} \cdot \nabla V^{\ddagger}(u^{\dagger}) + \Delta_x q^{\dagger}\,.$$

The conclusion follows from the definition (4.9) of $\overline{N}(\cdot)$. Lemma 4.2 is proved. $\qquad\square$

The next goal is to construct an appropriate *supersolution* above this subsolution $q^{\dagger}(\cdot,\cdot)$, for the same system (4.10).

4.2.4 Construction of a supersolution

To simplify the expressions below, let us introduce the quantities

$$\gamma' = \frac{1}{2}(\delta')^2 \quad \text{and} \quad \gamma'' = \frac{1}{2}(\delta'')^2,$$

and

$$(4.11) \qquad q_{\max} = \frac{1}{2}R_{\mathrm{coerc}}^2.$$

Observe that, according to the definition (4.7) of the quantity R_{coerc},

$$(4.12) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}} q^\dagger(x,0) \le q_{\max}.$$

Let us introduce the function $\eta_0 : \mathbb{R} \to \mathbb{R}$, defined as

$$(4.13) \qquad \eta_0(\rho) = \begin{cases} q_{\max} + (\gamma'' - \gamma') & \text{if} \quad \rho \le 0, \\[2mm] q_{\max} + (\gamma'' - \gamma')(1 - \rho) & \text{if} \quad 0 \le \rho \le \dfrac{q_{\max}}{\gamma'' - \gamma'}, \\[2mm] (\gamma'' - \gamma') \exp\left(-\rho + \dfrac{q_{\max}}{\gamma'' - \gamma'}\right) & \text{if} \quad \dfrac{q_{\max}}{\gamma'' - \gamma'} \le \rho, \end{cases}$$

see figure 4.1. Let c and $r_{\mathrm{Esc\text{-}init}}$ denote positive quantities, and let us introduce the
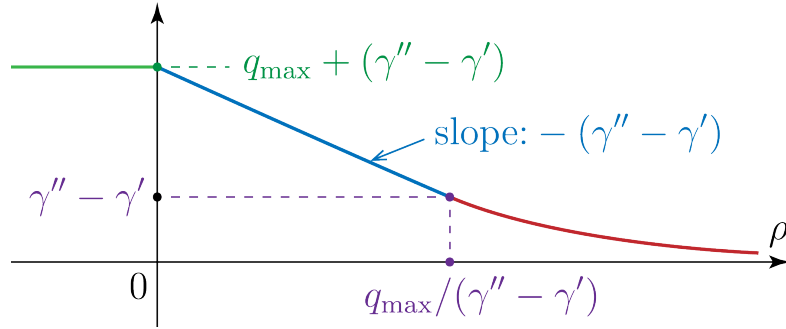
Figure 4.1: Graph of $\rho \mapsto \eta_0(\rho)$.

function

$$(4.14) \qquad \eta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}, \quad (r,t) \mapsto \eta_0\big(r - ct - r_{\mathrm{Esc\text{-}init}}\big),$$

see figure 4.2. Finally, let us introduce the function

$$\bar{\eta} : \mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty) \to \mathbb{R}, \quad (x,t) \mapsto \eta(|x|, t) + \gamma' \exp(-\lambda t),$$

and the quantity

$$(4.15) \qquad c_{\mathrm{no\text{-}esc}} = \max\left(\frac{1}{\gamma'' - \gamma'}\left(\lambda\gamma' + \sup_{q \in [0, q_{\max} + \gamma'']} \overline{N}(q)\right), 1 - \lambda\right).$$

15

Figure 4.2: Graph of $r \mapsto \eta(r,t)$.

Lemma 4.3 (supersolution). *If the positive quantity c is not smaller than $c_{\text{no-esc}}$, then the function $(x,t) \mapsto \bar{\eta}(x,t)$ is a supersolution of system* (4.10) *(for all (x,t) in $\mathbb{R}^{d_{\text{sp}}} \times [0,+\infty)$.*

Proof. This amounts to prove that, if c is not smaller than $c_{\text{no-esc}}$, then the following inequality holds for all (x,t) in $\mathbb{R}^{d_{\text{sp}}} \times [0,+\infty)$:

$$\bar{\eta}_t(x,t) \geq \overline{N}(\bar{\eta}(x,t)) + \Delta_x \bar{\eta}(x,t),$$

or equivalently, for all (r,t) in $[0,+\infty) \times [0,+\infty)$,

$$-c\partial_r \eta(r,t) - \lambda\gamma' \exp(-\lambda t) \geq \overline{N}(\eta(r,t) + \gamma' \exp(-\lambda t)) + \frac{d_{\text{sp}} - 1}{r}\partial_r \eta(r,t) + \partial_{rr}\eta(r,t).$$

Since $\partial_r \eta(\cdot,\cdot)$ is nonpositive, it is sufficient to prove this inequality without the curvature term, that is, for all (r,t) in $[0,+\infty) \times [0,+\infty)$,

$$(4.16) \qquad -c\partial_r \eta(r,t) - \lambda\gamma' \exp(-\lambda t) \geq \overline{N}(\eta(r,t) + \gamma' \exp(-\lambda t)) + \partial_{rr}\eta(r,t).$$

Let us introduce the three intervals

$$\begin{aligned} I_1 &= [0, r_{\text{Esc-init}} + ct], \\ \text{and} \quad I_2 &= [r_{\text{Esc-init}} + ct, r_{\text{Esc-init}} + ct + q_{\max}/(\gamma'' - \gamma')], \\ \text{and} \quad I_3 &= [r_{\text{Esc-init}} + ct + q_{\max}/(\gamma'' - \gamma'), +\infty), \end{aligned}$$

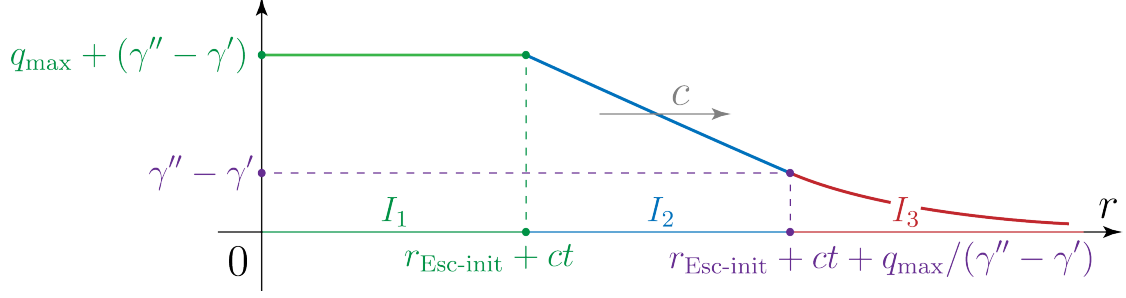see figure 4.2. Three cases are to be distinguished, depending on which among the three intervals above contains the argument r. Observe that the discontinuity of $\partial_r \eta(\cdot,t)$ at $r_{\text{Esc-init}} + ct$ can be ignored, since its contribution only enforces inequality (4.16) (it adds a negative Dirac mass to the right-hand side of this inequality).

Case 1: r is in I_1. In this case both quantities $\partial_r \eta(r,t)$ and $\partial_{rr}\eta(r,t)$ vanish and $\eta(r,t)$ equals $q_{\max} + (\gamma'' - \gamma')$, so that inequality (4.16) reduces to

$$\overline{N}(q_{\max} + (\gamma'' - \gamma') + \gamma' \exp(-\lambda t)) \leq -\lambda\gamma' \exp(-\lambda t),$$

which is implied by

$$\overline{N}(q_{\max} + (\gamma'' - \gamma') + \gamma' \exp(-\lambda t)) \leq -\lambda\gamma',$$

16

and thus by

$$\sup_{q \geq q_{\max}} \overline{N}(q) \leq -\lambda\gamma'. \tag{4.17}$$

According to the definition (4.11) of $q_{\max}$, for every quantity q not smaller than $q_{\max}$ and for every v in $\mathbb{R}^{d_{\mathrm{st}}}$ such that $v^2/2$ equals q, the following holds:

$$q \geq \frac{1}{2} R_{\mathrm{coerc}}^2 \text{ thus } |v| \geq R_{\mathrm{coerc}} \text{ thus, according to (4.8)}, \ -v \cdot \nabla V^{\ddagger}(v) \leq -\frac{\lambda(\delta')^2}{2} = -\lambda\gamma',$$

and inequality (4.17) follows.

Case 2: r is in I_2. In this case the quantity $\partial_{rr}\eta(r,t)$ vanishes and the quantity $\partial_r\eta(r,t)$ equals $-(\gamma'' - \gamma')$, so that inequality (4.16) reduces to

$$c(\gamma'' - \gamma') \geq \lambda\gamma' \exp(-\lambda t) + \overline{N}(\eta + \gamma' \exp(-\lambda t)),$$

which is implied by

$$c \geq \frac{1}{\gamma'' - \gamma'} \left(\lambda\gamma' + \sup_{q \in [0, q_{\max} + \gamma'']} \overline{N}(q) \right).$$

Case 3: r is I_3. In this case, it follows from the expressions (4.13) and (4.14) of η_0 and η that

$$\partial_r \eta(r,t) = -\eta(r,t) \quad \text{and} \quad \partial_{rr}\eta(r,t) = \eta(r,t).$$

Let us introduce the function q defined as

$$q(r,t) = \eta(r,t) + \gamma' \exp(-\lambda t).$$

It follows from the expressions of η_0 and η that $q(r,t)$ is not larger than γ''. As a consequence, for every v in $\mathbb{R}^{d_{\mathrm{sp}}}$, if $v^2/2$ is equal to $q(r,t)$ then

$$|v| = \sqrt{2q(r,t)} \leq \sqrt{2\gamma''} = \delta'',$$

thus it follows from (4.4) that

$$-v \cdot \nabla V^{\ddagger}(v) \leq -\frac{\lambda}{2}v^2 = -\lambda q, \quad \text{and it follows that} \quad \overline{N}(q(r,t)) \leq -\lambda q(r,t).$$

As a consequence, inequality (4.16) is implied in this case by the following inequality:

$$c\eta(r,t) - \lambda\gamma' \exp(-\lambda t) \geq -\lambda\eta(r,t) - \lambda\gamma' \exp(-\lambda t) + \eta(r,t),$$

or equivalently

$$(c + \lambda - 1)\eta(r,t) \geq 0, \quad \text{or equivalently} \quad c \geq 1 - \lambda.$$

17

Conclusion. In short, if c is not smaller than the quantity $c_{\text{no-esc}}$ defined in (4.15), then inequality (4.16) holds for all (r, t) in $[0, +\infty) \times [0, +\infty)$. Lemma 4.3 is proved. $\qquad\square$

4.2.5 Proof of Lemma 2.3

Proof of Lemma 2.3. Let us assume that the solution $(x, t) \mapsto u(x, t)$ under consideration satisfies hypothesis (2.1) of Lemma 2.3. Then, if the positive quantity $r_{\text{Esc-init}}$ is large enough,

$$\sup_{x \in \mathbb{R}^{d_{\text{SP}}}, \, |x| \geq r_{\text{Esc-init}}} \left| u^{\dagger}(x, t = 0) \right| \leq \delta',$$

(4.18) $\qquad$ or in other words $\qquad \displaystyle\sup_{x \in \mathbb{R}^{d_{\text{SP}}}, \, |x| \geq r_{\text{Esc-init}}} \left| q^{\dagger}(x, t = 0) \right| \leq \gamma'.$

Then it follows from inequalities (4.12) and (4.18) that, for all x in $\mathbb{R}^{d_{\text{SP}}}$,

$$q^{\dagger}(x, 0) \leq \bar{\eta}(x, 0).$$

Since $q^{\dagger}(\cdot, \cdot)$ is a subsolution of system (4.10) and since, according to Lemma 4.3 and provided that c is not smaller than the quantity $c_{\text{no-esc}}$, the function $\bar{\eta}(\cdot, \cdot)$ is a supersolution of the same system, it follows from the maximum principle that, for all (x, t) in $\mathbb{R}^{d_{\text{SP}}} \times [0, +\infty)$,

$$q^{\dagger}(x, t) \leq \bar{\eta}(x, t).$$

Since

$$\lim_{r \to +\infty} \sup_{x \in \mathbb{R}^{d_{\text{SP}}}, \, |x| \geq r} \bar{\eta}(x, t) \to 0 \quad \text{as} \quad t \to +\infty,$$

it follows that the solution $u(\cdot, \cdot)$ is stable close to m at infinity (Definition 2.1). Lemma 2.3 is proved. $\qquad\square$

4.2.6 Supersolution with uniform features

The aim of this sub-subsection is to formulate a more uniform version of Lemma 4.3 leading to a proof of Lemma 2.5. Let us introduce the potential function $V_{\text{att}}^{\ddagger}$ defined as

$$V_{\text{att}}^{\ddagger}(v) = \chi(|v| - R_{\text{att}, \infty} - |m|) V^{\dagger}(v) + \left(1 - \chi(|v| - R_{\text{att}, \infty} - |m|) \right) \frac{|v|^2}{2},$$

and the function

$$\overline{N}_{\text{att}} : [0, +\infty) \to \mathbb{R}, \quad q \mapsto \max\left\{ -v \cdot \nabla V_{\text{att}}^{\ddagger}(v) : \ v \in \mathbb{R}^{d_{\text{st}}} \text{ and } \frac{1}{2} v^2 = q \right\},$$

and the system

(4.19) $$q_t = \overline{N}_{\text{att}}(q) + \Delta_x q,$$

and the quantities

$$R_{\text{coerc-att}} = R_{\text{att},\infty} + |m| + 1 + \sqrt{\frac{\lambda(\delta')^2}{2}} \quad \text{and} \quad q_{\text{att}} = \frac{1}{2} R_{\text{coerc-att}}^2,$$

and the function $\eta_{0\text{-att}} : \mathbb{R} \to \mathbb{R}$, defined as

$$\eta_{0\text{-att}}(\rho) = \begin{cases} q_{\text{att}} + (\gamma'' - \gamma') & \text{if} \quad \rho \leq 0, \\ q_{\text{att}} + (\gamma'' - \gamma')(1 - \rho) & \text{if} \quad 0 \leq \rho \leq \dfrac{q_{\text{att}}}{\gamma'' - \gamma'}, \\ (\gamma'' - \gamma') \exp\left(-\rho + \dfrac{q_{\text{att}}}{\gamma'' - \gamma'}\right) & \text{if} \quad \dfrac{q_{\text{att}}}{\gamma'' - \gamma'} \leq \rho, \end{cases}$$

and the quantity

(4.20) $$c_{\text{no-esc-att}} = \max\left(\frac{1}{\gamma'' - \gamma'}\left(\lambda\gamma' + \sup_{q \in [0, q_{\text{att}} + \gamma'']} \overline{N}(q)\right), 1 - \lambda\right),$$

and the functions

$$\eta_{\text{att}} : \times [0, +\infty) \times [0, +\infty) \to \mathbb{R}, \qquad (r, t) \mapsto \eta_{0\text{-att}}\left(r - c_{\text{no-esc-att}}t - r_{\text{Esc-init}}\right)$$
$$\text{and} \quad \bar{\eta}_{\text{att}} : \mathbb{R}^{d_{\text{sp}}} \times [0, +\infty) \to \mathbb{R}, \qquad (x, t) \mapsto \eta_{\text{att}}\left(|x|, t\right) + \gamma' \exp(-\lambda t).$$

Observe that the quantity $c_{\text{no-esc-att}}$ depends on V and d_{sp} and the choice of δ and δ' and δ'' (in other words on m) but not on the solution $u(\cdot, \cdot)$ under consideration.

Lemma 4.4 (supersolution)**.** *Whatever the value of the positive quantity $r_{\text{Esc-init}}$, the function $(x, t) \mapsto \bar{\eta}_{\text{att}}(x, t)$ is a supersolution of system (4.19) (for all (x, t) in $\mathbb{R}^{d_{\text{sp}}} \times [0, +\infty)$).*

Proof. The proof is the same as that of Lemma 4.3. $\qquad\square$

4.2.7 Proof of Lemma 2.5

Lemma 2.5 follows from the next more precise lemma.

Lemma 4.5 ("explicit" upper bound on the invasion speed)**.** *If the solution $(x, t) \mapsto u(x, t)$ under consideration is stable close to m at infinity, then*

(4.21) $$c_{\text{inv}}[u] \leq c_{\text{no-esc-att}}.$$

Proof. Let us assume that the solution $(x, t) \mapsto u(x, t)$ under consideration is stable close to m at infinity. Then, according to Proposition 3.1, there exists a positive time t_{init} such that, for every time t greater than or equal to t_{init},

(4.22) $$\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att},\infty}.$$

In addition, according to Definition 2.1 of a solution stable at infinity, it may be assumed, up to replacing t_{init} by a greater quantity, that for every time t greater than or equal to t_{init}, the limit

$$\lim_{r \to +\infty} \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \, |x| \geq r} \left| u^\dagger(x,t) \right|$$

is not larger than any given positive quantity; for instance, it may be assumed that this limit is not larger than the quantity δ defined in sub-subsection 4.2.1. As a consequence, if the positive quantity $r_{\mathrm{Esc\text{-}init}}$ is chosen large enough, then

$$(4.23) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \, |x| \geq r_{\mathrm{Esc\text{-}init}} + c_{\mathrm{no\text{-}esc\text{-}att}} t_{\mathrm{init}}} \left| u^\dagger(x, t_{\mathrm{init}}) \right| \leq \delta',$$

and it follows from inequalities (4.22) and (4.23) that, for all x in $\mathbb{R}^{d_{\mathrm{sp}}}$,

$$q^\dagger(x, t_{\mathrm{init}}) \leq \bar{\eta}_{\mathrm{att}}(x, t_{\mathrm{init}}).$$

Besides, the functions $q^\dagger(\cdot, \cdot)$ and $\bar{\eta}_{\mathrm{att}}(\cdot, \cdot)$ are, according to Lemmas 4.2 and 4.4, a subsolution and a supersolution of system (4.19), respectively. As a consequence, it follows from the maximum principle that, for all (x, t) in $\mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty)$,

$$q^\dagger(x, t) \leq \bar{\eta}_{\mathrm{att}}(x, t).$$

It follows from the definition of $\bar{\eta}_{\mathrm{att}}$ that, for every positive quantity c larger than $c_{\mathrm{no\text{-}esc\text{-}att}}$,

$$\sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \, |x| \geq ct} \bar{\eta}_{\mathrm{att}}(x, t) \to 0 \quad \text{as} \quad t \to +\infty,$$

$$\text{thus} \quad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \, |x| \geq ct} u^\dagger(x, t) \to 0 \quad \text{as} \quad t \to +\infty,$$

and it follows that $c_{\mathrm{inv}}[u]$ cannot be larger than c. This shows that $c_{\mathrm{inv}}[u]$ is actually not larger than $c_{\mathrm{no\text{-}esc\text{-}att}}$. Lemma 4.5 is proved. $\qquad \square$

4.3 Firewall function

4.3.1 Preliminaries

Let us keep the notation of subsection 4.1. It follows from inequality (3.18) satisfied by w_{en} that, for all v in $\mathbb{R}^{d_{\mathrm{st}}}$,

$$(4.24) \qquad\qquad w_{\mathrm{en}} V^\dagger(v) + \frac{v^2}{4} \geq 0,$$

and it follows from inequalities (3.15) and (3.16) that, for all v in $\mathbb{R}^{d_{\mathrm{st}}}$ satisfying $|v| \leq \delta_{\mathrm{Esc}}(m)$,

$$(4.25) \qquad\qquad v \cdot \nabla V^\dagger(v) \geq \frac{\lambda_{\min}(m)}{2} v^2,$$

$$(4.26) \qquad\qquad \text{and} \quad v \cdot \nabla V^\dagger(v) \geq V^\dagger(v).$$

4.3.2 Definition

Let κ_0 denote a positive quantity, small enough so that

$$(4.27) \qquad \frac{w_{\mathrm{en}}\,\kappa_0^2}{4} \leq \frac{1}{2} \quad \text{and} \quad \frac{\kappa_0^2}{2} \leq \frac{\lambda_{\min}(m)}{8}$$

(those properties will be used to prove inequality (4.33) below); this quantity may be chosen as

$$\kappa_0 = \min\left(\sqrt{\frac{2}{w_{\mathrm{en}}}}, \frac{\sqrt{\lambda_{\min}(m)}}{2}\right).$$

For every x in $\mathbb{R}^{d_{\mathrm{sp}}}$ and every nonnegative time t, let

$$(4.28) \qquad \begin{aligned} E^\dagger(x,t) &= \frac{1}{2}\left|\nabla_x u^\dagger(x,t)\right|^2 + V^\dagger(u^\dagger(x,t)),\\ \text{and} \quad F^\dagger(x,t) &= E^\dagger(x,t) + \frac{1}{2}u^\dagger(x,t)^2. \end{aligned}$$

Let us introduce the weight function $\psi_0 : \mathbb{R}^{d_{\mathrm{sp}}} \times \mathbb{R}$ defined as

$$\psi_0(x) = \exp(-\kappa_0\,|x|),$$

and, for every $\bar{x}$ in $\mathbb{R}^{d_{\mathrm{sp}}}$, its translate $T_{\bar{x}}\psi_0$ defined as

$$T_{\bar{x}}\psi_0(x) = \psi_0(x - \bar{x}),$$

and let us introduce the "firewall" function defined, for every $\bar{x}$ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and every positive time t, as

$$(4.29) \qquad \mathcal{F}_0(\bar{x},t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}}\psi_0(x)F^\dagger(x,t)\,dx.$$

Remark. The subscript "0" (in κ_0 and ψ_0 and $\mathcal{F}_0$) is here to distinguish these objects from others, similar but unequal, introduced in subsection 6.3.

4.3.3 Coercivity

Lemma 4.6 (coercivity of firewall function). *For all t in $(0,+\infty)$ and $\bar{x}$ in $\mathbb{R}$,*

$$(4.30) \qquad \mathcal{F}_0(\bar{x},t) \geq \min\left(\frac{w_{\mathrm{en}}}{2}, \frac{1}{4}\right) \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}}\psi_0(x)\left(\left|\nabla_x u^\dagger(x,t)\right|^2 + u^\dagger(x,t)^2\right)\,dx.$$

Proof. Inequality (4.30) follows from inequality (4.24). $\qquad\square$

4.3.4 Linear decrease up to pollution.

For every nonnegative time t, let us introduce the set:

$$(4.31) \qquad \Sigma_{\mathrm{Esc}}(t) = \{x \in \mathbb{R}^{d_{\mathrm{sp}}} : \left| u^\dagger(x,t) \right| > \delta_{\mathrm{Esc}}(m) \} .$$

Lemma 4.7 (firewall linear decrease up to pollution). *There exist positive quantities $\nu_{\mathcal{F}_0}$ and $K_{\mathcal{F}_0}$ such that, for every $\bar{x}$ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and every positive time t,*

$$(4.32) \qquad \partial_t \mathcal{F}_0(\bar{x},t) \leq -\nu_{\mathcal{F}_0} \, \mathcal{F}_0(\bar{x},t) + K_{\mathcal{F}_0} \int_{\Sigma_{\mathrm{Esc}}(t)} T_{\bar{x}}\psi_0(x) \, dx \, .$$

The quantity $\nu_{\mathcal{F}_0}$ depends on V and m (only), whereas $K_{\mathcal{F}_0}$ depends additionally on the upper bound $R_{\mathrm{max},\infty}$ on the L^∞-norm of the solution.

Proof. It follows from expressions (3.7) and (3.8) that, for every $\bar{x}$ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and every positive time t,

$$\partial_t \mathcal{F}_0(\bar{x},t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[T_{\bar{x}}\psi_0 \left(-w_{\mathrm{en}}(u_t^\dagger)^2 - u^\dagger \cdot \nabla V^\dagger(u^\dagger) - \left| \nabla_x u^\dagger \right|^2 \right) \right.$$
$$\left. - w_{\mathrm{en}} \nabla_x T_{\bar{x}}\psi_0 \cdot \nabla_x u^\dagger \cdot u_t^\dagger + \frac{\Delta_x T_{\bar{x}}\psi_0}{2}(u^\dagger)^2 \right] dx \, .$$

Since (for every x in $\mathbb{R}^{d_{\mathrm{sp}}}$)

$$\left| \nabla_x T_{\bar{x}}\psi_0(x) \right| \leq \kappa_0 T_{\bar{x}}\psi_0(x) \quad \text{and} \quad \Delta_x T_{\bar{x}}\psi_0(x) \leq \kappa_0^2 T_{\bar{x}}\psi_0(x)$$

(indeed the measure with support at $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ involved in $\Delta_x \psi_0$ is negative), it follows that

$$\partial_t \mathcal{F}_0(\bar{x},t) \leq$$
$$\int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}}\psi_0 \left(-w_{\mathrm{en}}(u_t^\dagger)^2 - u^\dagger \cdot \nabla V^\dagger(u^\dagger) - \left| \nabla_x u^\dagger \right|^2 + w_{\mathrm{en}}\kappa_0 \left| \nabla_x u^\dagger \right| \left| u_t^\dagger \right| + \frac{\kappa_0^2}{2}(u^\dagger)^2 \right) dx \, ,$$

thus, using the inequality

$$\kappa_0 \left| \nabla_x u^\dagger \right| \left| u_t^\dagger \right| \leq (u_t^\dagger)^2 + \frac{\kappa_0^2}{4} \left| \nabla_x u^\dagger \right|^2 \, ,$$

it follows that

$$\partial_t \mathcal{F}_0(\bar{x},t) \leq \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}}\psi_0 \left(\left(\frac{w_{\mathrm{en}}\kappa_0^2}{4} - 1 \right) \left| \nabla_x u^\dagger \right|^2 - u^\dagger \cdot \nabla V^\dagger(u^\dagger) + \frac{\kappa_0^2}{2}(u^\dagger)^2 \right) dx \, ,$$

and according to inequalities (4.27) satisfied by the quantity κ_0,

$$(4.33) \qquad \partial_t \mathcal{F}_0(\bar{x},t) \leq \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}}\psi_0 \left(-\frac{1}{2} \left| \nabla_x u^\dagger \right|^2 - u \cdot \nabla V^\dagger(u^\dagger) + \frac{\lambda_{\min}(m)}{8}(u^\dagger)^2 \right) dx \, .$$

Let $\nu_{\mathcal{F}_0}$ denote a positive quantity to be chosen below. It follows from the previous inequality and from the definition (4.29) of $\mathcal{F}(\bar{x}, t)$ that

$$
\begin{aligned}
(4.34) \quad \partial_t \mathcal{F}_0(\bar{x}, t) + \nu_{\mathcal{F}_0} \mathcal{F}_0(\bar{x}, t) &\leq \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}} \psi_0 \Big[-\frac{1}{2} (1 - \nu_{\mathcal{F}_0} w_{\mathrm{en}}) \big| \nabla_x u^\dagger \big|^2 - u^\dagger \cdot \nabla V^\dagger(u^\dagger) \\
&\quad + \nu_{\mathcal{F}_0} w_{\mathrm{en}} V^\dagger(u^\dagger) + \Big(\frac{\lambda_{\min}(m)}{8} + \frac{\nu_{\mathcal{F}_0}}{2} \Big) (u^\dagger)^2 \Big] \, dx \, .
\end{aligned}
$$

In view of this expression and of inequalities (4.25) and (4.26) on page 20, let us assume that $\nu_{\mathcal{F}_0}$ is small enough so that

$$
(4.35) \qquad \nu_{\mathcal{F}_0} w_{\mathrm{en}} \leq 1 \quad \text{and} \quad \nu_{\mathcal{F}_0} w_{\mathrm{en}} \leq \frac{1}{2} \quad \text{and} \quad \frac{\nu_{\mathcal{F}_0}}{2} \leq \frac{\lambda_{\min}(m)}{8} \, ;
$$

the quantity $\nu_{\mathcal{F}_0}$ may be chosen as

$$
\nu_{\mathcal{F}_0} = \min \Big(\frac{1}{2 w_{\mathrm{en}}}, \frac{\lambda_{\min}(m)}{4} \Big) \, .
$$

Then, it follows from inequalities (4.34) and (4.35) that
(4.36)
$$
\partial_t \mathcal{F}_0(\bar{x}, t) + \nu_{\mathcal{F}_0} \mathcal{F}_0(\bar{x}, t) \leq \int_{\mathbb{R}^{d_{\mathrm{sp}}}} T_{\bar{x}} \psi_0 \Big[-u^\dagger \cdot \nabla V^\dagger(u^\dagger) + \frac{1}{2} \big| V^\dagger(u^\dagger) \big| + \frac{\lambda_{\min}(m)}{4} (u^\dagger)^2 \Big] \, dx \, .
$$

According to inequalities (4.25) and (4.26), the integrand of the integral at the right-hand side of this inequality is nonpositive as long as x is *not* in $\Sigma_{\mathrm{Esc}}(t)$. Therefore this inequality still holds if the domain of integration of this integral is changed from $\mathbb{R}^{d_{\mathrm{sp}}}$ to $\Sigma_{\mathrm{Esc}}(t)$. Besides, observe that, in terms of the "initial" potential V and solution $u(x, t)$, the factor of $T_{\bar{x}} \psi_0$ under the integral of the right-hand side of this last inequality reads

$$
-(u - m) \cdot \nabla V(u) + \frac{1}{2} |V(u) - V(m)| + \frac{\lambda_{\min}(m)}{4} (u - m)^2 \, .
$$

Thus, if $K_{\mathcal{F}_0}$ denotes the maximum of this expression over all possible values for u, that is (according to the L^∞ bound (4.1) on the solution) the quantity

$$
(4.37) \quad K_{\mathcal{F}_0} = \max_{v \in \mathbb{R}^{d_{\mathrm{st}}}, \, |v| \leq R_{\max, \infty}} \Big[-(v - m) \cdot \nabla V(v) + \frac{1}{2} |V(v) - V(m)| + \frac{\lambda_{\min}(m)}{4} (v - m)^2 \Big] \, ,
$$

then inequality (4.32) follows from inequality (4.36) (with the domain of integration of the integral on the right-hand side restricted to $\Sigma_{\mathrm{Esc}}(t)$). This finishes the proof of Lemma 4.7. □

4.3.5 Exponential decrease and proof of Lemma 2.6

Let c_1 and c_2 denote two positive quantity with c_1 smaller than c_2. The proof of Lemma 2.6 will follow from the next lemma.

Lemma 4.8 (exponential decrease of firewall). *Assume that there exists a positive quantity t_{init} such that, for every t in $[t_{\mathrm{init}}, +\infty)$,*

$$(4.38) \qquad \Sigma_{\mathrm{Esc}}(t) \subset B(c_1 t) \,,$$

and let us introduce the quantities $\nu'_{\mathcal{F}_0}$ and $K'_{\mathcal{F}_0}$ defined as
(4.39)
$$\nu'_{\mathcal{F}_0} = \min\left(\nu_{\mathcal{F}_0}, \frac{\kappa_0(c_2 - c_1)}{2}\right)\,, \quad and$$

$$K'_{\mathcal{F}_0} = \exp(\nu'_{\mathcal{F}_0} t_{\mathrm{init}}) \left(\sup_{x' \in \mathbb{R}^{d_{\mathrm{sp}}},\, |x'| \geq c_2 t_{\mathrm{init}}} \mathcal{F}_0(x', t_{\mathrm{init}})\right) + \frac{2^{d_{\mathrm{sp}}} K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1} c_1^{d_{\mathrm{sp}}-1} (d_{\mathrm{sp}} - 1)!}{\kappa_0^{d_{\mathrm{sp}}+1} (c_2 - c_1)^{d_{\mathrm{sp}}}}\,.$$

Then, for every t in $[t_{\mathrm{init}}, +\infty)$, the following inequality holds:

$$(4.40) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}},\, |x| \geq c_2 t} \mathcal{F}_0(x, t) \leq K'_{\mathcal{F}_0} \exp(-\nu'_{\mathcal{F}_0} t)\,.$$

Proof. According to inclusion (4.38) and to inequality (4.32) of Lemma 4.7, for all x in $\mathbb{R}$,

$$\partial_t \mathcal{F}_0(x, t) + \nu_{\mathcal{F}_0} \mathcal{F}_0(x, t) \leq K_{\mathcal{F}_0} \int_{B(c_1 t)} \exp(-\kappa_0 |x - y|)\, dy$$

$$\leq K_{\mathcal{F}_0} e^{-\kappa_0 |x|} \int_{B(c_1 t)} e^{\kappa_0 |y|}\, dy$$

$$= K_{\mathcal{F}_0} e^{-\kappa_0 |x|} \int_0^{c_1 t} S_{d_{\mathrm{sp}}-1} r^{d_{\mathrm{sp}}-1} e^{\kappa_0 r}\, dr$$

$$= K_{\mathcal{F}_0} e^{-\kappa_0 |x|} S_{d_{\mathrm{sp}}-1} (c_1 t)^{d_{\mathrm{sp}}-1} \int_0^{c_1 t} e^{\kappa_0 r}\, dr$$

$$= K_{\mathcal{F}_0} e^{-\kappa_0 |x|} S_{d_{\mathrm{sp}}-1} (c_1 t)^{d_{\mathrm{sp}}-1} \frac{1}{\kappa_0} \exp(\kappa_0 c_1 t)\,,$$

so that, if in addition it is assumed that $|x|$ is not smaller than $c_2 t$, then

$$(4.41) \qquad \partial_t \mathcal{F}_0(x, t) + \nu_{\mathcal{F}_0} \mathcal{F}_0(x, t) \leq \frac{K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1}}{\kappa_0} (c_1 t)^{d_{\mathrm{sp}}-1} \exp(-\kappa_0(c_2 - c_1)t)\,.$$

For every x in $\mathbb{R}^{d_{\mathrm{sp}}}$ such that $|x|$ is not smaller than $c_2 t_{\mathrm{init}}$, and for every time t in the interval $[t_{\mathrm{init}}, x/c_2]$, let us introduce the quantity

$$(4.42) \qquad \mathcal{G}_0(x, t) = \exp(\nu'_{\mathcal{F}_0} t) \mathcal{F}_0(x, t)\,,$$

so that

$$\partial_t \mathcal{G}_0(x, t) = \exp(\nu'_{\mathcal{F}_0} t)\left(\partial_t \mathcal{F}_0(x, t) + \nu'_{\mathcal{F}_0} \mathcal{F}_0(x, t)\right)\,.$$

It follows from inequality (4.41) that

$$\partial_t \mathcal{G}_0(x, t) \leq$$
$$\exp(\nu'_{\mathcal{F}_0} t)\left((\nu'_{\mathcal{F}_0} - \nu_{\mathcal{F}_0})\mathcal{F}_0(x, t) + \frac{K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1}}{\kappa_0}(c_1 t)^{d_{\mathrm{sp}}-1} \exp(-\kappa_0(c_2 - c_1)t)\right)\,.$$

Since according to the coercivity inequality (4.30) of Lemma 4.6 the quantity $\mathcal{F}_0(x,t)$ is nonnegative, it follows from the definition (4.39) of $\nu'_{\mathcal{F}_0}$ that

$$(4.43) \qquad \partial_t \mathcal{G}_0(x,t) \leq \frac{K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1}}{\kappa_0} (c_1 t)^{d_{\mathrm{sp}}-1} \exp\left(-\frac{\kappa_0(c_2-c_1)}{2}t\right),$$

and thus, integrating this inequality between t_{init} and t, it follows that

$$\mathcal{G}_0(x,t) \leq \mathcal{G}_0(x,t_{\mathrm{init}}) + \frac{K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1}}{\kappa_0} c_1^{d_{\mathrm{sp}}-1} \int_{t_{\mathrm{init}}}^{+\infty} (t')^{d_{\mathrm{sp}}-1} \exp\left(-\frac{\kappa_0(c_2-c_1)}{2}t'\right) dt'$$

$$\leq \exp(\nu'_{\mathcal{F}_0} t_{\mathrm{init}}) \sup_{x'\in\mathbb{R}^{d_{\mathrm{sp}}}, |x'|\geq c_2 t_{\mathrm{init}}} \mathcal{F}_0(x', t_{\mathrm{init}})$$

$$+ \frac{K_{\mathcal{F}_0} S_{d_{\mathrm{sp}}-1}}{\kappa_0} c_1^{d_{\mathrm{sp}}-1} \left(\frac{2}{\kappa_0(c_2-c_1)}\right)^{d_{\mathrm{sp}}} \int_0^{+\infty} \tau^{d_{\mathrm{sp}}-1} e^{-\tau} d\tau.$$

According to (3.19), the quantity at the right-hand side of this last inequality is equal to the quantity K introduced in (4.39). Thus it follows from the definition (4.42) of $\mathcal{G}_0(x,t)$ that inequality (4.40) holds. Lemma 4.8 is proved. $\qquad \square$

Proof of Lemma 2.6. Let us assume that the solution $(x,t) \mapsto u(x,t)$ under consideration is stable close to m at infinity (Definition 2.1), and let c denote a positive quantity, larger than than the invasion speed $c_{\mathrm{inv}}[u]$. Let us write

$$c_2 = c \quad \text{and} \quad c_1 = \frac{1}{2}(c + c_{\mathrm{inv}}[u]), \quad \text{so that} \quad c_{\mathrm{inv}}[u] < c_1 < c_2.$$

According to Definition 2.2 of $c_{\mathrm{inv}}[u]$, there exists a positive time t_{init} such that, for every time t greater than or equal to t_{init},

$$(4.44) \qquad \Sigma_{\mathrm{Esc}}(t) \subset B(c_1 t),$$

so that assumption (4.38) of the previous Lemma 4.8 is fulfilled. The conclusion (2.2) of Lemma 2.6 follows from the conclusion (4.40) of Lemma 4.8, the coercivity (4.30) of $\mathcal{F}_0(\cdot,\cdot)$, and the bounds (3.5) on the solution. Lemma 2.6 is proved. $\qquad \square$

5 Asymptotic energy

As everywhere else, let us consider a function V in $\mathcal{C}^2(\mathbb{R}^{d_{\mathrm{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis $(\mathrm{H}_{\mathrm{coerc}})$.

5.1 Definition, proof of Proposition 2.7

Proof of Proposition 2.7. Let m be a point of $\mathcal{M}$, and let $(x,t) \mapsto u(x,t)$ be a solution of system (1.1) which is stable close to m at infinity. For every positive quantity c and

every positive time t, with the notation $V^\dagger$ and $u^\dagger$ introduced in (4.2) and the notation $E^\dagger(x,t)$ introduced in (4.28), let

$$\mathcal{E}_c(t) = \int_{B(ct)} E^\dagger(x,t)\, dx\,,$$

(5.1)
$$\text{and} \quad \mathcal{D}_c(t) = \int_{B(ct)} u_t^\dagger(x,t)^2\, dx\,,$$

$$\text{and} \quad \mathcal{B}_c(t) = \int_{\partial B(ct)} E^\dagger(x,t)\, dx\,.$$

It follows from system (1.1) that, for every positive time t,

(5.2)
$$\mathcal{E}_c'(t) = -\mathcal{D}_c(t) + c\mathcal{B}_c(t)\,.$$

According to Lemma 2.6, for every positive quantity c' larger than $c_{\text{inv}}[u]$, the quantity

(5.3)
$$\sup_{x\in\mathbb{R}^{d_{\text{SP}}},\,|x|\ge c't} \left| u^\dagger(x,t) \right|$$

goes to 0 at an exponential rate as t goes to $+\infty$. And according to the bounds (3.5), the same is true for the quantity

(5.4)
$$\sup_{x\in\mathbb{R}^{d_{\text{SP}}},\,|x|\ge c't} \left| \nabla_x u^\dagger(x,t) \right|\,.$$

It follows that, if c is positive larger than $c_{\text{inv}}[u]$, then $\mathcal{B}_c(t)$ goes to 0 as t goes to $+\infty$. As a consequence, the quantity

$$\mathcal{E}_{\text{asympt}}[u] = \liminf_{t\to+\infty} \mathcal{E}_c(t)$$

is in $\{-\infty\}\cup\mathbb{R}$ and $\mathcal{E}_c(t)$ goes to $\mathcal{E}_{\text{asympt}}[u]$ as t goes to $+\infty$. The convergences at exponential rates of the quantities (5.3) and (5.4) also show that, for every couple (c,c') of positive quantities both larger than $c_{\text{inv}}[u]$, the difference

$$\mathcal{E}_c(t) - \mathcal{E}_{c'}(t)$$

goes to 0 as t goes to $+\infty$. In other words, the limit $\mathcal{E}_{\text{asympt}}[u]$ does not depend on the choice of c in the interval $(c_{\text{inv}}[u], +\infty)$. Proposition 2.7 is proved. $\qquad\square$

5.2 Upper semi-continuity, proof of Proposition 2.9

Proof of Proposition 2.9. Let m be a point of $\mathcal{M}$, let $(u_{0,n})_{n\in\mathbb{N}}$ denote a sequence of functions in the set $X_{\text{stab-infty}}(m)$ of initial conditions that are stable close to m at infinity (sub-subsection 2.2.2), and let $u_{0,\infty}$ denote a function in $X_{\text{stab-infty}}(m)$ such that

$$\|u_{0,n} - u_{0,\infty}\|_X \to 0 \quad \text{as} \quad n\to+\infty\,.$$

26

The goal is to prove that

$$(5.5) \qquad \mathcal{E}_{\text{asympt}}[u_{0,\infty}] \geq \limsup_{n \to +\infty} \mathcal{E}_{\text{asympt}}[u_{0,n}] \,.$$

For every n in $\mathbb{N} \cup \{\infty\}$, let $u_n(\cdot,\cdot)$ denote the solution of system (1.1) with initial condition $u_{0,n}$, and let us introduce the "normalized" solution $u_n^\dagger(\cdot,\cdot)$ and the function $q_n^\dagger(\cdot,\cdot)$ defined as

$$u_n^\dagger(x,t) = u(x,t) - m \quad \text{and} \quad q_n^\dagger(x,t) = \frac{1}{2} u_n^\dagger(x,t)^2 \,,$$

and let us define the potentials $V^\dagger$ and $V^\ddagger$ as in (4.2) and sub-subsection 4.2.2. Let us introduce three positive quantities δ and δ' and δ'' with the same properties as in sub-subsection 4.2.1. According to Proposition 3.1, there exists a positive quantity $R_{\text{max},\infty}$ such that, for every t in $[0,+\infty)$,

$$\sup_{x \in \mathbb{R}^{d_{\text{SP}}}} \left| u_\infty^\dagger(x,t) \right| \leq R_{\text{max},\infty} - 1 \,;$$

and since $u_{0,\infty}$ was assumed to be stable close to m at infinity, there exist positive quantities t_{init} and $r_{\text{Esc-init}}$ such that

$$\sup_{x \in \mathbb{R}^{d_{\text{SP}}}, |x| \geq r_{\text{Esc-init}}} \left| u_\infty^\dagger(x,t_{\text{init}}) \right| \leq \delta \,.$$

By continuity of the semi-flow $(S_t)_{t \geq 0}$ of system (1.1) with respect to initial conditions in X, there exists n_0 in $\mathbb{N}$ such that, for every integer n greater than or equal to n_0,

$$(5.6) \qquad \sup_{x \in \mathbb{R}^{d_{\text{SP}}}} \left| u_n^\dagger(x,t_{\text{init}}) \right| \leq R_{\text{max},\infty} \,,$$

$$(5.7) \qquad \text{and} \qquad \sup_{x \in \mathbb{R}^{d_{\text{SP}}}, |x| \geq r_{\text{Esc-init}}} \left| u_n^\dagger(x,t_{\text{init}}) \right| \leq \delta' \,.$$

Let us define the quantities λ, R_{coerc}, γ', γ'', q_{max}, $c_{\text{no-esc}}$, and the functions $\overline{N}$ and $\eta_0(\cdot)$ as in subsection 4.2, together with the functions $\eta(\cdot,\cdot)$ and $\bar{\eta}(\cdot,\cdot)$ with the parameter c chosen to be equal to $c_{\text{no-esc}}$. Finally, let us introduce the function

$$\tilde{\eta} : \mathbb{R}^{d_{\text{SP}}} \times [t_{\text{init}},+\infty) \to \mathbb{R}, \quad (x,t) \mapsto \bar{\eta}(x,t - t_{\text{init}}) \,.$$

According to Lemma 4.2, for every n in $\mathbb{N} \cup \{\infty\}$, the function $q_n^\dagger(\cdot,\cdot)$ is a subsolution of system (4.10); and according to Lemma 4.3, the function $\tilde{\eta}$ is a supersolution of the same system. And it follows from inequalities (5.6) and (5.7) that, for every n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$ and for every x in $\mathbb{R}^{d_{\text{SP}}}$,

$$q_n^\dagger(x,t_{\text{init}}) \leq \tilde{\eta}(x,t_{\text{init}}) \,.$$

It follows that, for every time t greater than or equal to t_{init}, the same inequality still holds at time t; that is, for every n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$ and for every x in $\mathbb{R}^{d_{\text{SP}}}$,

$$(5.8) \qquad q_n^\dagger(x,t) \leq \tilde{\eta}(x,t) \,.$$

Let us introduce the quantity
$$c' = c_{\text{no-esc}} + 1 \,,$$

and, for every n in $\mathbb{N} \cup \{+\infty\}$ and t in $[0, +\infty)$, let us introduce the quantities

$$\mathcal{E}_{c',n}(t) \quad \text{and} \quad \mathcal{B}_{c',n}(t)$$

defined exactly as in (5.1) for the solution u_n and the parameter c'. It follows from (5.2) that

$$(5.9) \qquad\qquad \mathcal{E}_{\text{asympt}}[u_{0,n}] - \mathcal{E}_{c',n}(t) \le c' \int_t^{+\infty} \mathcal{B}_{c',n}(s) \, ds \,.$$

Besides, it follows from inequality (5.8) and from the definition of the function $\tilde{\eta}(\cdot, \cdot)$ that, for every n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$, the quantity

$$\sup_{x \in \mathbb{R}^{d_{\text{SP}}} \, |x| \ge c't} \left| u_n^{\dagger}(x, t) \right|$$

goes to as t goes to $+\infty$, at an exponential rate and uniformly with respect to n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$. And thus, according to the bounds (3.5), the same is true for the quantity

$$\sup_{x \in \mathbb{R}^{d_{\text{SP}}} \, |x| \ge c't} \left| \nabla_x u_n^{\dagger}(x, t) \right| \,,$$

and thus the same is true for the quantity $\mathcal{B}_{c',n}(t)$. As a consequence, there exist positive quantities ν and K such that, for every n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$ and for every t greater than or equal to t_{init},

$$\int_t^{+\infty} \mathcal{B}_{c',n}(s) \, ds \le K \exp(-\nu t) \,.$$

Thus it follows from inequality (5.9) that, for every n in $\{n_0, n_0 + 1, \dots\} \cup \{\infty\}$ and for every t greater than or equal to t_{init},

$$\mathcal{E}_{c',n}(t) \ge \mathcal{E}_{\text{asympt}}[u_{0,n}] - K \exp(-\nu t) \,.$$

Passing to the limit as n goes to $+\infty$, it follows from the continuity of the semi-flow $(S_t)_{t \ge 0}$ of system (1.1) with respect to initial conditions in X that, for every t greater than or equal to t_{init},

$$\mathcal{E}_{c'}^{(\infty)}(t) \ge \limsup_{n \to +\infty} \mathcal{E}_{\text{asympt}}[u_{0,n}] - K \exp(-\nu t) \,.$$

Finally, passing to the limit as time goes to $+\infty$, inequality (5.5) follows. Proposition 2.9 is proved. $\qquad\square$

6 No invasion implies relaxation

6.1 Definitions and hypotheses

As everywhere else, let us consider a function V in $\mathcal{C}^2(\mathbb{R}^{d_{\mathrm{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis ($\mathrm{H_{coerc}}$). Let us consider a point m in $\mathcal{M}$ and a solution $(x, t) \mapsto u(x, t)$ of system (1.1). In this section it will not be assumed that the solution stable at infinity, but that it satisfies, instead, the following (less restrictive) hypothesis ($\mathrm{H_{hom}}$). The more general statement issued from this setting is appropriate to be applied (in the radially symmetric case [43]) to the behaviour (relaxation) of a solution behind a chain of fronts travelling to infinity (in space).

($\mathrm{H_{hom}}$) There exist a positive quantity c_{hom} and a $\mathcal{C}^1$-function

$$r_{\mathrm{hom}} : [0, +\infty) \to [0, +\infty) \quad \text{satisfying} \quad r'_{\mathrm{hom}}(t) \to c_{\mathrm{hom}} \quad \text{as} \quad t \to +\infty$$

such that, for every positive quantity L,

$$\sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \; r_{\mathrm{hom}}(t) - L \leq |x| \leq r_{\mathrm{hom}}(t) + L} |u(x, t) - m| \to 0 \quad \text{as} \quad t \to +\infty.$$

If the solution $(x, t) \mapsto u(x, t)$ is stable close to m at infinity then hypothesis ($\mathrm{H_{hom}}$) holds (for every positive quantity c_{hom} larger than the invasion speed of the solution and a function r_{hom} defined as $t \mapsto c_{\mathrm{hom}} t$), but the converse is not true. For every nonnegative
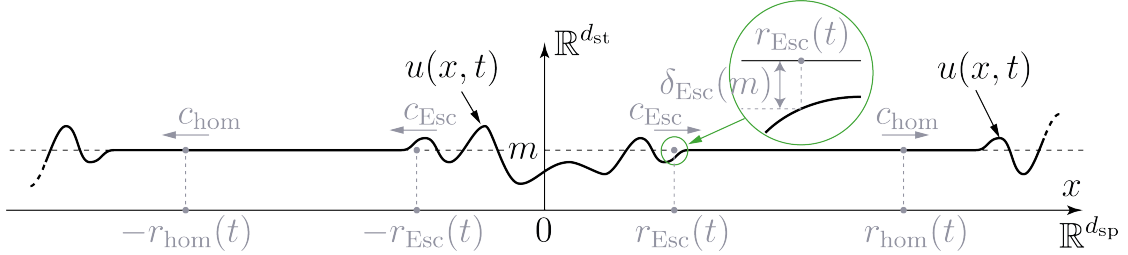
Figure 6.1: Illustration of hypothesis ($\mathrm{H_{hom}}$) and of the notation $r_{\mathrm{hom}}(t)$ and c_{hom} and $r_{\mathrm{Esc}}(t)$ (for simplicity the figure is made with a space dimension d_{sp} equal to 1, although in the paper d_{sp} is assumed to be not smaller than 2).

time t, let us denote by $r_{\mathrm{Esc}}(t)$ the supremum of the set

$$\left\{ r \in \big[0, r_{\mathrm{hom}}(t)\big) : \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}, \; |x| = r} |u(x, t) - m| \geq \delta_{\mathrm{Esc}}(m) \right\},$$

with the convention that $r_{\mathrm{Esc}}(t)$ equals 0 if this set is empty (see figure 6.1). Let us assume that the following "no invasion" assumption holds.

($\mathrm{H_{no\text{-}inv}}$) $\dfrac{r_{\mathrm{Esc}}(t)}{t} \to 0$ as $t \to +\infty$.

6.2 Statement

Recall (see sub-subsection 2.3.2) that, for every positive quantity r, $B(r)$ denotes the (open) ball of radius r and centred at the origin in $\mathbb{R}^{d_{\mathrm{sp}}}$. The goal of section 6 is to prove the following theorem, which is a variant of conclusion 1 of Theorem 1 in the slightly more general setting considered here.

Theorem 2 (no invasion implies relaxation). *Let V denote a function in $\mathcal{C}^2(\mathbb{R}^{d_{\mathrm{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis ($\mathrm{H_{coerc}}$). Then, for every point m of $\mathcal{M}$ and every solution $(x,t) \mapsto u(x,t)$ of system (1.1) satisfying hypotheses ($\mathrm{H_{hom}}$) and ($\mathrm{H_{no\text{-}inv}}$), the following conclusions hold.*

1. *There exists a nonnegative quantity $\mathcal{E}_{\mathrm{res\text{-}asympt}}[u]$ ("residual asymptotic energy") such that, for every quantity c in $(0, c_{\mathrm{hom}})$,*

$$\int_{B(ct)} \left(\frac{1}{2} |\nabla_x u(x,t)|^2 + V(u(x,t)) - V(m) \right) dx \to \mathcal{E}_{\mathrm{res\text{-}asympt}}[u] \quad as \quad t \to +\infty \,.$$

2. *The quantity*

$$\sup_{x \in B\left(r_{\mathrm{hom}}(t)\right)} |u_t(x,t)|$$

   goes to 0 as time goes to $+\infty$.

3. *For every quantity c in $(0, c_{\mathrm{hom}})$, the function*

$$t \mapsto \int_{B(ct)} u_t(x,t)^2 \, dx$$

   is integrable on a neighbourhood of $+\infty$.

Remark. Conclusion 3 of this theorem is somehow redundant with conclusion 1, therefore it could have been omitted in this statement. The reason why this additional conclusion 3 is added to the statement above is that this turns out to be convenient in the proof of the main result of the companion paper [43] (see Proposition 5.1 in this reference).

6.3 Relaxation scheme in a standing or almost standing frame

6.3.1 Set-up

Let us keep the notation and assumptions of subsection 6.1, and let us assume that the hypotheses ($\mathrm{H_{coerc}}$) and ($\mathrm{H_{hom}}$) of Theorem 2 hold. According to Proposition 3.1 and Corollary 3.2, it may be assumed, up to changing the origin of time, that, for all t in $[0, +\infty)$,

$$(6.1) \qquad \qquad \|x \mapsto u(x,t)\|_{L^\infty\left(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}}\right)} \le R_{\mathrm{att},\infty}$$

$$(6.2) \qquad \text{and} \qquad \|x \mapsto u(x,t)\|_{H^1_{\mathrm{ul}}\left(\mathbb{R}^{d_{\mathrm{sp}}}, \mathbb{R}^{d_{\mathrm{st}}}\right)} \le R_{\mathrm{att},1,\mathrm{ul}} \,.$$

Let us introduce the "normalized potential" $V^\dagger$ and the "normalized solution" $u^\dagger$ defined in (4.2).

6.3.2 Notation for the travelling frame

Let c denote a vector of $\mathbb{R}^{d_{\mathrm{sp}}}$. It may be written as

$$c = |c|\,\vec{n}\,,$$

where $\vec{n}$ is in $\mathbb{R}^{d_{\mathrm{sp}}}$ and its euclidean norm is equal to 1. Actually the choice of $\vec{n}$ will be of no importance, thus this vector may very well be chosen as $(1, 0, \dots, 0)$.

Let us consider the solution viewed in a frame travelling at velocity c, that is the function $(\xi, t) \mapsto v(\xi, t)$ defined for every ξ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and nonnegative time t by

$$v(\xi, t) = u^\dagger(x, t) \quad \text{for} \quad x = ct + \xi\,.$$

This function is a solution of the differential system

$$v_t - c \cdot \nabla_\xi v = -\nabla V^\dagger(v) + \Delta_\xi v\,.$$

In the forthcoming calculations, the notation r is used to refer to $|x|$ and ρ to refer to $|\xi|$.

6.3.3 Choice of the parameters to define energy and firewalls

Let κ (rate of decrease of the weight functions) and c_{cut} (speed of cut-off radius) be positive quantities, small enough so that

$$(6.3) \quad w_{\mathrm{en}}\kappa\left(\frac{c_{\mathrm{cut}}}{2} + \frac{\kappa}{4}\right) \le \frac{1}{2} \quad \text{and} \quad w_{\mathrm{en}}\,\kappa\,c_{\mathrm{cut}} \le \frac{1}{4} \quad \text{and} \quad \frac{\kappa(d_{\mathrm{sp}}\kappa + c_{\mathrm{cut}})}{2} \le \frac{\lambda_{\min}(m)}{8}\,.$$

Conditions (6.3) will be used to prove inequality (6.28) on page 37. These quantities may, for instance, be chosen as

$$(6.4) \quad \kappa = \min\left(\frac{1}{\sqrt{w_{\mathrm{en}}}}, \frac{\sqrt{\lambda_{\min}(m)}}{4\sqrt{d_{\mathrm{sp}}}}\right) \quad \text{and} \quad c_{\mathrm{cut}} = \min\left(\frac{1}{4\sqrt{w_{\mathrm{en}}}}, \frac{\sqrt{\lambda_{\min}(m)}}{4}, \frac{c_{\mathrm{hom}}}{2}\right).$$

Let us assume that

$$(6.5) \qquad |c| \le \frac{\sqrt{\lambda_{\min}(m)}}{4} \quad \text{and} \quad |c| \le \frac{\kappa}{20} \quad \text{and} \quad |c| \le \frac{c_{\mathrm{cut}}}{6}\,.$$

According to hypotheses ($\mathrm{H}_{\mathrm{hom}}$) and ($\mathrm{H}_{\mathrm{no\text{-}inv}}$) and to the value of the quantity c_{cut} chosen above, there exists a nonnegative time T such that, for every time t greater than or equal to T,

$$(6.6) \qquad r_{\mathrm{Esc}}(t) \le \frac{1}{6}c_{\mathrm{cut}}t \quad \text{and} \quad r_{\mathrm{hom}}(t) \ge \frac{11}{6}c_{\mathrm{cut}}t\,.$$
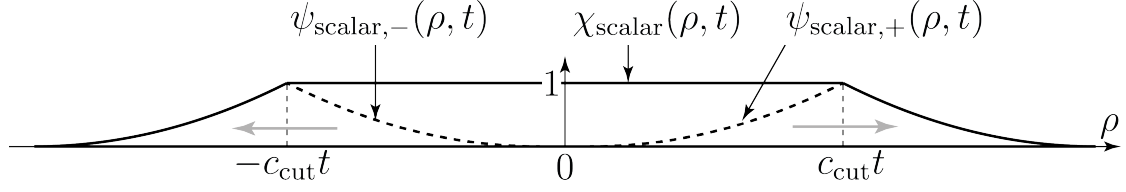
Figure 6.2: Graphs of functions $\rho \mapsto \chi_{\text{scalar}}(\rho, t)$ and $\rho \mapsto \psi_{\text{scalar},+}(\rho, t)$ and $\rho \mapsto \psi_{\text{scalar},-}(\rho, t)$.

6.3.4 Localized energy

Let us introduce the function $(\rho, t) \mapsto \chi_{\text{scalar}}(\rho, t)$ defined on $\mathbb{R} \times [0, +\infty)$ as

$$\chi_{\text{scalar}}(\rho, t) = \begin{cases} 1 & \text{if} \quad |\rho| \leq c_{\text{cut}}t \\ \exp\bigl(-\kappa(|\rho| - c_{\text{cut}}t)\bigr) & \text{if} \quad |\rho| \geq c_{\text{cut}}t \,, \end{cases}$$

see figure 6.2; and let us introduce the functions $(\xi, t) \mapsto \chi_{\text{stand}}(\xi, t)$ and $(\xi, t) \mapsto \chi(\xi, t)$ defined on $\mathbb{R}^{d_{\text{sp}}} \times [0, +\infty)$ as

$$(6.7) \qquad \chi_{\text{stand}}(\xi, t) = \chi_{\text{scalar}}(|\xi|, t) \quad \text{and} \quad \chi(\xi, t) = \exp(c \cdot \xi) \chi_{\text{stand}}(\xi, t) \,.$$

In this notation the index "scalar" refers to the scalar argument of the function, and the index "stand" refers to the fact that χ_{stand} equals χ when the velocity c vanishes ("standing" frame). For every nonnegative time t, let us introduce the following "localized energy":

$$(6.8) \qquad \mathcal{E}(t) = \int_{\mathbb{R}^{d_{\text{sp}}}} \chi(\xi, t) \left(\frac{1}{2} |\nabla_\xi v(\xi, t)|^2 + V^\dagger\bigl(v(\xi, t)\bigr) \right) d\xi \,.$$

6.3.5 Time derivative of localized energy

For every nonnegative time t, let us introduce the "dissipation" defined as

$$(6.9) \qquad \mathcal{D}(t) = \int_{\mathbb{R}^{d_{\text{sp}}}} \chi(\xi, t) \, v_t(\xi, t)^2 \, d\xi \,.$$

Lemma 6.1 (time derivative of localized energy)**.** *For every nonnegative time t,*

$$(6.10) \qquad \mathcal{E}'(t) \leq -\frac{1}{2}\mathcal{D}(t) + \kappa \int_{\mathbb{R}^{d_{\text{sp}}} \setminus B(c_{\text{cut}}t)} \chi \left(\frac{c_{\text{cut}} + \kappa}{2} |\nabla_\xi v|^2 + c_{\text{cut}} V^\dagger(v) \right) d\xi \,.$$

Proof. It follows from expression (3.9) on page 10 (time derivative of localized energy) that, for every nonnegative time t,

$$(6.11) \qquad \mathcal{E}'(t) = -\mathcal{D}(t) + \int_{\mathbb{R}^{d_{\text{sp}}}} \left[\chi_t \left(\frac{1}{2} |\nabla_\xi v|^2 + V^\dagger(v) \right) + (\chi c - \nabla_\xi \chi) \cdot \nabla_\xi v \cdot v_t \right] d\xi \,.$$

32

It follows from the definitions of χ_{stand} and χ that:

$$\chi_t(\xi, t) = \begin{cases} 0 & \text{if} \quad |\xi| \leq c_{\text{cut}}t \\ \kappa c_{\text{cut}}\chi(\xi, t) & \text{if} \quad |\xi| \geq c_{\text{cut}}t, \end{cases}$$

and

$$\nabla_\xi \chi = \chi c + e^{c \cdot \xi} \nabla_\xi \chi_{\text{stand}} \quad \text{thus} \quad \chi c - \nabla_\xi \chi = -e^{c \cdot \xi} \nabla_\xi \chi_{\text{stand}},$$

and

$$\nabla_\xi \chi_{\text{stand}}(\xi, t) = \begin{cases} 0 & \text{if} \quad |\xi| \leq c_{\text{cut}}t \\ -\kappa \dfrac{\xi}{|\xi|} \chi_{\text{stand}}(\xi, t) & \text{if} \quad |\xi| \geq c_{\text{cut}}t, \end{cases}$$

thus

$$(\chi c - \nabla_\xi \chi)(\xi, t) = \begin{cases} 0 & \text{if} \quad |\xi| \leq c_{\text{cut}}t \\ \kappa \dfrac{\xi}{|\xi|} \chi(\xi, t) & \text{if} \quad |\xi| \geq c_{\text{cut}}t. \end{cases}$$

Thus it follows from (6.11) that

$$\mathcal{E}'(t) = -\mathcal{D}(t) + \kappa \int_{\mathbb{R}^{d_{\text{SP}}} \setminus B(c_{\text{cut}}t)} \chi \left[c_{\text{cut}} \left(\frac{1}{2} |\nabla_\xi v|^2 + V^\dagger(v) \right) + \frac{\xi}{|\xi|} \cdot \nabla_\xi v \cdot v_t \right] d\xi$$

$$\leq -\mathcal{D}(t) + \kappa \int_{\mathbb{R}^{d_{\text{SP}}} \setminus B(c_{\text{cut}}t)} \chi \left[c_{\text{cut}} \left(\frac{1}{2} |\nabla_\xi v|^2 + V^\dagger(v) \right) + |\nabla_\xi v \cdot v_t| \right] d\xi.$$

Using the polar inequality

$$\kappa |\nabla_\xi v \cdot v_t| \leq \frac{1}{2} v_t^2 + \frac{\kappa^2}{2} |\nabla_\xi v|^2,$$

inequality (6.10) follows. Lemma 6.1 is proved. □

6.3.6 Firewall function

A firewall function will now be introduced to control the last term of the right-hand side of inequality (6.10) above. First let us introduce the functions $\psi_{\text{scalar},-}$ and $\psi_{\text{scalar},+}$ and ψ_{scalar} defined on $\mathbb{R} \times [0, +\infty)$ as

$$\psi_{\text{scalar},-}(r, t) = \exp\left(-\kappa \left| -c_{\text{cut}}t - r \right| \right),$$
$$\text{and} \quad \psi_{\text{scalar},+}(r, t) = \exp\left(-\kappa \left| c_{\text{cut}}t - r \right| \right),$$
$$\text{and} \quad \psi_{\text{scalar}}(r, t) = \psi_{\text{scalar},+}(r, t) + \psi_{\text{scalar},-}(r, t),$$

see figure 6.2, and the functions $(\xi, t) \mapsto \psi_{\text{stand}}(\xi, t)$ and $(\xi, t) \mapsto \psi(\xi, t)$ defined on $\mathbb{R}^{d_{\text{SP}}} \times [0, +\infty)$ by

$$(6.12) \qquad \psi_{\text{stand}}(\xi, t) = \psi_{\text{scalar}}(|\xi|, t) \quad \text{and} \quad \psi(\xi, t) = e^{c \cdot \xi} \psi_{\text{stand}}(\xi, t).$$

The meaning of the indexes "scalar" and "stand" is the same as for the notation involving the symbol "χ" in sub-subsection 6.3.4. Observe that, for all ξ in $\mathbb{R}^{d_{\mathrm{sp}}} \setminus B(c_{\mathrm{cut}}t)$,

$$(6.13) \qquad \chi(\xi,t) \leq \psi(\xi,t)$$

(although the difference is tiny). The definition of ψ_{scalar} as the sum of the two functions $\psi_{\mathrm{scalar},-}$ and $\psi_{\mathrm{scalar},+}$ is more convenient than if $\psi_{\mathrm{scalar}}(r,t)$ was defined as the supremum of these two functions (it avoids a singularity at $r = 0$, which would induce a inconvenient Dirac mass of positive weight at $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ for the Laplacian of ψ, see for instance expression (3.10) on page 10).

Let us introduce the "firewall" function defined, for every nonnegative time t, as

$$(6.14) \qquad \mathcal{F}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \psi(\xi,t) \left[w_{\mathrm{en}} \left(\frac{1}{2} |\nabla_\xi v(\xi,t)|^2 + V^\dagger(v(\xi,t)) \right) + \frac{1}{2} v(\xi,t)^2 \right] d\xi \, .$$

6.3.7 Energy decrease up to firewall

Lemma 6.2 (decrease up to firewall for localized energy). *There exists a nonnegative quantity $K_{\mathcal{E},\mathcal{F}}$, depending only on V and m and d_{sp}, such that, for every nonnegative time t,*

$$(6.15) \qquad \mathcal{E}'(t) \leq -\frac{1}{2} \mathcal{D}(t) + K_{\mathcal{E},\mathcal{F}} \mathcal{F}(t) \, .$$

Proof. Since χ takes only nonnegative values, it follows from (6.10) that, for every nonnegative time t,

$$(6.16) \quad \mathcal{E}'(t) \leq -\frac{1}{2} \mathcal{D}(t) + \kappa \int_{\mathbb{R}^{d_{\mathrm{sp}}} \setminus B(c_{\mathrm{cut}}t)} \chi \left(\frac{c_{\mathrm{cut}} + \kappa}{2} |\nabla_\xi v|^2 + c_{\mathrm{cut}} \left(V^\dagger(v) + \frac{1}{2w_{\mathrm{en}}} v^2 \right) \right) d\xi \, .$$

It follows from inequality (4.24) on page 20 derived from the definition of w_{en} that the quantity

$$V^\dagger(v) + \frac{1}{2w_{\mathrm{en}}} v^2 = \frac{1}{w_{\mathrm{en}}} \left(w_{\mathrm{en}} V^\dagger(v) + \frac{1}{2} v^2 \right)$$

is nonnegative; as a consequence, inequality (6.16) remains true if the following three changes to the integral of the right-hand side are made:

- in the integrand the factor c_{cut} is replaced by the larger factor $c_{\mathrm{cut}} + \kappa$,

- χ is replaced with ψ,

- the integration domain is extended to $\mathbb{R}^{d_{\mathrm{sp}}}$ (after χ is replaced with ψ).

After these three changes inequality (6.16) reads:

$$(6.17) \qquad \mathcal{E}'(t) \leq -\frac{1}{2} \mathcal{D}(t) + \frac{\kappa(c_{\mathrm{cut}} + \kappa)}{w_{\mathrm{en}}} \mathcal{F}(t) \, .$$

Thus, introducing the following quantity (depending only on V and m and d_{sp}):

$$K_{\mathcal{E},\mathcal{F}} = \frac{\kappa(c_{\mathrm{cut}} + \kappa)}{w_{\mathrm{en}}},$$

inequality (6.15) in Lemma 6.2 follows from inequality (6.17). Lemma 6.2 is proved. $\square$

Remark: according to the definitions (6.4) of κ and c_{cut}, Lemma 6.2 and inequality (6.15) still hold if the quantity $K_{\mathcal{E},\mathcal{F}}$ is chosen equal to $2/w_{\mathrm{en}}^2$, which depends only on V; thus the dependence of $K_{\mathcal{E},\mathcal{F}}$ on m and d_{sp} could be removed from the statement of Lemma 6.2 (this change would have no implication on the following).

6.3.8 Firewall linear decrease up to pollution

For every nonnegative time t, let

$$\Sigma_{\mathrm{Esc,\,stand}}(t) = \{x \in \mathbb{R}^{d_{\mathrm{sp}}} : |u(x,t)| > \delta_{\mathrm{Esc}}(m)\}$$

$$\text{and} \quad \Sigma_{\mathrm{Esc,\,trav}}(t) = \{\xi \in \mathbb{R}^{d_{\mathrm{sp}}} : |v(\xi,t)| > \delta_{\mathrm{Esc}}(m)\}\,.$$

According to the notation of sub-subsection 6.3.2, for all x and ξ in $\mathbb{R}^{d_{\mathrm{sp}}}$ satisfying $x = ct + \xi$,

$$x \in \Sigma_{\mathrm{Esc,\,stand}}(t) \iff \xi \in \Sigma_{\mathrm{Esc,\,trav}}(t)\,.$$

Lemma 6.3 (firewall decrease up to pollution)**.** *There exist positive quantities $\nu_{\mathcal{F}}$ and $K_{\mathcal{F}}$, depending only on V and m, such that, for every nonnegative time t,*

$$(6.18) \qquad \mathcal{F}'(t) \le -\nu_{\mathcal{F}}\mathcal{F}(t) + K_{\mathcal{F}} \int_{\Sigma_{\mathrm{Esc,\,trav}}(t)} \psi(\xi,t)\,d\xi\,.$$

Proof. According to expressions (3.9) and (3.10) on page 10 for the time derivatives of localized energy and L^2 functional, for every nonnegative time t,

$$(6.19)$$
$$\mathcal{F}'(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left[\psi\left(-w_{\mathrm{en}}v_t^2 - v \cdot \nabla V^{\dagger}(v) - |\nabla_{\xi}v|^2\right) + w_{\mathrm{en}}\psi_t\left(\frac{1}{2}|\nabla_{\xi}v|^2 + V^{\dagger}(v)\right) \right.$$
$$\left. + w_{\mathrm{en}}(\psi c - \nabla_{\xi}\psi) \cdot \nabla_{\xi}v \cdot v_t + \frac{\psi_t + \Delta_{\xi}\psi - c \cdot \nabla_{\xi}\psi}{2}v^2 \right] d\xi\,.$$

The proof of Lemma 6.3 resumes after the statement and the proof of the following intermediary result.

Lemma 6.4 (bounds on combinations of derivatives of the weight function ψ)**.** *For every nonnegative time t and for all ξ in $\mathbb{R}^{d_{\mathrm{sp}}}$,*

$$(6.20) \qquad\qquad\qquad |\psi_t(\xi,t)| \le \kappa c_{\mathrm{cut}}\psi(\xi,t)\,,$$

$$(6.21) \qquad\qquad \text{and} \quad |\psi c - \nabla_{\xi}\psi| \le \kappa\psi\,,$$

$$(6.22) \qquad\qquad \text{and} \quad \Delta_{\xi}\psi - c \cdot \nabla_{\xi}\psi \le \kappa(d_{\mathrm{sp}}\kappa + |c|)\psi\,.$$

Proof of Lemma 6.4. For every nonnegative time t and for every real quantity r,

$$|\partial_t \psi_{\text{scalar},\pm}(r,t)| = \kappa c_{\text{cut}} \psi_{\text{scalar},\pm}(r,t),$$

thus

$$|\partial_t \psi_{\text{scalar}}(r,t)| \leq \kappa c_{\text{cut}} \psi_{\text{scalar}}(r,t),$$

and as a consequence, for every ξ in $\mathbb{R}^{d_{\text{sp}}}$,

$$|\partial_t \psi_{\text{stand}}(\xi,t)| \leq \kappa c_{\text{cut}} \psi_{\text{stand}}(\xi,t),$$

and (6.20) follows. Besides, it follows from the definitions (6.12) of ψ_{stand} and ψ that

$$\nabla_\xi \psi = \psi c + e^{c \cdot \xi} \nabla_\xi \psi_{\text{stand}} \quad \text{thus} \quad \psi c - \nabla_\xi \psi = -e^{c \cdot \xi} \nabla_\xi \psi_{\text{stand}},$$

and that

$$\nabla_\xi \psi_{\text{stand}}\ (\xi,t) = \begin{cases} \kappa \big(\psi_{\text{scalar},+}(|\xi|,t) - \psi_{\text{scalar},-}(|\xi|,t)\big) \dfrac{\xi}{|\xi|} & \text{if} \quad |\xi| < c_{\text{cut}} t \\ - \kappa \psi_{\text{stand}}(\xi,t) \dfrac{\xi}{|\xi|} & \text{if} \quad |\xi| > c_{\text{cut}} t, \end{cases}$$

and as a consequence, for all ξ in $\mathbb{R}^{d_{\text{sp}}}$,

$$(6.23) \qquad\qquad |\nabla_\xi \psi_{\text{stand}}(\xi,t)| \leq \kappa \psi_{\text{stand}}(\xi,t),$$

and (6.21) follows. Besides, it again follows from the definitions (6.12) of ψ_{stand} and ψ that

$$(6.24) \qquad\qquad \Delta_\xi \psi = c \cdot \nabla_\xi \psi + e^{c \cdot \xi}(c \cdot \nabla_\xi \psi_{\text{stand}} + \Delta_\xi \psi_{\text{stand}})$$
$$\text{thus} \qquad \Delta_\xi \psi - c \cdot \nabla_\xi \psi = e^{c \cdot \xi}(c \cdot \nabla_\xi \psi_{\text{stand}} + \Delta_\xi \psi_{\text{stand}}),$$

and that

$$(6.25) \qquad \Delta_\xi \psi_{\text{stand}}(\xi,t) = \partial_r^2 \psi_{\text{scalar}}(|\xi|,t) + \frac{d_{\text{sp}} - 1}{|\xi|} \partial_r \psi_{\text{scalar}}(|\xi|,t).$$

For all ρ in $\mathbb{R}$,

$$(6.26) \qquad\qquad \partial_\rho^2 \psi_{\text{scalar}}(\rho,t) \leq \kappa^2 \psi_{\text{scalar}}(\rho,t)$$

(indeed the difference $\partial_\rho^2 \psi_{\text{scalar}}(\rho,t) - \kappa^2 \psi_{\text{scalar}}(\rho,t)$ is made of two Dirac masses of negative weight, at $\pm c_{\text{cut}} t$ for the argument ρ). On the other hand,

$$\psi_{\text{scalar}}(\rho,t) = \begin{cases} e^{\kappa(\rho - c_{\text{cut}} t)} + e^{\kappa(-\rho - c_{\text{cut}} t)} = 2\, e^{-\kappa c_{\text{cut}} t} \cosh(\kappa \rho) & \text{if} \quad \rho < c_{\text{cut}} t \\ e^{-\kappa(\rho - c_{\text{cut}} t)} + e^{\kappa(-\rho - c_{\text{cut}} t)} = 2 e^{-\kappa \rho} \cosh(\kappa c_{\text{cut}} t) & \text{if} \quad \rho > c_{\text{cut}} t, \end{cases}$$

36

and as a consequence,

$$(6.27) \qquad \frac{\partial_\rho \psi_{\text{scalar}}(\rho, t)}{\rho \psi_{\text{scalar}}(\rho, t)} = \begin{cases} \dfrac{\kappa \sinh(\kappa\rho)}{\rho \cosh(\kappa\rho)} = \kappa^2 \, \dfrac{\tanh(\kappa\rho)}{\kappa\rho} \le \kappa^2 & \text{if} \quad \rho < c_{\text{cut}} t \\[2mm] -\dfrac{\kappa}{\rho} \le 0 & \text{if} \quad \rho > c_{\text{cut}} t\,. \end{cases}$$

It follows from (6.25) to (6.27) that, for all ξ in $\mathbb{R}^{d_{\text{sp}}}$,

$$\Delta_\xi \psi_{\text{stand}}(\xi, t) \le d_{\text{sp}} \kappa^2 \psi_{\text{stand}}(\xi, t)\,,$$

and according to (6.23) and (6.24), (6.22) follows. $\qquad\square$

End of the proof of Lemma 6.3. It follows from inequality (6.19) and from Lemma 6.4 that, for every nonnegative time t,

$$\mathcal{F}'(t) = \int_{\mathbb{R}^{d_{\text{sp}}}} \psi \left[-w_{\text{en}} v_t^2 - v \cdot \nabla V^\dagger(v) - |\nabla_\xi v|^2 + w_{\text{en}} \kappa c_{\text{cut}} \left(\frac{1}{2} |\nabla_\xi v|^2 + \left| V^\dagger(v) \right| \right) \right.$$
$$\left. + w_{\text{en}} \kappa |\nabla_\xi v \cdot v_t| + \frac{\kappa(d_{\text{sp}} \kappa + |c| + c_{\text{cut}})}{2} v^2 \right] d\xi\,.$$

Using the inequality

$$w_{\text{en}} \kappa |\nabla_\xi v \cdot v_t| \le w_{\text{en}} v_t^2 + w_{\text{en}} \frac{\kappa^2}{4} |\nabla_\xi v|^2\,,$$

it follows that

$$\mathcal{F}'(t) \le \int_{\mathbb{R}^{d_{\text{sp}}}} \psi \left[\left(w_{\text{en}} \kappa \left(\frac{c_{\text{cut}}}{2} + \frac{\kappa}{4} \right) - 1 \right) |\nabla_\xi v|^2 - v \cdot \nabla V^\dagger(v) \right.$$
$$\left. + w_{\text{en}} \, \kappa \, c_{\text{cut}} \left| V^\dagger(v) \right| + \frac{\kappa(d_{\text{sp}} \kappa + |c| + c_{\text{cut}})}{2} v^2 \right] d\xi\,,$$

and according to the conditions (6.3) and (6.5) on page 31 satisfied by w_{en} and κ and c_{cut} and c, it follows that

$$(6.28) \qquad \mathcal{F}'(t) \le \int_{\mathbb{R}^{d_{\text{sp}}}} \psi \left[-\frac{1}{2} |\nabla_\xi v|^2 - v \cdot \nabla V^\dagger(v) + \frac{1}{4} \left| V^\dagger(v) \right| + \frac{\lambda_{\min}(m)}{8} v^2 \right] d\xi$$

Let $\nu_{\mathcal{F}}$ be a positive quantity to be chosen below. It follows from the previous inequality and from the definition (6.14) of $\mathcal{F}(t)$ that

$$(6.29) \qquad \begin{aligned} \mathcal{F}'(t) + \nu_{\mathcal{F}} \mathcal{F}(t) &\le \int_{\mathbb{R}^{d_{\text{sp}}}} \psi \left[-\frac{1}{2}(1 - \nu_{\mathcal{F}} \, w_{\text{en}}) |\nabla_\xi v|^2 - v \cdot \nabla V^\dagger(v) \right. \\ &\left. + \left(\frac{1}{4} + \nu_{\mathcal{F}} w_{\text{en}} \right) \left| V^\dagger(v) \right| + \left(\frac{\lambda_{\min}(m)}{8} + \frac{\nu_{\mathcal{F}}}{2} \right) v^2 \right] d\xi\,. \end{aligned}$$

37

In view of this expression and of inequalities (3.15) and (3.16) on page 11, let us assume that $\nu_{\mathcal{F}}$ is small enough so that

$$(6.30) \qquad \nu_{\mathcal{F}} w_{\text{en}} \leq 1 \quad \text{and} \quad \nu_{\mathcal{F}} w_{\text{en}} \leq \frac{1}{4} \quad \text{and} \quad \frac{\nu_{\mathcal{F}}}{2} \leq \frac{\lambda_{\min}(m)}{8}\,;$$

the quantity $\nu_{\mathcal{F}}$ may be chosen as

$$\nu_{\mathcal{F}} = \min\Big(\frac{1}{4w_{\text{en}}}, \frac{\lambda_{\min}(m)}{4}\Big)\,.$$

Then, it follows from (6.29) and (6.30) that

$$(6.31) \qquad \mathcal{F}'(t) + \nu_{\mathcal{F}}\mathcal{F}(t) \leq \int_{\mathbb{R}^{d_{\text{sp}}}} \psi \left[-v \cdot \nabla V^{\dagger}(v) + \frac{1}{2} \left| V^{\dagger}(v) \right| + \frac{\lambda_{\min}(m)}{4} v^2 \right] d\xi\,.$$

According to (3.15) and (3.16), the integrand of the integral at the right-hand side of this inequality is nonpositive as long as ξ is *not* in $\Sigma_{\text{Esc, trav}}(t)$. Therefore this inequality still holds if the domain of integration of this integral is changed from $\mathbb{R}$ to $\Sigma_{\text{Esc, trav}}(t)$. Thus, according to the uniform bound (4.1) on page 12 on the solution, if $K_{\mathcal{F}}$ is chosen as the quantity $K_{\mathcal{F}_0}$ defined in (4.37), then inequality (6.18) follows from (6.31) — with the domain of integration of the integral on the right-hand side restricted to $\Sigma_{\text{Esc, trav}}(t)$. This finishes the proof of Lemma 6.3. $\qquad \square$

6.3.9 Control over the pollution in the time derivative of the firewalls

The following lemma calls upon the notation T introduced to state inequalities (6.6) on page 31.

Lemma 6.5 (firewall linear decrease up to pollution, continuation). *There exists a positive quantity $K_{\mathcal{F}}'$, depending only on V and m and d_{sp}, such that, for every time t greater than or equal to T,*

$$(6.32) \qquad \mathcal{F}'(t) \leq -\nu_{\mathcal{F}}\mathcal{F}(t) + K_{\mathcal{F}}' \exp\Big(-\frac{\kappa c_{\text{cut}}}{2} t\Big)\,.$$

Proof. Let t denote a time greater than or equal to T. According to the definition of $r_{\text{Esc}}(t)$,

$$(6.33) \qquad \Sigma_{\text{Esc, stand}}(t) \subset B\big(r_{\text{Esc}}(t)\big) \sqcup \Big(\mathbb{R}^{d_{\text{sp}}} \setminus B\big(r_{\text{hom}}(t)\big)\Big)\,,$$

thus

$$\Sigma_{\text{Esc, trav}}(t) \subset B\big(r_{\text{Esc}}(t) + |c|\, t\big) \sqcup \Big(\mathbb{R}^{d_{\text{sp}}} \setminus B\big(r_{\text{hom}}(t) - |c|\, t\big)\Big)\,.$$

Since according to the conditions (6.5) on page 31 the quantity $|c|$ is not larger than $c_{\text{cut}}/6$, it follows from inequalities (6.6) that

$$r_{\text{Esc}}(t) + |c|\, t \leq \frac{c_{\text{cut}}}{3} t \quad \text{and} \quad \frac{5 c_{\text{cut}}}{3} t \leq r_{\text{hom}}(t) - |c|\, t\,,$$

thus
$$\Sigma_{\mathrm{Esc,\,trav}}(t) \subset B\Big(\frac{c_{\mathrm{cut}}}{3}t\Big) \sqcup \Big[\mathbb{R}^{d_{\mathrm{sp}}} \setminus B\Big(\frac{5c_{\mathrm{cut}}}{3}t\Big)\Big].$$

As a consequence, it follows from inequality (6.18) of Lemma 6.3 that
$$\mathcal{F}'(t) + \nu_{\mathcal{F}}\mathcal{F}(t) \le \int_{B\left(\frac{c_{\mathrm{cut}}}{3}t\right)} \psi(\xi,t)\,d\xi + \int_{\mathbb{R}^{d_{\mathrm{sp}}}\setminus B\left(\frac{5c_{\mathrm{cut}}}{3}t\right)} \psi(\xi,t)\,d\xi\,.$$

According to the definition of ψ,
$$\psi(\xi,t) \le \begin{cases} 2e^{|c|\cdot|\xi|-\kappa(c_{\mathrm{cut}}t-|\xi|)} & \text{for} \quad \xi \in B\Big(\frac{c_{\mathrm{cut}}}{3}t\Big) \\[2mm] 2e^{|c|\cdot|\xi|+\kappa(c_{\mathrm{cut}}t-|\xi|)} & \text{for} \quad \xi \in \mathbb{R}^{d_{\mathrm{sp}}} \setminus B\Big(\frac{5c_{\mathrm{cut}}}{3}t\Big)\,. \end{cases}$$

It follows from these upper bounds that (with the notation of sub-subsection 3.4.3)
$$e^{\frac{\kappa c_{\mathrm{cut}}}{2}t}\big(\mathcal{F}'(t) + \nu_{\mathcal{F}}\mathcal{F}(t)\big) \le 2S_{d_{\mathrm{sp}}-1}\big(\mathcal{I}_{\mathrm{centre}}(t) + \mathcal{I}_{\mathrm{periph}}(t)\big)\,,$$

where
$$\mathcal{I}_{\mathrm{centre}}(t) = e^{-\frac{\kappa c_{\mathrm{cut}}}{2}t} \int_0^{\frac{c_{\mathrm{cut}}}{3}t} \rho^{d_{\mathrm{sp}}-1}e^{(\kappa+|c|)\rho}\,d\rho$$

$$\text{and} \qquad \mathcal{I}_{\mathrm{periph}}(t) = e^{\frac{3\kappa c_{\mathrm{cut}}}{2}t} \int_{\frac{5c_{\mathrm{cut}}}{3}t}^{+\infty} \rho^{d_{\mathrm{sp}}-1}e^{-(\kappa-|c|)\rho}\,d\rho\,.$$

All what remains to be done is to prove that the two quantities $\mathcal{I}_{\mathrm{centre}}(t)$ and $\mathcal{I}_{\mathrm{periph}}(t)$ are bounded from above by quantities depending only on V and m and d_{sp}. On the one hand,
$$\mathcal{I}_{\mathrm{centre}} \le e^{-\frac{\kappa c_{\mathrm{cut}}}{2}t}\Big(\frac{c_{\mathrm{cut}}}{3}t\Big)^{d_{\mathrm{sp}}-1}\frac{1}{\kappa+|c|}e^{(\kappa+|c|)\frac{c_{\mathrm{cut}}}{3}t}\,,$$

and since according to the assumptions (6.5) the quantity $|c|$ is not larger than $\kappa/4$,
$$\begin{aligned} \mathcal{I}_{\mathrm{centre}}(t) &\le \frac{1}{\kappa}\Big(\frac{c_{\mathrm{cut}}}{3}t\Big)^{d_{\mathrm{sp}}-1}e^{-\frac{\kappa c_{\mathrm{cut}}}{12}t} \\ &\le \frac{4^{d_{\mathrm{sp}}-1}}{\kappa^{d_{\mathrm{sp}}}}\Big[\max_{\tau\in\mathbb{R}}\tau^{d_{\mathrm{sp}}-1}e^{-\tau}\Big] \\ &= \frac{4^{d_{\mathrm{sp}}-1}}{\kappa^{d_{\mathrm{sp}}}}\cdot(d_{\mathrm{sp}}-1)^{d_{\mathrm{sp}}-1}e^{-(d_{\mathrm{sp}}-1)}\,. \end{aligned}$$

(6.34)

On the other hand, writing $\tilde{\rho} = (\kappa-|c|)\rho$ in the expression of $\mathcal{I}_{\mathrm{periph}}(t)$ yields
$$\mathcal{I}_{\mathrm{periph}}(t) = \frac{e^{\frac{3\kappa c_{\mathrm{cut}}}{2}t}}{(\kappa-|c|)^{d_{\mathrm{sp}}}} \int_{\frac{5(\kappa-|c|)c_{\mathrm{cut}}}{3}t}^{+\infty} \tilde{\rho}^{d_{\mathrm{sp}}-1}e^{-\tilde{\rho}}\,d\tilde{\rho}\,,$$

thus, with the notation of sub-subsection 3.4.3,
$$\mathcal{I}_{\mathrm{periph}}(t) = \frac{e^{\frac{3\kappa c_{\mathrm{cut}}}{2}t}}{(\kappa-|c|)^{d_{\mathrm{sp}}}}(d_{\mathrm{sp}}-1)!\, e^{-\frac{5(\kappa-|c|)c_{\mathrm{cut}}}{3}t}\, e_{d_{\mathrm{sp}}-1}\Big(\frac{5c_{\mathrm{cut}}(\kappa-|c|)}{3}t\Big)\,,$$

39

and since according to the assumptions (6.5) the quantity $|c|$ is not larger than $\kappa/20$,

$$
\begin{aligned}
\mathcal{I}_{\mathrm{periph}}(t) &\leq \frac{(d_{\mathrm{sp}}-1)!}{(\kappa-|c|)^{d_{\mathrm{sp}}}}\, e^{-\frac{\kappa c_{\mathrm{cut}}}{12}t}\, e_{d_{\mathrm{sp}}-1}\Big(\frac{5(\kappa-|c|)c_{\mathrm{cut}}}{3}t\Big) \\
(6.35) \qquad &\leq \frac{2^{d_{\mathrm{sp}}-1}(d_{\mathrm{sp}}-1)!}{\kappa^{d_{\mathrm{sp}}}}\, e^{-\frac{\kappa c_{\mathrm{cut}}}{12}t}\, e_{d_{\mathrm{sp}}-1}\Big(\frac{5\kappa c_{\mathrm{cut}}}{3}t\Big)\,.
\end{aligned}
$$

Both quantities (6.34) and (6.35) are bounded from above by quantities depending only on V and m and d_{sp}. Lemma 6.5 is proved. $\qquad\square$

6.3.10 Nonnegativity of firewall

Lemma 6.6 (nonnegativity of firewall). *For every nonnegative time t,*

$$
(6.36) \qquad\qquad\qquad \mathcal{F}(t) \geq 0\,.
$$

Proof. Inequality (6.36) follows from inequality (4.24) on page 20 and from the definition (6.14) of the firewall. $\qquad\square$

6.3.11 Energy decrease up to pollution

Lemma 6.7 (energy decrease up to pollution). *There exist positive quantities $K_{\mathcal{E}}$ and $\nu_{\mathcal{E}}$ (depending only on V and m and d_{sp}) such that, for every time t greater than or equal to T,*

$$
(6.37) \qquad \mathcal{E}'(t) \leq -\frac{1}{2}\mathcal{D}(t) + K_{\mathcal{E}}\exp(-\nu_{\mathcal{E}}(t-T))\,.
$$

Proof. Let

$$
\nu_{\mathcal{E}} = \min\Big(\nu_{\mathcal{F}}, \frac{\kappa c_{\mathrm{cut}}}{4}\Big)\,.
$$

According to Grönwall's inequality, it follows from inequalities (6.32) of Lemma 6.5 that, for every time t greater than or equal to T,

$$
\begin{aligned}
\mathcal{F}(t) &\leq \exp(-\nu_{\mathcal{F}}(t-T))\mathcal{F}(T) + K'_{\mathcal{F}}\int_T^t \exp(-\nu_{\mathcal{F}}(t-s))\exp\Big(-\frac{\kappa c_{\mathrm{cut}}}{2}s\Big)\,ds \\
&\leq \exp(-\nu_{\mathcal{E}}(t-T)\Big(\mathcal{F}(T) + K'_{\mathcal{F}}\exp\Big(-\frac{\kappa c_{\mathrm{cut}}}{2}T\Big)\times \\
&\qquad \int_T^t \exp\Big(-(\nu_{\mathcal{F}}-\nu_{\mathcal{E}})(t-s)\Big)\exp\Big(-\Big(\frac{\kappa c_{\mathrm{cut}}}{2}-\nu_{\mathcal{E}}\Big)(s-T)\Big)\,ds\Big) \\
&\leq \exp(-\nu_{\mathcal{E}}(t-T))\Big(\mathcal{F}(T) + K'_{\mathcal{F}}\int_T^t \exp\Big(-\frac{\kappa c_{\mathrm{cut}}}{4}(s-T)\Big)\,ds\Big) \\
(6.38) \qquad &\leq \Big(\mathcal{F}(T) + \frac{4K'_{\mathcal{F}}}{\kappa c_{\mathrm{cut}}}\Big)\exp(-\nu_{\mathcal{E}}(t-T))\,.
\end{aligned}
$$

According to the H^1_{ul}-bound (6.2) on page 30 for the solution, there exists a positive quantity $\mathcal{F}_{\mathrm{init}}$, depending only on V and m and d_{sp}, such that $\mathcal{F}(T)$ is not larger than $\mathcal{F}_{\mathrm{init}}$. Thus, introducing the quantity

$$K_{\mathcal{E}} = K_{\mathcal{E},\mathcal{F}} \left(\mathcal{F}_{\mathrm{init}} + \frac{4K'_{\mathcal{F}}}{\kappa c_{\mathrm{cut}}} \right) ,$$

inequality (6.37) follows from inequality (6.38) and from inequality (6.15) of Lemma 6.2. Lemma 6.7 is proved. $\qquad\square$

6.4 Nonnegative residual asymptotic energy

6.4.1 Notation

Let us keep the notation and hypotheses of the previous subsection. For every velocity c in $\mathbb{R}^{d_{\mathrm{sp}}}$ close enough to $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ so that conditions (6.5) on page 31 be satisfied, let us denote by

$$v^{(c)}(\cdot,\cdot) \quad \text{and} \quad \chi^{(c)}(\cdot,\cdot) \quad \text{and} \quad \mathcal{E}^{(c)}(\cdot) \quad \text{and} \quad \Sigma^{(c)}_{\mathrm{Esc,\,trav}}(\cdot) \quad \text{and} \quad \mathcal{D}^{(c)}(\cdot),$$
$$\text{and} \quad \psi^{(c)}(\cdot,\cdot) \quad \text{and} \quad \mathcal{F}^{(c)}(\cdot)$$

the objects that were defined in subsection 6.3 (with the same notation except the "(c)" superscript that is here to remind that these objects depend on c). For every such c, let us introduce the quantity $\mathcal{E}^{(c)}_{\mathrm{res\text{-}asympt}}[u]$ in $\{-\infty\} \cup \mathbb{R}$ defined as

$$\mathcal{E}^{(c)}_{\mathrm{res\text{-}asympt}}[u] = \liminf_{t \to +\infty} \mathcal{E}^{(c)}(t) \,,$$

and let us call "residual asymptotic energy at velocity c" this quantity. According to estimate (6.37) above, for every such c,

$$(6.39) \qquad\qquad \mathcal{E}^{(c)}(t) \to \mathcal{E}^{(c)}_{\mathrm{res\text{-}asympt}}[u] \quad \text{as} \quad t \to +\infty \,.$$

6.4.2 Statement

The aim of this subsection is to prove the following proposition.

Proposition 6.8 (nonnegative residual asymptotic energy at zero velocity)**.** *The quantity* $\mathcal{E}^{(0)}_{\mathrm{res\text{-}asympt}}[u]$ *(the residual asymptotic energy at velocity zero) is nonnegative.*

The proof proceeds through the following lemmas and corollaries, that are rather direct consequences of the relaxation scheme set up in the previous subsection 6.3, and in particular of the inequality (6.37) for the time derivative of the energy.

6.4.3 Nonnegative residual asymptotic energy for small nonzero velocities

Lemma 6.9 (nonnegative residual asymptotic energy for small nonzero velocities)**.** *For every* nonzero *velocity c close enough to $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ so that conditions* (6.5) *on page 31 be satisfied,*

$$\mathcal{E}^{(c)}_{\text{res-asympt}}[u] \geq 0 \,. \tag{6.40}$$

Proof. Let c be a *nonzero* vector of $\mathbb{R}^{d_{\mathrm{sp}}}$, close enough to $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ so that conditions (6.5) on page 31 be satisfied. For every nonnegative time t, according to the definition of $\mathcal{E}$ (6.8) on page 32,

$$
\begin{aligned}
\mathcal{E}^{(c)}(t) &= \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \chi^{(c)}(\xi,t) \left(\frac{1}{2} \left| \nabla_\xi v^{(c)}(\xi,t) \right|^2 + V^\dagger \big(v^{(c)}(\xi,t) \big) \right) d\xi \\
&\geq \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \chi^{(c)}(\xi,t) V^\dagger \big(v^{(c)}(\xi,t) \big) \, d\xi \\
&\geq \int_{\Sigma^{(c)}_{\mathrm{Esc,\,trav}}(t)} \chi^{(c)}(\xi,t) V^\dagger \big(v^{(c)}(\xi,t) \big) \, d\xi \,.
\end{aligned}
$$

Thus, considering the global minimum value of $V^\dagger$:

$$V^\dagger_{\min} = \min_{u \in \mathbb{R}^{d_{\mathrm{sp}}}} V^\dagger(u) \leq 0 \,,$$

it follows that

$$
\begin{aligned}
\mathcal{E}^{(c)}(t) &\geq V^\dagger_{\min} \int_{\Sigma^{(c)}_{\mathrm{Esc,\,trav}}(t)} \chi^{(c)}(\xi,t) \, d\xi \\
&= V^\dagger_{\min} \int_{\Sigma_{\mathrm{Esc,\,stand}}(t)} \chi^{(c)}(x - ct, t) \, dx \,.
\end{aligned}
\tag{6.41}
$$

As already mentioned in (6.33), according to the definition of $r_{\mathrm{Esc}}(t)$ and to the inequalities (6.6) on page 31,

$$\Sigma_{\mathrm{Esc,\,stand}}(t) \subset B\big(r_{\mathrm{Esc}}(t)\big) \sqcup \left(\mathbb{R}^{d_{\mathrm{sp}}} \setminus B\big(r_{\mathrm{hom}}(t)\big) \right) , \tag{6.42}$$

thus

$$\mathcal{E}^{(c)}(t) \geq V^\dagger_{\min} \big(\mathcal{J}_{\mathrm{centre}}(t) + \mathcal{J}_{\mathrm{periph}}(t) \big)$$

where

$$\mathcal{J}_{\mathrm{centre}}(t) = \int_{B\left(r_{\mathrm{Esc}}(t)\right)} \chi^{(c)}(x - ct, t) \, dx$$

and

$$\mathcal{J}_{\mathrm{periph}}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}} \setminus B\left(r_{\mathrm{hom}}(t)\right)} \chi^{(c)}(x - ct, t) \, dx \,.$$

All what remains to be done is to prove that the two (nonnegative) quantities $\mathcal{J}_{\mathrm{centre}}(t)$ and $\mathcal{J}_{\mathrm{periph}}(t)$ go to 0 as t goes to $+\infty$.

- For every x in $B\big(r_{\text{Esc}}(t)\big)$, according to the definition of χ (6.7) on page 32,

$$\chi^{(c)}(x - ct, t) \leq \exp\big(c \cdot (x - ct)\big) \leq \exp\big(|c| \cdot |x| - c^2 t\big) \leq \exp\big(|c|\ r_{\text{Esc}}(t) - c^2 t\big),$$

thus, with the notation of sub-subsection 3.4.3,

$$\begin{aligned}
\mathcal{J}_{\text{centre}}(t) &\leq \exp\big(|c|\ r_{\text{Esc}}(t) - c^2 t\big) \int_{B\big(r_{\text{Esc}}(t)\big)} dx \\
&= \exp\big(|c|\ r_{\text{Esc}}(t) - c^2 t\big) \int_0^{r_{\text{Esc}}(t)} S_{d_{\text{sp}}-1}\, r^{d_{\text{sp}}-1}\, dr \\
&= \exp\big(|c|\ r_{\text{Esc}}(t) - c^2 t\big) \frac{S_{d_{\text{sp}}-1}\, r_{\text{Esc}}(t)^{d_{\text{sp}}}}{d_{\text{sp}}} \\
&= t_{\text{sp}}^{d} \exp\left(t\left(-c^2 + |c|\frac{r_{\text{Esc}}(t)}{t}\right)\right) \frac{S_{d_{\text{sp}}-1}}{d_{\text{sp}}} \left(\frac{r_{\text{Esc}}(t)}{t}\right)^{d_{\text{sp}}}.
\end{aligned}$$

Since according to hypothesis $(\mathrm{H}_{\text{no-inv}})$ the ratio $r_{\text{Esc}}(t)/t$ goes to 0 as t goes to $+\infty$ and sine c is not equal to $0_{\mathbb{R}^{d_{\text{sp}}}}$, it follows that $\mathcal{J}_{\text{centre}}(t)$ goes to 0 as t goes to $+\infty$.

- For every x in $\mathbb{R}^{d_{\text{sp}}} \setminus B\big(r_{\text{hom}}(t)\big)$, according to the definition of χ (6.7) on page 32,

$$\begin{aligned}
\chi^{(c)}(x - ct, t) &\leq \exp\big(c \cdot (x - ct) - \kappa(|x - ct| - c_{\text{cut}} t)\big) \\
&\leq \exp\big(-(\kappa - |c|)(|x| - |c|\, t) + \kappa c_{\text{cut}} t\big) \\
&\leq \exp\big(-(\kappa - |c|)\, |x| + \kappa(c_{\text{cut}} + |c|)t\big).
\end{aligned}$$

It follows that, with the notation of sub-subsection 3.4.3,

$$\mathcal{J}_{\text{periph}}(t) \leq \exp\big(\kappa(c_{\text{cut}} + |c|)t\big) \int_{r_{\text{hom}}(t)}^{+\infty} S_{d_{\text{sp}}-1}\, r^{d_{\text{sp}}-1} \exp\big(-(\kappa - |c|)r\big)\, dr,$$

or in other words, writing $\tilde{r} = (\kappa - |c|)r$ in the integral on the right-hand side,

$$\mathcal{J}_{\text{periph}}(t) \leq \exp\big(\kappa(c_{\text{cut}} + |c|)t\big) \frac{S_{d_{\text{sp}}-1}}{(\kappa - |c|)^{d_{\text{sp}}}} \int_{(\kappa-|c|)r_{\text{hom}}(t)}^{+\infty} \tilde{r}^{d_{\text{sp}}-1} e^{-\tilde{r}}\, d\tilde{r}$$

$$= \frac{(d_{\text{sp}} - 1)!\, S_{d_{\text{sp}}-1}}{(\kappa - |c|)^{d_{\text{sp}}}} \exp\big[\kappa(c_{\text{cut}} + |c|)t - (\kappa - |c|)r_{\text{hom}}(t)\big]\ e_{d_{\text{sp}}-1}\big((\kappa - |c|)r_{\text{hom}}(t)\big),$$

and since according to inequality (6.6) on page 31 the quantity $r_{\text{hom}}(t)$ is not smaller than $11 c_{\text{cut}} t / 6$ for t greater than or equal to T, and in view of the conditions (6.5) on page 31 on c, it follows that $\mathcal{J}_{\text{periph}}(t)$ goes to 0 as t goes to $+\infty$.

Lemma 6.9 is proved. $\qquad\square$

6.4.4 Almost nonnegative energy at small nonzero velocities

Corollary 6.10 (almost nonnegative energy at small nonzero velocities)**.** *For every nonzero velocity c close enough to $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ so that conditions (6.5) on page 31 be satisfied and for every time t greater than or equal to T,*

$$\mathcal{E}^{(c)}(t) \geq -\frac{K_{\mathcal{E}}}{\nu_{\mathcal{E}}} \exp\left(-\nu_{\mathcal{E}}(t-T)\right).$$

Proof. This lower bound follows from the nonnegativity of $\mathcal{E}^{(c)}_{\mathrm{res\text{-}asympt}}[u]$ (Lemma 6.9) and the upper bound (6.37) of Lemma 6.7 for the time derivative of the energy $\mathcal{E}^{(c)}(t)$. $\quad\square$

6.4.5 Continuity of energy with respect to the velocity at $c = 0_{\mathbb{R}^{d_{\mathsf{sp}}}}$

Lemma 6.11 (continuity of energy with respect to the velocity at $c = 0_{\mathbb{R}^{d_{\mathrm{sp}}}}$)**.** *For every nonnegative time t,*

$$\mathcal{E}^{(c)}(t) \to \mathcal{E}^{(0)}(t) \quad as \quad c \to 0.$$

Proof. For every nonnegative time t,

$$\mathcal{E}^{(0)}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \chi^{(0)}(x,t)\left(\frac{1}{2}\left|\nabla_x u^{\dagger}(x,t)\right|^2 + V^{\dagger}(u^{\dagger}(x,t))\right) dx,$$

and, for every velocity c close enough to $0_{\mathbb{R}^{d_{\mathrm{sp}}}}$ so that conditions (6.5) on page 31 be satisfied (substituting the notation ξ used in the definition (6.8) of $\mathcal{E}(\cdot)$ with x),

$$\mathcal{E}^{(c)}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \chi^{(c)}(ct+x,t)\left(\frac{1}{2}\left|\nabla_x u^{\dagger}(ct+x,t)\right|^2 + V^{\dagger}(u^{\dagger}(ct+x,t))\right) dx,$$

and the result follows from the continuity of $\chi^{(c)}(\cdot,\cdot)$ with respect to c, the exponential decrease to 0 of $\chi^{(c)}(x,t)$ when $|x| \to +\infty$, and the H^1_{ul}-bounds (6.2) on page 30 for the solution. $\quad\square$

6.4.6 Almost nonnegative energy in a standing frame

Corollary 6.12 (almost nonnegative energy in a standing frame)**.** *For every time t greater than or equal to T,*

$$\mathcal{E}^{(0)}(t) \geq -\frac{K_{\mathcal{E}}}{\nu_{\mathcal{E}}} \exp\left(-\nu_{\mathcal{E}}(t-T)\right).$$

Proof. This lower bound follows from Corollary 6.10 and Lemma 6.11. $\quad\square$

Proposition 6.8 ("nonnegative residual asymptotic energy at zero velocity") follows from Corollary 6.12.

6.5 End of the proof of Theorem 2

Lemma 6.13 (integrability of dissipation in the standing frame)**.** *The function $t \mapsto \mathcal{D}^{(0)}(t)$ is integrable on a neighbourhood of $+\infty$.*

Proof. The statement follows from Proposition 6.8 ("nonnegative residual asymptotic energy at zero velocity") and from the upper bound (6.37) on the time derivative of energy. □

Lemma 6.14 (relaxation)**.** *The following limit holds:*

$$\sup_{x \in B(r_{\mathrm{hom}}(t))} |u_t(x,t)| \to 0 \quad as \quad t \to +\infty\,.$$

Proof. For every positive quantity L, it follows from inclusion (6.33) and the first inequality of (6.6) and hypothesis $(\mathrm{H_{hom}})$ that, for every large enough positive time t,

$$\Sigma_{\mathrm{Esc,\,stand}}(t) \subset B\left(\frac{1}{6}c_{\mathrm{cut}}t\right) \sqcup \mathbb{R}^{d_{\mathrm{sp}}} \setminus B(r_{\mathrm{hom}}(t) + L)\,.$$

Proceeding as in the proof of Lemma 4.8, it follows, using the notation $\mathcal{F}_0(\cdot,\cdot)$ introduced in (4.29), that

$$\sup_{x \in B(r_{\mathrm{hom}}(t)) \setminus B(c_{\mathrm{cut}}t)} \mathcal{F}_0(x,t) \to 0 \quad as \quad t \to +\infty\,.$$

As a consequence, according to the bounds (3.5) on the solution,

$$\sup_{x \in B(r_{\mathrm{hom}}(t)) \setminus B(c_{\mathrm{cut}}t)} |u(x,t)| + |\nabla_x u(x,t)| + |\Delta_x u(x,t)| \to 0 \quad as \quad t \to +\infty\,.$$

Thus, according to system (1.1),

(6.43)
$$\sup_{x \in B(r_{\mathrm{hom}}(t)) \setminus B(c_{\mathrm{cut}}t)} |u_t(x,t)| \to 0 \quad as \quad t \to +\infty\,.$$

It remains to prove that

$$\sup_{x \in B(c_{\mathrm{cut}}t)} |u_t(x,t)| \to 0 \quad as \quad t \to +\infty\,.$$

Let us proceed by contradiction and assume that the converse assumption holds. Then, there exists a positive quantity ε and a sequence $(x_n, t_n)_{n \in \mathbb{N}}$ such that $t_n \to +\infty$ as $n \to +\infty$, and such that, for every n in $\mathbb{N}$,

(6.44)
$$|u_t(x_n, t_n)| \geq \varepsilon\,.$$

According to (6.43), it may be assumed (up to dropping the first terms of the sequence $(x_n, t_n)_{n \in \mathbb{N}}$) that, for every n in $\mathbb{N}$, x_n is in the ball $B(c_{\mathrm{cut}}t_n)$. By compactness

(Lemma 3.3), there exists an entire solution $\bar{u}$ of system (1.1) such that, up to replacing the sequence $(x_n, t_n)_{n\in\mathbb{N}}$ by a subsequence, with the notation of (3.6),

(6.45) $$D^{2,1}u(x_n + \cdot, t_n + \cdot) \to D^{2,1}\bar{u} \quad \text{as} \quad n \to +\infty\,,$$

uniformly on every compact subset of $\mathbb{R}^{d_{\mathrm{sp}}} \times \mathbb{R}$. It follows from (6.44) and (6.45) that the quantity $|\bar{u}_t(0,0)|$ is positive, so that the quantity

$$\int_0^1 \left(\int_{\mathbb{R}^{d_{\mathrm{sp}}}} e^{-\kappa|\xi|} \bar{u}_t(\xi, s)^2 \, d\xi \right) ds$$

is also positive. This quantity is less than or equal to the quantity

$$\liminf_{n\to+\infty} \int_0^1 \mathcal{D}^{(0)}(t_n + s) \, ds\,,$$

which is therefore also positive, a contradiction with the integrability of $t \mapsto \mathcal{D}^{(0)}(t)$ (Lemma 6.13). Lemma 6.14 is proved. $\qquad\square$

The following lemma calls upon the notation $\mathcal{E}_c(t)$ introduced in (5.1), and its proof upon the notation $E^\dagger(x,t)$ introduced in (4.28).

Lemma 6.15 (convergence towards residual asymptotic energy). *For every quantity c in the interval $(0, c_{\mathrm{hom}})$,*

(6.46) $$\mathcal{E}_c(t) \to \mathcal{E}^{(0)}_{\mathrm{res\text{-}asympt}}[u]$$

as time goes to $+\infty$.

Proof. Let c be a quantity in the interval $(0, c_{\mathrm{hom}})$, and, for every nonnegative time t, let us introduce the set

$$\Sigma(t) = B\big(\max(c_{\mathrm{cut}}, c)t\big) \setminus B\big(\min(c_{\mathrm{cut}}, c)t\big)$$

and the quantity

$$\delta\mathcal{E}(t) = \mathcal{E}^{(0)}(t) - \mathcal{E}_c(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}\setminus B(c_{\mathrm{cut}}t)} \chi^{(0)}(x,t)E^\dagger(x,t)\,dx + \int_{\Sigma(t)} E^\dagger(x,t)\,dx\,.$$

According to (6.39), the quantity $\mathcal{E}^{(0)}(t)$ goes to $\mathcal{E}^{(0)}_{\mathrm{res\text{-}asympt}}[u]$ as t goes to $+\infty$. As a consequence, all what remains to be proved is that $\delta\mathcal{E}(t)$ goes to 0 as t goes to $+\infty$. Let us introduce the integrals

$$\mathcal{I}(t) = \int_{B(r_{\mathrm{hom}}(t))\setminus B(c_{\mathrm{cut}}t)} \chi^{(0)}(x,t)\left|E^\dagger(x,t)\right| dx\,,$$

$$\text{and} \quad \mathcal{J}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}\setminus B(r_{\mathrm{hom}}(t))} \chi^{(0)}(x,t)\left|E^\dagger(x,t)\right| dx\,,$$

$$\text{and} \quad \mathcal{K}(t) = \int_{\Sigma(t)} \left|E^\dagger(x,t)\right| dx\,.$$

According to this notation,

(6.47) $$|\delta\mathcal{E}(t)| \leq \mathcal{I}(t) + \mathcal{J}(t) + \mathcal{K}(t)\,.$$

- It follows from the definition of $r_{\text{Esc}}(t)$, from the first inequality of (6.6), and from hypothesis ($H_{\text{no-inv}}$) that, for every large enough positive time t,

$$\text{for every } x \text{ in } B\big(r_{\text{hom}}(t)\big) \setminus B(c_{\text{cut}}t), \quad \left|u^\dagger(x,t)\right| \le \delta_{\text{Esc}}(m),$$

so that, actually,

$$\mathcal{I}(t) = \int_{B\left(r_{\text{hom}}(t)\right) \setminus B(c_{\text{cut}}t)} \chi^{(0)}(x,t) E^\dagger(x,t)\, dx.$$

Thus, according to inequality (6.13),

$$
\begin{aligned}
\mathcal{I}(t) &\le \int_{B\left(r_{\text{hom}}(t)\right) \setminus B(c_{\text{cut}}t)} \psi^{(0)}(x,t) E^\dagger(x,t)\, dx \\
&= \frac{1}{w_{\text{en}}} \int_{B\left(r_{\text{hom}}(t)\right) \setminus B(c_{\text{cut}}t)} \psi^{(0)}(x,t) w_{\text{en}} E^\dagger(x,t)\, dx \\
&\le \frac{1}{w_{\text{en}}} \int_{B\left(r_{\text{hom}}(t)\right) \setminus B(c_{\text{cut}}t)} \psi^{(0)}(x,t) \left(w_{\text{en}} E^\dagger(x,t) + \frac{1}{2}\left|u^\dagger(x,t)^2\right|\right)\, dx \\
&\le \frac{1}{w_{\text{en}}} \mathcal{F}^{(0)}(t),
\end{aligned}
$$

so that, according to inequality (6.38), $\mathcal{I}(t)$ goes to 0 as t goes to $+\infty$.

- According to the bounds (6.1) and (6.2) on the solution, there exists a (positive) quantity $E^\dagger_{\max}$ such that, for every real quantity x and every nonnegative time t,

$$E^\dagger(x,t) \le E^\dagger_{\max}.$$

It follows that

$$
\begin{aligned}
\mathcal{J}(t) &\le E^\dagger_{\max} \int_{\mathbb{R}^{d_{\text{sp}}} \setminus B\left(r_{\text{hom}}(t)\right)} \chi^{(0)}(x,t)\, dx \\
&= E^\dagger_{\max} \int_{r_{\text{hom}}(t)}^{+\infty} S_{d_{\text{sp}}-1} r^{d_{\text{sp}}-1} \exp\big(-\kappa(r - c_{\text{cut}}t)\big)\, dr \\
&= E^\dagger_{\max} S_{d_{\text{sp}}-1} \exp(c_{\text{cut}}t) \int_{r_{\text{hom}}(t)}^{+\infty} r^{d_{\text{sp}}-1} \exp(-\kappa r)\, dr \\
&= \frac{E^\dagger_{\max} S_{d_{\text{sp}}-1}}{\kappa^{d_{\text{sp}}}} \exp(c_{\text{cut}}t) \int_{r_{\text{hom}}(t)}^{+\infty} \rho^{d_{\text{sp}}-1} \exp(-\rho)\, d\rho \\
&= \frac{E^\dagger_{\max} S_{d_{\text{sp}}-1}(d_{\text{sp}} - 1)!}{\kappa^{d_{\text{sp}}}} \exp\big(c_{\text{cut}}t - r_{\text{hom}}(t)\big) \cdot e_{d_{\text{sp}}-1}\big(r_{\text{hom}}(t)\big),
\end{aligned}
$$

and it follows from the second inequality of (6.6) that $\mathcal{J}(t)$ goes to 0 as t goes to $+\infty$.

- Since c is positive and smaller than c_{hom}, it follows, proceeding as in the proof of Lemma 4.8, that the quantity

$$\sup_{x \in \Sigma(t)} \mathcal{F}_0(x, t)$$

  goes to 0 at an exponential rate when t goes to $+\infty$. As a consequence, according to the bounds (3.5), the same is true for the quantities

$$(6.48) \qquad \sup_{x \in \Sigma(t)} |u(x, t)| \quad \text{and} \quad \sup_{x \in \Sigma(t)} |\nabla_x u(x, t)| \,,$$

  so that $\mathcal{K}(t)$ goes to 0 as t goes to $+\infty$.

In view of inequality (6.47), Lemma 6.15 is proved. $\qquad\square$

The following lemma calls upon the notation $\mathcal{D}_c(t)$ and $\mathcal{B}_c(t)$ introduced in (5.1).

Lemma 6.16 (integrability of dissipation, 2). *For every quantity c in the interval $(0, c_{\text{hom}})$, the function $t \mapsto \mathcal{D}_c(t)$ is integrable on a neighbourhood of $+\infty$.*

Proof. The conclusion follows from the limit (6.46), from equality (5.2), and from the fact that, since the quantities (6.48) go to 0 at an exponential rate, the same is true for the quantity $\mathcal{B}_c(t)$. $\qquad\square$

In view of Proposition 6.8 and Lemmas 6.14 to 6.16, Theorem 2 is proved.

7 Proof of Theorem 1

As everywhere else, let us consider a function V in $\mathcal{C}^2(\mathbb{R}^{d_{\text{st}}}, \mathbb{R})$ satisfying the coercivity hypothesis (H_{coerc}). Let m be a point of $\mathcal{M}$ and $(x, t) \mapsto u(x, t)$ be a solution of system (1.1) stable close to m at infinity.

7.1 Asymptotics of derivatives beyond invasion speed

The following lemma will be called upon in the next two subsections.

Lemma 7.1 (asymptotics of time derivative beyond invasion speed). *For every positive quantity c larger than $c_{\text{inv}}[u]$, there exists positive quantities ν' and $K'[u]$ such that, for every nonnegative time t,*

$$\sup_{x \in \mathbb{R}^{d_{\text{sp}}}, |x| \geq ct} \left(|u_x(x, t)| + |u_{xx}(x, t)| + |u_t(x, t)| \right) \leq K'[u] \exp(-\nu' t) \,.$$

The quantity ν' depends on V and m and the difference $c - c_{\text{inv}}[u]$ (only), whereas $K'[u]$ depends additionally on u.

Proof. According to Lemma 2.6 and to the bounds (3.5) on the solution, the conclusion holds for the quantity

$$\sup_{x\in\mathbb{R}^{d_{\mathrm{sp}}},\,|x|\ge ct}\left(|u_x(x,t)|+|u_{xx}(x,t)|\right),$$

and in view of system (1.1), the conclusion also holds for the quantity

$$\sup_{x\in\mathbb{R}^{d_{\mathrm{sp}}},\,|x|\ge ct}|u_t(x,t)|\,.$$

Lemma 7.1 is proved. □

7.2 Proof of conclusion 1 of Theorem 1

Let us assume that the invasion speed $c_{\mathrm{inv}}[u]$ is equal to 0. Then, introducing the function $r_{\mathrm{hom}}:[0,+\infty)\to[0,+\infty)$ defined as

$$r_{\mathrm{hom}}(t)=t\,,$$

it follows from the definition of the invasion speed (Definition 2.2) that both hypotheses ($\mathrm{H}_{\mathrm{hom}}$) and ($\mathrm{H}_{\mathrm{no\text{-}inv}}$) of Theorem 2 hold. According to Proposition 2.7 and to conclusion 1 of Theorem 2, the asymptotic energy defined in Proposition 2.7 is equal to the residual asymptotic energy defined in Theorem 2, and according to conclusion 1 of Theorem 2 this asymptotic energy is nonnegative. In addition, according to conclusion 2 of Theorem 2,

$$(7.1)\qquad\qquad \sup_{x\in B(t)}|u_t(x,t)|\to 0\quad\text{as}\quad t\to+\infty\,.$$

Besides, it follows from Lemma 7.1 that, for every positive quantity ε,

$$(7.2)\qquad\qquad \sup_{x\in\mathbb{R}^{d_{\mathrm{sp}}}\setminus B(\varepsilon t)}|u_t(x,t)|\to 0\quad\text{as}\quad t\to+\infty\,,$$

and the limit (2.4) follows from (7.1) and (7.2). Conclusion 1 of Theorem 1 is proved.

7.3 Proof of conclusion 2 of Theorem 1

Let us assume that the invasion speed $c_{\mathrm{inv}}[u]$ is positive, and let $\mathcal{E}_{\mathrm{asympt}}[u]$ denote the asymptotic energy of the solution $(x,t)\mapsto u(x,t)$. The task is to prove that $\mathcal{E}_{\mathrm{asympt}}[u]$ equals $-\infty$. Let us proceed by contradiction and assume that

$$(7.3)\qquad\qquad \mathcal{E}_{\mathrm{asympt}}[u]>-\infty\,.$$

7.3.1 Uniform convergence towards zero of the time derivative of the solution

The aim of this sub-subsection is to prove the following proposition.

Proposition 7.2 (the time derivative of the solution goes to 0 uniformly in space). *The following limit hold:*

$$(7.4) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}} |u_t(x,t)| \to 0 \quad as \quad t \to +\infty .$$

Proof. Let us introduce the quantity

$$c = c_{\mathrm{inv}}[u] + 1 .$$

and let us use the notation $V^\dagger$ and $u^\dagger$ introduced in (4.2).

Objects similar to but different from some of them introduced in sub-subsections 6.3.4 and 6.3.5 will now be introduced. They will be denoted similarly, with an additional tilde ("˜") to avoid any confusion. Let us introduce the function $(\rho, t) \mapsto \tilde{\chi}_{\mathrm{scalar}}(\rho, t)$ defined on $\mathbb{R} \times [0, +\infty)$ as

$$\tilde{\chi}_{\mathrm{scalar}}(\rho, t) = \begin{cases} 1 & \text{if} \quad |\rho| \leq t \\ \exp\big(-(|\rho| - t)\big) & \text{if} \quad |\rho| \geq t , \end{cases}$$

and let us introduce the function $(\xi, t) \mapsto \tilde{\chi}(\xi, t)$ defined on $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty)$ as

$$(7.5) \qquad \tilde{\chi}(\xi, t) = \tilde{\chi}_{\mathrm{scalar}}(|\xi|, t) .$$

Let us recall the notation $E^\dagger(x, t)$ introduced in (4.28). For every nonnegative quantity t, let us define the quantities

$$(7.6) \qquad \tilde{\mathcal{E}}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \tilde{\chi}(\xi, t) E^\dagger(x, t) \, dt \quad \text{and} \quad \tilde{\mathcal{D}}(t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \tilde{\chi}(\xi, t) u_t^\dagger(x, t)^2 \, dt .$$

The continuation of the proof calls upon the following intermediary results.

Lemma 7.3 (time derivative of localized energy). *For every positive time t,*

$$(7.7) \qquad \tilde{\mathcal{E}}'(t) \leq -\frac{1}{2}\tilde{\mathcal{D}}(t) + \int_{\mathbb{R}^{d_{\mathrm{sp}}} \backslash B(t)} \tilde{\chi}(x, t) \left(\left| \nabla_x u^\dagger(x, t) \right|^2 + V^\dagger\big(u^\dagger(x, t)\big) \right) dx .$$

Proof of Lemma 7.3. Apply the proof of Lemma 6.1 substituting κ and c_{cut} with 1 and c with 0. $\qquad \square$

Corollary 7.4 (integrability of the dissipation). *The function $t \mapsto \tilde{\mathcal{D}}(t)$ is integrable on $[1, +\infty)$.*

Proof of Corollary 7.4. According to inequality (7.7) and to Lemma 7.1, the function

$$t \mapsto \tilde{\mathcal{E}}'(t) + \frac{1}{2}\tilde{\mathcal{D}}(t)$$

is bounded from above by a function going to 0 at an exponential rate as t goes to $+\infty$. Besides, it follows from Proposition 2.7 and Lemma 7.1 that $\tilde{\mathcal{E}}(t)$ converges towards the finite quantity $\mathcal{E}_{\mathrm{asympt}}[u]$ as t goes to $+\infty$. The statement of Corollary 7.4 follows. $\qquad \square$

End of the proof of Proposition 7.2. The end of the proof of Proposition 7.2 is identical to the proof of Lemma 6.14. $\qquad \square$

7.3.2 Contradiction with the positivity of the invasion speed

The following corollary of Proposition 7.2 calls upon the notation $\mathcal{F}_0(x, t)$ introduced in (4.29).

Corollary 7.5 (the time derivative of the firewall goes to 0 uniformly in space)**.** *The following limit hold:*

$$(7.8) \qquad \sup_{x \in \mathbb{R}^{d_{\mathrm{sp}}}} |\partial_t \mathcal{F}_0(x, t)| \to 0 \quad as \quad t \to +\infty\,.$$

Proof. It follows from equality (3.7) that, for every $\bar{x}$ in $\mathbb{R}^{d_{\mathrm{sp}}}$ and for every positive time t,

$$\partial_t \mathcal{F}_0(\bar{x}, t) = \int_{\mathbb{R}^{d_{\mathrm{sp}}}} \left(T_{\bar{x}} \psi_0 \left(-w_{\mathrm{en}}(u_t^\dagger)^2 + u^\dagger \cdot u_t^\dagger \right) - w_{\mathrm{en}} \nabla_x T_{\bar{x}} \psi_0 \cdot \nabla_x u^\dagger \cdot u_t^\dagger \right) dx\,,$$

and the conclusion (7.8) follows from Proposition 7.2 and from the bounds (3.5) on the solution. $\qquad \square$

Lemma 7.6 ($\mathcal{F}_0(x, t)$ controls $|u(x, t)|$)**.** *There exists a positive quantity $\delta_{\mathrm{esc}}(m)$ such that, for every x in $\mathbb{R}^{d_{\mathrm{sp}}}$ and for every time t greater than or equal to 1, the following implication holds:*

$$(7.9) \qquad \mathcal{F}_0(x, t) \le \delta_{\mathrm{esc}}(m)^2 \implies |u(x, t)| \le \delta_{\mathrm{Esc}}(m)\,.$$

Proof. Implication (7.9) follows from the coercivity (4.30) of $\mathcal{F}_0(\cdot, \cdot)$ and the bounds (3.5) on the solution (note that $\delta_{\mathrm{Esc}}(m)$ and $\delta_{\mathrm{esc}}(m)$ do not denote the same quantity). $\qquad \square$

The following lemma calls upon the quantities $\nu_{\mathcal{F}_0}$ and $K_{\mathcal{F}_0}$ introduced in Lemma 4.7.

Lemma 7.7 (control over pollution)**.** *There exists a positive quantity L such that the following inequality holds:*

$$(7.10) \qquad K_{\mathcal{F}_0} \int_{\mathbb{R}^{d_{\mathrm{sp}}} \setminus B(L)} \psi_0(x)\, dx \le \nu_{\mathcal{F}_0} \frac{\delta_{\mathrm{esc}}(m)^2}{8}\,.$$

Proof. For every positive quantity L, using the notation introduced in (3.19),

$$\begin{aligned}
\int_{\mathbb{R}^{d_{\mathrm{sp}}} \setminus B(L)} \psi_0(x)\, dx &= \int_L^{+\infty} S_{d_{\mathrm{sp}}-1} \exp(-\kappa_0 r)\, dr \\
&= \frac{S_{d_{\mathrm{sp}}-1}}{\kappa_0^{d_{\mathrm{sp}}}} \int_{\kappa_0 L}^{+\infty} \rho^{d_{\mathrm{sp}}-1} \exp(-\rho)\, d\rho \\
&= \frac{S_{d_{\mathrm{sp}}-1}(d_{\mathrm{sp}}-1)!}{\kappa_0^{d_{\mathrm{sp}}}} \exp(-\kappa_0 L) e_{d_{\mathrm{sp}}-1}(\kappa_0 L)\,.
\end{aligned}$$

Since this last expression goes to 0 as L goes to $+\infty$, the conclusion follows. $\qquad \square$

Let us introduce the function $\tilde{\eta}_0 : \mathbb{R} \to \mathbb{R}$ defined as

$$\tilde{\eta}_0(\rho) = \begin{cases} \dfrac{\delta_{\mathrm{esc}}(m)^2}{2} - \dfrac{\delta_{\mathrm{esc}}(m)^2}{2L}\rho & \text{if} \quad \rho \leq 0 \,, \\ \dfrac{\delta_{\mathrm{esc}}(m)^2}{2} & \text{if} \quad \rho \geq 0 \,, \end{cases}$$

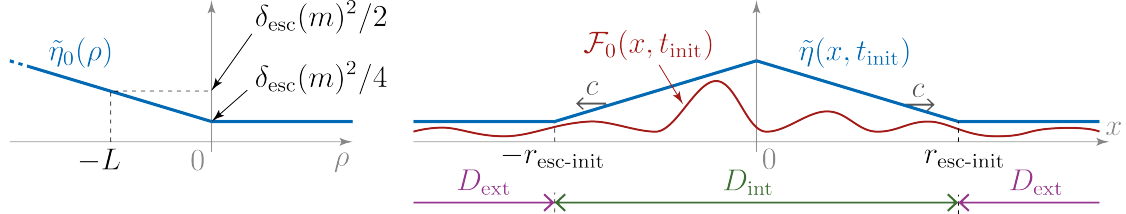see figure 7.1. Let c denote a (fixed, arbitrarily small) positive quantity, let t_{init} and

Figure 7.1: Graphs of the functions $\rho \mapsto \tilde{\eta}_0(\rho)$ (left) and $x \mapsto \tilde{\eta}(x, t_{\mathrm{init}})$ (right), illustration of Lemma 7.8 (right) and of the sets D_{int} and D_{ext} at t equal to t_{init} (right).

$r_{\mathrm{esc\text{-}init}}$ denote two positive quantities to be chosen below, and let us introduce the function

$$\tilde{\eta} : \mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty) \to \mathbb{R} \,, \quad (x, t) \mapsto \tilde{\eta}_0\Big(|x| - r_{\mathrm{esc\text{-}init}} - c(t - t_{\mathrm{init}})\Big) \,,$$

see figure 7.1.

Lemma 7.8 (choice of the quantities t_{init} and $r_{\mathrm{esc\text{-}init}}$). *If the quantity t_{init} is large enough positive and if the quantity $r_{\mathrm{esc\text{-}init}}$ is also large enough positive (depending on the choice of t_{init}), the following inequalities hold: for every x in $\mathbb{R}^{d_{\mathrm{sp}}}$,*

$$(7.11) \qquad\qquad \mathcal{F}_0(x, t_{\mathrm{init}}) \leq \tilde{\eta}(x, t_{\mathrm{init}}) \,,$$

and, for every x in $\mathbb{R}^{d_{\mathrm{sp}}}$ and t in $[t_{\mathrm{init}}, +\infty)$,

$$(7.12) \qquad\qquad \partial_t \mathcal{F}_0(x, t) \leq c\frac{\delta_{\mathrm{esc}}(m)^2}{4L} \,.$$

Proof. The fact that inequality (7.12) holds, if t_{init} is large enough positive, follows from Corollary 7.5, and the fact that inequality (7.11) holds, if t_{init} is large enough positive and $r_{\mathrm{esc\text{-}init}}$ is large enough positive once t_{init} has been chosen, follows from Lemma 7.1 and from the bounds (3.5) on the solution. $\square$

From now on, let us assume that the quantities t_{init} and $r_{\mathrm{esc\text{-}init}}$ are chosen so that the conclusions of Lemma 7.8 hold.

Lemma 7.9 ($\mathcal{F}_0(x, t)$ remains below $\tilde{\eta}(x, t)$ forever from time t_{init} on). *For every time t greater than or equal to t_{init},*

$$\mathcal{F}_0(x, s) \leq \tilde{\eta}(x, t) \quad \text{for all } x \text{ in } \mathbb{R}^{d_{\mathrm{sp}}}.$$

52

Proof. Let us introduce the function

$$\Delta : \mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty) \to \mathbb{R}, \quad (x,t) \mapsto \mathcal{F}_0(x,t) - \tilde{\eta}(x,t).$$

Proving Lemma 7.9 amounts to prove that this function is nonpositive on $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty)$. Let us introduce the domains

$$D_{\mathrm{int}} = \left\{ (x,t) \in \mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty) : |x| < r_{\mathrm{esc\text{-}init}} + c(t - t_{\mathrm{init}}) \right\},$$

$$\text{and} \quad \partial D_{\mathrm{int}} = \left\{ (x,t) \in \mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty) : |x| = r_{\mathrm{esc\text{-}init}} + c(t - t_{\mathrm{init}}) \right\},$$

$$\text{and} \quad D_{\mathrm{ext}}(t) = \left\{ (x,t) \in \mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{init}}, +\infty) : |x| > r_{\mathrm{esc\text{-}init}} + c(t - t_{\mathrm{init}}) \right\}.$$

The following observations can be made concerning the function Δ:

- according to inequality (7.11), it is nonpositive on $\mathbb{R}^{d_{\mathrm{sp}}} \times \{t_{\mathrm{init}}\}$;

- it is continuous on $\mathbb{R}^{d_{\mathrm{sp}}} \times [0, +\infty)$;

- its partial derivative $\partial_t \Delta$ is defined on $D_{\mathrm{int}} \sqcup D_{\mathrm{ext}}$;

- for every (x,t) in D_{int}, since, according to the definition of $\tilde{\eta}$,

$$\partial_t \tilde{\eta}(x,t) = c\frac{\delta_{\mathrm{esc}}(m)^2}{4L},$$

  it follows from inequality (7.12) that $\partial_t \Delta(x,t)$ is nonpositive.

- for every (x,t) in D_{ext}, according to inequality (4.32) of Lemma 4.7,

$$(7.13) \qquad \partial_t \Delta(x,t) \leq -\nu_{\mathcal{F}_0} \mathcal{F}_0(x,t) + K_{\mathcal{F}_0} \int_{\Sigma_{\mathrm{Esc}}(t)} T_x \psi_0(y) \, dy.$$

Let us proceed by contradiction and assume that the set

$$\{t \in [t_{\mathrm{init}}, +\infty) : \text{ there exists } x \text{ in } \mathbb{R}^{d_{\mathrm{sp}}} \text{ such that } \Delta(x,t) > 0\}$$

is nonempty. Let us denote by t_{exit} the infimum of this set, which is a nonnegative quantity, and let us introduce the quantity

$$(7.14) \qquad \tau = \frac{L}{c}, \quad \text{so that} \quad \tau c\frac{\delta_{\mathrm{esc}}(m)^2}{2L} \leq \frac{\delta_{\mathrm{esc}}(m)^2}{2}.$$

The following lemma conflicts the definition of t_{exit} (and thus completes the proof).

Lemma 7.10 ($\Delta(\cdot, \cdot)$ cannot reach any positive value on $\mathbb{R} \times [t_{\mathrm{exit}}, t_{\mathrm{exit}} + \tau]$). *For every x in $\mathbb{R}^{d_{\mathrm{sp}}}$,*

$$(7.15) \qquad \Delta(x, t_{\mathrm{exit}}) \leq 0,$$

and for every (x,t) in $\mathbb{R}^{d_{\mathrm{sp}}} \times [t_{\mathrm{exit}}, t_{\mathrm{exit}} + \tau]$,

$$(7.16) \qquad \left(\Delta(x,t) \geq -\frac{\delta_{\mathrm{esc}}(m)^2}{8} \quad \text{and} \quad (x,t) \notin \partial D_{\mathrm{int}} \right) \implies \partial_t \Delta(x,t) \leq 0.$$

Proof. Since the function $\Delta(\cdot,\cdot)$ is continuous, inequality (7.15) must hold (or else the would be a contradiction with the definition of t_{exit}). Besides, since $\partial_t\Delta(x,t)$ is nonpositive on D_{int}, implication (7.16) readily holds for (x,t) in D_{int}. It remains to prove that this implication also holds when (x,t) is in D_{ext}.

Observe that, according to inequality in (7.14), to inequality (7.12), to inequality (7.9) of Lemma 7.6 and to the definition of $\tilde{\eta}$, for every (x,t) in D_{ext} with t in $[t_{\text{exit}}, t_{\text{exit}} + \tau]$,

$$\Sigma_{\text{Esc}}(t) \cap B(x,L) = \emptyset\,,$$

so that, according to inequality (7.13),

$$\partial_t\Delta(x,t) \leq -\nu_{\mathcal{F}_0}\mathcal{F}_0(x,t) + K_{\mathcal{F}_0}\int_{\mathbb{R}^{d_{\text{SP}}}\setminus B(x,L)} T_x\psi_0(y)\,dy\,,$$

and thus, according to inequality (7.10) of Lemma 7.7,

$$\partial_t\Delta(x,t) \leq -\nu_{\mathcal{F}_0}\left(\mathcal{F}_0(x,t) - \frac{\delta_{\text{esc}}(m)^2}{8}\right) = -\nu_{\mathcal{F}_0}\left(\Delta(x,t) + \frac{\delta_{\text{esc}}(m)^2}{8}\right)\,,$$

so that implication (7.16) holds when (x,t) is in D_{ext}. Lemma 7.10 is proved. □

End of the proof of Lemma 7.9. It follows from Lemma 7.10 and from the continuity of $\Delta(\cdot,\cdot)$ that $\Delta(x,t)$ remains nonpositive on $\mathbb{R}^{d_{\text{SP}}} \times [t_{\text{exit}}, t_{\text{exit}} + \tau]$, a contradiction with the definition of t_{exit}. Lemma 7.9 is proved. □

It follows from Lemma 7.9, from the definition of $\tilde{\eta}(\cdot,\cdot)$, and from inequality (7.9) that, for every time t greater than or equal to t_{init},

$$\Sigma_{\text{Esc}}(t) \subset D_{\text{int}}(t)\,.$$

Then it follows from Lemma 4.8 and from the bounds (3.5) on the solution that the invasion speed $c_{\text{inv}}[u]$ cannot be larger than c. Since the positive quantity c was chosen arbitrarily small, the invasion speed $c_{\text{inv}}[u]$ must be equal to 0, a contradiction (indeed $c_{\text{inv}}[u]$ was assumed to be positive, see the beginning of subsection 7.3). This shows that inequality (7.3) cannot occur, in other words the asymptotic energy $\mathcal{E}_{\text{asympt}}[u]$ must be equal to $-\infty$. Conclusion 2 of Theorem 1 is proved.

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References

[1] N. D. ALIKAKOS, G. FUSCO, and P. SMYRNELIS. *Elliptic Systems of Phase Transition Type.* Vol. 91. Progress in Nonlinear Differential Equations and Their Applications. Cham: Springer International Publishing, 2018. DOI: 10.1007/978-3-319-90572-3 (cit. on p. 2).

[2] N. D. Alikakos and N. I. Katzourakis. "Heteroclinic travelling waves of gradient diffusion systems". In: *Trans. Am. Math. Soc.* 363.03 (2011), pp. 1365–1365. doi: 10.1090/S0002-9947-2010-04987-6. arXiv: 0712.2800 (cit. on p. 2).

[3] D. G. Aronson and H. F. Weinberger. "Multidimensional nonlinear diffusion arising in population genetics". In: *Adv. Math. (N. Y).* 30.1 (1978), pp. 33–76. doi: 10.1016/0001-8708(78)90130-5 (cit. on p. 3).

[4] J. Bouhours and T. Giletti. "Extinction and spreading of a species under the joint influence of climate change and a weak Allee effect: a two-patch model". In: *arXiv* (2016), pp. 1–33. arXiv: 1601.06589 (cit. on p. 2).

[5] J. Bouhours and T. Giletti. "Spreading and Vanishing for a Monostable Reaction – Diffusion Equation with Forced Speed". In: *J. Dyn. Differ. Equations* (2018). doi: 10.1007/s10884-018-9643-5 (cit. on p. 2).

[6] J. Bouhours and G. Nadin. "A variational approach to reaction-diffusion equations with forced speed in dimension 1". In: *Discret. Contin. Dyn. Syst. - A* 35.5 (2015), pp. 1843–1872. doi: 10.3934/dcds.2015.35.1843. arXiv: 1310.3689 (cit. on p. 2).

[7] G. Chapuisat. "Existence and nonexistence of curved front solution of a biological equation". In: *J. Differ. Equ.* 236.1 (2007), pp. 237–279. doi: 10.1016/j.jde.2007.01.021 (cit. on p. 2).

[8] G. Chapuisat and R. Joly. "Asymptotic profiles for a travelling front solution of a biological equation". In: *Math. Model. Methods Appl. Sci.* 21.10 (2011), pp. 2155–2177. doi: 10.1142/S0218202511005696. arXiv: 1005.1729 (cit. on p. 2).

[9] C.-N. Chen and V. Coti Zelati. "Traveling wave solutions to the Allen–Cahn equation". In: *Ann. l'Institut Henri Poincaré C, Anal. non linéaire* 39 (2022), pp. 905–926. doi: 10.4171/aihpc/23. arXiv: 2003.08248 (cit. on p. 2).

[10] C. C. Chen, H. Y. Chien, and C. C. Huang. "A variational approach to three-phase traveling waves for a gradient system". In: *Discret. Contin. Dyn. Syst. Ser. A* 41.10 (2021), pp. 4737–4765. doi: 10.3934/dcds.2021055 (cit. on p. 2).

[11] Y. Du and H. Matano. "Convergence and sharp thresholds for propagation in nonlinear diffusion problems". In: *J. Eur. Math. Soc.* 12.2 (2010), pp. 279–312. doi: 10.4171/JEMS/198 (cit. on p. 3).

[12] Y. Du and H. Matano. "Radial terrace solutions and propagation profile of multistable reaction-diffusion equations over $\mathbb{R}^N$". In: *arXiv* (2020), pp. 4–21. arXiv: 1711.00952 (cit. on p. 3).

[13] Y. Du and P. Polacik. "Locally uniform convergence to an equilibrium for nonlinear parabolic equations on $\mathbb{R}^N$". In: *Indiana Univ. Math. J.* 64.3 (2015), pp. 787–824. doi: 10.1512/iumj.2015.64.5535 (cit. on p. 3).

[14] P. C. Fife and J. B. McLeod. "The approach of solutions of nonlinear diffusion equations to travelling front solutions". In: *Arch. Ration. Mech. Anal.* 65.4 (1977), pp. 335–361. doi: 10.1007/BF00250432 (cit. on pp. 2, 3).

[15] P. C. Fife and J. B. McLeod. "A phase plane discussion of convergence to travelling fronts for nonlinear diffusion". In: *Arch. Ration. Mech. Anal.* 75.4 (1981), pp. 281–314. DOI: 10.1007/BF00256381 (cit. on p. 3).

[16] P. Fife. "Long time behavior of solutions of bistable nonlinear diffusion equations". In: *Arch. Ration. Mech. Anal.* 70.1 (1979), pp. 31–36. DOI: 10.1007/BF00276380 (cit. on p. 3).

[17] T. Gallay and R. Joly. "Global stability of travelling fronts for a damped wave equation with bistable nonlinearity". In: *Ann. Sci. l'École Norm. Supérieure* 42.1 (2009), pp. 103–140. DOI: 10.24033/asens.2091. arXiv: 0710.0794 (cit. on p. 2).

[18] T. Gallay and E. Risler. "A variational proof of global stability for bistable travelling waves". In: *Differ. Integr. Equations* 20.8 (2007), pp. 901–926. arXiv: math/0612684 (cit. on p. 2).

[19] F. Hamel and H. Ninomiya. "Localized and Expanding Entire Solutions of Reaction–Diffusion Equations". In: *J. Dyn. Differ. Equations* (2021), pp. 4–21. DOI: 10.1007/s10884-020-09936-2. arXiv: arXiv:2005.07420v1 (cit. on p. 3).

[20] F. Hamel and L. Rossi. "Spreading speeds and one-dimensional symmetry for reaction-diffusion equations". In: *arXiv* (2021), pp. 1–88. arXiv: 2105.08344 (cit. on p. 3).

[21] S. Heinze. *A variational approach to travelling waves.* Tech. rep. Leipzig: Technical Report 85, Max Planck Institute for Mathematical Sciences, 2001 (cit. on p. 2).

[22] D. B. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Vol. 840. Lecture Notes in Mathematics. Berlin, New-York: Springer Berlin Heidelberg, 1981. DOI: 10.1007/BFb0089647 (cit. on pp. 3, 8).

[23] C. K. R. T. Jones. "Asymptotic behavior of a reaction-diffusion equation in higher space dimensions". In: *Rocky Mt. J. Math.* 13.2 (1983), pp. 355–364. DOI: 10.1216/RMJ-1983-13-2-355 (cit. on p. 3).

[24] C. K. R. T. Jones. "Spherically symmetric solutions of a reaction-diffusion equation". In: *J. Differ. Equ.* 49.1 (1983), pp. 142–169. DOI: 10.1016/0022-0396(83)90023-2 (cit. on p. 3).

[25] C. Luo. "Global stability of travelling fronts for a damped wave equation". In: *J. Math. Anal. Appl.* 399 (2013), pp. 260–278. DOI: 10.1016/j.jmaa.2012.05.089 (cit. on p. 2).

[26] H. Matano and P. Poláčik. "Dynamics of nonnegative solutions of one-dimensional reaction–diffusion equations with localized initial data. Part I: A general quasiconvergence theorem and its consequences". In: *Commun. Partial Differ. Equations* 41.5 (2016), pp. 785–811. DOI: 10.1080/03605302.2016.1156697 (cit. on p. 3).

[27] H. Matano and P. Poláčik. "An entire solution of a bistable parabolic equation on R with two colliding pulses". In: *J. Funct. Anal.* 272.5 (2017), pp. 1956–1979. DOI: 10.1016/j.jfa.2016.11.006 (cit. on p. 9).

[28] H. MATANO and P. POLÁČIK. "Dynamics of nonnegative solutions of one-dimensional reaction-diffusion equations with localized initial data. Part II: Generic nonlinearities". In: *Commun. Partial Differ. Equations* 45.6 (2020), pp. 483–524. DOI: 10.1080/03605302.2019.1700273 (cit. on p. 3).

[29] C. B. MURATOV. "A global variational structure and propagation of disturbances in reaction-diffusion systems of gradient type". In: *Discret. Contin. Dyn. Syst. - Ser. B* 4.4 (2004), pp. 867–892. DOI: 10.3934/dcdsb.2004.4.867 (cit. on p. 2).

[30] C. B. MURATOV and M. NOVAGA. "Front propagation in infinite cylinders. I. A variational approach". In: *Commun. Math. Sci.* 6.4 (2008), pp. 799–826. DOI: 10.4310/CMS.2008.v6.n4.a1 (cit. on p. 2).

[31] C. B. MURATOV and M. NOVAGA. "Front propagation in infinite cylinders. II. The sharp reaction zone limit". In: *Calc. Var. Partial Differ. Equ.* 31.4 (2008), pp. 521–547. DOI: 10.1007/s00526-007-0125-6 (cit. on p. 2).

[32] C. B. MURATOV and M. NOVAGA. "Global exponential convergence to variational traveling waves in cylinders". In: *SIAM J. Math. Anal.* 44.1 (2012), pp. 293–315. DOI: 10.1137/110833269. arXiv: 1105.1190 (cit. on p. 2).

[33] C. B. MURATOV and X. ZHONG. "Threshold phenomena for symmetric decreasing solutions of reaction-diffusion equations". In: *Nonlinear Differ. Equations Appl. NoDEA* 20.4 (2013), pp. 1519–1552. DOI: 10.1007/s00030-013-0220-7. arXiv: 1203.4623 (cit. on p. 3).

[34] C. B. MURATOV and X. ZHONG. "Threshold phenomena for symmetric-decreasing radial solutions of reaction-diffusion equations". In: *Discret. Contin. Dyn. Syst. Ser. A* 37.2 (2017), pp. 915–944. DOI: 10.3934/dcds.2017038. arXiv: 1506.09146 (cit. on p. 3).

[35] R. OLIVER-BONAFOUX. "Heteroclinic traveling waves of 1D parabolic systems with degenerate stable states". In: *arXiv* (2021), pp. 1–37. arXiv: 2111.12546 (cit. on p. 2).

[36] R. OLIVER-BONAFOUX. "Heteroclinic traveling waves of 2D parabolic Allen-Cahn systems". In: *arXiv* (2021), pp. 1–67. arXiv: 2106.09441 (cit. on p. 2).

[37] R. OLIVER-BONAFOUX and E. RISLER. "Global convergence towards pushed travelling fronts for parabolic gradient systems". 2023 (cit. on p. 2).

[38] P. POLÁČIK. "Convergence and Quasiconvergence Properties of Solutions of Parabolic Equations on the Real Line: An Overview". In: *Patterns Dyn.* Ed. by P. GUREVICH, J. HELL, B. SANDSTEDE, and A. SCHEEL. Springer, Cham, 2017, pp. 172–183. DOI: 10.1007/978-3-319-64173-7_11 (cit. on p. 3).

[39] P. POLÁČIK. "On bounded radial solutions of parabolic equations on $\mathbb{R}^N$: Quasi-convergence for initial data with a stable limit at infinity (preprint)". 2023 (cit. on p. 3).

[40] E. Risler. "Global convergence toward traveling fronts in nonlinear parabolic systems with a gradient structure". In: *Ann. l'Institut Henri Poincare. Ann. Anal. Non Lineaire/Nonlinear Anal.* 25.2 (2008), pp. 381–424. DOI: `10.1016/j.anihpc.2006.12.005` (cit. on p. 2).

[41] E. Risler. "Global behaviour of bistable solutions for gradient systems in one unbounded spatial dimension". In: *arXiv* (2022), pp. 1–91. arXiv: `1604.02002` (cit. on p. 3).

[42] E. Risler. "Global behaviour of bistable solutions for hyperbolic gradient systems in one unbounded spatial dimension". In: *arXiv* (2022), pp. 1–75. arXiv: `1703.01221` (cit. on p. 3).

[43] E. Risler. "Global behaviour of radially symmetric solutions stable at infinity for gradient systems". In: *arXiv* (2022), pp. 1–52. arXiv: `1703.02134` (cit. on pp. 3, 29, 30).

[44] E. Risler. "Global relaxation of bistable solutions for gradient systems in one unbounded spatial dimension". In: *arXiv* (2022), pp. 1–69. arXiv: `1604.00804` (cit. on pp. 3, 9, 11).

[45] J.-M. Roquejoffre and V. Roussier-Michon. "Sharp large time behaviour in N-dimensional reaction-diffusion equations of bistable type". In: *arXiv* (2021), pp. 1–15. arXiv: `2101.07333` (cit. on p. 3).

[46] V. Roussier-Michon. "Stability of radially symmetric travelling waves in reaction–diffusion equations". In: *Ann. l'Institut Henri Poincare Non Linear Anal.* 21.3 (2004), pp. 341–379. DOI: `10.1016/j.anihpc.2003.04.002` (cit. on p. 3).

[47] K. Uchiyama. "Asymptotic behavior of solutions of reaction-diffusion equations with varying drift coefficients". In: *Arch. Ration. Mech. Anal.* 90.4 (1985), pp. 291–311. DOI: `10.1007/BF00276293` (cit. on p. 3).

[48] A. Zlatoš. "Sharp transition between extinction and propagation of reaction". In: *J. Am. Math. Soc.* 19.1 (2005), pp. 251–263. DOI: `10.1090/S0894-0347-05-00504-7`. arXiv: `0504333 [math]` (cit. on p. 3).

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