Corso (con Renato Lucà) 2023-2024

"Introd. to incompr. fluid dynamics"

BCAM – SO courses

Programma Parte 1: Ideal fluid flows

(i) The Euler equations and basic properties.
(ii) Strong solutions theory
(iii) Beale - Kato - Majda ~~automation~~ criterion and applications: global existence in 2D.
(iv) More on the 2D case: Yudovich theory and the study of vortex patches.
(v) The case of the ideal MHD: general theory in any space dimension; the 2D case.

N° lezioni: 8 da 2h → 16 h

26/02,	28/02
04/03,	06/03
11/03,	13/03
18/03	20/03

• References:

(i) Bahouri, Chemin, Danchin: "Fourier Analysis and nonlinear ..." (2010).

(ii) Bertozzi, Majda: "Vorticity and incompressible flow" (2003)

Many other texts on NS and Euler eq:

- Marchioro, Pulvirenti (1996)
- Chemin: "Perfect incompressible fluids" (1995, traduz. ingl. 1998).

(1) The Euler equations.

Goal: describe fluid motion.

We will use Newton's law $F = m \cdot a$ on an infinitesimal fluid parcel/on a fluid particle.

Assumptions: (a) fluid is homogeneous; density $\rho = 1$

(b) fluid is incompressible: $\operatorname{div} u = \sum_{j=1}^{d} \partial_j u^j = 0$

2 points of view: (i) Eulerian: field equations: $\left\{\begin{array}{l} \text{velocity } u \\ \text{density } \rho \\ \text{pressure } \pi \end{array}\right.$ as funct. of (t,x)

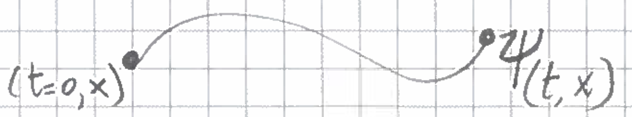
(ii) Lagrangian: following fluid particles.

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$, Ω = fluid domain (fixed).

For us, $\Omega = \mathbb{R}^d$ always (but everything goes well also with $\Omega = \mathbb{T}^d$).

Remark: (a) $\Rightarrow$ (b). Viceversa is not true.

So, take a fluid particle that at time $t = 0$ stays at some $x \in \Omega$. Define the map $\psi = \psi(t,x) = \psi_t(x)$ = position at time $t > 0$.



ψ = flow

$\partial_t \psi_{(t,x)}$ = velocity = $u(t, \psi(t,x))$.

Now, $\left\{\begin{array}{l} \text{acceleration} = \frac{d}{dt} u = \partial_t u + \partial_t \psi \cdot \nabla u = \partial_t u + u \cdot \nabla u \\ \text{forces: pressure, viscosity...} \end{array}\right.$

Here: viscous friction = 0.

In addition: pressure comes form boundary integral

$\Rightarrow$ pressure force $= -\nabla \pi$

$\frac{1}{|B|} \int_{\partial B(x,r)} -p \, \mathrm{Id} \cdot n = \int \operatorname{div}(p \, \mathrm{Id}) = \int \nabla p$

Thus, incompressible Euler equations:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = 0 \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

Here, interested in initial value problem: $u_0 \rightsquigarrow \exists, !,$ stability, qualitative behaviour.

Solutions: ∘ classical (say, C^∞)

∘ strong (all terms well-def. a.e.)

∘ weak: $\forall \psi \in \mathcal{D}([0,T[\times\mathbb{R}^d)$, $\operatorname{div}\psi = 0$, one has

$$-\int_0^T\!\!\int_{\mathbb{R}^d} u\,\partial_t\psi - \int_0^T\!\!\int_{\mathbb{R}^d} u\otimes u : \nabla\psi = \int_{\mathbb{R}^d} u_0(x)\cdot\psi(0,x)\,dx,$$

where

$$u\otimes u : \nabla\psi = \sum_{j,k} u^j u^k \partial_j \psi^k$$

$\boxed{\text{Remark: } (u\cdot\nabla)u = \operatorname{div}(u\otimes u)}$

Remark on the pressure function

$\nabla\pi$ = Lagrangian multiplier associated to the constraint $\operatorname{div} u = 0$.

It can be recovered from u by solving

$$-\Delta\pi = \operatorname{div}(u\cdot\nabla u) = \sum_{j,k} \partial_j(u^k\partial_k u^j) = \sum_{j,k} \partial_j u^k \partial_k u^j =$$

$$= \nabla u : \nabla u \Rightarrow \nabla\pi \approx u.$$

$\boxed{\nabla\pi = \text{right quantity, rather than } \pi}$

This implies $\nabla\pi = \nabla(-\Delta)^{-1}\operatorname{div}(u\cdot\nabla u)$ in the sense of Fourier multipliers

Rem: $(-\Delta)^{-1}$ = conv. operator with fundam. sol. of the Laplacian.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } d=2 \\ \frac{1}{d(d-2)\,\alpha(d)} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3, \end{cases}$$

where $\alpha(d)$ = volume of the unit ball in $\mathbb{R}^d$.

<u>Rem on Fourier multipliers:</u>

$m : \mathbb{R}^d \to \mathbb{R}$ smooth.

$$m(\xi) \rightsquigarrow M : X \to X$$

$X \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$

$$f \mapsto Mf = \mathcal{F}^{-1}\left(m(\xi)\, \hat{f}(\xi)\right)$$

$$\overset{!}{=} k * f(x),$$

where $k = \mathcal{F}^{-1}(m)$.

Define $\mathbb{Q} \doteq -\nabla(-\Delta)^{-1}\,\mathrm{div}$

$= \mathbb{P} \doteq \mathrm{Id} - \mathbb{Q}$, whose symbol is

$\mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$

$$\mathbb{P}(\xi) \doteq 1 - \frac{1}{|\xi|^2}\,\xi\,(\xi\cdot \quad .$$

$\mathbb{P} \doteq$ Leray-Helmholtz projector. It is the L^2-orthogonal projector onto the div-free vector fields

Remark: $\mathbb{P}$ and $\mathbb{Q}$ are singular integral operators, with symbols homogeneous of order 0

$\Rightarrow$ Calderón-Zygmund theory applies.

In particular, $\mathbb{P}, \mathbb{Q} : L^p \longrightarrow L^p \quad \forall p \in \,]1, +\infty[$.

$\boxed{\text{Conserved quantities: the energy}}$ (see Sect. 1.7 of [Bert M

Assume to have a smooth solution $(u, \nabla\pi)$ on $[0, T[\times\mathbb{R}^d$, which vanishes sufficiently rapidly for $\|x\| \to +\infty$.

Then we can test the Euler eq. on u and get

$$\frac{d}{dt}\int|u|^2 = 0 \Rightarrow \forall t>0,\ \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 .$$

Fundamental computation: $\int(u\cdot\nabla)u\cdot u = \frac{1}{2}\int u\cdot\nabla|u|^2 = 0$

This requires $\int$ by parts $\Rightarrow$ regularity.

chimico e fisico $\leftarrow$ Onsager (1949): threshold for energy conservation is Hölder $\frac{1}{3}$-regularity

Positive part: [Constantin, E, Titi] (1994), [Eyink] (1994).

Negative part: many recent efforts

([De Lellis, Székelyhidi] 2009, 2010, 2013,

[Buckmaster, DL, Sz K] 2015, ..., [I 2018

Another problem:

$u \in L^\infty_T L^2$, $\nabla\pi$?

and reg is not enough to pass to the limit ~ weak formulation.

One needs to gain some info on derivatives of u.
Easiest way: look for strong sol.

We will mainly work in H^s spaces for simplicity.
Notice, however, that, roughly speaking, Euler eq. = coupling of transport eq. So, L^p-based spaces are good also, $\forall p \in [1, +\infty]$.

<u>Remark</u> (warning): consider the Euler eq.

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = 0 \\ \operatorname{div} u = 0. \end{cases}$$

Let $f \in \mathcal{C}^\infty(\mathbb{R}_+)$, $f = f(t)$ and define

$$u(t,x) = \big(f(t), 0, 0 \ldots 0\big) \quad \text{and} \quad \pi(t,x) = -f'(t)\,x_1 + c,$$
$$c \in \mathbb{R}.$$

Then the couple $(u, \nabla\pi)$ solves the Euler system.
In particular, if $f(0) = 0$
$\Rightarrow$ non-uniqueness (even in a regular functional framework)!

Problem hidden in the elliptic eq.

$$-\Delta\pi = \operatorname{div}(u\cdot\nabla u)$$

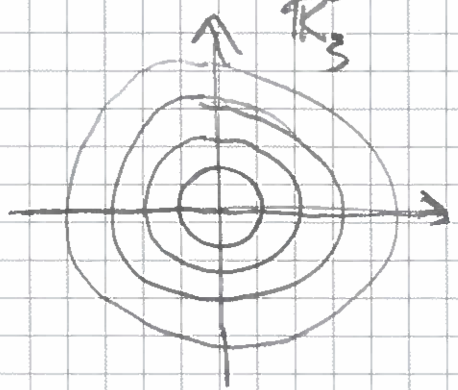
and in solving in a functional space based on L^∞.

See [Cobb-F., 2023, JMPA], [Cobb, 2024, JFA], [Elgindi, Masmoudi, 2020, ARMA].

(2) Elements of Littlewood-Paley theory and paradiff. calc. (6)

([B-C-D], Chap 2; [Mét. 2008]; [Alinhac-Gérard 90

We introduce the non-homog. Littlewood-Paley decompos.
The idea is to construct a special (dyadic) partition of unity in Fourier space.



Prop:
$\exists\, \chi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, radial and decreasing in radial directions,
with $\chi \equiv 1$ on $B(0,\frac{3}{4})$ and
$\chi \equiv 0$ out of $B(0,\frac{4}{3})$,
such that, if we define

$$\varphi(\xi) = \chi(\xi) - \chi(2\xi) \quad \text{and}, \ \forall j \geq 0, \quad \varphi_j(\xi) \doteq \varphi(2^{-j}\xi),$$

one has $\forall \xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j=0}^{+\infty} \varphi_j(\xi) = 1$.

(without proof) $\longrightarrow$ considerazioni su φ, φ_j, supporti. Guardare dietro

Now, define $\Delta_{-1} \doteq \chi(D)$ and, $\forall j \geq 0$, $\Delta_j = \varphi_j(D)$.

Then, $\forall u \in \mathcal{S}'(\mathbb{R}^d)$, one has

$$u = \sum_{j \geq -1} \Delta_j u \quad \text{in the sense of } \mathcal{S}'.$$

For later use, set, $\forall j \geq 0$,

$$S_j \doteq \sum_{k \leq j-1} \Delta_k = \chi(2^{-j}D)$$

low-frequency cut-off operator.

Rem: $\Delta_j \Delta_k \equiv 0$ if $|j-k| \geq 2$.

Remark:

$\varphi \equiv 0$ if $|\xi| \leq \frac{3}{8}$ or $|\xi| \geq \frac{4}{3}$; $\varphi \equiv 1$ if $\frac{2}{3} \leq |\xi| \leq \frac{3}{4}$

Then, $\operatorname{supp} \varphi_J \subset \left\{ \frac{3}{8} 2^J \leq |\xi| \leq \frac{4}{3} 2^J \right\}$.

Thus, $\varphi_J \varphi_k \equiv 0$ if $\frac{4}{3} 2^J \leq \frac{3}{8} 2^k$ or viceversa.

$$2^{J-k} \leq \quad < \frac{1}{6} < \frac{1}{4} = 2^{-2} \Rightarrow \quad J-k \leq -2$$

$$k - J \geq 2.$$

$$J-k \geq 2$$

$$k-J \leq -2$$

$\Rightarrow \varphi_J \varphi_k = 0$ if $|k-J| \geq 2$.

Remark: Let us set $K_0 = \mathcal{F}^{-1}\chi$.

Then $K_J = \mathcal{F}^{-1}(\chi(2^{-}\cdot)) = 2 \; K_0($

one has $S_J u = K_J * u$.

As $\|K_J\|_{L^1} = \|K_0\|_{L^1}$ $\quad S_J : L^p \to L^p \quad \forall$

$\forall \in [\quad +\infty]$

The same holds true also for the operators

$\boxed{\text{Bernstein's inequalities}}$

Rem: $\left.\begin{array}{l}\forall J \geq -1 \\ \forall u \in \mathcal{S}'\end{array}\right\}$, $\Delta_J u$ and $S_J u$ are analytic on (by Paley–Wiener rem)

Prop: $\exists C > 0$, $C_1 > 0$, $C_2 > 0$ $\mid$ $\forall (p,q) \in [1,+\infty] \quad [\; , \;]$ ith $\forall k \geq 0$, $\forall u \in L^p$, one has

(i) $\forall J \geq 0$,

$$\|D^k S_J u\|_{L^q} \doteq \sup_{|\alpha| = k} \quad {}_J u\|_{L^q} \leq C\, 2^{Jd - \frac{}{q}} \quad K J$$

(recall that $S_0 = \Delta_{-1}$)

$\operatorname{supp} \hat{u} \subset \lambda B$, λ

(ii) $\forall J \geq 0$,

$$C_1 2^{Jk} \|\Delta_J u\|_{L^p} \leq \|D^k \Delta u\|_{L^p} \leq C_2 2^{Jk} \|\Delta_J u\|_{L^p}$$

Rem: $\operatorname{supp} \hat{u} \subset \lambda \mathcal{C}$, $\lambda \sim 2^J$

Mention that they are true for general spectrally cpt distrib. (important for next part (3)).

Fai forma generale! (guarda dietro)

<u>Lemma</u> (Bernstein's inequalities):

$\mathcal{B} = \{\xi \in \mathbb{R}^d \mid |\xi| \leq R\}$

$\mathcal{C} = \{\xi \in \mathbb{R}^2 \mid 0 < r_1 \leq |\xi| \leq r_2\}.$

$\exists\, C > 0 \mid \forall k \in \mathbb{N},\ \forall (p,q) \in [1,+\infty] \mid p \leq q.$
$\forall u \in L^p,\ \forall \lambda > 0,$

on has:

(i) if $\operatorname{supp} \hat{u} \subset \lambda \mathcal{B}$, then

$$\|D^k u\|_{L^q} \doteq \sup_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p};$$

(ii) if $\operatorname{supp} \hat{u} \subset \lambda \mathcal{C}$, then

$$C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

<u>Rem:</u> for us, $\lambda = 2^J$ (almost) always.

<u>Lemma:</u> $\mathcal{C}$ an annulus, $m \in \mathbb{R}$.
Let σ smooth on $\mathbb{R}^d \setminus \{0\} \mid \forall \alpha \in \mathbb{N}^d,\ |\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{m-|\alpha|}$
$\forall \xi \in \mathbb{R}^d \setminus \{0\}.$

Then, $\exists\, C > 0 \mid \forall p \in [1,+\infty],\ \forall \lambda > 0,$
$\forall u \in L^p \mid \operatorname{supp} \hat{u} \subset \lambda \mathcal{C},$

one has $\|\sigma(D) u\|_{L^p} \leq C \lambda^m \|u\|_{L^p}.$

Dyadic (Besov) characterization of Sobolev spaces

Let $s \in \mathbb{R}$. $H^s(\mathbb{R}^d) \doteq \left\{ u \in \mathcal{S}' \mid \hat{u} \in L^2_{loc} \text{ and } \|u\|^2_{H^s} \doteq \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\}$

Prop: ~~Let~~ $u \in \mathcal{S}'$, $s \in \mathbb{R}$.

Then $u \in H^s \Longleftrightarrow$ two facts: (a) $\forall J \geq -1$, $\Delta_J u \in L^2(\mathbb{R}^d)$

(b) The sequence $\left(2^{Js} \|\Delta_J u\|_{L^2}\right)_{J \geq -1} \in \ell^2(\mathbb{N} \cup \{-1\})$

Moreover, $\exists C > 0$ "universal" $\mid$

$$\frac{1}{C} \sum_{J \geq -1} 2^{2Js} \|\Delta_J u\|^2_{L^2} \leq \|u\|^2_{H^s} \leq C \sum_{J=-1}^{+\infty} 2^{2Js} \|\Delta_J u\|^2_{L^2}.$$

We say that $H^s \equiv B^s_{2,2}$.

Also, non-lin. est. We will need tame est.

Prop: $\forall \sigma > 0$, $L^\infty \cap H^\sigma$ is a Banach algebra.

In addition, $\exists C > 0 \mid \forall f, g \in L^\infty \cap H^\sigma$, one has

$$\|fg\|_{H^\sigma} \lesssim \|f\|_{L^\infty} \|g\|_{H^\sigma} + \|f\|_{H^\sigma} \|g\|_{L^\infty}.$$

Remark:

- $s > \frac{d}{2} \Rightarrow H^s \hookrightarrow L^\infty \Rightarrow H^s$ is a Banach alg.
- $s > 1 + \frac{d}{2} \Rightarrow H^s \hookrightarrow W^{1,\infty}$.

Typical reg. needed for solving quasi-lin. hyperb. problems

(3) Strong solutions theory.

$$(E)\quad\begin{cases}\partial_t u+(u\cdot\nabla)u+\nabla\pi=0.\\ \operatorname{div} u=0\\ u_{|t=0}=u_0\ ,\quad \text{with}\quad \operatorname{div} u_0=0.\end{cases}$$

Comment! $H^s\hookrightarrow W^{1,\infty}$ in $\mathbb{R}^d$ for $s>1+\frac{d}{2}$. Necessary for non-lin. hyperb. problem

Th: Let $s>1+\frac{d}{2}$. Fix $u_0\in H^s(\mathbb{R}^d)\ |\ \operatorname{div} u_0=0$.
Then, $\exists\, T>0$ and $\exists!$ solution $(u,\nabla\pi)$ to (E) on $[0,T]\times\mathbb{R}^d$ | $u\in \mathcal{C}([0,T];H^s(\mathbb{R}^d))$ and $\nabla\pi\in\mathcal{C}([0,T];H^s(\mathbb{R}^d))$.
In addition, let T^* be the lifespan of the solution.
Then $\exists\, C>0$ universal |

$$T^*\geq\frac{C}{\|u_0\|_{H^s}}.$$

Finally, if $T^*<+\infty$, then one has

$$\int_0^T\|\nabla u(t)\|_{L^\infty}\,dt=+\infty.$$

The rest of this part is devoted to the proof of the th.

(1) A priori estimates.

Assume to dispose of a global smooth solution $(u,\nabla\pi)$ on $\mathbb{R}_+\times\mathbb{R}^d$, decaying sufficiently fast at ∞.

First of all, energy conservation:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2=0\ \Rightarrow\ \|u(t)\|_{L^2}\leq\|u_0\|_{L^2}\quad \boxed{\forall t\geq 0}$$

Next, propagation of high regularity norms.

Goal: find a uniform bound on some interval $[0,T]$.

Why this is important? See point (2) (actual existence proof).

Take $J \geq -1$ and apply Δ_J to the eq: we find

$$\partial_t \Delta_J u + u \cdot \nabla \Delta_J u + \Delta_J \nabla \pi = +[u \cdot \nabla, \Delta_J] u$$

where we have denoted $[A,B] = AB - BA$ the commutator between two operators, namely

$$[v \cdot \nabla, \Delta_J] f = v \cdot \nabla \Delta_J f - \Delta_J (v \cdot \nabla f).$$

Multiply the eq. by $\Delta_J u$, an $\int_{\mathbb{R}^d}$ and 2^{2JS}: one gets

$$\frac{1}{2} \frac{d}{dt} \left(2^{2JS} \|\Delta_J u\|_{L^2}^2 \right) = \int_{\mathbb{R}^d} 2^{JS} [u \cdot \nabla, \Delta_J] u \cdot 2^{JS} \Delta_J u$$

$$\leq 2^{JS} \|\Delta_J u\|_{L^2} \, 2^{JS} \| [u \cdot \nabla, \Delta_J] u \|_{L^2}$$

$$\Rightarrow \frac{d}{dt} \left(2^{JS} \|\Delta_J u\|_{L^2} \right) \leq 2^{JS} \| [u \cdot \nabla, \Delta_J] u \|_{L^2}.$$

<u>LEMMA</u> (see Lemma 2.100 of [BCD]):

Let $\sigma > 0$, or $\sigma > -1$ if (as in this case) v is div-free.

$$R_J \doteq [v \cdot \nabla, \Delta_J] f \underset{\boxed{\text{if } \operatorname{div} v = 0}}{=} \operatorname{div}([v, \Delta_J] f).$$

$$\left\| 2^{J\sigma} \|R_J\|_{L^2} \right\|_{\ell^2} \lesssim \|\nabla v\|_{L^\infty} \|f\|_{H^\sigma} + \|\nabla f\|_{L^\infty} \|\nabla v\|_{H^{\sigma-1}}.$$

<u>NB</u> (x Fanu): explain the meaning of the lemma.

Do the basic comm. est. with proof? (see behind the page)

Basic commutator est.

LEMMA (Lemma 2.97 of [BCD]):

$\theta \in C^1(\mathbb{R}^d) \mid (1+|\cdot|)\hat{\theta} \in L^1$.

Then $\exists\, C>0 \mid \forall\, a \in W^{1,p}$,
$\forall\, b \in L^q$,
$\forall\, \lambda > 0$,

$$\| [\theta(\lambda^{-1}D), a]\, b\|_{L^r} \lesssim \lambda^{-1} \|\nabla a\|_{L^p} \|b\|_{L^q},$$

where we have set $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof: $\theta(D) f = k * f$, $\quad k = \check{\theta}$.

Hence $\theta(\lambda^{-1}D) f = \lambda^d k(\lambda\,\cdot) * f$.

Thus $[\theta(\lambda^{-1}D), a] b = \theta(\lambda^{-1}D)(ab) - a\,\theta(\lambda^{-1}D) b =$

$$= \lambda^d \int k(\lambda(x-y))\,\big(a(y) - a(x)\big)\, b(y)\, dy =$$

$$= \lambda^d \int_{[0,1]\times\mathbb{R}^d} k(\lambda(x-y))\, \nabla a_{((1-\tau)x+\tau y)} \cdot (y-x)\, b(y)\, dy\, d\tau$$

and take the L^r norm.

$\square$

Using the commutator lemma, we find

$$\frac{d}{dt}\left(2^{js}\|\Delta_j u\|_{L^2}\right) \lesssim \|\nabla u\|_{L^\infty}\|u\|_{H^s}\,\delta_j,$$

with $(\delta_j)_j \subset \ell^2$, $\|\delta_j\|_{\ell^2}\leq 1$.

Taking $\int_0^t$ and $\Big(\sum_{j\geq -1}\Big)_{\ell^2}$ + Minkowski we thus find

$$\boxed{\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C\int_0^t \|\nabla u\|_{L^\infty}\|u(\tau)\|_{H^s}\,d\tau.}$$

$\Rightarrow$ by Grönwall's lemma, we get

$$\boxed{\forall t\geq 0,\quad \|u(t)\|_{H^s}\leq C\|u_0\|_{H^s}\exp\left(C\int_0^t\|\nabla u(\tau)\|_{L^\infty}\,d\tau\right)}$$

Let $T \doteq \sup\left\{t>0 \;\Big|\; \int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau \leq \log 2\right\}$.

Remark that $T>0$.

Then, on $[0,T]$, one has $\|u(t)\|_{H^s}\leq \tilde{C}\|u_0\|_{H^s}$.

Estimates for the pressure

for lemma?

$$\partial_t u + u\cdot\nabla u + \nabla\pi = 0 \Rightarrow -\Delta\pi = \operatorname{div}(u\cdot\nabla u)$$

By Lax-Milgram theorem, we immediately deduce

$$\|\nabla\pi\|_{L^2} \lesssim \|u\|_{L^2}\|\nabla u\|_{L^\infty} \lesssim \|u_0\|_{L^2}\|u\|_{H^s},$$

as $H^s\hookrightarrow W^{1,\infty}$ for $s>1+\frac{d}{2}$.

Higher order est. slightly more involved.

LEMMA: Consider the elliptic equation

$$-\operatorname{div}\left(a\,\nabla\pi\right) = \operatorname{div} F,$$

with $a(x) \geq a_* > 0$ and $F \in L^2(\mathbb{R}^d)$.

Then $\exists\, \pi \in \mathcal{D}'(\mathbb{R}^d)$, unique up to constant functions, $\nabla\pi \in L^2(\mathbb{R}^d)$ and eq. is solved. In addition, $\exists\, C > 0 \mid$

$$a_* \|\nabla\pi\|_{L^2} \leq C\, \|F\|_{L^2}.$$

LEMMA:

$$\|\nabla \pi\|_{H^s} \lesssim \|\nabla u\|_{L^\infty} \|u\|_{H^s}.$$

Proof: $\nabla\pi = \Delta_{-1}\nabla\pi + (Id-\Delta_{-1})\nabla\pi$ cutting into low and high freq

$$\|\nabla\pi\|^2_{H^s} = \sum_{j\geq -1} 2^{2js}\|\Delta_j\nabla\pi\|^2_{L^2} = 2^{-2s}\|\Delta_{-1}\nabla\pi\|^2_{L^2} + \sum_{j\geq 0}\cdots$$

$$\leq C\|\nabla\pi\|^2_{L^2} + \sum_{j\geq 0} 2^{2j(s-1)} \underbrace{\|\Delta_j \nabla^2\pi\|^2_{L^2}}$$

Bernstein's ineq.

Calderón-Zygmund theory

$$\lesssim C\left(\|\nabla\pi\|^2_{L^2} + \|\Delta\pi\|^2_{H^{s-1}}\right)$$

Now, $-\Delta\pi = \operatorname{div}(u\cdot\nabla u) = \nabla u : \nabla u$

$$\Rightarrow \|\Delta\pi\|_{H^{s-1}} \lesssim \|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{s-1}} \lesssim \|\nabla u\|_{L^\infty}\|u\|_{H^s}. \qquad \square$$

2 $\exists$ of solutions

We first need a couple of preliminaries.

LEMMA 1: $T \in S'(\mathbb{R}^d;\mathbb{R}^d) \mid \langle T,\varphi\rangle_{\mathcal{D}'\times\mathcal{D}} = 0 \quad \forall \varphi\in\mathcal{D} \mid \operatorname{div}\varphi = 0.$

Then $\exists.\ \Phi \in S'(\mathbb{R}^d) \mid T = \nabla\Phi.$

(without proof. See Lemma 3.11 of [Gobb-F.],
Prop. 1.2.1 of [Chemin, 1998].)

Now, consider $u_0 \mid \operatorname{div} u_0 = 0$ and

$$(E)\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = 0 \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \end{cases} \qquad (\mathbb{P}E)\begin{cases} \partial_t u + \mathbb{P}\big((u\cdot\nabla)u\big) = 0 \\ u_{|t=0} = u_0 \end{cases}$$

LEMMA 2: Let $T>0$ and $\sigma > 1+\frac{d}{2}$.
Let $u \in L^\infty_{loc}([0,T[; H^\sigma)$ solve $(\mathbb{P}E)$ on $[0,T[\times\mathbb{R}^d$.
Then $\exists\ \nabla\pi \in L^\infty_{loc}([0,T[, H^\sigma)$ | $(u, \nabla\pi)$ solve (E) on $[0,T[\times\mathbb{R}^d$.

Proof: First of all, ~~Next,~~ observe that, by applying div operator to $(\mathbb{P}E)$, we see that $\partial_t \operatorname{div} u = 0$ in $\mathcal{D}'$.
Then $\operatorname{div} u(t) = \operatorname{div} u_0 = 0 \quad \forall t \in [0,T[$.

$\boxed{\text{Observe that } u\cdot\nabla u \in L^\infty_{loc}([0,T[; H^{\sigma-1}) \Rightarrow \mathbb{P}(u\cdot\nabla)u \text{ belongs to the same space} \Rightarrow \text{so does } \partial_t u.}$

Now, set $T_{(t)} := \partial_t u + (u\cdot\nabla) u_{|t} \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$. This makes sense $\forall t$ as $u \in C_t(H^\sigma)$

Then, $\langle T, \varphi\rangle = 0 \quad \forall \varphi \in \mathcal{D}$ | $\operatorname{div}\varphi = 0$ (decomp. $(u\cdot\nabla)u = \mathbb{P}(\ldots) + \mathbb{Q}(\ldots)$)

$\Rightarrow$ by Lemma 1, $\exists\ \pi(t) \in \mathcal{S}'(\mathbb{R}^d)$ |

$$\partial_t u + (u\cdot\nabla) u = -\nabla\pi.$$

Regularity of $\nabla\pi$ comes by following the analysis of the a priori est.

$\square$

So, we are now going to solve $(\mathbb{P}E)$ by using Friedrichs' method.

For $m \in \mathbb{N}\setminus\{0\}$, define the operator P_m such that

$$\forall f \in \mathcal{S}', \quad \widehat{P_m f}(\xi) = \mathbb{1}_{\{|\xi|\le m\}} \hat{f}(\xi).$$

Then, we solve the eq.

$$(\mathrm{PE}_n) \quad \begin{cases} \partial_t u_n + P_n \mathbb{P}\left(P_n u_n \cdot \nabla P_n u_n\right) = 0 \\ u_n\big|_{t=0} = P_n u_0 \end{cases}$$

Let $\mathcal{H}_n = P_n L^2_{(\mathbb{R}^d;\mathbb{R}^d)} = \{ f \in L^2 \mid \operatorname{supp} \hat{f} \subset B(0,n) \} \subset \bigcap_{\sigma \geq 0} H^\sigma$

We solve the system of ODEs

$$\dot{v} = -P_n \mathbb{P}\left(P_n v \cdot \nabla P_n v\right)$$

Th. (Cauchy-Lipschitz in Banach spaces): (see Th. 3.1 of [Bert-Majda])

B Banach space, $O \subset B$ open.

$F : O \to B$ be a map | F is locally Lip:

$\forall x \in O$, $\exists L_x > 0$ and a neighbourhood $U_x \subset O$

$\forall y, z \in U_x$, $\|F(y) - F(z)\|_B \leq L_x \|y - z\|_B$.

Then, $\forall X_0 \in O$, $\exists T > 0$ and $\exists!\, X \in C^1(]-T, T[; O)$ sol. to

$$\begin{cases} \dfrac{dX}{dt} = F(X) \\ X\big|_{t=0} = X_0 \end{cases}$$

By appl. of the Cauchy-Lip. th. in Banach spaces, we get, $\forall n \in \mathbb{N} \setminus \{0\}$, $\exists T_n > 0$ and $\exists!\, u_n \in C^\infty([0, T_n[; \mathcal{H}_n)$

Rem: ~~$T_n = T(n, \|P_n u_0\|_{H^s}) \to T(n, \|u_0\|_{H^s})$,~~

~~as $\|P_n u_0\|_{H^s} \leq \|u_0\|_{H^s}$.~~

Observe that $\operatorname{div} u_n = 0$

$\Rightarrow P_n u_n = u_n = \mathbb{P} u_n$.

Observe also that $(P_n)^2 = P_n$

Remarks: i have to verify that i can apply Cauchy-Lip. th.

$$\dot{v} = F_n(v), \quad \text{where}$$

$$F_n : \mathcal{H}_n \longrightarrow \mathcal{H}_n$$
$$v \longmapsto -P_n \mathbb{P}(P_n v \cdot \nabla P_n v)$$

(a) Let $v \in \mathcal{H}_n$. Then $P_n v = v$. In addition, v is smooth

$$\Rightarrow \|v \cdot \nabla v\|_{L^2} \leq \|v\|_{L^2} \|\nabla v\|_{L^\infty} \underset{\text{Bernst.}}{\leq} C_n \|v\|_{L^2}^2$$

Then $\mathbb{P}(v \cdot \nabla v) \in L^2$ and then i apply P_n again

$$\Rightarrow F_n(v) \in \mathcal{H}_n.$$

Thus, F_n well def.

(b) I have to verify that F_n is locally Lip. on $\mathcal{H}_n$. Let $v, z \in \mathcal{H}_n$.

$$F_n(v) - F_n(z) = -P_n \mathbb{P}\left[v \cdot \nabla v - z \cdot \nabla z\right] \ldots$$

Then F_n is Lipschitz cont. on any open ball $O_M = \{ w \in \mathcal{H}_n \mid \|w\|_{L^2} < M \}$.

As a conclusion, we can apply Cauchy-Lip. th.

Next, we prove that $T_n = +\infty$ $\forall n \in \mathbb{N}\setminus\{0\}$.
We use the following classical theorem.

Th. (continuation of auton. ODE in Banach spaces):
B Banach, $O \subset B$ open, $F: O \to X$ locally Lipschitz.
Then, the unique $C^1([0,T[\,; O)$ sol. to the autonomous ODE system

$$\begin{cases} \frac{d}{dt} X = F(X) \\ X_{|t=0} = X_0 \in O \end{cases}$$

either exists globally in time, i.e. $T = +\infty$,
or $T < +\infty$ and $X(t)$ leaves the open set O when $t \nearrow T$.

Now, perform a priori est. for $u_n \in C^\infty([0,T_n[\,; H_n)$. In part, u_n smooth both in time and space variables.
First of all, by mult. eq. by u_n by $\mathbb{P}_n u_n = u_n = \mathbb{P} u_n$
we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2} = 0 \Rightarrow \|u_n(t)\|^2_{L^2} \leq \|u_n(0)\|^2_{L^2}$$
$$\leq C\|u_0\|^2_{L^2}.$$

Next, we bound

$$\|\partial_t u_n\|_{L^2} \leq \|P_n \mathbb{P}(u_n\cdot\nabla u_n)\|_{L^2} \leq \|u_n\cdot\nabla u_n\|_{L^2} \leq \|u_n\|_{L^2}\|\nabla u_n\|_{L^\infty}$$
$$\leq C_n \|u_n\|^2_{L^2} \leq K_n \|u_0\|^2_{L^2}.$$

Bernstein's ineq.

$$\|u_n(t_1) - u_n(t_2)\|_{L^2} \leq \int_{t_1}^{t_2} \|\partial_t u_n\|_{L^2}\, d\tau$$
$$\lesssim K\, |t_2 - t_1|$$

$\forall \varepsilon > 0,\ \exists \delta > 0 \mid \forall\, t_1, t_2 \in]T_n - \delta\, T_n[\,,\ \|u_n(t_1) - u_n(t_2)\|_{L^2} \leq \varepsilon$

$\Rightarrow \exists \lim_{t \to T_n^-} u_n(t) \in \mathcal{H}_n$

$\Rightarrow$ the solution cannot escape from any compact set $K \subset \mathcal{H}_n$. $\Rightarrow T_n = +\infty$.

Therefore, we have proved that $u_n \in \mathcal{C}^{\infty}([0,+\infty[\,; \mathcal{H}_n)$ and solve (PE_n). (*) State the proposition and use this as a proof.

Proof of the Thm: At this point, we perform energy est. on (PE_n), or on (E_n). Following the computations of the "a priori est." section, we arrive at a bound of the form:

$$\forall t \geq 0,\ \|u_n(t)\|_{H^s} \leq \|u_n(0)\|_{H^s} + C \int_0^t \|\nabla u_n(\tau)\|_{L^\infty} \|u_n(\tau)\|_{H^s}\, d\tau$$
$$\leq \|u_0\|_{H^s} + C \int_0^t \|\nabla u_n(\tau)\|_{L^\infty} \|u_n(\tau)\|_{H^s}\, d\tau.$$

$\forall n \geq 1$, set $T^n \doteq \sup \left\{ t > 0 \mid \int_0^t \|\nabla u_n\|_{L^\infty} d\tau \leq \log 2 \right\}$

Then, on $[0, T^n]$, one has
$$\forall t \in [0, T^n],\ \|u_n(t)\|_{H^s} \leq \tilde{C}\, \|u_0\|_{H^s}.$$

In part.
$$\log 2 = \int_0^{T^n} \|\nabla u_n(\tau)\|_{L^\infty} d\tau \leq \tilde{C}\, \|u_0\|_{H^s}\, T^n$$

by continuity

$$\Rightarrow T^n \geq \boxed{T \doteq \frac{k}{\|u_0\|_{H^s}}}.$$

□

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We have thus proved the following result. (17)

Proposition: $\exists\, T>0 \mid (u_n)_n$ is uniformly bounded in $L^\infty([0,T]; H^s(\mathbb{R}^d))$.
Also $(\nabla \pi_n)_n$ is bounded in the same space.
In addition, $T \geq \dfrac{k}{\|u_0\|_{H^s}}$, for $k>0$ universal.

Thanks to the previous proposition, we can prove $\exists$.
Indeed, by Banach-Alaoglu th.,
$\exists$ subsequence (not relabelled here)

$$(u_n)_n \subset L^\infty([0,T]; H^s), \quad \exists\, u \in L^\infty([0,T]; H^s) \mid$$

$$u_n \xrightarrow{*} u \quad \text{in that space.}$$

Rem: this is not enough to pass to the lim. in the weak form of the eqs: $\forall \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\operatorname{div} \psi = 0$,

$$-\iint u_n \, \partial_t \psi - \iint u_n \otimes u_n : \nabla R_n \psi = \int u_n(0)\, \psi(0)\, dx.$$

We need compactness, i.e. strong convergence.
For this, we use the equations:

$\|(R_n - \mathrm{Id})\psi\|_{L^2} \leq \frac{1}{n^\sigma}\|\nabla\psi\|$ $\Rightarrow$ strong conv. in $L^\infty_t L^2$

$$\partial_t u_n = -u_n \cdot \nabla u_n - \nabla \pi_n = -\operatorname{div}(u_n \otimes u_n) - \nabla \pi_n$$

$$\Rightarrow (\partial_t u_n)_n \text{ unif. bounded in } L^\infty([0,T]; H^{s-1})$$

$$\Rightarrow (u_n)_n \subset L^\infty([0,T]; H^s) \cap W^{1,\infty}([0,T]; H^{s-1}).$$

By Ascoli-Arzelà Th. in Banach spaces, we get (up to extraction)

Th. (Ascoli-Arzelà in Banach spaces):

I = compact metric space, Y reflexive Banach space.
Let $F \subset \mathcal{C}(I;Y)$ |
(a) $\forall t \in I$, the set $F(t) \stackrel{\text{def}}{=} \{ f(t) \mid f \in F \}$ is relatively cpt in Y, i.e. it has cpt closure in Y;
(b) F is equicont: $\forall \varepsilon > 0$, $\exists \delta > 0$ | $\forall t, s \in I$, $d(t,s) < \delta$, then $\|f(t) - f(s)\|_Y < \varepsilon$ $\forall f \in F$.
Then F is relatively cpt in $\mathcal{C}(I;Y)$.

Remark: alternative proof of compactness: stability estimates in L^2 / H^{s-1}.

Remark: after uniqueness is proved, our argument gives as a byproduct, convergence of the whole sequence to u. Indeed, we have shown that every weak limit has to solve the incompressible Euler (for which we have uniqueness) strong convergence in $\mathcal{C}([0,T]; L^2_{loc})$.

Thanks to this property, it is easy to pass to the lim. in the previous eq. and get that u solves

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$ in the weak sense.

As $u \in L^\infty([0,T], H^s)$, also $\nabla\pi$ satisfies this and $(u, \nabla\pi)$ is a strong sol.

By standard cont. properties of transp. eq., $u \in \mathcal{C}([0,T]; H^s)$ and so does $\nabla\pi$.

Finally, $T \geq \frac{C}{\|u_0\|_{H^s}}$. This completes the $\exists$ proof.

(3) Uniqueness. It's a consequence of the next statement.

Th: (stability):
Let $(u^1, \nabla\pi^1)$ and $(u^2, \nabla\pi^2)$ two sol. to (E) on $[0,T] \times \mathbb{R}^d$.
Assume that,

(i) $\forall J \in \{1,2\}$, $u^J \in \mathcal{C}^1([0,T]; L^2(\mathbb{R}^d))$ and $\nabla\pi^J \in \mathcal{C}([0,T]; L^2(\mathbb{R}^d))$

(ii) $\forall J$, $\nabla u^J \in L^1([0,T]; W^{1,\infty}(\mathbb{R}^d))$.

Then $\exists C > 0 \,|$

$$\forall t \in [0,T], \quad \|u^1(t) - u^2(t)\|^2_{L^2} \leq \|u^1_0 - u^2_0\|^2_{L^2} \exp\left(C\int_0^t \|\nabla u^2\|_{L^\infty} d\tau\right)$$

Rem: regularity assumpt. could be weakened

Proof: Set $\delta u \doteq u^1 - u^2$ and $\nabla \delta \pi \doteq \nabla \pi^1 - \nabla \pi^2$.
Taking the difference of the equations, we find

$$\partial_t \delta u + u^1 \cdot \nabla \delta u + \nabla \delta \pi = - \delta u \cdot \nabla u^2 .$$

Owing to the reg assumptions, we can multiply the previous eq. by δu in L^2 and find

$$\frac{1}{2} \frac{d}{dt} \|\delta u\|_{L^2}^2 \leq \|\delta u\|_{L^2}^2 \, \|\nabla u^2\|_{L^\infty} .$$

An application of the Grönwall Lemma yields the result.

$\square$

Corollary: The solutions in the th. of pag. 9 are unique.

4 Blow-up criterion.

We want to prove that, if $\boxed{T^* \text{ is the maximal } \exists \text{ time}}$ and $T^* < +\infty$, then

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} \, dt = +\infty .$$

= lifespan = lifespan as H^s sol.

Namely, this means that, if $T^* < +\infty$,

$$\limsup_{t \to (T^*)^-} \|u(t)\|_{H^s} = +\infty .$$

The blow-up criterion will be proved by contraposition, namely by proving the following continuation criterion.

Th (continuation criterion): Let $d \geq 2$, $s > 1 + \frac{d}{2}$.
Let $(u, \nabla\pi)$ be a sol to (E) on $[0,T[\times \mathbb{R}^d$, ~~for some T>0~~ such that for all $t_0 < T$, one has

(i) $u \in \mathcal{C}([0,t_0]; H^s(\mathbb{R}^d))$

(ii) $\nabla\pi \in \mathcal{C}([0,t_0]; H^s(\mathbb{R}^d))$.

Assume that

$$(A) \qquad \int_0^T \|\nabla u(t)\|_{L^\infty} \, dt < +\infty.$$

Then $(u, \nabla\pi)$ can be continued beyond T into a sol. to (E) having the same regularity.

Proof: Given $(u, \nabla\pi)$ as in the assumptions, we have seen (see the a priori est. part) that

$$\forall t \in [0,T[, \quad \begin{cases} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \exp\left(c \int_0^t \|\nabla u\|_{L^\infty} d\tau\right) \\ \|\nabla\pi\|_{H^s} \lesssim \|\nabla u\|_{L^\infty} \|u\|_{H^s}. \end{cases}$$

$\Rightarrow$ under assumption (A), we have

$$K \doteq \sup_{t \in [0,T[} \left(\|u(t)\|_{H^s} + \|\nabla\pi\|_{H^s} \right) < +\infty$$

We know that, for any initial datum of size K, the corresp. sol. $\exists$ until a time $T_K \doteq \frac{c}{K}$, for $c > 0$ a universal constant.

Therefore, let us solve the Euler eq. with in. datum $u(T - \frac{T_k}{2})$, starting from time $T - \frac{T_k}{2}$.

By the existence part, we know that $\exists!$ sol $(\tilde{u}, \nabla\tilde{\pi})$, defined on $[T - \frac{T_k}{2}, T + \frac{T_k}{2}]$, having the required regularity properties.

By uniqueness, $(\tilde{u}, \nabla\tilde{\pi}) \equiv (u, \nabla\pi)$ on $[T - \frac{T_k}{2}, T[$.

Then, we can define a new sol. on $[0, T + \frac{T_k}{2}[$. $\square$

Remark: nothing specific of the condition $p=2$.

One could solve in spaces based on L^p, $1 < p < +\infty$, provided one is able to guarantee $\nabla u \in L^1_T(L^\infty)$.

The case $p = +\infty$ also works, with some caution (recall the warning at pag. 5.).

(4) Vorticity

Given u vel. field, we set Du = Jacob. matrix $\begin{pmatrix} \nabla u^1 \\ \nabla u^2 \\ \vdots \end{pmatrix}$

and $\nabla u = {}^t(Du) = (\nabla u^1 | \nabla u^2 | \cdots)$.

$(Du)_{Jk} = \partial_k u^J$ and $(\nabla u)_{Jk} = \partial_J u^k$

The vorticity of u is the skew-symm. matrix

$$\Omega \doteq Du - \nabla u, \quad \text{i.e.} \quad \Omega_{Jk} = \partial_k u^J - \partial_J u^k.$$

Then, if u solves Euler, straightforward computations yield that Ω satisfy

$$\boxed{\partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot Du + \nabla u \cdot \Omega = 0}$$

In addition, $\Omega_{Jk} = \partial_k u^J - \partial_J u^k$

Apply ∂_k and $\sum_k$ $\Rightarrow$ $-\Delta u^J = -\sum_{k=1}^{d} \partial_k \Omega_{Jk}$

$$\Rightarrow \left[\begin{aligned} u^J &= -(-\Delta)^{-1} \sum_k \partial_k \Omega_{Jk} \\ &= +C_d \sum_k \int_{\mathbb{R}^d} \frac{(x-y)^k}{|x-y|^d} \Omega_{Jk}(y)\, dy \end{aligned} \right] \quad \text{Biot-Savart law}$$

In dimension $\boxed{d=3}$, $\Omega \longleftrightarrow \begin{cases} \omega = \mathrm{curl}(u) = \nabla \times u. \\ u = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega(y)\, dy \end{cases}$

and ω satisfies

$$\boxed{\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u}$$

In dimension $\boxed{d=2}$, instead, $\Omega \leftrightarrow \begin{cases} \omega = \mathrm{curl}(u) \\ \quad = \partial_1 u^2 - \partial_2 u^1, \\ u = -\nabla^{\perp}(-\Delta)^{-1}\omega \end{cases}$

spiega la notazione

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\,\omega(y)\,dy$$

and ω satisfies $\boxed{\partial_t \omega + u\cdot\nabla\omega = 0}$

(pure transport eq., active scalar).

Remark: In dim. $d=2$, at least formally, one has

$$\forall t \geq 0, \quad \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}, \quad \forall p\in[1,+\infty].$$

Remark: from Calderón-Zygmund theory, we know that

$$\forall d\geq 2, \quad \exists\, C = C_d > 0 \;\Big|\; \|\nabla u\|_{L^p} \leq C\frac{p^2}{p-1}\|\Omega\|_{L^p}$$

$\forall\, 1 < p < +\infty$,

$\forall u$ div-free vector field and

$\Omega = \mathrm{vort}(u)$.

More estimates proved later on.

We are now going to explore the role of the vorticity in the dynamics.

Th. (Beale-Kato-Majda contin. criterion).

$T>0$, $s > 1+\frac{d}{2}$.

s.t. $(u, \nabla\pi)$ sol. of (E) on $[\,T[\times\mathbb{R}^d \;|\; \forall t<T,\ u \in C_t H^s$

Assume that

$$\int_0^T \|\Omega(t)\|_{L^\infty}\,dt < +\infty \qquad \text{(BKM)}$$

Then $(u, \nabla\pi)$ can be continued beyond T.

LEMMA (logarithmic interpol. ineq.):

$$\|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{B^0_{\infty,\infty}} \left(\log\left(e + \frac{\|u\|_{H^s}}{\|\nabla u\|_{B^0_{\infty,\infty}}}\right)\right),$$

for an implicit mult. const. $c = c_{(s,d)} > 0$, where $\|\nabla u\|_{B^0_{\infty,\infty}} \doteq \sup_{J\geq -1} \|\Delta_J \nabla u\|_{L^\infty}$

Proof of the lemma:

$$\nabla u = \sum_{J\geq -1} \Delta_J \nabla u$$

$\boxed{\forall N \in \mathbb{N},}$

$$\Rightarrow \|\nabla u\|_{L^\infty} \leq \sum_{J\leq N} \|\Delta_J \nabla u\|_{L^\infty} + \sum_{J\geq N+1} \|\Delta_J \nabla u\|_{L^\infty}$$

$$\leq \|\nabla u\|_{B^0_{\infty,\infty}} (N+1) + \sum_{J\geq N+1} 2^{J(1+\frac{d}{2})} \|\Delta_J u\|_{L^2}\, 2^{\pm J}$$

$$\leq \|\nabla u\|_{B^0_{\infty,\infty}} (N+1) + c\,\|u\|_{H^s} \sum_{J\geq N+1} 2^{-J(s-(1+\frac{d}{2}))}\, \delta_J,$$

for a suitable sequence $(\delta_J)_J \in \ell^2$; $\|\delta_J\|_{\ell^2} \leq 1$.

For notational convenience, set $\sigma \doteq s-(1+\frac{d}{2}) > 0$.

Then $\|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{B^0_{\infty,\infty}} (N+1) + \|u\|_{H^s}\, 2^{-(N+1)\sigma}$

In order to balance the 2 terms, we choose

$$N+1 = \left[\frac{1}{\sigma} \log_2\left(1+ \frac{\|u\|_{H^s}}{\|\nabla u\|_{B^0_{\infty,\infty}}}\right)\right].$$

and $\log(e+\dots) \geq 1$, so I can avoid the $[1+\dots]$ in front of the log $\square$.

Proof of the theorem:

First of all, we refine the previous ineq.

$$\|\nabla u\|_{B^0_{\infty,\infty}} = \sup_{j\geq -1} \|\Delta_j \nabla u\|_{L^\infty} \leq \|\Delta_{-1}\nabla u\|_{L^\infty} + \sup_{j\geq 0}\|\Delta_j \nabla u\|_{L^\infty}$$

$$\leq \|u\|_{L^2} + \|\Omega\|_{L^\infty}.$$

$$\leq \|u_0\|_{L^2} + \|\Omega\|_{L^\infty}$$

Now, the function $z \mapsto z\log(1+\frac{c}{z})$ is increasing over $\mathbb{R}_+$

Then ~~we get~~, from the previous lemma we get

(*) $$\|\nabla u\|_{L^\infty} \lesssim \left(\|u_0\|_{L^2} + \|\Omega\|_{L^\infty}\right)\log\left(e + \frac{\|u\|_{H^s}}{\|u_0\|_{L^2}+\|\Omega\|_{L^\infty}}\right).$$

(i) Case $d=2$: $\|\Omega\|_{L^\infty} = \|\omega\|_{L^\infty} = \|\omega_0\|_{L^\infty}$.

$$\Rightarrow \|\nabla u\|_{L^\infty} \leq c\log\left(e + \|u\|_{H^s}\right).$$

Next, we go back to the est. (see page 11)

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s}\exp\left(c\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\right)$$

$$\Rightarrow \|\nabla u\|_{L^\infty} \leq C\left(1 + \int_0^t \|\nabla u\|_{L^\infty}\,d\tau\right).$$

$\Rightarrow$ by Grönwall's lemma, $\int_0^t \|\nabla u\|_{L^\infty} < +\infty$

and previous cont. criterion applies.

Prop: (<u>Osgood lemma</u>):

$\rho : [0,T] \to [0,a]$ measurable

$\gamma : [0,T] \to \mathbb{R}_+ \in L^1_{loc}$

$\mu : [0,a] \to \mathbb{R}_+ \in \mathcal{C}^0$, nondecreasing

Assume that $\exists\, c \geq 0 \mid$

$$\tilde{\forall} t \in [0,T], \quad \rho(t) \leq c + \int_0^t \mu(\rho(\tau))\, \gamma(\tau)\, d\tau$$

If $c > 0$, then

$$\tilde{\forall} t \in [0,T], \quad -\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_0^t \gamma(\tau)\, d\tau,$$

where $\mathcal{M}(z) \doteq \displaystyle\int_z^a \frac{1}{\mu(r)}\, dr$

If $c = 0$ and μ is Osgood, i.e. $\displaystyle\int_0^a \frac{1}{\mu(r)}\, dr = +\infty$,

then $\rho \equiv 0$.

<u>Rem:</u> $z \mapsto z \log\left(\frac{1}{z}\right)$ is Osgood.

$= -z \log z$

(ii') Case $d \geq 3$: recall that one has the following bound: (26)

$$\forall t \in [0,T[, \quad \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \exp\left(c \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right)$$

Plugging this into (*), we find

$$\|\nabla u\|_{L^\infty} \lesssim \left(1+\|\Omega(t)\|_{L^\infty}\right) \log\left(e+\frac{\|u_0\|_{H^s}}{\|u_0\|_{L^2}} \exp\left(c\int_0^t \|\nabla u\|_{L^\infty} d\tau\right)\right)$$

$$\lesssim \left(1+\|\Omega(t)\|_{L^\infty}\right)\left(1+\int_0^t \|\nabla u\|_{L^\infty} d\tau\right)$$

$\Rightarrow$ by Grönwall's lemma,

$$\forall t \in [0,T[, \quad \int_0^t \|\nabla u\|_{L^\infty} d\tau \leq c \exp\left(c\int_0^t (1+\|\Omega\|_{L^\infty}) d\tau\right).$$

□.

<u>Corollary</u>:

In $d=2$, the constructed solutions are global in time.

<u>Proof</u>: $\forall t \geq 0$, $\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$. Then, apply BKM crit. □.

<u>NB</u>: Make historical remark (BKM 1984 vs Wolibner 1933)

(3) <u>Flows with compactly supported vorticity</u>

A class of interest for us is the one of velocity fields having vorticity of compact support.

In $d=3$, a velocity field u | curl u is compactly supported, typically belongs to $L^2(\mathbb{R}^3)$.

In $d=2$, u <u>does not</u> $\in L^2(\mathbb{R}^2)$. <u>Problem</u>: weaker decay of Biot-Savart kernel.

In general, u in $\mathbb{R}^2$ has only <u>locally</u> finite energy.

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Proposition: Let u a 2-D incompressible velocity field | $\omega = \text{curl}\, u$ has compact support.
Then
$$\int_{\mathbb{R}^2} |u|^2\, dx < +\infty \Leftrightarrow \int_{\mathbb{R}^2} \omega(y)\, dy = 0.$$

Proof:
$$u(x) \underset{(|x|>>1)}{\approx} K_2(x) \int_{\mathbb{R}^2} \omega(y)\, dy + O(|x|^{-2}),$$
where
$$K_2(x) = \frac{1}{2\pi} \frac{1}{|x|^2} \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix}.$$
$\square$

(See Paragraph 3.1.3 of [Bertozzi-Majda] for more details)

Of course, $\int \omega = 0$ not verified if ω has constant sign.
However, in general one can derive a radial energy decomp.

Def: u = vector field on $\mathbb{R}^2$, smooth, $\nabla u = 0$.
Then u has a radial-en. decomposition if
$\exists$ vorticity $\bar{\omega}$, $\bar{\omega} = \bar{\omega}(|x|)$ radial smooth, |
$u = \tilde{u} + \bar{u}$, where: (i) $\tilde{u} \in L^2(\mathbb{R}^2)$, $\text{div}\, \tilde{u} = 0$
ii) $\bar{u}$ is the vel. field corresp. to vort. $\bar{\omega}$.

Remark:
$$\bar{u}(x) = \frac{1}{|x|^2} \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} \int_0^{|x|} r\, \bar{\omega}(r)\, dr.$$
(use the ... -f for getting the formula)

Observe that $\bar{u}$ is a steady, radially symmetric sol. to Euler in 2D.

Remarks: Radial en. decomp is not uni
It depends on the choice of $\bar{\omega}$.

Lemma: Let u be a smooth, incompressible el. field on $\mathbb{R}^2$.
Assume that curl $u \in L^1(\mathbb{R}$.
Then u has a radial-energy decomp.

$\boxed{\text{How to adapt the } H^s \text{ theory seen so far?}}$

Let $\omega_0 \in L^1(\mathbb{R}^2)$, $u_0 = -\nabla^\perp(-\Delta)^{-1}\omega_0$ corresp. vel. field
Let $\bar{\omega}_0$ the radial part of a radial-en. decomp. of (ω_0, u_0), and define $\bar{u}_0$ be the related vel. field given by the Biot-Savart law.
Observe that $(\bar{\omega}_0, \bar{u}_0)$ are stationary sol. to the 2D Euler
Define $\tilde{u}_0 \doteq u_0 - \bar{u}_0$ and look for a solution $\tilde{u}$ to

$$\begin{cases} \partial_t \tilde{u} + \tilde{u}\cdot\nabla\tilde{u} + \bar{u}_0\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u}_0 + \nabla\pi = 0 \\ \operatorname{div} \tilde{u} = 0 \end{cases}$$

with in. datum $\tilde{u}|_{t=0} = \tilde{u}_0$.

An energy est. yields

$$\|\tilde{u}(t)\|_{L^2}^2 \lesssim \|\tilde{u}_0\|_{L^2}^2 \exp\Big(c\int_0^t \|\nabla\bar{u}\|_{L^\infty}\Big).$$

H^s est. work similarly and, in turn, one [gets] a similar statement about local in time H^s well posedness and BKM thm.

(5) Yudovich theory and vortex patches.

Now, weak solutions theory, in space dimension $d=2$.

$$(E_\omega)\quad \begin{cases} \partial_t \omega + u\cdot\nabla\omega = 0 \\ u = -\nabla^\perp(-\Delta)^{-1}\omega = \int K(x-y)\,\omega(t,y)\,dy, \\ \qquad \text{where } K(x) = \frac{1}{2\pi}\frac{1}{|x|^2}\begin{pmatrix} -x^2 \\ x^1\end{pmatrix}. \\ \omega_{|t=0} = \omega_0. \end{cases}$$

Throughout this part, $\omega_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$

- **Th** (Yudovich - 1963):

Let $\omega_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then $\exists!$ (ω,u) $\boxed{\text{global (weak)}}$ sol. to (E_ω) |
$\omega \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$.

First, one needs static estimates.

Lemma: Let $\omega \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and $u = -\nabla^\perp(-\Delta)^{-1}\omega$ the div-free vel. field associated to it by Biot-Savart law.

Then $u \in L^\infty(\mathbb{R}^2)$ and $\nabla u \in L^p$ $\forall\ 1<p<+\infty$.

Proof of the lemma:

$$u(x) = \frac{1}{2\pi}\int \frac{1}{|y|^2}\begin{pmatrix} -y^2 \\ y^1\end{pmatrix}\omega(x-y)\,dy.$$

Split the $\int$ into $B(0,1)$ and $B^c_{(0,1)} = \mathbb{R}^2\setminus B(0,1)$.

$$\left|\int_{B(0,1)} K(y)\,\omega(x-y)\,dy\right| \lesssim \|\omega\|_{L^\infty}\int_{B(0,1)}\frac{1}{|y|}\,dy \lesssim \|\omega\|_{L^\infty}.$$

$$\left|\int_{B(0,1)^c} K(y)\,\omega(x-y)\,dy\right| \lesssim \|\omega\|_{L^1}$$ by Young ineq. for convol.

Observe that any L^p, $1\leq p<2$, would work.

Next, gradient est. Observe that $\omega \in L^p(\mathbb{R}^2)$ $\forall p \in [1,+\infty]$.

$$\nabla u = \underbrace{-\nabla \nabla^\perp (-\Delta)^{-1}}_{\text{Calderón-Zygmund op.}} \omega$$

$\Rightarrow$ $\forall\ 1<p<+\infty$, $\|\nabla u\|_{L^p} \lesssim \frac{p^2}{p-1} \|\omega\|_{L^p}$. $\square$

Rem: By Sobolev embeddings, taking $1<p<d$, u then belongs to $L^q(\mathbb{R}^2)$, with $1-d\left(\frac{1}{p}-\frac{1}{q}\right)=0$

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{d} = \frac{1}{p} - \frac{1}{2}$$

$\Rightarrow$ $q = \frac{2p}{p-2}$.

Observe that, for $p>1$, $q>2$.

This argument may be made rigorous by using cut-off function χ and writing $K_2 = \chi K_2 + (1-\chi) K_2$. See page 1 of Corso SO.

In part., $\nabla u \notin L^\infty$! In general, $u \in LL(\mathbb{R}^2)$.

Def: $f \in LL$ if $f \in L^\infty$ and $\exists\ C_{LL} > 0$ |

$$\forall x, y \in \mathbb{R}^2, \quad |f(x) - f(y)| \leq C_{LL} |x-y| \log\left(1+\frac{1}{|x-y|}\right).$$

Before proving that, we need more Littlewood-Paley theory.

DEF: $\forall \alpha \in \mathbb{R}$, define $B^\alpha_{\infty,\infty} \doteq \{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|_{B^\alpha_{\infty,\infty}} < +\infty \}$,

where one defines

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{j \geq -1} \left(2^{j\alpha} \|\Delta_j f\|_{L^\infty} \right).$$

Remark:

(i) $\forall\ 0 < \alpha < 1$, $B^\alpha_{\infty,\infty} \equiv C^{0,\alpha}$.

(ii) $\forall\ \alpha > 0$, $\alpha \notin \mathbb{N}$, $B^\alpha_{\infty,\infty} \equiv C^{k,\theta}$, where $k = [\alpha]$ and $\theta = \alpha - k$.

(iii) If $\alpha = m \in \mathbb{N}$, $C^{m-1,1} \underset{\neq}{\subset} C^m \hookrightarrow B^m_{\infty,\infty} \doteq C^m_*$.

(iv) If $\alpha < 0$, $C^\alpha \doteq B^\alpha_{\infty,\infty}$

Lemma: $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. $u \doteq -\nabla^\perp(-\Delta)^{-1}\omega$.
Then $u \in B^1_{\infty,\infty}$. In particular, $u \in LL(\mathbb{R}^2)$.

Proof of the lemma:
We have already seen that $u \in L^\infty$. Then, it is enough to prove that $\nabla u \in B^0_{\infty,\infty}$.

$$\nabla u = \sum_{J \geq -1} \Delta_J \nabla u.$$

Now, for $J = -1$, we estimate

$$\| \Delta_{-1} \nabla u \|_{L^\infty} \leq C \| u \|_{L^\infty}.$$

Next, if $J \geq 0$, we have

$$\| \Delta_J \nabla u \|_{L^\infty} = \| \nabla^\perp (-\Delta)^{-1} \nabla \Delta_J \omega \|_{L^\infty} \lesssim \| \omega \|_{L^\infty}. \quad \square$$

Remark: $B^1_{\infty,\infty} \hookrightarrow LL$, with $C_{LL} = \| \nabla f \|_{B^0_{\infty,\infty}}$.

Proof is based on a decomp. to low-high freqs:

$$f(x) - f(y) = \sum_{J \leq N} \Delta_J f(x) - \Delta_J f(y) + \sum_{J > N} \Delta_J f(\;) - \Delta_J f(\;)$$

and optimisation procedure in N.

Proof of Yudovich theorem (existence part)
Let $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and let u_0 the associated vel. field by the Biot-Savart law.

For $n \in \mathbb{N}$, take a low-frequency cut-off operator S_n of a LP decomposition. Define $u_0^n \doteq S_n u_0$.

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Observe that curl $u_0 \equiv \omega_0^n \doteq S_n \omega_0$.

One has

$\forall p \in [1,+\infty]$, $\quad \|\omega_0^n\|_{L^p} \lesssim \|\omega_0\|_{L^p}$,

$\forall q \in]2,+\infty]$, $\quad \|u_0^n\|_{L^q} \lesssim \|u_0\|_{L^q}$.

In addition, $\begin{cases} \omega_0^n \longrightarrow \omega_0 & \text{in } L^p \quad \forall\, 1 \leq p < +\infty. \\ u_0^n \longrightarrow u_0 & \text{in } L^q \quad \forall\, 2 < q < +\infty. \end{cases}$

[In fact, for u_0 one could get more (based on $B^1_{\infty,\infty}$ condition)]

Observe that, $\forall n \in \mathbb{N}$ fixed, both ω_0^n and u_0^n are smooth.

Then, we can construct a ! global in time solution u^n on $\mathbb{R}_+ \times \mathbb{R}^2$ to the Euler system, related to u_0^n (either by L^p theory, or by using the result about radial-energy decomposition).

Observe that u^n is smooth in $\mathbb{R}_+ \times \mathbb{R}^d$ and $\operatorname{curl} u^n \doteq \omega^n$ solves (E_ω) with in. datum ω_0^n.

By classical L^p est. for transport eq., we can derive

$$\forall t \geq 0, \quad \|\omega^n(t)\|_{L^p} \lesssim \|\omega_0^n\|_{L^p} \lesssim \|\omega_0\|_{L^p}.$$

It follows that $(\omega^n)_n \subset_b L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$

where $\subset_b$ means "included and bounded".

From the static estimates (of the lemmas), we get that $(u^n)_n \subset_b L^\infty(\mathbb{R}_+; B^1_{\infty,\infty})$

Then $\exists\ \omega \in L^\infty(\mathbb{R}_+; \bigcap_{p \geq 1} L^p(\mathbb{R}^2) \cap \mathcal{M}(\mathbb{R}^2))$ (Radon meas.) and $u \in L^\infty(\mathbb{R}_+; B^1_{\infty,\infty})$

up to extraction,

$\omega^n \overset{*}{\rightharpoonup} \omega$ in $L^\infty(\mathbb{R}_+; L^p)$ $\forall\, 1 < p \leq +\infty$ and

$u^n \longrightarrow u$ in $\mathcal{S}'$.

classical separable version of Banach-Alaoglu th. (see Chapter 3 of [Brezis])

Fatou property of Besov spaces (see Th. 2.72 of [BCD])

~~Def~~:

$$\mathcal{M}_b(\mathbb{R}^d) = \{\text{Radon measures on } \mathbb{R}^d\}$$
$$= (\mathcal{C}_c(\mathbb{R}^d))^*, \quad \text{where}$$

$$\mathcal{C}_c(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ cont., } \operatorname{supp} f \text{ cpt} \}.$$

In $\Omega \subset \mathbb{R}^d$ bounded,

$$\mathcal{M}_b(\overline{\Omega}) = (\mathcal{C}(\overline{\Omega}))^*.$$

We need 3 properties:

(i) passing to the limit in the weak form of the eq.;

(ii) verifying that $u = -\nabla^\perp(-\Delta)^{-1}\omega$

(iii) $\omega \in L^\infty(\mathbb{R}_+; L^1)$.

We will not prove (iii), but only the first two points. For (iii), see [Bertozzi-Majda] (Chap 8), also for properties of the flow...

Here, we will follow a different strategy.

(ii) For $n \in \mathbb{N}$, we know that

$$u_n = \int k(x-y)\,\omega_n(y)\,dy = K * \omega_n.$$

By uniqueness of (weak) limit, it is enough to prove that $u = K * \omega$ in a weak sense.

Let $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)$ and compute

$$\langle u_n, \psi \rangle = \langle K * \omega_n, \psi \rangle = \langle \omega_n, \check{K} * \psi \rangle,$$

where $\check{K}(x) \doteq K(-x)$.

Now, $\langle u_n, \psi \rangle \longrightarrow \langle u, \psi \rangle$, because $u_n \overset{S'}{\rightharpoonup} u$ (and $\overset{*}{\rightharpoonup}$ in $L^\infty_{t,x}$

Notice that $\check{K} * \psi \in L^\infty_T(L^\infty \cap L^p)$, for $p > 2$.

Then, $\langle \omega_n, \check{K} * \psi \rangle \longrightarrow \langle \omega, \check{K} * \psi \rangle = \langle K * \omega, \psi \rangle$.

Thus, $\forall \psi \in C^\infty_0$, $\langle u, \psi \rangle = \langle K * \omega, \psi \rangle$

$\Rightarrow$ $u = K * \omega$ in $\mathcal{D}'$ hence a.e. (as they are functions def. a.e. in $\mathbb{R}_+ \times \mathbb{R}^2$).

ok(ii).

(i) We need compactness properties (34)

($(\omega^n u^n)_n$ is a non-linear term).

Indeed, we have to pass to the lim. in the equality

$$-\iint \omega^n \partial_t \varphi - \iint \omega^n u^n \cdot \nabla \varphi = \int \omega_0^n(x) \varphi(0,x)\, dx,$$

where φ is an arbitrary test-function in $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$.

First of all, observe that first term on the left and the one on the right converge to the expected limit.

Given $k \in \mathbb{N}$ large (to be fixed later), we can write

$$\omega^n = S_k \omega^n + (\mathrm{Id} - S_k)\, \omega^n, \quad \text{where } S_k = \text{low-freq. cut-off op.}$$

As k is fixed, the seq. $(S_k \omega^n)_n$ is a seq. of smooth functions

$$(S_k \omega^n)_n \subset_{b(n)} L^\infty(\mathbb{R}_+; W^{1,p}(\mathbb{R}^2)) \quad \forall\, 1 < p \leq +\infty,$$

$$\subset_{b(n,k)} L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^2)).$$

Next, we write $\partial_t S_k \omega^n = -\operatorname{div} S_k(u^n \omega^n)$.

$$(\omega^n u^n)_n \subset_b L^\infty(\mathbb{R}_+; L^p \cap L^\infty) \quad \forall p > 1$$

$$\Rightarrow (\partial_t S_k \omega^n)_n \subset_b L^\infty(\mathbb{R}_+; L^\infty \cap L^p)$$

$$\Rightarrow (S_k \omega^n)_n \subset_b W^{1,\infty}(\mathbb{R}_+; L^\infty \cap L^p) \cap L^\infty(\mathbb{R}_+; W^{1,p} \cap W^{1,\infty})$$

Then, by Ascoli-Arzelà th., up to extraction,

$(S_k \omega^n)_n$ is cpt in $L^\infty_T(L^p_{loc})$ $\forall T > 0$.

i.e. $\forall T > 0$, $\forall K \subset\subset \mathbb{R}^2$, $S_k \omega^n \longrightarrow S_k \omega$ in $L^\infty_T(L^p(K))$ (up to extraction).

At this point, write

$$-\iint \omega^n u^n\cdot\nabla\varphi = -\iint S_k\omega^n \; u^n\cdot\nabla\varphi - \iint (\mathrm{Id}-S_k)\,\omega^n \; u^n\cdot\nabla\varphi$$

Observe that $(u^n\cdot\nabla\varphi)_n \subset_b L^\infty([0,T]\times K)$, where $\mathrm{supp}\,\varphi\subset[0,T]\times K$

and $(S_k\omega^n)_n$ is cpt in $L^1([0,T]\times K)$.

Then

$$-\iint S_k\omega^n u^n\cdot\nabla\varphi \xrightarrow[n\to+\infty]{} -\iint S_k\omega \; u\cdot\nabla\varphi .$$

On the other hand, $\mathrm{Id}-S_k$ is a self-adjoint operator,

so $\quad -\iint (\mathrm{Id}-S_k)\omega^n \; u^n\cdot\nabla\varphi = -\iint \omega^n\,(\mathrm{Id}-S_k)\left[u^n\cdot\nabla\varphi\right].$

Lemma:

(a) $(u^n\cdot\nabla\varphi)_n \subset_b L^\infty(\mathbb{R}_+;B^1_{\infty,\infty})$

(b) If $g\in B^1_{\infty,\infty}$, then $\forall k\in\mathbb{N}$ one has

$$\|(\mathrm{Id}-S_k)g\|_{L^\infty}\lesssim 2^{-k}\,\|\nabla g\|_{B^0_{\infty,\infty}} .$$

Proof of the lemma (only (b)).

$$\|(\mathrm{Id}-S_k)g\|_{L^\infty}\lesssim \sum_{j\geq k+1} 2^{\pm j}\,\|\Delta_j g\|_{L^\infty}\lesssim \sum_{j\geq k+1} 2^{-j}\,\|\nabla\Delta_j g\|_{L^\infty}$$

$$\lesssim 2^{-k}\,\|\nabla g\|_{B^0_{\infty,\infty}}\sum_{\ell\geq -1} 2^{-\ell} . \qquad \square .$$

Then, we deduce that

$$\left|\iint(\mathrm{Id}-S_k)\,\omega^n \; u^n\cdot\nabla\varphi\right|\lesssim \|\omega^n\|_{L^\infty_{t,x}}\,\|u^n\|_{L^\infty_t(B^1_{\infty,\infty})}\,2^{-k}\lesssim 2^{-k}.$$

Next, by LP decomposition, we know that $S_k\omega \xrightarrow[k\to+\infty]{} \omega$ in $\mathcal{S}'$ and in w-* top. of $L^\infty(\mathbb{R}_+;L^p)$ $\forall\, 1<p<+\infty$

As $(u\cdot\nabla\varphi)\in L^\infty(\mathbb{R}_+;L^\infty\cap L^1)$

we deduce that

$$-\iint S_k \omega \cdot u \cdot \nabla\varphi \xrightarrow[k\to+\infty]{} -\iint \omega\, u \cdot \nabla\varphi$$

All in all, we have proved that

$$\lim_{n\to+\infty} -\iint \omega^n \cdot u^n \cdot \nabla\varphi = -\iint \omega\, u \cdot \nabla\varphi.$$

Therefore, the couple $(\omega\ u)$ solves in $\mathcal{D}'$ the problem

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = 0 \\ u = -\nabla^\perp(-\Delta)^{-1}\omega, \end{cases}$$

hence it is a sol. of (E_ω) satisfying the required properties. ok (i) $\square(\exists)$.

(up to $\omega \in L^\infty(L^\infty)$)

Proof of Yudovich theorem (uniqueness part).

Assume that (ω^1, u^1) and $(\omega^2\ u^2)$ are two sol. to (E_ω) $\left|\ \begin{cases} \omega^J \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \\ u^J \in L^\infty(\mathbb{R}_+; B^1_{\infty,\infty}(\mathbb{R}^2)) \end{cases}\right.$

and both related to the same in. datum ω_0.

For each $J\in\{1,2\}$, u^J solves (in $\mathcal{D}'$)

$$\begin{cases} \partial_t u^J + u^J\cdot\nabla u^J + \nabla\pi^J = \\ \operatorname{div} u^J = 0. \end{cases}$$

As the in. datum is the same, this entails the same radial smooth function $\bar\omega$ in the radial en. decomp, so the same $\bar u$ in the radial en. decomp. of u^J.

As $\bar\omega$ is stationary, so is $\bar u$. It follows that

$$u^J = \bar u + \tilde u^J$$

$$\Rightarrow \delta u = u^1 - u^2 = \delta\tilde u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2)$$

Th. (see Theorem 3, Section 5.9 of [Evans]):

Let $u \in L^2([0,T]; H^1) \,|\, \partial_t u \in L^2([0,T]; H^{-1})$.

Then: (i) $u \in \mathcal{C}([0,T]; L^2)$;

(ii) the mapping $t \mapsto \|u(t)\|_{L^2}^2$ is $AC([0,T])$,

with

$$\tilde{\forall} t \in [0,T], \quad \frac{d}{dt}\|u(t)\|_{L^2}^2 = 2 \langle \partial_t u(t), u(t) \rangle .$$

Next, compute the eq. for δu:

$$\partial_t \delta u + u^1\cdot\nabla\delta u + \delta u\cdot\nabla u^2 + \nabla\delta\pi = 0$$

|| wrt y that $\nabla\delta\pi \in L^2$.

By transp, $\delta u \in L^\infty_T$

$\Rightarrow \delta u \in L^\infty H^1$

$\partial_t \delta u \in L^\infty H^{-1}$

By an en. est., we get

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|^2_{L^2} \leq \left|\int \delta u\cdot\nabla u^2\cdot\delta u\right|$$

$$\lesssim \|\nabla u^2\|_{L^p}\left(\int|\delta u|^{2p'}\right)^{\frac{1}{p'}}, \quad \text{with } \frac{1}{p}+\frac{1}{p'}=1,\ p<+\infty.$$

Now, by Calderón-Zygmund theory, we have

$$\|\nabla u^2\|_{L^p} \lesssim p\,\|\omega^2\|_{L^p} \lesssim p\,\|\omega_0\|_{L^p} \lesssim p\,\|\omega_0\|_{L^1\cap L^\infty}.$$

In addition,

$$2p' = \frac{2p}{p-1}, \text{ so } \int|\delta u|^{\frac{2p}{p-1}} \lesssim \|\delta u\|_{L^\infty}^{\frac{2}{p-1}}\,\|\delta u\|^2_{L^2}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\underbrace{\|\delta u\|^2_{L^2}}_{=:E(t)} \lesssim p\,\|\omega_0\|_{L^1\cap L^\infty}\,\|\delta u\|_{L^\infty}^{2/p}\,\|\delta u\|_{L^2}^{\frac{2(p-1)}{p}}$$

So, we have

$$(*)\quad \frac{d}{dt}E(t) \leq C_0\,p\,E(t)^{1-\frac{1}{p}}.$$

Problem: this ineq. does not have a unique sol.

Maximal sol. is the one with $=$.

$$E_M^{\frac{1}{p}-1}\frac{d}{dt}E_M = C_0\,p \qquad \frac{d}{dt}E_M^{\frac{1}{p}} = C_0 \Rightarrow E_M(t) = (C_0 t)^p.$$

$$\boxed{E_M|_{t=0} = 0}$$

$\forall$ sol. di $(*)$, one has

$$E(t) \leq \bar{E}(t) \quad \forall\, t\geq 0.$$

Now, fix $T^*>0 \mid (C_0 T^*) \leq \frac{1}{2}$. Then, $\forall t\in[0,T^*]$, one has

$$E(t) \leq \left(\frac{1}{2}\right)^p \xrightarrow[p\to+\infty]{} 0 \Rightarrow E\equiv 0 \text{ on } [0,T^*].$$

local ! implies global uniqueness on $[0,T]$, $\forall T>0$ fixed. $\square$ (!)

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- Vortex patches

DEF: A vortex patch is a vorticity function ω of the form $\omega = \mathbb{1}_{D_0}$, for some $D_0 \subset \mathbb{R}^2$, D_0 bounded.

By Yudovich theorem, if ω_0 is a vortex patch
$$\omega_0 = \mathbb{1}_{D_0},$$
then $\exists!$ global $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$ sol. to (E_ω).

In addition, the vel. field u is LL_x and it is possible to define a ! flow ψ,
$$\psi_t(x) = x + \int_0^t u(\tau, \psi_\tau(x))\, d\tau, \quad \text{i.e.} \quad \begin{cases} \partial_t \psi(t,x) = u(t, \psi_t(x)) \\ \psi_{|t=0} = Id \end{cases}$$

However, as $u \in LL_x$, ψ is not regular in x, namely
$$\forall t > 0, \qquad \psi(t,\cdot) - Id \in \mathcal{C}^{\beta(t)}, \quad \beta(t) = \exp\left(-c\|\omega_0\|_{L^1 \cap L^\infty} t\right)$$
So, only Hölder regularity, which deteriorates in time.

In particular, it is possible to see that
$$\forall t \geq 0, \quad \omega(t) = \mathbb{1}_{D_t}, \quad \text{where } D_t = \psi_t(D_0),$$
hence the vortex patch structure is preserved.

However, the regularity of the domain D_0 is not necessarily propagated.

For a long time, conjecture of finite time singularity formation at the boundary (cusp, spirals, branches...), also corroborated by numerical simulations.

Th. (Chemin - 1991, 1993):
Let $\varepsilon \in]0,1[$. Assume that $\partial D_0 \in C^{1,\varepsilon}$.
Then $\forall t \geq 0$, $\partial D_t \in C^{1,\varepsilon}$ and $\nabla u \in L^\infty_{loc}(\mathbb{R}_+; L^\infty)$.

Basic idea: measure the regularity by the regularity of tangential vector fields.

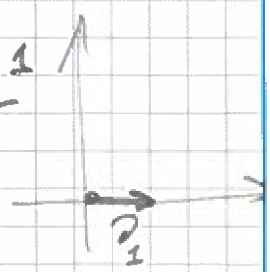
Vortex patch: $\omega = 1_D$. Then $\partial_\tau \omega \text{ "=" } \partial_X \omega = 0$,

$X \in C^\varepsilon$, $\operatorname{div} X \in C^\varepsilon$.

only $\partial_n \omega \neq 0$.

$\boxed{\text{Tangential reg is enough to reconstruct } \nabla u \in L^\infty.}$

Example (the flat case):
u vel. field such that $\begin{cases} \omega = \operatorname{curl} u \in L^\infty \cap L^1 \\ \operatorname{div} u = 0 \end{cases}$



Assume $\partial_1 \omega \in B^{\varepsilon-1}_{\infty,\infty} = C^{\varepsilon-1}$. Then $\nabla u \in L^\infty$.

$$\partial_1 u = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} (-\Delta)^{-1} \partial_1 \omega \in B^\varepsilon_{\infty,\infty} = C^\varepsilon \hookrightarrow L^\infty.$$

$$\partial_2 u^2 = \partial_2 \partial_1 (-\Delta)^{-1} \omega \quad \text{(same as for)}.$$

$$\partial_1 u^2 = \omega + \partial_2 u^1 \in L^\infty.$$

Other proofs: Bertozzi - Constantin (1993) $\rightsquigarrow$ contour dynamic approach.
Ph. Serfati (1993)

Remark: tangential/striated reg approach robust even when vortex patch structure is not preserved. $\triangleright$ Ex: $d \geq 3$, viscous fluids, non-homog fluids.

For proving Chemin's result, we are going to prove propagation of striated regularity.
We first need some preliminaries.

(a) The flow of u

$$\omega \in L^1 \cap L^\infty \Rightarrow u = -\nabla^\perp(-\Delta)^{-1}\omega \in LL$$
$$\Rightarrow \exists!\ \text{flow map } \psi_t(x) = \psi(t,x):$$
$$\psi_t(x) = x + \int_0^t u(\tau, \psi_\tau(x))\, d\tau, \text{ i.e.}$$
$$\partial_t \psi_t(x) = u(t, \psi_t(x)).$$

(b) Non degenerate families of vector fields

DEF: $\mathcal{X} = (X_\lambda)_{1\le\lambda\le m}$ finite family of vector fields.
$\mathcal{X}$ is non-degenerate if
$$I(\mathcal{X}) \doteq \inf_{x\in\mathbb{R}^2} \sup_\lambda |X_\lambda(x)| > 0.$$

(explain the meaning of the previous def).

In what follows, we will consider $X_\lambda \in C^\varepsilon \mid \operatorname{div} X_\lambda \in C^\varepsilon$ (L^∞ would be enough)

Define
$$\begin{cases} \widetilde{\|X_\lambda\|}_\varepsilon \doteq \|X_\lambda\|_{C^\varepsilon} + \|\operatorname{div} X_\lambda\|_{C^\varepsilon} \\ \widetilde{|||\mathcal{X}|||}_\varepsilon \doteq \sup_{1\le\lambda\le m} \widetilde{\|X_\lambda\|}_\varepsilon . \end{cases}$$

($|||\cdot||| = \sup_\lambda$ corresponding norms)

(c) Striated regularity

DEF: (i) Take a vector field Y over $\mathbb{R}^2$ | $Y \in \mathcal{C}^\varepsilon$, $\operatorname{div} Y \in \mathcal{C}^\varepsilon$.
Fix $\eta \in [\varepsilon, 1+\varepsilon]$ and let $f \in L^\infty(\mathbb{R}^2) \cap \mathcal{C}^{\eta-1}(\mathbb{R}^2)$, serve se η-1
Then f is said to be $\mathcal{C}^\eta$ along Y, and we write $f \in \mathcal{C}^\eta_Y$,
if

$$\partial_Y f = Y \cdot \nabla f \doteq \operatorname{div}(fY) - f \operatorname{div} Y \in \mathcal{C}^{\eta-1}$$

(ii) If $\mathcal{X} = (X_\lambda)_{1 \le \lambda \le m}$ is a non-degenerate family of vector fields as above, we define

$$\mathcal{C}^\eta_{\mathcal{X}} \doteq \bigcap_{1 \le \lambda \le m} \mathcal{C}^\eta_{X_\lambda}, \quad \text{with norm}$$

$$\|f\|_{\mathcal{C}^\eta_{\mathcal{X}}} \doteq \frac{1}{I(\mathcal{X})} \left(\|f\|_{L^\infty \cap \mathcal{C}^{\eta-1}} \widetilde{|||\mathcal{X}|||} \quad |||\operatorname{div}(f\mathcal{X})|||_{\mathcal{C}^{\eta-1}} \right).$$

(d) Propagating vector fields

Take a vector field X_0 over $\mathbb{R}^2$ and $u \in L^1([0,T]; L(\mathbb{R}^2))$.
Let ψ = flow associated to u.
For t $[0,T]$ define, $\forall x \in \mathbb{R}^2$, the vector field transported by u, i.e.

$$\boxed{X(t,x) = X_t(x) \doteq \partial_{X_0} \psi_t \left(\psi_t^{-1}(x) \right)}$$

Then X_t solves

$$\begin{cases} (\partial_t + u \cdot \nabla) X = \partial_X u \\ X_{|t=0} = X_0 . \end{cases}$$

Remark:

$$[\partial_X, \partial_t + u \cdot \nabla] = 0$$

This having been done, we are to prove Chemin's theorem, which is a straightforward conseq. of the following statement.

Th: Fix $\varepsilon>0$ and a non-deg family $\mathcal{X}_0=(X_{0,\lambda})_{1\leq\lambda\leq m}$ of vector fields in C^ε with their div.
Let $\omega_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Assume that $\omega_0\in C^\varepsilon_{\mathcal{X}_0}$.
Then $\exists!$ global in time sol. (ω,u) to (E_∞) |

(i) $\omega\in L^\infty(\mathbb{R}_+;L^1\cap L^\infty)$;

(ii) $u\in L^\infty(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^2))\cap\bigcap_{p>2}L^\infty(\mathbb{R}_+;L^p(\mathbb{R}^2))$.

Moreover, the family $\mathcal{X}(t)$ remains non-degenerate, with
$$X_\lambda,\ \mathrm{div}\,X_\lambda\in L^\infty(\mathbb{R}_+;C^\varepsilon)\quad\forall\lambda\in\{1\ldots m\},$$
and striated regularity is propagated: $\forall t\in\mathbb{R}_+$,
$$\omega(t)\in C^\varepsilon_{\mathcal{X}(t)}\quad\text{uniformly in time.}$$

Proof (only a priori est.):

We focus on striated reg. only.

First of all, it is possible to prove that
$$I(\mathcal{X}(t))\geq I(\mathcal{X}_0)\exp\left(-c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right).$$
Next, by transport eq. one has
$$\|X_\lambda(t)\|_{C^\varepsilon}\leq\left(\|X_{0,\lambda}\|_{C^\varepsilon}+\int_0^t\|\partial_{X_\lambda}u\|_{C^\varepsilon}\,d\tau\right)\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)$$

In addition, we can compute

$$(\partial_t + u\cdot\nabla)\,\mathrm{div}\, X_\lambda = 0$$

$$\Rightarrow \quad \|\mathrm{div}\, X_\lambda\|_{\mathcal{C}^\varepsilon} \lesssim \|\mathrm{div}\, X_{0,\lambda}\|_{\mathcal{C}^\varepsilon} \exp\left(c\int_0^t \|\nabla u\|_{L^\infty} d\tau\right).$$

Let us consider the vorticity.

$$(\partial_t + u\cdot\nabla)\,\omega = 0 \;\Rightarrow\; (\partial_t + u\cdot\nabla)\,\partial_{X_\lambda}\omega = 0$$

$$\Rightarrow \quad \|\partial_{X_\lambda}\omega\|_{\mathcal{C}^\varepsilon} \lesssim \|\partial_{X_\lambda}\omega_0\|_{\mathcal{C}^\varepsilon} \exp\left(c\int_0^t \|\nabla u\|_{L^\infty} d\tau\right).$$

Next, we bound $\partial_{X_\lambda} u$ in $\mathcal{C}^\varepsilon$.

$$\|\partial_{X_\lambda} u\|_{\mathcal{C}^\varepsilon} \lesssim \|X_\lambda\|_{\mathcal{C}^\varepsilon}\|\nabla u\|_{L^\infty} + \|T_{X_\lambda}\nabla u\|_{\mathcal{C}^\varepsilon}$$

by Bony's paraproduct decomposition.

$$T_{X_\lambda}\nabla u = -T_{X_\lambda}\nabla\nabla^\perp(-\Delta)^{-1}\omega$$

$$= \underbrace{-\nabla^\perp(-\Delta)^{-1}T_X\nabla\omega}_{} + [T_X, -\nabla^\perp(-\Delta)^{-1}]\nabla\omega$$

$$\|\cdot\|_{\mathcal{C}^\varepsilon} \lesssim \|T_{X_\lambda}\nabla\omega\|_{\mathcal{C}^{\varepsilon-1}}$$

$$\lesssim \|\partial_{X_\lambda}\omega\|_{\mathcal{C}^{\varepsilon-1}} + \|X_\lambda\|_{\mathcal{C}^\varepsilon}\|\nabla u\|_{L^\infty}$$

Commutator:

$$\|[T_{X_\lambda}, \nabla^\perp(-\Delta)^{-1}]\nabla\omega\|_{\mathcal{C}^\varepsilon} \lesssim \|X_\lambda\|_{\mathcal{C}^\varepsilon}\|\omega\|_{L^\infty}$$

All in all, we have proved that

$$\left[\;\|\partial_{X_\lambda} u\|_{\mathcal{C}^\varepsilon} \lesssim \|X_\lambda\|_{\mathcal{C}^\varepsilon}\|\nabla u\|_{L^\infty} + \|\partial_{X_\lambda}\omega\|_{\mathcal{C}^{\varepsilon-1}}\;\right]$$

In order to complete the a priori bounds, we need to find a bound for ∇u in L^∞.

Write

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$$\|\nabla u\|_{L^\infty} \leq \|\Delta_{-1}\nabla u\|_{L^\infty} + \|(Id-\Delta_{-1})\nabla\nabla^\perp(-\Delta)^{-1}\omega\|_{L^\infty}$$

$$\lesssim \|\omega\|_{L^1\cap L^\infty} + \|\nabla\nabla^\perp(-\Delta)^{-1}(Id-\Delta_{-1})\omega\|_{L^\infty}.$$

Lemma (without proof):

$\varepsilon\in]0,1[$, $m\geq d-1=1$. Let $Y=(Y_\lambda)_{1\leq\lambda\leq m}$ be a non-deg fam of vector fields on $\mathbb{R}^2$ |

$\forall\lambda$, Y_λ, $\operatorname{div} Y_\lambda \in C^\varepsilon$.

Then, $\forall i,j\in\{1\ldots d=2\}$, $\exists$ C^ε functions a_{ij}, $b_{ij}^{k\lambda}$, with $1\leq k\leq d$ and $1\leq\lambda\leq m$, such that, $\forall (x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2$, one has

$$\xi^i\xi^j = a_{ij}(x)\,|\xi|^2 + \sum_{k,\lambda} b_{ij}^{k\lambda}(x)\,Y_\lambda(\,)\,\xi)\,\xi_k$$

Applying this lemma, we find

$$\partial_i\partial_j(-\Delta)^{-1}(Id-\Delta_{-1})\omega = (Id-\Delta_{-1})(a_{ij}\omega) +$$

$$+\sum_{k,\lambda,l}(-\Delta)^{-1}\partial_k\partial_l(Id-\ {}_{-1}\ b^{k\lambda}Y^l_\lambda\,\omega)$$

$$\Rightarrow \|\cdot\|_{L^\infty} \lesssim \|\omega\|_{L^\infty} + \underbrace{\|\sum_{k,\lambda}\ \|_{L^\infty}}_{0}.$$

For them, logarithmic interpol. ineq

$$\|f\|_{L^\infty} \lesssim_\varepsilon \|f\|_{B^0_{\infty,\infty}}\ \log\left(1+\frac{\|f\|_{B^\varepsilon_{\infty,\infty}}}{\|f\|_{B^0_{\infty,\infty}}}\right)$$

(which implies the lemma at page 24.)

$$\Rightarrow \|\cdot\|_{L^\infty} \lesssim \|\omega\|_{L^\infty}\log\left(1+\frac{\|\partial_X\omega\|_{C^{\varepsilon-1}}}{\|\omega\|_{L^\infty}}\right).$$

The log is the key to get global (by Osgood lemma). □

<u>Remark</u>: th. about propagation of striated regularity implies Chemin's theorem of persistence of boundary regularity of vortex patches

Indeed:

$\partial D_0 \in \mathcal{C}^{1,\varepsilon} \Rightarrow$ we produce a non-deg family X_0 of $\mathcal{C}^\varepsilon$ vector fields having $\mathcal{C}^\varepsilon$ divergence

Then, use propagation of striated regularity to prove that $\nabla u \in L^\infty_{loc}(\mathbb{R}_+; L^\infty)$.

This implies that $\psi \in L^\infty_{loc}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^2))$, ψ diffeomorphism.

Then $D_t = \psi_t(D_0) \Rightarrow \partial D_t = \psi_t(D_0)$

$\Rightarrow \partial D_t \in \mathcal{C}^{1,\varepsilon}$.

The claim of the remark is thus proved.

Concluding remarks about vortex patches

(1) $\exists$ even more singular solutions, namely vortex sheets, i.e. $\omega = \mu_\Sigma$ measure supported on a C^1 compact curve.
More in general, a vortex sheet is a velocity discont. in an inviscid flow. Then, the vorticity is a measure (δ Dirac) concentrated on a surface of co-dim. 1.
Delort's theory of vortex sheets in $d=2$: 1991.

(2) $\exists$ V-states for Euler and other active scalar eq.
V-state: vortex patch which rotates with constant angular velocity around its center of mass, i.e.

$$D_t = e^{it\Omega} D_0, \quad \Omega \text{ depending on } D_0.$$

Many works by Hmidi, De la Hoz, Mateu, Soler, Verdera, J.C. García ...

(3) Vortex patches used to "desingularise" point vortices.
(Marchioro, Pulvirenti, Iftimie ...)

(4) Vortex patches with corners:
Danchin [1997, 2000], Elgindi & Jeong [2023]

(5) • Non-homogeneous fluids, compressible fluids with discont. densities
$\boxed{\text{[F. 2012], [Hmidi - Zerguine, 2014]}}$
[Danchin - F. - Paicu, 2020],
density patches for incompr. NS
[Danchin, Zhang, 2017], [Liao - Zhang 2016, 2019].

Tangential regularity approach used also for

(6) The ideal MHD, fluid which is electrically conducting

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi = \operatorname{curl} b \times b & \text{in } 3D \\ \partial_t b - \operatorname{curl}(u \times b) = 0 \\ \operatorname{div} u = \operatorname{div} b = 0. \end{cases}$$

$b^\perp$ in $2D$,

$=$)

Assumptions:
- fluid is homog.
- electrostatic approx.

Now, $\begin{cases} \operatorname{curl} b \times b = b\cdot\nabla b - \frac{1}{2}\nabla|b|^2 \\ +\operatorname{curl}(u\times b) = -u\cdot\nabla b + b\cdot\nabla u \end{cases}$

From Maxwell's eq under electrostatic approximation

$\Rightarrow$ ideal MHD eq:

$$(\text{IMHD}) \begin{cases} \partial_t u + u\cdot\nabla u + \nabla\tilde{\pi} = b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b = b\cdot\nabla u \\ \operatorname{div} u = \operatorname{div} b = 0. \end{cases} \qquad \tilde{\pi} = \pi + \frac{1}{2}|b|$$

Remark: overdet. system. But in fact cond. $\operatorname{div} b = 0$ is redundant.

Indeed, $\partial_t \operatorname{div} b = 0 \Rightarrow$ it is enough to take $b_0 \mid \operatorname{div} b_0 = 0$

We are interested in well-posedness theory.

Rem: the equations do not have the structure of system of transport equations.

Indeed, r.h.s. makes a loss of derivatives appear.

However, structure of a quasi-lin. hyperbolic system.

Indeed, from a L^2 est. we see a special cancellation appearing $\Rightarrow$ $\frac{d}{dt}\left(\frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|b\|_{L^2}^2\right) = 0.$

NB (x Fan):
do the computations

Cancellations appear also for H est, with 0 (47)

Thm $(u_0, b_0) \in H^s \times H^s$ $\quad s > 1+\frac{d}{2}$ $\quad \mathrm{div}\, u_0 = \mathrm{div}\, b_0 = 0$.

Then $\exists T>0$ and $\exists ! (u, b, \nabla\pi)$ sol. to (IMHD) $|$

$u, b, \nabla\pi \in C_T(H^s)$

In addition if T^* is the lifespan of this sol., then $\exists c>0 \,|$

$$T^* \geqslant \frac{c}{\|u_0, b_0\|_{H^s}}.$$

Finally, if $0<T<+\infty$ is such that

$$\int_0^T \left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{\infty}\right) dt < +\infty,$$

then $(u, b, \nabla\pi)$ can be continued beyond T into a sol. of the same regularity

NB (x Fanny): do the a priori estimates ?

Remark: $\left.\begin{array}{l}\|\nabla u\|_{L^\infty} \rightsquigarrow \|\omega\|_{L^\infty} \\ \|\nabla b\|_{L^\infty} \rightsquigarrow \|J\|_{L^\infty}\end{array}\right\}$ in the cont. criterion.

Previous statement not completely satisfactory

- one would like to have also theory
- we see that, for $b \to 0$, one recovers Euler eq, which are globally w hen d
  however, $T^* \underset{b_0\to 0}{\not\longrightarrow} +\infty$.

Define the <u>Elsässer variables</u>

$$\alpha = u + b \qquad \text{and} \qquad \beta = u - b.$$

If (u, b) solves (IMHD), then (α, β) solves

$$(\mathrm{Els}) \quad \begin{cases} \partial_t \alpha + \beta \cdot \nabla \alpha + \nabla \pi_1 = 0 \\ \partial_t \beta + \alpha \cdot \nabla \beta + \nabla \pi_2 = 0 \\ \operatorname{div} \alpha = \operatorname{div} \beta = 0 \end{cases}$$

, with, in addition, $\nabla \pi_1 = \nabla \pi_2$.

If one aims at solving (Els), then one has to assume $\nabla \pi_1$ and $\nabla \pi_2$ a priori distinct (if not, overdet. system).

Not every sol. of (Els) also solves (IMHD)

(see [Cobb & F. - 2023, JMPA]).

From (Els), transport structure apparent. Then, one can solve in L^p-based spaces, for $p \neq 2$ (actually, $\forall p \in]1, +\infty]$).

For instance, for $p = +\infty$, one can state the following result (only for Hölder spaces, for simplicity).

<u>Thm 1</u>: $d \geq 2$, u_0, b_0 div.-free vector fields ~~in~~ $L^2 \cap C^{1,\gamma}$, $\gamma > 0$.
Then $\exists\, T > 0$ and $\exists!$ $(u, b, \nabla \pi)$ sol. to (IMHD) (and (Els)) on $[0, T] \times \mathbb{R}^d$ and belonging to $L^2 \cap C^{1,\gamma}$.
In addition, the same cont. criterion as above holds true.

<u>Thm 2</u>: $d = 2$. Take in. datum $(u_0, b_0) \in \left(L^2 \cap C^{1,\gamma}\right)^2$, div-free.
Let $(u, b, \nabla \pi)$ the corresp. sol. and let T^* its lifespan.
Then $T^* \geq f(u_0, b_0)$
with the property that

$$f(u_0, b_0) \xrightarrow[b_0 \to 0]{} +\infty.$$

In the following, we sketch the proof of the second theorem.

Proof of Th. 2 (sketch):

Let (u_0, b_0) as in the assumption.
Let $(u, b, \nabla\pi)$ be the corresp. sol. to (IMHD) on $[0,T] \times \mathbb{R}^2$ in $L^2 \cap C^{2,\gamma}$.

Thanks to the cont. criterion, it is enough to bound the gradient of (α, β) in L^∞.

recasted in Els. variables

Now, we use

$$\|\nabla\alpha\|_{L^\infty} \lesssim \|\Delta_{-1}\alpha\|_{L^2} + \sum_{j\geq 0}' \|\Delta_j \nabla\alpha\|_{L^\infty}$$
$$\lesssim \|\alpha\|_{L^2} + \|\mathrm{curl}(\alpha)\|_{B^0_{\infty,1}}$$

NB (α Form define norm)

and the same for β.

Call $\begin{cases} X \doteq \mathrm{curl}(\alpha) \\ Y \doteq \mathrm{curl}(\beta). \end{cases}$

They solve

explain these transport est!

$$\begin{cases} \partial_t X + \beta\cdot\nabla X = \mathcal{L}(\nabla\alpha, \nabla\beta) \\ \partial_t Y + \alpha\cdot\nabla Y = \mathcal{L}(\nabla\beta, \nabla\alpha) = -\mathcal{L}(\nabla\alpha, \nabla\beta), \end{cases}$$

where $\mathcal{L}(\nabla\alpha, \nabla\beta) = \partial_1\alpha^1 (\partial_1\beta^2 + \partial_2\beta^1) + \partial_2\beta^2(\partial_1\alpha^2 + \partial_2\alpha^1)$.

$$\Rightarrow \|X\|_{B^0_{\infty,1}} + \|Y\|_{B^0_{\infty,1}} \lesssim \left(\|(X_0,Y_0)\|_{B^0_{\infty,1}} + \int_0^t \|\mathcal{L}(\nabla\alpha,\nabla\beta)\|_{B^0_{\infty,1}} d\tau\right) \cdot \left(1 + \int_0^t \|(\nabla\alpha,\nabla\beta)\|_{L^\infty} d\tau\right).$$

Thus, if we introduce the "energy" associated to the sol. as

$$E(t) \doteq \|\alpha\|_{L^2} + \|\beta\|_{L^2} + \|X\|_{B^0_{\infty,1}} + \|Y\|_{B^0_{\infty,1}}$$

we find

$$E(t) \lesssim \left(1 + \int_0^t E(\tau)\, d\tau\right)\left(E_0 + \int_0^t \|\mathcal{L}(\nabla\alpha\ \nabla\beta)\|_{B^0_{\infty,1}}\, d\tau\right)$$

Observe that $\mathcal{L}$ is skew-symmetric (in $d=2$):

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = -\mathcal{L}(\nabla\beta, \nabla\alpha)$$

$$\Rightarrow \boxed{\mathcal{L}(\nabla\alpha, \nabla\beta) = 2\,\mathcal{L}(\nabla u, \nabla b)}.$$

Another obstacle: $B^0_{\infty,1}$ is not an algebra, but $\mathcal{L}$ possesses a commutator structure which allows to bound

$$\|\mathcal{L}(\nabla u, \nabla b)\|_{B^0_{\infty,1}} \lesssim \|b\|_{B^1_{\infty,1}} \|u\|_{B^1_{\infty,1}} \lesssim \|b\|_{B^1_{\infty,1}} E(t).$$

Now, we bound $B^1_{\infty,1}$ norm of b, but loss of derivatives (L^∞ space, no use of (L^2) hyperb. theory):

$$\|b\|_{B^1_{\infty,1}} \lesssim \|b_0\|_{B^1_{\infty,1}} \exp\left(\int_0^t \|u(\tau)\|_{B^2_{\infty,1}}\, d\tau\right).$$

Then, if we denote by

$$H(t) \doteq \|\alpha\|_{L^2 \cap B^2_{\infty,1}} + \|\beta\|_{L^2 \cap B^2_{\infty,1}}$$

the high order norms we get

$$E(t) \lesssim \left(E_0 + \|b_0\|_{B^1_{\infty,1}} \int_0^t E(\tau) \exp\left(c\int_0^\tau H(\sigma)\, d\sigma\right) d\tau\right) \times$$

$$\times \left(1 + \int_0^t E(\tau)\, d\tau\right).$$

Working similarly as for obtaining the cont. crit. we see that $H(t)$ is bounded by a suitable function of $E(t)$:

$$H(t) \lesssim H_0 \exp\left(c \int_0^t e^{c\int_0^\tau E(\sigma)d\sigma} d\tau\right).$$

At this point, we can define

$$T^* \doteq \sup\left\{ t>0 \;\middle|\; \|b_0\|_{B^1_{\infty,1}} \int_0^t E(\tau) \exp\left(c\int_0^\tau H(\sigma)d\sigma\right) d\tau \leq E_0\, e^{cH_0 t} \right\}.$$

$T^* > 0$ and in $[0, T^*]$,

$$E(t) \lesssim E_0\, e^{cH_0 t}\left(1 + \int_0^t E(\tau)\,d\tau\right)$$

$$\lesssim E_0\, e^{cH_0 t}\, e^{c E_0 e^{cH_0 t}}.$$

This inequality yields bounds for $H(t)$ in $[0, T^*]$.

By continuity, in $t = T^*$, one must have

$$\|b_0\|_{B^1_{\infty,1}} \int_0^{T^*} E(t)\, e^{c\int_0^t H(\tau)\,d\tau}\, dt = E_0\, e^{cH_0 T^*}.$$

Using the previous bounds one gets

$$\|b_0\|_{B^1_{\infty,1}}\, e^{e^{e^{\cdots}}}\Big|_{t=T^*} \geq C.$$

$\square$.

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