Math ICJ Institut Camille Jordan

Laboratoire de recherche en mathématiques Lyon/Saint-Étienne

Heterogeneity and low regularity issues in PDEs arising from Fluid Mechanics

Francesco Fanelli

Habilitation à diriger des recherches

Université Claude Bernard Lyon 1

Université Claude Bernard Lyon 1
Spécialité: **Mathématiques**

Heterogeneity and low regularity issues in PDEs arising from Fluid Mechanics

Habilitation à diriger des recherches

Soutenue publiquement le 30 mai 2023 par

Francesco Fanelli

devant le Jury composé de:

M.	Thomas Alazard	CNRS, École Normale Supérieure Paris-Saclay	
M.	Jean-Yves Chemin	Sorbonne Université	Membre invité
M.me	Anne-Laure Dalibard	Sorbonne Université	Rapporteuse
M.	Raphaël Danchin	Université Paris-Est Créteil	
M.me	Isabelle Gallagher	École Normale Supérieure	Membre invitée
M.	Dragoş Iftimie	Université Claude Bernard Lyon 1	
M.	David Lannes	CNRS, Université de Bordeaux	Rapporteur
M.	Nader Masmoudi	New York University Abu Dhabi	Rapporteur
M.me	Anna Laura Mazzucato	Pennsylvania State University	
M.me	Laure Saint-Raymond	Institut des Hautes Études Scientifiques	

Contents

List of publications

Last update: 16/05/2023

Published and accepted papers[1]

(P.1) * F. Colombini, F. Fanelli: *A note on non-homogeneous hyperbolic operators with low regularity coefficients.* **Rend. Istit. Mat. Univ. Trieste**, 42 (2010), suppl., 1-25.

(P.2) * R. Danchin, F. Fanelli: *The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces.* **J. Math. Pures Appl.** (9), 96 (2011), n. 3, 253-278.

(P.3) * F. Fanelli: *Conservation of geometric structures for non-homogeneous inviscid incompressible fluids.* **Comm. Partial Differential Equations**, 37 (2012), n. 9, 1553-1595.

(P.4) * F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *Time-dependent loss of derivatives for hyperbolic operators with non-regular coefficients.* **Comm. Partial Differential Equations**, 38 (2013), n. 10, 1791-1817.

(P.5) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *A well-posedness result for strictly hyperbolic operators with Zygmund coefficients.* **J. Math. Pures Appl.** (9), 100 (2013), n. 4, 455-475.

(P.6) F. Fanelli, E. Zuazua: *Weak observability estimates for* 1*-D wave equations with rough coefficients.* **Ann. Inst. H. Poincaré Anal. Non Linéaire**, 32 (2015), n. 2, 245-277.

(P.7) F. Fanelli, X. Liao: *The well-posedness issue for an inviscid zero-Mach number system in general Besov spaces.* **Asymptot. Anal.**, 93 (2015), n. 1-2, 115-140.

(P.8) F. Fanelli, X. Liao: *Analysis of an inviscid zero-Mach number system in endpoint Besov spaces for finite-energy initial data.* **J. Differential Equations**, 259 (2015), n. 10, 5074-5114.

(P.9) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients.* **Comm. Partial Differential Equations**, 40 (2015), n. 11, 2082-2121.

(P.10) F. Fanelli: *Highly rotating viscous compressible fluids in presence of capillarity effects.* **Math. Ann.**, 366 (2016), n. 3-4, 981-1033.

(P.11) F. Fanelli: *A singular limit problem for rotating capillary fluids with variable rotation axis.* **J. Math. Fluid Mech.**, 18 (2016), n. 4, 625-658.

(P.12) F. Fanelli: *Some local questions for hyperbolic systems with non-regular time dependent coefficients.* **J. Hyperbolic Differ. Equ.**, 14 (2017), n. 2, 301-322.

[1]The symbol $*$ next to a publication denotes that the work was accomplished during the Ph.D. thesis.

(P.13) F. Fanelli: *Corrigendum to "Highly rotating viscous compressible fluids in presence of capillarity effects".* **Math. Ann.**, 370 (2018), n. 3-4, 1789-1797.

(P.14) F. De Anna, F. Fanelli: *Global well-posedness and long-time dynamics for a higher order Quasi-Geostrophic type equation.* **J. Funct. Anal.**, 274 (2018), n. 8, 2291-2355.

(P.15) J. Bartsch, A. Borzì, F. Fanelli, S. Roy: *A theoretical investigation of Brockett's ensemble optimal control problems.* **Calc. Var. Partial Differential Equations**, 58 (2019), n. 5, Paper n. 162.

(P.16) F. Fanelli, I. Gallagher: *Asymptotics of fast rotating density-dependent incompressible fluids in two space dimensions.* **Rev. Mat. Iberoam.**, 35 (2019), n. 6, 1763-1807.

(P.17) R. Danchin, F. Fanelli, M. Paicu: *A well-posedness result for viscous compressible fluids with only bounded density.* **Anal. PDE**, 13 (2020), n. 1, 275-316.

(P.18) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *On the Cauchy problem for microlocally symmetrizable hyperbolic systems with log-Lipschitz coefficients.* **Indiana Univ. Math. J.**, 69 (2020), n. 3, 785-836.

(P.19) F. Fanelli, E. Feireisl: *Statistical solutions to the barotropic Navier–Stokes system.* **J. Stat. Phys.**, 181 (2020), n. 1, 212-245.

(P.20) D. Cobb, F. Fanelli: *Rigorous derivation and well-posedness of a quasi-homogeneous ideal MHD system.* **Nonlinear Anal. Real World Appl.**, 60 (2021), Paper n. 103284.

(P.21) F. Fanelli, E. Feireisl: *Some remarks on steady solutions to the Euler system in* $\mathbb{R}^d$. **Appl. Math. Lett.**, 116 (2021), Paper n. 107031.

(P.22) D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska: *A multi-scale problem for viscous heat-conducting fluids in fast rotation.* **J. Nonlinear Sci.**, 31 (2021), n. 1, Paper n. 21.

(P.23) D. Cobb, F. Fanelli: *On the fast rotation asymptotics of a non-homogeneous incompressible MHD system.* **Nonlinearity**, 34 (2021), n. 4, 2483-2526.

(P.24) F. Fanelli, E. Feireisl, M. Hofmanová: *Ergodic theory for energetically open compressible fluid flows.* **Phys. D**, 423 (2021), Paper n. 132914.

(P.25) J. Bartsch, A. Borzì, F. Fanelli, S. Roy: *A numerical investigation of Brockett's ensemble optimal control problems.* **Numer. Math.**, 149 (2021), n. 1, 1-42.

(P.26) F. Fanelli: *Incompressible and fast rotation limit for barotropic Navier-Stokes equations at large Mach numbers.* **Phys. D**, 428 (2021), Paper n. 133049.

(P.27) E. Bocchi, F. Fanelli, C. Prange: *Anisotropy and stratification effects in the dynamics of fast rotating compressible fluids.* **Ann. Inst. H. Poincaré C Anal. Non Linéaire**, 39 (2022), n. 3, 647-704.

(P.28) F. Fanelli, R. Granero-Belinchón: *Finite time blow-up for some parabolic systems arising in turbulence theory.* **Z. Angew. Math. Phys.**, 73 (2022), n. 5, Paper n. 180.

(P.29) D. Cobb, F. Fanelli: *Elsässer formulation of the ideal MHD and improved lifespan in two space dimensions.* **J. Math. Pures Appl.** (9), 169 (2023), 189-236.

(P.30) F. Fanelli, E. Zatorska: *Low Mach number limit for degenerate Navier-Stokes equations in presence of strong stratification.* **Comm. Math. Phys.** accepted for publication (2022).

Submitted papers

(S.1) F. Fanelli: *Geophysical flows and the effects of a strong surface tension.* Submitted for the Proceedings of a conference (2016), `http://arxiv.org/abs/1605.09210`.

(S.2) F. Fanelli, R. Granero-Belinchón: *Well-posedness and singularity formation for the Kolmogorov two-equation model of turbulence in 1-D.* Submitted (2021).

(S.3) M. Bravin, F. Fanelli: *Fast rotating non-homogeneous fluids in thin domains and the Ekman pumping effect.* Submitted (2022).

(S.4) F. Fanelli, R. Granero-Belinchón, S. Scrobogna: *Well-posedness theory for non-homogeneous incompressible fluids with odd viscosity.* Submitted (2022).

(S.5) F. Colombini, D. Del Santo, F. Fanelli: *Well-posedness results for hyperbolic operators with coefficients rapidly oscillating in time.* Submitted (2023).

Proceedings of conferences and seminars, book chapters

(C.1) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *A note on complete hyperbolic operators with log-Zygmund coefficients.* In "Fourier Analysis", Trends Math., Birkhäuser/Springer, Cham (2014), 47-72.

(C.2) F. Fanelli: *A note on viscous capillary fluids in fast rotation.* Bruno Pini Math. Anal. Semin., Univ. Bologna, Bologna (2015), 86-102.

(C.3) F. Fanelli: *A few remarks on hyperbolic systems with Zygmund in time coefficients.* In "Shocks, singularities and oscillations in Nonlinear Optics and Fluid Mechanics", Springer-INdAM series, **17**, Springer, Cham (2017), 75-102.

(C.4) F. Colombini, D. Del Santo, F. Fanelli: *No loss of derivatives for hyperbolic operators with Zygmund-continuous coefficients in time.* In "Anomalies in Partial Differential Equations", Springer-INdAM Series, **43**, Springer (2021), 127-148.

(C.5) D. Cobb, F. Fanelli: *Symmetry breaking in ideal magnetohydrodynamics: the role of the velocity.* **J. Elliptic Parabol. Equ.**, 7 (2021), n. 2, 273-295.

(C.6) D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska: *On the influence of gravity in the dynamics of geophysical flows.* **Math. Eng.**, 5 (2023), n. 1, Paper n. 008.

Dissertations

(D.1) F. Fanelli: *"Para-differential calculus and its applications to hyperbolic problems"*. Master thesis dissertation, prepared under the supervision of Ferruccio Colombini. Università degli Studi di Trieste & SISSA.

(D.2) F. Fanelli: *"Mathematical analysis of models of non-homogeneous fluids and of hyperbolic equations with low regularity coefficients"*. Ph.D. thsis dissertation. Supervisors: A. Agrachev, F. Colombini, R. Danchin. SISSA & Université Paris-Est.

Introduction

This habilitation thesis is intended to be an overview of the problems I have been studying since 2012, immediately after defending my Ph.D. thesis, and the results I have obtained in those directions.

The manuscript is composed of three main parts:

(I) the study of linear hyperbolic operators with low regularity coefficients;

(II) the investigation of the well-posedness of some non-linear models describing the dynamics of non-homogeneous fluids;

(III) the asymptotic analysis of singular perturbation problems arising in geophysics.

Each part contains three chapters. The first one is an introductory chapter, which aims at giving an overview of the problems we inted to study in that specific part and the related literature; there, we also give other general information, like *e.g.* a flavour of the expected results and the techniques which are employed, the delicate points of the analysis, the differences between various approaches... In the other two chapters, we focus on some specific problems and present the results we have obtained in those directions. Those two chapters end with a list of some open problems and questions which capture our interest and which we would like to consider in the future.

The content at a glance

The general theme of this manuscript is the study of some evolutionary partial differential equations, with a particular emphasis on those describing models of fluid mechanics, presenting some sort of heterogeneity.

Heterogeneity

Heterogeneity here has to be understood in a broad sense. From a general perspective, at the mathematical level heterogeneity often translates in the presence of variable coefficients in the differential operator describing the dynamics. A natural question is then to understand how the qualitative properties of the solutions are affected by the regularity of those variable coefficients. The case of hyperbolic operators looks especially interesting, as no regularising effect can be expected in the dynamics and then the regularity of the solution is dictated only by the smoothness/oscillation properties of the coefficients. It turns out that the study of this problem presents non-trivial aspects already at the linear level. This is why, in the Part I of this work, we devote attention to the study of the well-posedness of the Cauchy problem for linear hyperbolic operators having variable, low regularity coefficients.

Having fluid mechanics models in mind, however, heterogeneity may come into play at various levels. The first situation we can think about is when variations occur in the inner properties of the fluid: this may concern the density and temperature of the fluid, or a self-induced magnetic field in the case of an electrically conducting fluid... In some models arising in the theory of turbulence, small-scale quantities (like the mean turbulent kinetic energy, for instance) are treated as independent variables, which interact with the (large-scale) mean motion: this is, roughly speaking, the essence of the eddy viscosity assumption by Boussinesq and, later, Prandtl. All this results, of course, in a strong non-linear coupling of the equations describing the various models. Thus, in Part II we turn our attention to the study of the well-posedness of several non-linear systems describing models of non-homogeneous fluids.

On the other hand, heterogeneity can be related also to anisotropy. In this respect, a prototypical example comes from the context of geophysical flows, like currents in the oceans and in the atmosphere. As a matter of fact, the dynamics of geophysical flows is characterised by the action of both the Coriolis force and gravity, which are highly anisotropic forces. In addition, anisotropy appears also at the level of the physical domain where the dynamics takes place, inasmuch as, for oceanic or atmospheric flows, the aspect ratio between horizontal dimensions and vertical dimensions (depth) is very large. In Part III we focus precisely on the study of models for geophysical fluid dynamics; in particular, the main goal of that part is the derivation of reduced models through asymptotic analysis and singular limit problems.

To conclude this discussion, we mention that heterogeneities can be induced also by boundary effects: think *e.g.* to boundary layer phenomena (like the ones which appear for geophysical flows), interactions with the exterior through non-homogeneous boundary conditions... While most of the study will be performed in simple geometries, where Fourier analysis tools are available (as a matter of fact, our approach will be mostly based on their use), in some specific situations, appearing in Parts II and III, we will also deal with non-trivial boundary effects.

Low regularity

The *leitmotif* of this manuscript is to carry out the analysis in a low regularity framework. This means that we deal either with weak solutions, or with strong solutions having somehow critical regularity, where "critical" means with respect to the constraints imposed by the problem under study (for instance the scaling, the hyperbolic nature of the system...). There are several reasons for that, apart from the mathematical sake of generality.

First of all, we move from the principle that "propagating regularity costs". This can be seen in the very simple example of a linear transport equation: consider the problem

$$\begin{cases} \partial_t f \,+\, v\cdot\nabla f \,=\, g \\ f_{|t=0} \,=\, f_0 \end{cases}$$

on $\mathbb{R}_+\times\mathbb{R}^d$, and assume for simplicity that v and g have all the required smoothness and integrability on $\mathbb{R}_+\times\mathbb{R}^d$. We also suppose that the divergence-free condition $\operatorname{div} v=0$ on v holds. Then, for any $p\in[1,+\infty]$ one has

$$\|f(t)\|_{L^p}\,\lesssim\,\|f_0\|_{L^p}\,+\,\int_0^t\|g(\tau)\|_{L^p}\,\,d\tau\,,$$

but, if we want to propagate even a small amount of regularity, an exponential growth appears in the Lipschitz norm of the transport field v: working in $H^s(\mathbb{R}^d)$ for simplicity, with $0<s<1$, we have

$$\|f(t)\|_{H^s}\,\lesssim\,\left(\|f_0\|_{H^s}\,+\,\int_0^t\|g(\tau)\|_{H^s}\,\,d\tau\right)\,\exp\left(C\int_0^t\|\nabla v(\tau)\|_{L^\infty}\right)\,, \tag{1}$$

Nonetheless, we observe that, whenever we want to estimate f in a Besov norm of index of regularity 0, namely $\|f\|_{B^0_{p,r}}$, then we get very close to the L^p framework and the exponential factor has to be replaced by a factor which is instead *linear* in the Lipschitz norm of v (see [217, 153]).

Of course, the exponential growth in (1) looks catastrophic for studying non-linear problems, at least if one aims at obtaining global well-posedness results. It is fair to point out that, in general, for non-linear equations one disposes of continuation criteria ensuring that the lifespan of the solutions does not depend on the considered level of regularity. Despite this, our first attempt will be always to solve the systems in low regularity spaces, in order to put in evidence the right quantities which need to be controlled in view of proving global results. Besides, in certain situations this will enable us to get improved lower bounds for the lifespan of the solutions (more details about this will be given in the next section).

Considering solutions at low regularity is also needed for the description of some real life phenomena, where irregularities are tied to the presence of heterogeneities in the flow. Indeed, in many situations some hydrodynamical quantities present jump discontinuities across an interface: this is the case, for instance, of multi-phase flows, or of flows in which fluids with very different densities cohexist (like in oil-water mixtures). It is clear that the theory of strong solutions cannot capture those phenomena, while the weak solutions theory is often poor of qualitative information and does not allow for a description of the dynamics of the interface. This motivates the necessity for a theory of solutions at (low) critical regularity.

Finally, the kind of problems we face often imposes constraints on the regularity of the solutions. For instance, in linear hyperbolic problems, the limited smoothness of the variable coefficients imposes constraints on the regularity of the solutions which can be propagated in the dynamics; thus, having a smooth initial datum does not allow to deduce any better information on the corresponding solution. Another example appears in singular limit problems: it turns out that, in certain situations, the singular perturbation operator is skew-symmetric with respect to the L^2 scalar product, but ceases to be skew-symmetric whenever one considers the H^s scalar product, for $s > 0$; then, working with finite energy weak solutions to the model under consideration becomes necessary and one has no other available choices. We will be more specific about all this in the text, whenever encoutering similar difficulties.

Zoom on the specific topics

We now explore a bit more in detail the contents of the thesis. We divide our discussion into three sections, corresponding to the three parts of the manuscript identified above.

Part I: linear hyperbolic problems

Part I is devoted to the analysis of linear hyperbolic operators with low regularity coefficients. We are mainly interested in well-posedness questions, but observability and controllability will be also matter of study at some point.

Here, "low regularity" means non-Lipschitz with respect to the time variable. Lipschitz continuity in time of the coefficients (L^∞ in space is enough for wave operators, while $W^{1,\infty}$ is needed for hyperbolic systems) is a necessary and sufficient condition in order to obtain well-posedness and observability properties for the corresponding hyperbolic operator. Of course, this may be improved in the case of some special structure of the operator, like in the case of the transport equation, but here we want to keep our discussion as general as possible. Whenever the Lipschitz condition fails to hold, in general one obtains weaker results, which involve a loss of derivatives in the energy and observability estimates. Then, minimal regularity conditions also in the space variable may be considered.

However, quite surprisingly, it turns out that the so-called Zygmund-type conditions escape from that classification. Zygmund regularities are second order conditions; given some function $a = a(t)$, this type of conditions writes as follows:

$$\forall\, \tau \in]0,1[\,, \qquad \big|a(t+\tau) \,+\, a(t-\tau) \,-\, 2\,a(t)\big| \,\lesssim\, \sigma(\tau)\,,$$

for some modulus of continuity $\sigma : [0,1] \longrightarrow \mathbb{R}_+$. Thus, Zygmund regularities are weaker than the corresponding first order conditions, written in terms of the difference $\big|a(t+\tau) - a(t)\big|$. Nonetheless, by using a lower order corrector in the definition of the energy (an idea which goes back to Tarama [215]), it is possible to improve the well-posedness results (with and without loss of derivatives) by passing from Lipschitz-type to Zygmund-type regularity assumptions on the coefficients.

Part I will elaborate on Zygmund-type conditions in time and their interplay with the space regularity of the coefficients which is needed to recover well-posedness and observability results. After **Chapter 1**, which is an extended introduction on the subject, we will develop this approach in Chapters 2 and 3.

More precisely, in **Chapter 2** we start by considering the case of wave operators with variable coefficients. In Section 2.1 we review the literature on the subject, by giving details on the interplay between time and space regularities of the coefficients and on the Zygmund-type conditions. In the next two sections, we expose our main results in this direction.

In Section 2.2 we show a well-posedness result with *no loss of derivatives* for coefficients which are isotropically Zygmund continuous (namely, Zygmund both in time and space variables). Notice that this is a weaker condition than the Lipschitz one both in time and space. However, the result without loss holds true only in the space $H^{1/2} \times H^{-1/2}$, as it relies on very special cancellations which occur (only at this level of regularity) in symbolic calculus, both for the principal and subprincipal symbols of some bad remainder which appears in the computations. This result was obtained in [54] in collaboration with F. Colombini, D. Del Santo and G. Métivier.

Section 2.3, on the other hand, considers the analogous problem from the angle of control and observability properties of the corresponding operator. After reviewing the specific literature on this subject, we present some results on observability estimates for 1-D wave operators with non-Lipschitz coefficients, with and without loss of derivatives; in addition, by constructing suitable counterexamples, we are able to give a full characterisation on the dependence of observability estimates on the regularity of the coefficients. These results correspond to paper [120] and were obtained in collaboration with E. Zuazua. The technique which is employed there, however, is purely one-dimensional and the extension to higher dimensions is not clear at present.

In **Chapter 3**, we turn our attention to the well-posedness issue for first-order hyperbolic systems. Again, the first section of this chapter presents an overview of the literature devoted to this specific problem. In addition, there we introduce the fundamental notion of *microlocal symmetrizability*, due to Ivrĭ and Petkov [162] and, later, Métivier [193]. The rest of the chapter contains two main sections, corresponding to results obtained in collaboration with F. Colombini, D. Del Santo and G. Métivier.

In Section 3.2, corresponding to the outcomes of paper [57], we dismiss Zygmund-type conditions for a while and consider instead the case of coefficients which are log-Lipschitz continuous both in time and space variables. The goal is to extend the study of previous works [61, 62] for the wave equations to the case of first-order systems. The first main result of this part is the attainment of an energy estimate with time-dependent loss of derivatives in $\mathbb{R}_+ \times \mathbb{R}^d$. The second important point of the analysis is the study of local in space well-posedness questions associated to the hyperbolic operator, which turns out to be delicate in such a low regularity framework.

In Section 3.3, instead, we consider the case of Zygmund and log-Zygmund in time coefficients; no dependence on the space variable is assumed. We show estimates, respectively, with no loss of derivatives and with a finite, time dependent loss. This study corresponds to the content of [56]. Dealing with such weak regularity conditions requires to introduce suitable correctors in the definition of the energy, in the spirit of Tarama's work [215]. As a consequence, the main point of the analysis is the construction of a suitable symmetrizer for the operator under study. Because of that, we have to dismiss here the assumption of microlocal symmetrizability and place ourselves, instead, in the context of hyperbolic operators with constant multiplicities. As a matter of fact, in this situation a classical procedure makes it possible to construct a microlocal symmetrizer. It is important to notice that this classical procedure has to be deeply revisited in our framework, in order to find an *ad hoc* symmetrizer which yields the sought energy estimates.

Part II: well-posedness of non-linear problems

Part II is devoted to the study of the well-posedness of some non-linear models describing the dynamics of fluid flows. The underlying themes of this part are various.

First of all, all the considered systems of equations present some sort of non-homogeneity. In most of the cases, the effects of the non-homogeneity are encoded by variations of the density function or of other inner properties of the fluid; in some specific cases, instead, they are given by the presence of non-homogeneous boundary conditions.

Next, as already mentioned above, we look for solutions having critical regularity. We work in either the class of weak solutions, or of strong solutions but under minimal regularity requirements. Sometimes, this allows us to capture special configurations (like density functions presenting jump discontinuities), some other times this enables us to deduce improved lower bounds for the lifespan of the solutions, implying an "asymptotically global" well-posedness result (in a sense to be specified below).

Another aspect of this part is the relation of the proposed study with turbulence theory. This relation appears either in the specific models we consider, or in the proposed approach (based on the notion of statistical solutions).

Chapter 4, which is an introductory chapter, aims at clarifying all those points and gives some unified overview of this part, which remains quite vast and heterogeneous with respect to the considered models and proposed studies. We decided to divide the rest of the material of this part into two main chapters, corresponding to viscous models (treated in Chapter 5) and inviscid ones (considered in Chapter 6).

Thus, in **Chapter 5** we deal with models for viscous non-homogeneous fluids. The key player of this chapter is the barotropic Navier-Stokes system. In Section 5.1, we recall the existing theories of weak solutions with finite energy for this system, namely the Lions-Feireisl theory for large data (analogous, in spirit, to the Leray solutions for the incompressible Navier-Stokes equations) and the Hoff theory for shock data (these are weak solutions corresponding to initial data having *small* energy).

In Section 5.2 we present a well-posedness result for densities being *discontinuous* across a hypersurface. This result generalises Hoff's theory, inasmuch as it holds true for general pressure laws, it gives a precise description of the evolution of the discontinuity region in any space dimension and it also guarantees uniqueness of solutions. This result was obtained in [84] in collaboration with R. Danchin and M. Paicu. The key point of the analysis is to combine an elementary approach based on maximal regularity estimates for the heat equation with tangential regularity *à la Chemin* to describe the evolution of the discontinuity region and to prove Lipschitz continuity of the velocity field.

In Section 5.3, instead, we move from the Lions-Feireisl theory to develop a theory of *statistical solutions* for the barotropic Navier-Stokes equations, in presence of general in-flow/out-flow boundary conditions. In a first time, by using a semiflow selection procedure, we introduce a new notion (with respect to the previous literature on the incompressible Navier-Stokes equations) of statistical solutions, defined as push-forward measures of the measure initially fixed on the space of data. In particular, the constructed statistical solutions extend the notion of finite energy weak solutions, enjoy a sort of semigroup property and satisfy a certain continuity property if they are supported on the set of regular data. Of course, the selection beeing not unique, statistical solutions suffer from the same lack of uniqueness as weak solutions do. This part corresponds to the study of [111], done in collaboration with E. Feireisl. After that, we focus on the special class of statistical solutions which are *stationary*, with the aim of shading some light on the validity of the so-called ergodic hypothesis in turbulence theory. We adopt a dynamical system approach on the space of global trajectories with globally bounded energy and define the dynamics as given by time shifts. Our result in this direction is only partial and shows that the validity of the ergodic hypothesis for one trajectory is strictly connected with the structure of its ω-limit set. This was done in collaboration with E. Feireisl and M. Hofmanová in [112].

In Section 5.4 we continue our investigation of questions linked to turbulence, this time by directly studying the well-posedness of a system proposed by Kolmogorov to describe a fully-developed turbulent flow, which is now known under the name of Kolmogorov two-equation model of turbulence. Actually, we focus our attention on a one-dimensional reduction of it and consider the special (degenerate) case in which the initial mean turbulent kinetic energy is supposed to vanish in some point of the domain. Under suitable assumptions, we prove that a unique local in time solution exists, but in general solutions blow up in finite time. We show two different blow-up mechanisms, one of Burgers type, the other one based on the blow-up of the curvature of the turbulent kinetic energy (actually, the second type of blow-up is shown only for a toy-model, and not for the original system of equations). This part of the manuscript corresponds to the results obtained in [114, 115] in collaboration with R. Granero-Belinchón.

Chapter 6 is devoted instead to the study of the well-posedness for some inviscid models. The first section of this chapter aims at recalling some basic (and well-known) facts about the incompressible Euler equation, both in its homogeneous and non-homogeneous versions. Our main focus here is the global in time well-posedness of the homogeneous system in the case of space dimension $d = 2$, as this results completely breaks down for any kind of non-homogeneous perturbation of the (homogeneous) incompressible Euler equations.

In Sections 6.2 and 6.3 (corresponding, respectively, to the results obtained in [118, 117] with X. Liao and in [50] with D. Cobb), we investigate the well-posedness of two systems of this kind, namely a quasi-incompressible Euler system in the former section and the ideal MHD system in the latter one. In both cases, the final goal is to show an "asymptotically global" well-posedness result, in the following sense: we obtain an explicit lower bound of the lifespan of the solution in terms of the norms of the initial datum, showing that, for small size of the non-homogeneity (the density in the quasi-incompressible system, the magnetic field in the case of the ideal MHD system), the lifespan tends to be larger and larger. While previous results of this kind required to work in endpoint critical spaces, the argument of Section 6.3 shows that this is not strictly necessary: what one really needs for obtaining such a result is a solid well-posedness theory in high regularity spaces, plus a continuation criterion in terms of minimal regularity norms (minimal here means that those norms are controlled by some $B^0_{p,r}$ regularity, for which one can use the improved transport estimates of [217, 153]).

After that, in Section 6.4 we investigate the well-posedness of a system for incompressible fluids with variable density, exhibiting viscosity effects which are *non-dissipative*. Because of this, the corresponding system of equations behaves much more like a hyperbolic system rather than a parabolic one. As a matter of fact, any kind of smoothing effect is completely absent here; on the contrary, the odd viscosity term (namely, the term encoding the non-dissipative nature of the viscous stress tensor) consumes derivatives of the solutions, making any attempt of finding well-posedness results inconclusive, at first sight. However, by resorting to suitable *good unknowns* for the system, we show that the system is indeed well-posed, locally in time, in high regularity Sobolev spaces. The role of the good unknowns is to put in evidence an underlying hyperbolic structure: more precisely, the good unknowns satisfy mere transport equations, although with complicated forcing terms. This hyperbolic structure is crucial to propagate regularity without losing derivatives; actually, persistence of regularity for the good unknowns allows to highlight a sort of smoothing effect on the density function. On the other hand, the analysis of the pressure is particularly involved; for accomplishing it, the key remark is that there is a sort of "effective viscous flux" effect hidden in the system: a particular quantity, linking the pressure and the vorticity functions, is more regular than those two quantities separately. Using this piece of information in the vorticity equation shows that the vorticity is in fact transported not by the velocity field u, but by an effective velocity, depending both on u and on the density ρ; taking advantage of this transport structure, one can close the estimates for the vorticity function. This part corresponds to work [116], written in collaboration with R. Granero-Belinchón and S. Scrobogna.

Part III: singular perturbation problems

In the last part of the manuscript, Part III, we focus on the derivation of reduced models for geophysical flows through asymptotic analysis, and more precisely through the study of singular limit problems.

There exists a number of models which are pertinent to describe the dynamics of geophysical flows, depending on the physical characters of the phenomenon one would like to study. For instance, one may want to focus on the description of atmospheric currents, in which case a compressible model is well-suited, or instead of oceanic currents, for which an incompressible model looks to be more adapted. There may be, however, other features which affect the choice of the system of equations to focus on. In addition to that, there are several physical parameters which come into play in the physical process. For geophysical flows, the most important ones are the Mach number, linked with the property of (weak) compressibility of the fluid, the Rossby number, linked to the influence of the Earth rotation, and the Froude number, which measures the importance of gravity. The values of those parameters, and more importantly their relative orders of magnitude, are not fixed *a priori* and in fact vary depending on the specific phenomenon one looks at. From the mathematical point of view, it is desirable to have robust methods which allow to compute reduced models for a large variety of choices of the orders of magnitude of those parameters.

All this results in a very rich zoology of models and studies. In the first chapter of this part, **Chapter 7**, we review those aspects in detail. An important part of that chapter is the discussion of the main differences which appear when considering the singular limit problem for compressible and incompressible (yet non-homogeneous) flows. The following two chapters are devoted, respectively, to treat the problem in those settings.

Thus, in **Chapter 8** we consider the fast rotation limit (low Rossby number) for compressible fluid flows, in the regime of low Mach number and (often) of low Froude number. In Section 8.1, which is an introductory section, we explain the asymptotic study in a model case, namely for fluids in quasi-geostrophic balance (in absence of gravity). Then, we discuss the multiscale problem, where all the three physical parameters are present at different orders of magnitudes: we review some previous results and show the main ideas used so far to deal with the presence of multiple scales in the system. The rest of the chapter is devoted to multiscale limits for various models: let us present them in detail.

In Section 8.2 we show a convergence result for a Navier-Stokes-Korteweg system, in which the Mach and Rossby numbers are penalised, together with the so-called Weber number (the physical adimensional parameter appearing in front of the Korteweg term in the equations); the latter may be penalised with different orders of magnitudes. This study corresponds to the outcome of works [106, 109]. The key to treat the multiscale problems reduces to three main keywords: dispersion (which is linked with the use of the RAGE theorem from scattering theory to get some strong convergence property), symmetrization (in the spirit of microlocal symmetrizability of Métivier) and perturbation (in order to treat terms "out of scaling", which would appear as large external forces in the wave system, and absorbe them as small perturbations of the singular perturbation operator).

In Section 8.3 we consider the multiscale limit problem for the Navier-Stokes-Fourier system with centrifugal force term. By resorting to compensated compactness arguments, we are able to perform the limit for a large (somehow sharp) choice of the parameters. Nonetheless, some restrictions still appear on the orders of magnitude which can be considered for the various parameters. The first one is linked with the presence of the centrifugal force: roughly speaking, for the full Navier-Stokes-Fourier system one needs the incompressible limit to act at a higher order than the rotation, so that the centrifugal force term remains small enough in the limit. The second one is linked with gravity and prevents us from taking a strong stratification regime in presence of the fast rotation. The results of this section correspond to the studies performed in [98, 99] in collaboration with D. Del Santo, G. Sbaiz and A. Wróblewska-Kamińska.

So far, all the results concerned the case in which the Mach number was either of higher order than the Rossby nymber, or of order equal to the Rossby number. In Section 8.4, we focus on the case in which the Mach number is of lower order than the Rossby number, a case which was treated in paper [110]. As, in this kind of problems, one needs the rotation to be compensated by a gradient term in order to get a non-trivial information in the limit, we imposed a penalisation on the bulk viscosity coefficient. The proof of the convergence uses again compensated compactness, but the anisotropy of scaling creates some complications also in this context. In particular, one is not able to avoid, in this kind of argument, the appearing of a non-linear term, which is of order $O(1)$. Thus, in order to prove its convergence, one needs some compactness properties on suitable quantities. The idea to get compactness is to resort to sharp decay estimates for solutions of the heat equation with a penalised viscosity coefficient, in order to prove that the potential part of the velocity goes to 0 fast enough; in turn, this property enables us to prove compactness of the vorticities, and this piece of information allows to compute the limit in the above mentioned non-linear term.

In **Chapter 9**, we deal with the study of the fast rotation limit for non-homogeneous incompressible flows. In Section 9.1 we set the problem in the framework of weak solutions. We also discuss the fact that, in the incompressible case, two different regimes can be considered: the quasi-homogeneous one, where the initial densities are assumed to be small perturbations of a constant state (a property which is transported by the flow), and the fully non-homogeneous one, in which the densities are small perturbations of a generic non-constant state (now, *a priori* this property is not transported anymore by the flow). We conclude this section by presenting the difficulties linked with the study of the fully non-homogeneous case in a three-dimensional framework.

After that, in Section 9.2 we specialise on the two-dimensional setting and we perform the limit in both regimes, the quasi-homogeneous and the fully non-homogeneous cases. The proof (quite easy in the former instance, rather involved in the latter one) relies again on the use of compensated compactness techniques. In particular, the study of the wave system allows us to infer fundamental quantitative smallness results for the density perturbations, which would otherwise be out of reach in the fully non-homogeneous case, and which turn out to be the key to compute the limit. We point out that our argument is able to treat the possible presence of vacuum on the initial density: in order to perform the asymptotic study, one essentially needs the same conditions required for the theory of existence of finite energy weak solutions. It is worth to point out that, in the fully non-homogeneous regime, the target system is underdetermined. Indeed, the information one disposes in order to pass to the limit is so poor that we miss an equation for the target density variation function (say) r, whereas we are able to pass to the limit in the vorticity formulation of the momentum equation; yet, the resulting equation turns out to mix both the target vorticity and the target density variation r and, besides, involves the presence of an additional Lagrangian multiplier (which is yet another unknown of the system). The study presented here corresponds to papers [113], in collaboration with I. Gallagher, and [47], in collaboration with D. Cobb.

In Section 9.3, we discuss a result obtained in [18] in collaboration with M. Bravin, where we extend the previous study to thin domains, and more precisely to thin infinite slabs. Although the general fast rotation asymptotics in 3-D remains unclear at present, considering a thin domain imposes a geometric rigidity to the problem, inasmuch as we force the flow to become purely planar in the limit. We thus recover a fundamental information on the target dynamics, which seems to be missing in the general 3-D situation. This is why we are able to perform the study of the fast rotation limit in this context. More importantly, in our study we impose Navier-slip conditions at the bottom boundary and at the top boundary of the infinite slabs: by choosing the friction parameters in a suitable way (depending on the thickness of the slabs), a boundary term remains in the limit equation. It turns out that this boundary term is a damping term, which encodes the well-known physical phenomenon known as *Ekman pumping effect*. Of course, in this way we completely miss the (quite complicated) analysis of the Ekman boundary layers, but the advantage of this approach is that it is nonetheless able to capture relevant boundary effects in the limit. In addition, this approach looks very flexible, as it does not require any anisotropy of the viscous stress tensor, which would be otherwise demanded by the study of the Ekman boundary layers. In this respect, we remark that, when imposed on the system of compressible flows, the anisotropy of the viscous stress tensor is dramatic for the theory of existence of weak solutions (see [177, 123], but also recent improvements in [25]); thus, our approach based on thin domains with Navier-slip boundary conditions allows one to rigorously derive a Navier-Stokes system with Ekman pumping term also from a compressible Navier-Stokes system.

Part I

Linear hyperbolic problems

Chapter 1

Overview of Part I

Part I of this manuscript focuses on the analysis of linear hyperbolic problems with variable *low regularity* coefficients. This kind of operators appears in several contexts: for instance, when looking at propagation of waves in highly heterogeneous media, or when linearising non-linear equations or systems around special non-constant solutions.

Here we will be concerned with wave-type operators, namely second-order scalar hyperbolic operators with variable coefficients (treated in Chapter 2), and with first-order hyperbolic systems (considered in Chapter 3). We will mainly study questions linked to the well-posedness of the related Cauchy problem. More precisely, the theme of those chapters is the following: finding minimal regularity assumptions on the coefficients of the operator such that the Cauchy problem is well-posed in the scale of Sobolev spaces $H^s(\mathbb{R}^d)$, with $d \geq 1$ and for suitable values of $s \in \mathbb{R}$, possibly admitting a finite loss of regularity (in a sense which we are going to specify) in the dynamics. Related issues will also be investigated: for instance, we will address local questions (the local Cauchy problem, finite propagation speed...) as well, and consider consequences of the low regularity of the coefficients on observability and controllability properties for the operator under study.

The expression "low regularity" typically refers to *non-Lipschitz regularity.* The Lipschitz condition represents indeed the minimal smoothness threshold for propagating the regularity of the initial datum. If the coefficients have less regularity than Lipschitz, then it is possible to see that the solution starts to lose smoothness in the time evolution, giving rise to the so-called phenomenon of the *loss of derivatives.* To the best of our knowledge, the first work in this direction was [51] by Colombini, De Giorgi and Spagnolo, who highlighted such loss of regularity in the context of wave equations with only time dependent coefficients.

As a matter of fact, it is worth emphasising that the problem relies above all on the *time regularity* of the coefficients. In other words, in general if the coefficients are smooth (even constant) with respect to the space variable, but non-Lipschitz with respect to time, then the solution loses derivatives (*i.e.* regularity) during time. This fact is in sharp contrast with what happens in the context of transport equations, where a $L^1_T(\mathrm{Lip}_x)$ condition on the velocity field is enough for proving persistence of (not too high) regularity of the solution, and the loss is produced whenever Lipschitz regularity fails with respect to the space variable (about this issue, see *e.g.* [7, 80] and, more recently, [5, 3, 69]). However, one has to remark that transport operators are a very special type of hyperbolic operators, where the coefficient matrices are not only symmetric, but even diagonal and equal to a multiple of the identity matrix. In particular, (as it is well-known) characteristics do not depend on the frequency.

The previous considerations allow us to introduce another crucial notion in all this business, namely the one of *symmetrizability.* In particular, we will always consider scalar wave operators

$$Wu := \partial_t^2 u - \sum_{j,k=1}^{d} \partial_j \big(a_{jk}(t,x)\,\partial_k u\big)$$

whose coefficient matrix $A = \big(a_{jk}\big)_{j,k}$ has real entries and is symmetric, *i.e.* satisfies the condition

$$(1.1) \qquad a_{jk}(t,x) = a_{kj}(t,x) \qquad \text{for all} \qquad 1 \leq j,k \leq d\,.$$

In the case of first-order systems, most of the time[1] we will deal with hyperbolic systems which are *microlocally symmetrizable* in the sense of Métivier [193]. This notion generalises the classical notion of Friederich's symmetrizable hyperbolic systems, inasmuch as the symmetriser is now allowed to depend also on the frequency variable.

Symmetrizability is somehow the "right" assumption to be made when studying well-posedness issues for first-order hyperbolic systems (see *e.g.* [193, 204] for additional explanations). If this assertion may appear not completely clear at first sight, the corresponding assumption (1.1) for second-order scalar operators looks absolutely natural, if one considers W as a generalisation of the classical wave operator $\partial_t^2 - \Delta$. However, one should aways retain that this is a *structural assumption* on our operator, which is at the basis of several well-posedness results (actually, very little is known whenever this assumption is dismissed).

In order to catch the flavour of the problems we are going to encounter in Part I, and the type of results we can hope for, let us report on one special result obtained in [51]. So, consider the following one-dimensional wave equation with the coefficient depending only on time and with no external force:

$$(1.2) \qquad \partial_t^2 u \,-\, a(t)\,\partial_x^2 u \,=\, 0\,.$$

We want to find *a priori* estimates for solutions of the previous equation in some time interval $[0,T]$, with $T>0$, where the coefficient a is defined. Up to extending a out of $[0,T]$ by constant values, we can assume that a belongs to $L^\infty\big(\mathbb{R}\big)$. We now suppose that a satisfies the following *log-Lipschitz* continuity assumption: there exists a constant $C_0>0$ such that

$$(1.3) \qquad \forall\,\tau>0\,, \qquad \sup_{t\in\mathbb{R}} |a(t+\tau)\,-\,a(t)| \,\leq\, C_0\,\tau\,\log\left(1\,+\,\frac{1}{\tau}\right)\,.$$

Finally, we assume that the operator in (1.2) is strictly hyperbolic, namely one has

$$(1.4) \qquad 0\,<\,a_*\,\leq\,a(t)\,\leq\,a^* \qquad \text{for all} \qquad t\in[0,T]\,.$$

We are now going to perform some formal computations, which however can be rigorously justified. First of all, we denote by $\mathcal{F}$ the Fourier transform with respect to the space variable; when convenient, we also write $\mathcal{F}u = \widehat{u}$. Then, applying $\mathcal{F}$ to both sides of (1.2), we get a second-order ODE for $\widehat{u}$, depending on the parameter ξ:

$$(1.5) \qquad \partial_t^2\widehat{u}(t,\xi)\,+\,a(t)\,|\xi|^2\,\widehat{u}(t,\xi)\,=\,0\,.$$

Next, we want to introduce the energy associated to the solution. Yet, performing energy estimates requires to deal with smooth functions. Therefore, first of all we smooth out the coefficient a, for instance by a convolution with a standard mollification kernel $\big(\rho_\varepsilon\big)_{\varepsilon>0}$. Skipping the details, we find a family of $C^\infty(\mathbb{R})$ coefficients $\big(a_\varepsilon\big)_{\varepsilon>0}$ such that

$$(1.6) \qquad \begin{aligned} &\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,, \qquad && a_*\,\leq\,a_\varepsilon(t)\,\leq\,a^*\,,\\ &\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,, && |a_\varepsilon(t)\,-\,a(t)|\,\leq\,C_0\,C\,\varepsilon\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\,,\\ &\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,, && |\partial_t a_\varepsilon(t)|\,\leq\,C_0\,C\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\,, \end{aligned}$$

[1] In Chapter 3 we will be faced to a special situation, for which the microlocal symmetrizability assumption will not be very pertinent, or at least not sufficient.

for a suitable constant $C>0$ related to the $W^{1,1}$ norm of ρ_1. Notice that, as a consequence of the fact that the coefficient a is *not* Lipschitz, the first-order derivatives of the functions a_ε are not uniformly bounded in $\varepsilon>0$.

This having been done, for any $\varepsilon>0$ and any $t\geq 0$, we can introduce the *approximate energy*

$$E_\varepsilon(t,\xi)\,=\,|\partial_t\widehat{u}(t,\xi)|^2\,+\,a_\varepsilon(t)\,|\xi|^2\,|\widehat{u}(t,\xi)|^2\,+\,|\widehat{u}(t,\xi)|^2\,.$$

It is easy to get convinced that the energy thus defined is equivalent (after integration with respect to the frequency ξ) to the classical norm $\|\partial_t u(t)\|_{L^2}^2+\|u(t)\|_{H^1}^2$. Now, a simple computation yields

$$\frac{d}{dt}E_\varepsilon(t,\xi)\,=\,2\,\mathrm{Re}\left(\partial_t\widehat{u}\cdot\big(a_\varepsilon(t)\,-\,a(t)\big)\,|\xi|^2\,\widehat{u}\right)\,+\,a_\varepsilon'(t)\,|\xi|^2\,|\widehat{u}(t,\xi)|^2\,+\,2\,\mathrm{Re}\left(\partial_t\widehat{u}\cdot\widehat{u}\right).$$

where we have used equation (1.5). Notice that the symmetry of the operator is hidden here by the fact that we are working in one space dimension, but, in the general case, it enters in a crucial way in the previous computations. Notice also that the right-hand side of the previous equality is bounded by $\alpha_\varepsilon(t,\xi)\,E_\varepsilon(t)$, where we have defined

$$\alpha_\varepsilon(t,\xi)\,:=\,1\,+\,\frac{a_\varepsilon'(t)}{a_\varepsilon(t)}\,+\,\frac{\big|a_\varepsilon(t)\,-\,a(t)\big|}{a_\varepsilon(t)}\,|\xi|\,.$$

Therefore, an application of Grönwall's lemma gives

$$E_\varepsilon(t,\xi)\,\leq\,E_\varepsilon(0,\xi)\,\exp\left(\int_0^t\alpha_\varepsilon(\tau,\xi)\,d\tau\right)\,.$$

To close the estimates, we have to find a bound for $\alpha_\varepsilon(\tau,\xi)$. By using the strict hyperbolicity condition (1.4) and the properties of the approximate coefficients $\big(a_\varepsilon\big)_\varepsilon$, we infer that, for any $t\in[0,T]$, any $\varepsilon>0$ and any $\xi\in\mathbb{R}$, one has

$$\alpha_\varepsilon(t,\xi)\,\leq\,C\,\left(1\,+\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\,+\,\varepsilon\,|\xi|\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\right)\,, \tag{1.7}$$

where the constant C depends on a_* and on C_0. At this point, the key idea of [51] was to *link* the approximation parameter $\varepsilon>0$ with the size of the dual variable $|\xi|$: this corresponds to performing different approximations of the coefficient in different regions of the phase space. Notice that the way of making this link is forced by the expression appearing on the right-hand side of (1.7) above. Thus, after setting

$$\varepsilon\,:=\,\frac{1}{|\xi|}\,, \tag{1.8}$$

for all $\xi\neq 0$ we get $\alpha_\varepsilon(t,\xi)\,\leq\,\beta\,\log\big(1\,+\,|\xi|\big)$, whence

$$E_\varepsilon(t,\xi)\,\leq\,E_\varepsilon(0,\xi)\,\big(1\,+\,|\xi|\big)^{\beta\,t}\,.$$

From the previous bound, it is easy to find the following estimate: there exists a constant $C>0$, depending only on a_*, a^*, C_0 and T, such that for any $t\in[0,T]$, one has

$$\|\partial_t u(t)\|_{H^{-\beta t}}\,+\,\|u(t)\|_{H^{1-\beta t}}\,\leq\,C\,\left(\|\partial_t u(0)\|_{L^2}\,+\,\|u(0)\|_{H^1}\right)\,.$$

It is apparent that the previous estimate exhibits a loss of derivatives of the solution with respect to the initial data; in other words, the solution loses the initial regularity in the time evolution. It is apparent from the computations that this fact is a consequence of the log-Lipschitz regularity assumption on a.

Before concluding this introductory part, two additional remarks are in order. First of all, we point out that our computations above are just a (rough) summary of what is done in the pioneering work [51]. Notice that, in that work, the log-Lipschitz regularity of the coefficient is measured in L^1, and not by taking the (stronger) sup norm as in our condition (1.3). This is a generic fact for hyperbolic problems with coefficients depending only on time: the weaker L^1-type condition sufficies in general, whereas the pointwise assumption is usually made when dealing with coefficients which depend also on the space variables (the obstruction to considering L^1 conditions in that case looks technical, but not removable at present). However, notice that the pointwise condition leads to a more precise estimate, since it entails a loss of regularity βt which is linearly increasing in time, in contrast with the loss one would have obtained under an integral condition (specifically, the loss $\delta > 0$ would have been fixed: the best one can say is that the solution immediately loses δ derivatives, no matter how close to the initial time we are).

The second remark concerns the case when the coefficients depend also on the space variable. In order to fix ideas, let us focus on the case of the wave operator W, defined above. Because of the loss of regularity of the solution, one has to require conditions on the regularity of the coefficients (this time, with respect to the space variable x) in order for the product with $\partial_k u$ to be well-defined: if $\partial_k u \in H^s$, for some $s \in \mathbb{R}$, one has to require that also $a_{jk}(t,x)\,\partial_k u$ belongs to the same space H^s. The consequence of this is twofold: on the one hand, the regularity of a_{jk} (again, with respect to x) cannot be too weak, otherwise the product would not belong to the desired space; on the other hand, requiring the coefficients to possess high smoothness in x does not help either, as it can be easily seen by paraproduct decomposition. Therefore, the second part of the game consists in finding minimal regularity assumptions for the coefficients *also* with respect to the space variable, in order for the Cauchy problem to be well-posed in suitable Sobolev classes. Notice that the origin of all those issues is the low regularity in time: when the coefficients are Lipschitz with respect to time, boundedness in x of the a_{jk}'s is enough to recover well-posedness in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ (see [160]).

This part consists of two chapters: Chapter 2 deals with the case of scalar wave-type equations, Chapter 3 with the case of first-order microlocally symmetrizable hyperbolic systems. It is important to stress the fact that those two cases are different, namely one cannot always reformulate (apart from very special cases) a wave equation as a first-order system, or *viceversa*.

The main focus of each chapter is to deal with the Cauchy problem, and highlight the previously mentioned loss of regularity of the solutions in those different contexts. However, the last part of Chapter 2 will be devoted to applications to control and observability problems.

Chapter 2

Wave equations with low-regularity coefficients

In this chapter we focus our attention on wave-type operators whose coefficients depend both on time and space variables. More precisely, we consider second order scalar hyperbolic operators W defined on $[0,T]\times\mathbb{R}^d$, for some $T>0$, having the form

$$Wu \,:=\, \partial_t^2 u \,-\, \sum_{j,k=1}^{d} \partial_j\big(a_{jk}(t,x)\,\partial_k u\big)\,, \tag{2.1}$$

under the symmetry condition

$$a_{jk}(t,x) = a_{kj}(t,x) \qquad \text{for all} \qquad 1\leq j,k\leq d\,. \tag{2.2}$$

We assume also that W is strictly hyperbolic with bounded coefficients: there exist two constants $0<\lambda\leq\Lambda$ such that

$$\forall\,(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\,, \qquad \lambda\,|\xi|^2 \,\leq\, \sum_{j,k=1}^{d} a_{jk}(t,x)\,\xi^j\,\xi^k \,\leq\, \Lambda\,|\xi|^2\,. \tag{2.3}$$

In addition, we will formulate *low regularity* assumptions on the coefficients of the operator W. As already pointed out in the introductory part, low regularity means here non-Lipschitz regularity. However, under our hypotheses, the coefficients turn out to be *continuous* over $[0,T]\times\mathbb{R}^d$, so imposing condition (2.3) pointwise makes sense.

Works presented in the chapter

(P.5) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *A well-posedness result for strictly hyperbolic operators with Zygmund coefficients.* **J. Math. Pures Appl.** (9), 100 (2013), n. 4, 455-475.

(P.6) F. Fanelli, E. Zuazua: *Weak observability estimates for 1-D wave equations with rough coefficients.* **Ann. Inst. H. Poincaré Anal. Non Linéaire**, 32 (2015), n. 2, 245-277.

(C.1) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *A note on complete hyperbolic operators with log-Zygmund coefficients.* In "Fourier Analysis", Trends Math., Birkhäuser/Springer, Cham (2014), 47-72.

(C.4) F. Colombini, D. Del Santo, F. Fanelli: *No loss of derivatives for hyperbolic operators with Zygmund-continuous coefficients in time.* In "Anomalies in Partial Differential Equations", Springer-INdAM Series, **43**, Springer (2021), 127-148.

Not mentioned, but in this context

(S.5) F. Colombini, D. Del Santo, F. Fanelli: *Well-posedness results for hyperbolic operators with coefficients rapidly oscillating in time.* Submitted (2023).

2.1 Introduction

It is well-known, since the work [160] by Hurd and Sattinger, that the Cauchy problem for second-order strictly hyperbolic operators like W in (2.1) are well-posed in the energy space $H^1 \times L^2$ as soon as the coefficients $\big(a_{j,k}\big)_{j,k}$ are Lipschitz continuous in time and (merely) bounded with respect to the space variable: namely, the solution u verifies $(u, \partial_t u) \in H^1 \times L^2$, if the initial datum, say, (u_0, u_1) belongs to that space.

In the pioneering paper [51], Colombini, De Giorgi and Spagnolo put in evidence, for the first time, the phenomenon of the *loss of derivatives* of the solution, in the case when the coefficients of the operator are of less regular than Lipschitz. Specifically, they considered coefficients depending on the time variable only, $a_{jk} = a_{jk}(t)$, and they proved the following facts:

(i) if the coefficients are log-Lipschitz, then the solution loses a finite number of derivatives in the time evolution;

(ii) if the coefficients belong to a Hölder class, then the solution loses an *infinite* number of derivatives.

In case (i), the Cauchy problem is well-posed in $H^\infty := \bigcap_{s\in\mathbb{R}} H^s$ with a finite loss of derivatives; in case (ii), instead, in order to absorbe the infinite loss, one is obliged to work in suitable Gevrey spaces.

In addition, the authors also exhibited explicit counterexamples, showing the sharpness of their results. Refinements of those counterexamples [64, 45] entail in particular that, if the coefficients are less regular than log-Lipschitz, then an infinite loss of derivatives occur, and well-posedness in Sobolev spaces in general fails. Similar pathologies related to the regularity of the coefficients arise also when studying uniqueness of solutions, see *e.g.* [60].

Throughout all this chapter, we will focus on the Cauchy problem for operator (2.1) in the framework of Sobolev spaces.

2.1.1 Space regularity of the coefficients

As already remarked, the result of [51] holds for coefficients depending only on time. The general situation where the coefficients depend also on the space variable stayed open for a long time, until work [61] by Colombini and Lerner.

The reason for this is that, when dependence of the coefficients on x is permitted, the problem becomes much more involved. First of all, owing to the loss of regularity of the solution, solving in the space $H^1 \times L^2$ as in [160] is out of reach; it seems then reasonable to solve in spaces $H^s \times H^{s-1}$, where the Sobolev exponent $s = s(t) < 1$ satisfies in addition $s'(t) < 0$ (this condition exactly means that the regularity of the solution deteriorates with time). On the other hand, a necessary condition for solving in the previous class of spaces $H^s \times H^{s-1}$ is that, for all $t \in [0, T]$, the multiplication operator by the coefficients $a_{jk}(t, \cdot)$ maps H^{s-1} (this is the space regularity of the terms $\partial_k u$ in (2.1) above) into itself. Now, if the coefficients are Lipschitz-continuous in space, then they obviously map H^σ into itself, for any $|\sigma| \leq 1$. However, as already remarked, the case $|\sigma| = 1$ is not very pertinent, due to the lack of regularity of the solution; thus, one can hope for considering coefficients which enjoy even less regularity than the Lipschitz one. This is indeed the case: as proved in [61] by resorting to Bony's paraproduct decomposition, if $a = a(x)$ is a log-Lipschitz function over $\mathbb{R}^d$, then the multiplication operator by a is a continuous self-map of H^σ, for all $|\sigma| < 1$.

Thus, in [61], the following *isotropic log-Lipschitz assumption* was formulated (up to extending the coefficients by constant value out of $[0,T]$, we can assume them to be defined on the whole $\mathbb{R}^{1+d} = \mathbb{R}_t \times \mathbb{R}^d_x$): there exists a constant $C_0 > 0$ such that, for all $1 \leq j,k \leq d$ and all $z \in \mathbb{R}^{1+d}$ with $|z| < 1$, one has

$$\sup_{y\in\mathbb{R}^{1+d}} \Big|a_{jk}(y+z) - a_{jk}(y)\Big| \,\leq\, C_0\,|z|\,\log\left(1+\frac{1}{|z|}\right)\,. \tag{2.4}$$

Under the previous assumption, one can prove the following result (see [61]; see also [62] for refinements and the study of local in space questions).

Theorem 2.1. *Let W be the wave operator defined in* (2.1)*. Assume that conditions* (2.2), (2.3) *and* (2.4) *are satisfied. Take $\theta \in]0,1[$.*

Then there exist $\beta > 0$, $T^* > 0$ and $C > 0$ such that, for all $u \in \mathcal{C}^2\big([0,T^*];H^\infty(\mathbb{R}^d)\big)$ and all $t \in [0,T^*[$, one has

$$\begin{aligned}\sup_{\tau\in[0,t]} \big\|\partial_t u(\tau)\big\|_{H^{-\theta-\beta\tau}} &+ \sup_{\tau\in[0,t]} \big\|u(\tau)\big\|_{H^{1-\theta-\beta\tau}} \\ &\leq C\left(\big\|\partial_t u(0)\big\|_{H^{-\theta}} + \big\|u(0)\big\|_{H^{1-\theta}} + \int_0^t \big\|(Wu)(\tau)\big\|_{H^{-\theta-\beta\tau}}\,d\tau\right)\,.\end{aligned}$$

In the previous inequality, β depends only on λ from (2.3) *and on C_0 from* (2.4)*; also, one has $T^* := (1-\theta)/\beta$.*

Before going on, several remarks are in order.

Remark 2.2. (i) The previous result is only local in time. Indeed, due to the loss of regularity and the product rules mentioned above, one has to make sure that $-\theta-\beta t > -1$.

(ii) The original result from [61] is stated for operators having also first-order and zeroth-order coefficients. For first-order coefficients, one usually requires Hölder regularity in space (and boundedness in time), for the zeroth-order coefficient uniform boundedness both in t and x is a sufficient condition (see also [59]). The Hölder regularity of the first-order terms imposes an additional constraint on the admissible Sobolev indices θ which can be considered (see [59, 55] for more details).

(iii) From Theorem 2.1, it is standard to deduce, for coefficients which are smooth in x, a well-posedness statement in the space $H^\infty(\mathbb{R}^d)$, with a finite loss of derivatives.

To conlude this part, let us give the basic ideas standing at the basis of the proof of Theorem 2.1. We will enter more into the details in Subsection 2.1.3, where we will use a similar approach to treat the case of coefficients satisfying Zygmund-type conditions in time.

The main difficulty here is to handle the low regularity of the coefficients both in time and space. The discussion of the introductory chapter suggests to resort to approximate energies $E_\varepsilon(t)$, defined in terms of approximate coefficients $a_{jk,\varepsilon}$, which are smooth with respect to time. Those approximate energies are, for any $t \in [0,T]$ and any $\varepsilon > 0$, equivalent to the Sobolev norms of the solutions which we want to estimate. However, owing to the dependence of the coefficients on the space variable, we can no more pass in Fourier variables in the equation and perform simple computations. At this point, the main idea of [61] is to use *Littlewood-Paley theory* and *paradifferential calculus* to handle the rough dependence of the a_{jk} with respect to x.

More precisely, the first step is to use the Littlewood-Paley characterisation of Sobolev spaces H^σ as the Besov spaces $B^\sigma_{2,2}$: every tempered distribution u in H^σ can be decomposed into

$$u \,=\, \sum_{\nu\geq-1}\Delta_\nu u\,,\qquad\text{with}\qquad \|u\|_{H^\sigma} \,\sim\, \left\|\Big(2^{\nu\sigma}\,\|\Delta_\nu u\|_{L^2}\Big)_{\nu\geq-1}\right\|_{\ell^2}\,.$$

We recall that, for any $\nu \geq 0$ the dyadic blocks $\Delta_\nu u$ are spectrally localised in dyadic annuli, where the size of the frequency ξ is proportional to 2^ν. On the other hand, $\Delta_{-1}u$ is the spectral localisation on a ball of center 0. We refer to [8] for more details.

Then, one looks at the growth of the localised energy $e_{\nu,\varepsilon}(t)$, namely the approximate energy related to each dyadic block $\Delta_\nu u$. Notice that the parameter $\varepsilon > 0$ appears because of the regularisation of the coefficients in time. A weighted sum allows to reconstruct the total eneryg $E_\varepsilon(t)$, which is equivalent to the $H^{-\theta-\beta t}$ norm that one wants to control. The strategy one follows for bounding $e_{\nu,\varepsilon}$ is standard, but requires an equation for $\Delta_\nu u$: therefore, one has to localise relation (2.1) by applying the operator Δ_ν. Of course, this procedure creates commutator terms; their study is based on Schur's lemma. Both parts of the analysis, namely the estimate of $e_{\nu,\varepsilon}$ and the bounds for the commutator terms, are based on a fine study of the properties of log-Lipschitz functions a and of their spectral localisations $\Delta_\nu a$, which relies on Bony's paraproduct decomposition and paradifferential calculus (see again [8]).

2.1.2 Second order conditions in time

Another breakthrough result in this context was achieved in [215] by Tarama, who proved that second-order regularity conditions, also called *Zygmund conditions*, on the coefficients are well adapted to the study of this kind of problems. Contrary to first-order conditions (namely, Lipschitz or Hölder type regularity assumptions), Zygmund conditions are conditions which rest on the second-order (or symmetric) difference of the coefficients.

To be more precise and fix ideas, let us consider the case, presented in Chapter 1, of space dimension $d = 1$ and coefficient $a = a(t)$. Tarama's assumptions on a then reads: either

$$\forall\,\tau \in]0,1[\,, \qquad \sup_{t\in\mathbb{R}} \big|a(t+\tau) + a(t-\tau) - 2\,a(t)\big| \leq C_0\,\tau\,, \tag{2.5}$$

in which case a is said to belong to the *Zygmund regularity* class, or

$$\forall\,\tau \in]0,1[\,, \qquad \sup_{t\in\mathbb{R}} \big|a(t+\tau) + a(t-\tau) - 2\,a(t)\big| \leq C_0\,\tau\,\log\left(1+\frac{1}{\tau}\right)\,, \tag{2.6}$$

which corresponds to the case of a having *log-Zygmund regularity*.

Observe that the previous conditions are weaker than the corresponding ones based on first-order diffence: in particular, we have

$$\mathrm{Lip} \quad\Longrightarrow\quad \mathrm{Zyg} \quad\Longrightarrow\quad \text{log-Lip} \quad\Longrightarrow\quad \text{log-Zyg}\,. \tag{2.7}$$

Indeed, the first and last implications are obvious by definition. The second one, instead, is slightly more tricky and is based on Littlewood-Paley decomposition: see for instance [8]. As a matter of fact, it turns out that one can give a characterisation of the Zygmund class

$$\mathcal{Z}(\mathbb{R}) := \Big\{a \in L^\infty(\mathbb{R})\;\Big|\quad a \quad \text{satisfies} \quad (2.5)\Big\}$$

as the Besov space $B^1_{\infty,\infty}(\mathbb{R})$: one has $\mathcal{Z} \equiv B^1_{\infty,\infty}$, and this is true in any dimension and also for logarithmic regularity, provided one passes to consider logarithmic Besov spaces. We refer to the Ph.D. dissertation **[D.2]** (see the section List of Publications at the beginning of the manuscript) and to [120, 56] for more details.

Now, if we introduce a smooth approximation $\big(a_\varepsilon\big)_{\varepsilon>0} \subset C^\infty(\mathbb{R})$ of the coefficient a, as done in Chapter 1, we get, besides (1.6), the inequalities

$$\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,,\qquad \big|a_\varepsilon(t) - a(t)\big| \leq C_0\,C\,\varepsilon\,\log^\gamma\left(1+\frac{1}{\varepsilon}\right)\,,$$

$$\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,,\qquad \big|\partial_t a_\varepsilon(t)\big| \leq C_0\,C\,\log^{1+\gamma}\left(1+\frac{1}{\varepsilon}\right)\,,$$

$$\forall\,\varepsilon>0\,,\quad \forall\,t\in\mathbb{R}\,,\qquad\qquad \left|\partial_t^2 a_\varepsilon(t)\right|\,\leq\,C_0\,C\,\frac{1}{\varepsilon}\,\log^\gamma\left(1\,+\,\frac{1}{\varepsilon}\right)\,,$$

for a suitable universal constant $C>0$, where $\gamma=0$ if a satisfies (2.5), $\gamma=1$ if a satisfies (2.6).

To fix ideas, take a Zygmund continuous, and then $\gamma=0$ in the bounds above: we observe that the estimate for the first-order derivative of a_ε loses one logarithmic factor, whereas the second order derivative start to "behave well" again, as it would do if a was Lipschitz continuous.

This remark prompted Tarama to define, in [215], a new energy, equivalent to the usual $H^1\times L^2$ (or $H^s\times H^{s-1}$) one, but in which a lower order corrector appeared. We will give more details in Section 2.2 about the precise form of the corrected energy. For the time being, we limit ourselves to point out that the role of this corrector was to erase bad terms appearing in the time derivative of the energy function, so to obtain terms in which $\partial_t u$ was multiplied only by a_ε or by $\partial_t^2 a_\varepsilon$: the treatement of those terms looks easier since one loses no logarithmic factors with respect to the classical case.

In the end, in the log-Zygmund case (2.6), one obtains an energy estimate with an increasing loss in time, as the one presented in Theorem 2.1 above, whereas for a Zygmund coefficient (2.5), one gets an energy estimate with no loss, hence well-posedness in $H^1\times L^2$. In particular, this improves the classical result for Lipschitz regularity. We observe that, in fact, as the coefficient is independent of the space variable, the energy estimates hold true in any space $H^s\times H^{s-1}$, for any $s\in\mathbb{R}$, and are global in time.

2.1.3 Extensions of Tarama's result

After Tarama's work [215], it was natural to try to generalise his previous well-posedness results to the case of coefficients depending also on the space variable.

In a first time, studies focused on the case of loss of derivatives, namely of log-Zygmund in time regularity assumptions on the coefficients of the operator W. The question was than to reduce as much as possible the space regularity assumptions, in order to keep a well-posedness result in Sobolev classes with finite loss of derivatives.

In light of the results of [61], log-Lipschitz regularity assumptions arose as a natural condition to impose. More precisely, one would like to consider coefficients which are log-Zygmund continuous in the time variable t, uniformly with respect to x, and log-Lipschitz continuous in the space variables, uniformly with respect to t. This hypothesis can be formulated in the following way: there exists a constant K_0 such that, for all $1\leq j,k\leq d$, all $\tau>0$ and all $y\in\mathbb{R}^d\setminus\{0\}$, one has

$$\sup_{(t,x)}\left|a_{jk}(t+\tau,x)+a_{jk}(t-\tau,x)-2a_{jk}(t,x)\right|\leq K_0\,\tau\,\log\left(1\,+\,\frac{1}{\tau}\right)\tag{2.8}$$

$$\sup_{(t,x)}\left|a_{jk}(t,x+y)-a_{jk}(t,x)\right|\leq K_0\,|y|\,\log\left(1\,+\,\frac{1}{|y|}\right).\tag{2.9}$$

A first result in this direction was the one of Colombini and Del Santo in [52] (see also [59]), who dealt with the one dimensional case $d=1$ and proved, under hypotheses (2.8) and (2.9) on the coefficients, a statement analogous to Theorem 2.1 above. The basic idea was to define localised energies $e_{\nu.\varepsilon}(t)$, where however, for each $\nu\geq-1$, the function $e_{\nu,\varepsilon}$ had the shape of Tarama's modified energy, in order to reproduce the special cancellation in the energy estimates which allows to deal with Zygmund-type regularity of the coefficients.

The key point was that, when $d=1$, Tarama's energy admits a straightforward generalisation to the case of coefficients $a(t,x)$ depending also on the space variable. This is no more true in the general case $d\geq2$. Indeed, when $d\geq2$ the natural generalisation of the functions used by Tarama to define the modified energy consists in using symbols (say) $\alpha=\alpha(t,x,\xi)$, which are smooth with respect to ξ, but have low regularity with respect to t and x. For the time variable, one then uses convolution as in the case of $\big(a_\varepsilon\big)_\varepsilon$ discussed above, whereas for the space variables one has to resort to paradifferential calculus (see *e.g.* [4, 8, 193]). In fact, one has to use *paradifferential calculus with parameters*, as developed by Métivier (see for instance [192, 195]), in order to recover positivity of certain operators used to define the energy. In the end, by employing the previous ingredients, the generalisation of [52] to the multi-dimensional case $d \geq 1$ was reached in [53, 55]. The result is of course the same, and the main ideas behind it too, but, as explained in the discussion above, the result is technically and conceptually more involved.

2.2 Isotropic Zygmund condition: estimates with no loss

At this point, a natural question arised as whether it is possible or not to assume Zygmund-type regularity conditions on the coefficients also with respect to the space variable, in order to recover well-posedness with no loss of derivatives of the related Cauchy problem.

As a matter of fact, the behaviour of the Zygmund class in somehow "ambiguous" in this context, in the following sense. On the one hand, noticing the embeddings (2.7) and keeping the counterexamples of [45] in mind, one expect that having coefficients a_{jk} which are Zygmund continuous in time would entail a finite loss of regularity (arbitrarily small, but non-zero) in the dynamics. On the other hand, we notice that, given any Zygmund function $a = a(x)$, for any $s \in]0,1[$ one has that

$$\forall\, u \in H^s(\mathbb{R}^d)\,, \qquad\qquad \partial_j\big(a\,\partial_k u\big) \,-\, \partial_j T_a \partial_k u \,\in\, H^{s-1}\,,$$

where T_a denotes the paradifferential operator associated to the symbol a (as $a = a(x)$, the paradifferential operator actually coincides with the classical paraproduct operator). Combining this fact with Tarama's result [215] prompts us to think that the derivative loss may not appear in presence of coefficients which are isotropically Zygmund continuous, *i.e.* which are Zygmund continuous with respect to (t,x).

In [54], we considered exactly this situation: we assumed the coefficientsa_{jk} to be isotropically Zygmund continuous, uniformly over $[0,T] \times \mathbb{R}^d$. This assumptions writes as follows: there exists a constant K_0 such that, fixed any $1 \,\leq\, i,j \,\leq\, d$, for all $\tau \geq 0$ and all $y \in \mathbb{R}^d$, one has

$$\sup_{(t,x)}\Big|a_{ij}(t+\tau,x+y) \,+\, a_{ij}(t-\tau,x-y) \,-\, 2\,a_{ij}(t,x)\Big| \;\leq\; K_0\left(\tau \,+\, |y|\right). \tag{2.10}$$

We were able to prove the following statement.

Theorem 2.3. *Let W be the wave operator defined by* (2.1)*. Assume that assumptions* (2.2) *of symmetry and* (2.3) *of strict hyperbolicity are in force. Moreover, suppose the coefficients a_{ij} to fulfill condition* (2.10)*.*

Then there exist constants $C > 0$ and $\gamma > 0$ such that the inequality

$$\sup_{0\leq t\leq T}\left(\|u(t)\|_{H^{1/2}} \,+\, \|\partial_t u(t)\|_{H^{-1/2}}\right) \,\leq$$
$$\leq\, C\,e^{\gamma T}\left(\|u(0)\|_{H^{1/2}} \,+\, \|\partial_t u(0)\|_{H^{-1/2}} \,+\, \int_0^T e^{-\gamma t}\,\|(Wu)(t)\|_{H^{-1/2}}\,dt\right)$$

holds true for all $u \in \mathcal{C}^2\big([0,T];H^\infty(\mathbb{R}^d)\big)$.

The proof of the previous result relies, as for [53, 55], on the use of Tarama's energy together with paradifferential calculus with parameters. However, the key point of the argument is a crucial cancellation which arises at the level of symbolic calculus, when replacing operator W by its paralinearisation: such a cancellation occurs not only at the level of the principal symbol, but also at the level of the sub-principal symbol of certain operators involved in the computations.

The effect of those cancellations is crucial: the sub-principal symbol in question would bring a contribution of order log, producing in this way a loss of derivatives in the energy estimates.

It is important to notice that the above mentioned fundamental cancellation only occurs at the $H^{1/2}$ level of the energy estimates. This is the reason why the result is stated in that specific functional class. It is not clear at all whether of not the previous well-posedness result without loss should hold true also in $H^s \times H^{s-1}$, for other values of $s \in]0,1[$.

2.3 Application to observability and control in 1-D

In this section we briefly mention, without entering into the details, that the previous well-posedness results and techniques were applied also to the problem of the control of wave operators with rough coefficients. The physical motivation for such a study is clear: trying to control waves which propagate in highly heterogeneous media, whose local properties are highly irregular.

As the property of control is intimately related (by Hilbert Uniqueness Method [175, 221]) to the one of observability, in order to simplify the presentation in what follows we will always speak about observability properties of the operator W.

It is well-known that, in general, the (both internal and boundary) observability properties of W are satisfied if and only if the observability region (the subset $\Omega_{\rm ob} \subset \overline{\Omega}$ where one wants to observe the waves, with Ω denoting the full spacial domain where the dynamics takes place) satisfy the so-called *Geometric Control Condition* (GCC in brief), see [9, 28]. The proof of such a fundamental result is based on tracing the rays of geometric optics, so it requires the bicharacteristic flow to be well-defined. In particular, this argument requires the coefficients of W to be smooth. On the other hand, dependence of the observability properties on the regularity of the coefficients have already been observed in *e.g.* [6], in the context of homogeneisation, and [34], where it was proven that observability and control may fail for Hölder continuous coefficients.

The one-dimensional case plays a special role in all this matter, as in 1-D waves can only travel sidewise, so the GCC is always satisfied. Without loss of generality, we can restrict our attention to the wave operator

$$W_1 u \,:=\, \omega(x)\,\partial_t^2 u \,-\, \partial_x^2 u \qquad\qquad \text{in} \qquad \Omega \,=\, [0,1]\,,$$

where ω satisfies $0 < \omega_* \leq \omega(x) \leq \omega^*$ for all $x \in \Omega$. For simplicity, let us focus on homogeneous boundary conditions

$$u(t,0) \,=\, u(t,1) \,=\, 0 \qquad\qquad \text{for all} \quad t \in [0,T]\,, \tag{2.11}$$

where the observability time $T > 0$ has to be chosen large enough, namely

$$T \,>\, 2\,T_\omega\,, \qquad\qquad \text{with} \qquad T_\omega \,:=\, \int_\Omega \sqrt{\omega(x)}\,\mathrm{d}x\,.$$

In [138], Fernández-Cara and Zuazua proved that the condition

$$\omega \,\in\, BV(\Omega)\,, \tag{2.12}$$

where $BV(\Omega)$ is the space of functions of bounded variation on $\Omega = [0,1]$, is sufficient for guaranteeing the observability inequality (see the inequality stated in Theorem 2.4 below for a precise form) to hold. The counterexamples of [34] for Hölder coefficients seemed to show that condition (2.12) is in fact sharp.

However, in [120] we were able to improve the result of [138] in two aspects. First of all, for observability estimtes to hold, it is in fact sufficient that the coefficient ω satisfies the integral Zygmund condition

$$\forall\, h \in \left]0,\frac{1}{2}\right[\,, \qquad\qquad \int_h^{1-h} \Big(\omega(x+h) \,+\, \omega(x-h) \,-\, 2\,\omega(x)\Big)\,\mathrm{d}x \,\leq\, K\,h\,. \tag{2.13}$$

This result can be roughly stated as follows.

Theorem 2.4. *Let the coefficient ω satisfy condition* (2.13) *and let $T > 2T_\omega$. Let us set $|\omega|_{\mathcal{Z}}$ to be the minimal constant $K > 0$ for which* (2.13) *is satisfied.*

Then, there exists a constanct $C > 0$, only depending on ω_*, ω^* and $|\omega|_{\mathcal{Z}}$, such that the boundary observability inequality

$$\|u(0)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} \leq C \int_0^T |\partial_x u(t,0)|^2 \, \mathrm{d}t$$

holds true for any u satisfying $W_1 u = 0$ in Ω, together with the boundary conditions (2.11).

Remark 2.5. Notice that Theorem 2.4 improves the result of [138], by extending the functions ω for which observability holds. Roughly speaking, this extension corresponds to filling the gap between $B^1_{1,1}$ to $B^1_{1,\infty}$.

Secondly, we considered less regular coefficients, namely functions ω satisfying an integral log-Lipschitz or log-Zygmund assumption. This latter condition means that, in relation (2.13) above, an additional logarithmic factor appears in the right-hand side. For such functions, we were able to prove observability estimates with a *finite loss of derivatives*. The statement can be roughly formulated in the following form. Notice the presence of the semi-norm $|\omega|_{\mathcal{LZ}}$, which can be defined analogously to $|\omega|_{\mathcal{Z}}$ above.

Theorem 2.6. *Let the coefficient ω satisfy an integral log-Zygmund condition and let $T > 2T_\omega$.*

Then, there exist a constanct $C > 0$ and an index $m \in \mathbb{N}$, only depending on ω_*, ω^* and $|\omega|_{\mathcal{LZ}}$, such that the boundary observability inequality

$$\|u(0)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} \leq C \int_0^T |\partial_t^m \partial_x u(t,0)|^2 \, \mathrm{d}t$$

holds true for any u satisfying $W_1 u = 0$ in Ω, together with the boundary conditions (2.11) *and some additional technical requirement on the initial data $u(0)$ and $\partial_t u(0)$.*

Finally, let us mention that, following some basic ideas of [61, 45], we were able to improve the counterexample exhibited in [34] and give a full characterisation on how observability and controllability properties depend on the modulus of continuity of the coefficients. In particular, we proved the following two facts:

(i) any modulus of continuity which is slightly worse than log-Lipschitz produces an infinite loss of derivatives in the observability estimates, which then fail;

(ii) for any modulus of continuity which is strictly between the Lipschitz and log-Lipschitz ones, observability estimates hold, but in general with a finite, but non-zero, loss of derivatives.

It is important to notice that all these results hold true *only* in the case of one space dimension. As a matter of fact, their proof relies in an essential way on the technique of *sidewise energy estimates* (see *e.g.* [66]), which consists in exchanging the role of the time and space variables, a fact which of course holds true only in dimension 1.

In higher space dimension, the problem of proving observability estimates for coefficients which have low regularity remains open. As a matter of fact, the microlocal analysis tools linked with the GCC condition requires the coefficients to be at least $\mathcal{C}^2(\overline{\Omega})$. On the other hand, important improvements have been obtained in the last years. For instance, by using refined Carleman estimates for hyperbolic operators with potential, in [103] Duyckaerts, Zhang and Zuazua were able to weaken the regularity conditions to $\mathcal{C}^1(\overline{\Omega})$. More recently, Dehman and Ervedoza [97] were able to prove observability estimates for coefficients which are merely $\mathcal{C}^0(\overline{\Omega})$, under the additional geometric assumption

$$\exists\, \alpha \in]0,2] \qquad \text{such that} \qquad x \cdot \nabla\omega(x) + (2-\alpha)\,\omega(x) \geq 0 \qquad \text{in the sense of } \mathcal{D}'(\Omega_1)\,,$$

where Ω_1 is a smooth domain such that $\overline{\Omega} \subset \Omega_1$. This is a geometric condition imposed on ω along the direction of the multiplier x, which is stronger than the GCC (in fact, the core of the proof of [97] consists in showing that, even without tracing rays, the previous condition implies GCC, so observability), but does not involve any additional regularity for ω.

Finally, we mention the very recent series of works [31, 29, 30] about observability properties for wave operators on Riemannian manifolds with and without boundary, for $\mathcal{C}^1$ metrics and $W^{1,\infty}$ perturbations of them.

2.4 Some open questions and perspectives

I list below some open problems in the context of hyperbolic operators with low regularity coefficients, in the two directions mentioned above.

Well-posedness

On the side of well-posedness, it is interesting to understand better the role of assumptions of Zygmund type on the regularity of the coefficients with respect to the space variable. The result of [54] hints that they may be relevant.

The primary question in this direction is to generalise the well-posedness result without loss for isotropic Zygmund coefficients (2.10) to H^s spaces, for values of s different from $1/2$. Motivated by simple algebraic considerations, it seems reasonable to trying to introduce weighted paraproduct operators, where weights have to be introduced depending on the size of the frequencies one is localising at.

Then, one may wonder whether or not it is possible to prove well-posedness with finite loss of regularity for coefficients which are log-Zygmund both in time and space.

Finally, we mention that there exists another "category" of well-posedness results, obtained under assumptions on the fast oscillations of the coefficients in one point (say, at $t = 0$) instead of the assumptions discussed here on the regularity of the coefficients. We refer *e.g.* to [58, 205, 219] for results in this direction. However, the proofs therein required high technicalities and high regularity of the coefficients in space (namely, $\mathcal{C}^\infty$). In could be interesting to reinterpret those results in terms of the modified energy of Tarama and see whether it is possible or not to obtain well-posedness with a simpler proof, and requiring less stringent assumptions on the space regularity of the coefficients.

Observability and control

On the side of the control and observability problem, the first important question is to tackle the higher dimensional case $d \geq 2$. The recent results of [31, 29, 30] have filled the gap between $\mathcal{C}^2$ and Lipschitz regularity conditions, although in the case of small perturbations of $\mathcal{C}^1$ metrics. Thus, they look as a good starting point to begin the study for less regular coefficients.

On the other hand, questions linked with observability estimates with loss of derivatives have not been tackled for transport equations with low regularity transport fields. It is well-known, see *e.g.* [7], that the solution to such linear equations loses regularity in the evolution, similarly to what happens for solutions to the wave equations. Whether those results have or not a counterpart in control theory seems to us an interesting question.

Chapter 3

First-order hyperbolic systems

In this chapter we continue the analysis of the Cauchy problem for hyperbolic operators with low regularity coefficients. Here, we focus on $m \times m$ first-order hyperbolic systems

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j(t,x)\,\partial_j u\,, \tag{3.1}$$

where, as in the previous chapter, $(t,x) \in [0,T] \times \mathbb{R}^d$, for some $T > 0$, while now the coefficients A_j are $m \times m$ real-valued matrices.

We immediately introduce the principal symbol associated with the operator L: for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, we set

$$\mathcal{A}(t,x,\xi) := \sum_{j=1}^{d} \xi_j\,A_j(t,x)\,. \tag{3.2}$$

Let us also note by $\big(\lambda_k(t,x,\xi)\big)_{1\leq k\leq m} \subset \mathbb{C}$ the set of its eigenvalues. Recall that the operator L is said to be hyperbolic if all the $\lambda_k(t,x,\xi)$ are real.

Works presented in the chapter

- (P.9) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients.* **Comm. Partial Differential Equations**, 40 (2015), n. 11, 2082-2121.
- (P.12) F. Fanelli: *Some local questions for hyperbolic systems with non-regular time dependent coefficients.* **J. Hyperbolic Differ. Equ.**, 14 (2017), n. 2, 301-322.
- (P.18) F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *On the Cauchy problem for microlocally symmetrizable hyperbolic systems with log-Lipschitz coefficients.* **Indiana Univ. Math. J.**, 69 (2020), n. 3, 785-836.

Not mentioned, but in this context

- (P.15) J. Bartsch, A. Borzì, F. Fanelli, S. Roy: *A theoretical investigation of Brockett's ensemble optimal control problems.* **Calc. Var. Partial Differential Equations**, 58 (2019), n. 5, Paper n. 162.
- (P.25) J. Bartsch, A. Borzì, F. Fanelli, S. Roy: *A numerical investigation of Brockett's ensemble optimal control problems.* **Numer. Math.**, 149 (2021), n. 1, 1-42.
- (C.3) F. Fanelli: *A few remarks on hyperbolic systems with Zygmund in time coefficients.* In "Shocks, singularities and oscillations in Nonlinear Optics and Fluid Mechanics", Springer-INdAM series, **17**, Springer, Cham (2017), 75-102.

3.1 Introduction

As already disclosed in Chapter 1, a very relevant notion in the context of well-posedness of the Cauchy problem for the operator L is the one of *microlocal symmetrizability.* Roughly speaking, this means that there exists a scalar product, defined microlocally, with respect to which the symbol matrix $\mathcal{A}$ is symmetric. The precise definition is the following one.

Definition 3.1. System (3.1) is *uniformly microlocally symmetrizable* if there exists a $m \times m$ matrix $S(t,x,\xi)$, homogeneous of degree 0 in ξ, such that:

- $\xi \mapsto S(t,x,\xi)$ is $\mathcal{C}^\infty$ for $\xi \neq 0$;
- for any point (t,x,ξ), the matrix $S(t,x,\xi)$ is self-adjoint;
- there exist constants $0 < \lambda \leq \Lambda$ such that $\lambda \,\mathrm{Id} \;\leq\; S(t,x,\xi) \;\leq\; \Lambda \,\mathrm{Id}$ for any (t,x,ξ);
- for any point (t,x,ξ), the matrix $S(t,x,\xi)\,\mathcal{A}(t,x,\xi)$ is self-adjoint.

The matrix valued function S is called a (bounded) *microlocal symmetrizer* for system (3.1).

Notice that, in the previous definition, the word "uniformly" refers to uniformity with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$.

The notion of microlocal symmetrizability extends the previous one of symmetrizability of system (3.1) *in the sense of Friedrichs*, which consists in assuming that the symmetrizer $S = S(t,x)$ does not depend on the frequency parameter ξ. A very special case is the one of symmetric systems, in which all the matrices A_j are symmetric (in which case, a microlocal symmetrizer is simply the identity matrix Id).

We also remark that the existence of a bounded microlocal symmetrizer for L has been proved [194] by Métivier to be equivalent to the strong hyperbolicity of operator L.

The relevance of Definition 3.1 in the study of the well-posedness of the Cauchy problem was highlighted in work [162] by Ivrĭ and Petkov, who proved that the existence of a bounded microlocal symmetrizer $S(t,x,\xi)$ is necessary for the well-posedness of L in $L^2(\mathbb{R}^d)$ to hold. However, this condition is far from being sufficient, even for $\mathcal{C}^\infty(\mathbb{R}^d)$ well-posedness, see *e.g.* the counterexamples in [214, 194, 63].

On the other hand, for hyperbolic systems (3.1) such that, for all j, one has $A_j = {}^tA_j$, L^2 well-posedness can be recovered straightforwardly by assuming Lipschitz continuity of all the A_j's over $[0,T] \times \mathbb{R}^d$. For this, it is enough to defined the energy of the solution as the L^2 norm of the solution and perform an estimate on its time derivative. The result can be easily extended to hyperbolic systems which are *symmetric in the sense of Friedrichs* [143, 144]; this means that there exists a symmetrizer $S = S(t,x)$ with respect to which every A_j becomes self-adjoint. Under a Lipschitz regularity assumption on both the coefficient matrices A_j and on the symmetrizer S, energy estimates with no loss, thus L^2 well-posedness, can be obtained by working on the modified energy

$$E(t) := \left\| \big(Su\big)(t) \right\|_{L^2}^2 . \tag{3.3}$$

The well-posedness result was later improved by Métivier [193], who was able to consider microlocally symmetrizable systems, still under the assumption that both coefficients and the symmetrizer $S(t,x,\xi)$ are Lipschitz with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$. We refer also to [194] for further developments in this context.

The fact that the regularity of the symmetrizer S is the same as the one of the coefficients A_j is quite natural and is a consequence of standard perturbation theory for linear operators, as S can be constructed in terms of the eigenvalues and eigenprojectors of the matrix $\mathcal{A}(t,x,\xi)$ (see *e.g.* [194, 204]). Notice however that some conditions could be relaxed, if one wants to keep the regularity of the coefficients of L separated from the one of the symmetrizer [108]. This is relevant in the context of transport equations and, more generally, of symmetric hyperbolic systems.

To conclude this introduction, we come back to the counterexamples of [63]. Analogously to what happens for the wave operator W from the previous chapter, those couterexamples show that the Cauchy problem for the operator L, assumed to possess *smooth* coefficients $A_j = A_j(t)$, is in general ill-posed in L^2 whenever the symmetrizer $S = S(t)$ is ω-continuous for some modulus of continuity ω which is worse than Lipschitz, ill-posed in $\mathcal{C}^\infty$ whenever ω is less regular than log-Lipschitz.

3.2 Isotropic log-Lipschitz regularity

In light of the results of [61, 62, 215, 53] for the wave operator W, it seems reasonable to try to fill the gap between the L^2 well-posedness result under a Lipschitz regularity assumption [193] and the $\mathcal{C}^\infty$ ill-posedness for regularities worse than log-Lipsschitz [63].

In [57], we considered the case in which system (3.1) is microlocally symmetrizable, under the isotropic log-Lipschitz regularity condition (2.4) with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$ on both the coefficients A_j and the symmetrizer S. Under those assumptions, we proved an energy estimate with time-dependent loss of derivatives, analogous to Theorem 2.1 above.

Theorem 3.2. *Let us consider the first-order system* (3.1)*, and assume it to be microlocally symmetrizable, in the sense of Definition 3.1. Suppose moreover that the coefficients* $\big(A_j\big)_{1\leq j\leq d}$ *and the symmetrizer* S *are bounded matrices which are moreover isotropically log-Lipschitz continuous.*

Then, for all $s \in \,]0,1[\,$*, there exist positive constants* C_1 *and* C_2*, a* $\beta > 0$ *and a time* $T_* \in \,]0,T]$*, with* $\beta\, T_* < s$*, such that the estimate*

$$\sup_{t\in[0,T_*]} \|u(t)\|_{H^{s-\beta t}} \leq C_1\, e^{C_2\, T} \left(\|u(0)\|_{H^s} + \int_0^{T_*} \big\|Lu(\tau)\big\|_{H^{s-\beta\tau}}\, d\tau\right) \tag{3.4}$$

holds true for any tempered distribution $u \in L^2\big([0,T];H^1(\mathbb{R}^d;\mathbb{R}^m)\big) \cap H^1\big([0,T];L^2(\mathbb{R}^d;\mathbb{R}^m)\big)$*.*

An analogous estimate holds true also for the adjoint operator $\widetilde{L}$*.*

From Theorem 3.2 it is a routine matter to derive $\mathcal{C}^\infty$ well-posedness for L with a finite loss of derivatives.

The proof of the previous theorem follows the main guidelines of the corresponding one for the wave operator W, as explained in Chapter 2. However, we observe that, owing to the possible singularity of the symmetrizer $S(t,x,\xi)$ at $\xi = 0$, dealing with low frequencies requires a special treatement.

As already noticed, at this stage the regularity conditions on the coefficients A_j could be somehow relaxed. We refer to Remark 4.8 of [57] for more details in this respect. However, the isotropic log-Lipschitz condition looks particularly convenient to perform a local analysis.

Thus, inspired by the study carried out in [62] for wave equations, in [57] we studied the local-in-space Cauchy problem. More precisely, given a smooth open bounded domain $\Omega \subset \mathbb{R}^{1+d}$, we considered the operator

$$P(z,\partial_z)u := \sum_{j=0}^{d} A_j(z)\, \partial_{z_j} u\,,$$

for $m\times m$ real-valued matrices A_j defined on Ω, log-Lipschitz continuous on Ω. We fixed a smooth hypersurface $\Sigma \subset \Omega$, with parametrisation $\Sigma = \big\{\varphi(z) = 0\big\}$, and a point $z_0 \in \Sigma$. We studied local existence and uniqueness of solutions to the Cauchy problem

$$\begin{cases} Pu = f \\ u_{|\Sigma} = u_0\,, \end{cases} \tag{CP}$$

where $u_0 \in H^s(\omega_0)$, with ω_0 being a neighbourhood of z_0 in Σ, and $f \in H^s(\Omega_0 \cap \{\varphi > 0\})$, with Ω_0 being a neighbourhood of z_0 in Ω. Here, $s \in]0,1[$ as in the global Cauchy problem.

As a matter of fact, our regularity assumptions and the hypothesis of microlocal symmetrizability are invariant under smooth change of variables. So, the local statements can in fact be reconducted to the global ones, after performing a suitable change of frame and after an application of the extension operator. Hence, one can prove the existence of a $s_0 \in]0, s[$ and of a unique solution $u \in H^{s_0}(\Omega_0 \cap \{\varphi > 0\})$ to (CP).

To conlude this part, we observe that, in our low regularity framework, the sense of the local Cauchy problem is not so clear *a priori*: for instance, it is not clear that we can give sense to the trace $u_{|\Sigma}$ of a local solution u belonging merely to H^{s_0}. Hence, the first part of the analysis was devoted to give sense to the formulation of (CP). We also explicitly point out that no assumption on the manifold Σ have to be made: the fact that the Cauchy problem is non-characteristic with respect to Σ is guaranteed by the assumption of *uniform strong hyperbolicity* [194] on P.

3.3 Zygmund-type conditions

Let us come back to the global-in-space Cauchy problem, hence to operator L defined in (3.1). Keeping in mind the result of Tarama [215] as well as the following extensions for the case of scalar wave operators, one may think that Zygmund-type assumptions are suitable also in the context of hyperbolic system, leading to substantial improvements of the results of [193] (*i.e.* well-posedness in L^2 for Lipschitz coefficients and symmetrizer) and of [57] (less regular coefficients in time could be considered, without qualitative changes in the statement of the previous Theorem 3.2).

Nonetheless, there is an intrinsic difficulty in such an argument. Pushing the parallel with Tarama's result further, we see that we need to include a lower order corrector in the definition of the energy; in turn, owing to (3.3), this boils down to performing suitable modifications in the definition of the symmetrizer, and more precisely to finding a symmetrizer which behaves "in the correct way" when performing energy estimates. How to do that in full generality seems to be out of reach; one has rather to work case by case on the specific form of the operator L, in order to find a suitable symmetrizer for it, where "suitable' of course refers to the capability of producing good cancellations in the energy estimates.

In [56], we considered operator (3.1) in the special case in which its coefficients only depend on the time variable. Namely, we focused our attention on the operator

$$\mathcal{L}u := \partial_t u + \sum_{j=1}^{d} A_j(t)\,\partial_j u\,, \tag{3.5}$$

and we formulated Zygmund-type regularity assumptions on the A_j's. In order to keep the discussion simple, we only consider the pure Zygmund case, which reads as follows: there exist $p \in [1, +\infty]$ and $K > 0$ such that, for all $1 \leq j \leq d$ and all $0 \leq \tau \leq T/2$, one has

$$\left\|A_j(\,\cdot\, + \tau) + A_j(\,\cdot\, - \tau) - 2\,A_j(\,\cdot\,)\right\|_{L^p([\tau,T-\tau];\mathcal{M}_m(\mathbb{R}))} \leq K\,\tau\,,$$

where $\mathcal{M}_m(\mathbb{R})$ denotes the space of $m \times m$ real-valued matrices, endowed with the classical sup-norm. The case of log-Zygmund coefficients could be considered as well, at the price of having a more elaborated (and of course qualitatively different, as a regularity loss is produced) statement.

In light of the discussion above, we dismessed the assumption of microlocal symmetrizability of $\mathcal{L}$. We supposed instead that $\mathcal{L}$ is *hyperbolic with constant multiplicities*. This means that the eigenvalues $\lambda_j(t,\xi)$, for $1 \leq j \leq m$, of the matrix $\mathcal{A} = \mathcal{A}(t,\xi)$ defined in (3.2) are all real, semi-simple and have constant multiplicities in t and ξ. In particular, the more classical strictly hyperbolic case, in which $\lambda_j(t,\xi) \neq \lambda_k(t,\xi)$ for all $j \neq k$ and any $(t,\xi) \in [0,T] \times \mathbb{R}^d$, is included as a special case.

Under the previous assumptions, in [56] we proved an energy estimate with *no loss* of regularity.

Theorem 3.3. *Let us consider the first-order system* (3.5)*, and let us assume it to be hyperbolic with constant multiplicities. Suppose moreover that the coefficients* $\big(A_j\big)_{1\leq j\leq n}$ *satisfy the Zygmund condition formulated above, for some* $p\in\,]1,+\infty]$.

Then, for all $s\in\mathbb{R}$*, there exist positive constants* C_1*,* C_2 *(just depending on* s *and on* K*) such that the estimate*

$$\sup_{t\in[0,T]}\|u(t)\|_{H^s}\;\leq\;C_1\,e^{C_2\,T}\left(\|u(0)\|_{H^s}\;+\;\int_0^T\big\|\mathcal{L}u(\tau)\big\|_{H^s}\,d\tau\right)$$

holds true for any $u\in\mathcal{C}^1\big([0,T];H^\infty(\mathbb{R}^d;\mathbb{R}^m)\big)$.

A statement in the same spirit can be proved also for log-Zygmund regularity assumptions. We refer to [56] for the precise result. We refer to that work also for comments on the assumption $p>1$ in the previous statement, as well as for a parallel with 1-D wave equations with Zygmund coefficients.

Let us comment a bit on the proof of the previous theorem.

It is well-known [193, 204] that, in the smooth case, hyperbolic systems with constant multiplicities are smoothly diagonalizable, so in particular microlocally symmetrizable. As a matter of fact, if we denote by Π_j the projection operators onto the eigenspaces E_j of the matrix $\mathcal{A}(t,\xi)$, then a symmetrizer for system (3.5) is defined as

$$S\;:=\;\sum_j\Pi_j^*\,\Pi_j\,.$$

So, the assumptions of Theorem 3.3 put ourselves in a context where it is possible to construct a microlocal symmetrizer for system (3.5). However, here the challange is to find a *suitable* symmetrizer, which is able to produce the sought cancellations (in the same spirit of Tarama's work [215]) in the energy estimates and, in the end, to give us an estimate without loss of derivatives.

Our idea was to look for an approximate symmetrizer of the form

$$S_\varepsilon(t,\xi)\;=\;S^0_\varepsilon(t,\xi)\;+\;|\xi|^{-1}\,S^1_\varepsilon(t,\xi)\,,$$

where both S^0_ε and S^1_ε are self-adjoint and bounded, S^0_ε being positive definite. The first term represents the principal part of S_ε, while the second term is a lower order corrector, whose role is to kill the terms out of control arising from the time derivative of S^0_ε. Finally, $\varepsilon>0$ is an approximation parameter, which is fixed afterwards according to the classical choice (1.8).

Let us omit the regularisation parameter $\varepsilon>0$ from the rest of the argument. Contrarily to the classical situation, this time, roughly speaking, we construct S^0 and S^1 of the following form:

$$S^0\;=\;\sum_j\Pi_j^*\,\Sigma^0_j\,\Pi_j\qquad\qquad\text{and}\qquad\qquad S^1\;=\;\sum_j\sum_{k\neq j}\Pi_j^*\,\Sigma^1_{jk}\Pi_k\,,$$

where Σ^0_j is a self-adjoint (diagonal, in fact) operator acting on the eigenspace E_j, whereas $\Sigma^1_{jk}:E_k\longrightarrow E_j$ has the role of bringing to the eigenspace E_j suitable corrections coming from the eigespace E_k, in order to produce the sought cancellations. In fact, Σ^1_{jk} can be recovered, from algebraic relations, in terms of Σ^0 and of the eigenprojectors, so the key point is to find Σ^0. Now, the construction of Σ^0 is reconducted to solving an ODE system in Zygmund classes, at least in an approximate way (smooth remainders will appear when cutting out the low frequencies in time). In goes without saying that, also in this context, the use of Littlewood-Paley theory (in particular to characterise Zygmund and log-Zygmund spaces as special Besov spaces) and paradifferential calculus with parameters came into play in a fundamental way.

As a last comment of this part, we mention that, in [108], we devoted attention to local in space questions (finite speed of propagation, local existence and uniqueness) for operator $\mathcal{L}$ defined in (3.5), for even less regular coefficients $A_j(t)$ (essentially, L^1 in time condition). The study is based on an analysis of the operator in the phase-space and on a suitable application of the Paley-Wiener theorem.

3.4 Some perspectives

We conclude this chapter by mentioning some questions which catch our attention and which we would like to consider in the future.

The primary question in this context is to extend the results of [56] to the case of coefficients also depending on the space variable, in the same spirit of [53] and [54] for the scalar wave equations. The problem looks quite hard to solve, as the ODE system used to identify the matrix Σ^0 is expected to be replaced, in this case, by a transport equation, which does not allow to retrive the initial Zygmund regularity of the coefficients (there is no elliptic effect in the transpot equation).

Also, it seems interesting to us to explore effects of the low regularity coefficients in the control and optimal control problems for operator (3.1), extending in this way the study of [11, 12], devoted to the regular case (actually, for the very special case of transport equations).

Finally, it could be interesting to apply the results about linear operators with low regularity coefficients to the study of non-linear problems (in the vein of [88], for instance) and to the study of propagation of oscillations in singular perturbation problems (see more about that in Part III).

Part II

Well-posedness of non-linear problems

Chapter 4

Overview of Part II

In Part II we start considering non-linear equations and systems, in particular those coming from fluid mechanics models.

Before entering into the core of the discussion, we should point out that the content of this part is widely heterogeneous. As a matter of fact, the general goal is to investigate well-posedness questions, but this will be done for a large variety of models. In what follows, we have decided to divide the material into two parts, the one concerning viscous models (see Chapter 5) and the one concerning inviscid models (see Chapter 6). However, in Chapter 6 we will encounter situations in which the fluid is viscous, but the viscosity tensor only presents a skew-symmetric component; as a consequence, the viscosity term does not contribute to the energy balance. Thus, a different classification of the material could be "dissipative *vs* non-dissipative models"[1], or alternatively "parabolic models *vs* hyperbolic models"[2]. Despite this rough classification, each chapter in itself will be heterogeneous, both concerning the systems of equations we will treat (for instance, compressible, quasi-incompressible and incompressible equations, models from turbulence theory, fluids with odd viscosity...) and the kind of solutions (regular, strong, weak, statistical) we will be interested in.

In this introductory chapter, we will try to make some order in all this material and put in evidence the common features and the main points which deserve attention. We refer to Chapters 5 and 6 for more details on the specific problems under consideration.

As already said, in Part II we are interested in well-posedness questions linked to various models from fluid mechanics. Thus, we will investigate existence and uniqueness of solutions and their lifespan; sometimes, we will also be interested in describing some qualitative properties of solutions.

The keyword here will be *non-homogeneity*, inasmuch as we will always consider flows of non-homogeneous fluids. This is the case, for instance, for (compressible or incompressible) fluids presenting density variations, or more generally variations of some of their inner properties. With this last expression, we refer in particular to models for turbulent flows, like the well-known k-ε models, in which the small-scale quantities (typically, the turbulent kinetic energy and the energy dissipation rate) are treated as independent unknowns of the system. We also have in mind electrically conducting fluids, which are characterised by the presence of a non-trivial magnetic field, which is self-induced by the fluid through its own motion. Sometimes we will consider situations where the heterogeneity is introduced by the interaction with the exterior (this is the case of open systems): at the mathematical level, this interaction is encoded by non-trivial in-flow/out-flow boundary conditions.

The *leitmotif* of the whole part will be *low regularity*, in the sense that we will perform our

[1] This classification is not completely satisfactory either, as it remains ambiguous about the dissipation mechanism which acts on the system.

[2] This classification, instead, is not completely correct, as hyperbolicity does not always hold *strictu senso*; indeed, many of the considered models are incompressible, thus non-local effects are also involved.

studies in a framework demanding minimal regularity assumptions on the initial data. So, in the viscous case, we will mainly work in the framework of weak solutions having finite energy, reminiscent in spirit of Leray's solutions to the incompressible Navier-Stokes system. We will also base our construction of statistical solutions on that class of solutions. In other cases, we will consider strong solutions, but having minimal regularity; we point out that, in general, we will not be at critical regularity in the sense of the scaling invariance of the equations, but slightly subcritical. This is the case in particular when dealing with hyperbolic models or degenerate parabolic models: as it is well-known, in those cases one typically needs to work in (subcritical) spaces which are embedded in the space $W^{1,\infty}$ of globally Lipschitz functions.

We point out that working with solutions having minimal regularity has not only a mathematical flavour, but is also important for describing special classes of solutions. A typical situation which catches our attention is the case of discontinuities in the density functions: in the case of two-phase flows, the density function experiences a jump at the interface separating the two phases; a similar situation appears when considering the prensence of two immiscible fluids (like oil-water, for instance) in the same region of space. Both for theoretical reasons and in view of applications, it is then important to capture, at the mathematical level, solutions which possess that structure and to understand their evolution. In order to describe such configurations, it is clear that one cannot rely on the theory of strong solutions, which are typically regular and do not allow for any kind of discontinuity of the unknowns. On the other hand, the notion of weak solutions (for instance, the ones having finite energy) is too weak, as one often misses uniqueness and any qualitative information on the solutions. In particular, a description of the evolution of the discontinuity interface seems out of reach in the weak solutions framework. Therefore, one needs a setting able to guarantee that the considered weak solutions possess some additional structure. In this respect, two settings come to our mind.

The first setting which is able to consider weak solutions with some additional qualitative properties is the Hoff theory [155, 156, 157, 158] of *shock data* for the compressible Navier-Stokes system: this is a theory of global in time finite energy weak solutions having small energy. The smallness of the energy allows Hoff to point out additional regularity properties for certain quantities, which are smoothed out in the dynamics: these are the vorticity of the fluid and the so-called *effective viscous flux* (see also the work [212] by Serre about the one-dimensional case). We refer to Section 5.2 below for more details about this, and to [159, 202] for extensions of Hoff's theory to the case of discontinuity regions of the density presenting corner-type singularities.

The second idea we will pursue here goes back to the pioneering works of Chemin [37, 38] about the vortex patches problem for the 2-D incompressible Euler equations. The point is to resort to some *tangential regularity assumption* over the initial density function and/or the velocity or vorticity of the fluid: even though those quantities present jumps at some interface (for instance, coming back to one of the examples discussed above, the density is discontinuous at the interface separating oil and water), so they are irregular in the normal direction, they are instead more regular in the tangential directions; sometimes, this information is all that one needs in order to deal with non-linear models. In order to make this discussion more clear and concrete, let us consider the 2-D Euler equations written in vorticity formulation:

$$(4.1)\qquad \begin{cases} \partial_t\omega \,+\, u\cdot\nabla\omega \,=\, 0 \\ u \,=\, -\,\nabla^\perp\,(-\Delta)^{-1}\,\omega\,, \end{cases}$$

where $u\in\mathbb{R}^2$ is the velocity field of the fluid and $\omega\,:=\,\mathrm{curl}\,(u)\,=\,\partial_1u^2-\partial_2u^1$ the (scalar) vorticity associated to u. The first equation appearing above says that ω is purely transported by u, which can in turn be recovered from ω by solving the second equation appearing in (6.2), called *Biot-Savart law*. For any 2-D vector $v\in\mathbb{R}^2$, $v=(v^1,v^2)$, we have denoted by $v^\perp\,=\,(-v^2,v^1)$ its rotation of angle $\pi/2$; so, in particular we have set $\nabla^\perp\,=\,(-\partial_2,\partial_1)$ in the Biot-Savart law. Assume to have initially a vortex patch, namely that the initial vorticity $\omega_0\,=\,\mathbb{1}_{D_0}$ is the characteristic function of a bounded domain D_0 of class (say) $\mathcal{C}^{1,\varepsilon}$. Then, Yudovich's theory ensures that a global in time weak solutions $\omega \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big)$ to (6.2) exists. Moreover, as the velocity field u is log-Lipschitz in space in this case, one can propagate the vortex patch structure by its flow: for any $t \geq 0$, one has $\omega(t) = \mathbb{1}_{D(t)}$, where the domain $D(t) = \psi_t(D_0)$ is the image of D_0 by the flow ψ_t associated to u. Remark that, in the previous situation, the vorticity ω presents a jump discontinuity in the direction which is normal to the boundary $\partial D(t)$ of the domain $D(t)$, but is regular in the tangential direction (the tangential derivative is actually 0). The fundamental idea of Chemin was to use tangential regularity to show that, under the previous assumption $D_0 \in \mathcal{C}^{1,\varepsilon}$ for some $\varepsilon \in]0,1[$, u is in fact Lipschitz and then $D(t) \in \mathcal{C}^{1,\varepsilon}$ for any later time $t \geq 0$. In order to understand the key role played by the tangential regularity of ω at the boundary $\partial D(t)$ of the domain, let us give a sketch of the proof in the flat case, namely in the situation where ω is more regular (say) in the x^1-direction. So, assume that

$$\operatorname{div} u = 0\,, \qquad \omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\,, \qquad \partial_1 \omega \in \mathcal{C}^{\varepsilon-1}\,,$$

where the negative Hölder space $\mathcal{C}^{\varepsilon-1}$ is defined as the Besov space $B^{\varepsilon-1}_{\infty,\infty}$. Notice that the last assumption above means that ω has ε-regularity more than expected in the x^1-direction. Our goal is to prove that $\nabla u \in L^\infty(\mathbb{R}^2)$: as a matter of fact, with this property at hand, propagating the regularity of the domain easily follows. Now, a direct computation shows that $\partial_1\omega = \Delta u^2$, so that $\nabla u^2 = -\nabla(-\Delta)^{-1}\partial_1\omega$ is easily seen to belong to L^∞ (notice that, here, one not only has to use the assumption on $\partial_1\omega$, but also to cut ∇u^2 into low and high frequencies and estimate the low frequencies using the Bernstein inequalities). Next, by using the divergence-free condition over u one immediately gets that $\partial_1 u^1$ is also bounded over $\mathbb{R}^2$. Finally, one may write $\partial_2 u^1 = \partial_1 u^2 - \omega$, so that also this component belongs to L^∞, as claimed. The core of Chemin's proof [37, 38] consists in generalising this argument to the non-flat situation. We also observe that the underlying idea, namely tangential regularity of the vorticity, generalises very well also to other situations in which there is no hope to transport patches structures, for instance to viscous fluids and to higher dimensions. We refer to Chapter 7 of [8] for further results and references for homogeneous fluids; we also refer to [105, 152, 154, 151, 172, 146, 90, 89, 147, 174, 87, 173, 148] for some implementations of that idea in the context of non-homogeneous fluids.

We will see in Section 5.2 below how Hoff's theory and Chemin's idea combines in the study of the compressible Navier-Stokes equations. For the time being, let us stop here the discussion about low regularity issues and move forward in the presentation of the material of Part II.

Another question which occupies a central role in Part II is the study of the *lifespan* of solutions to the various models under consideration. In fact, we will always deal with local in time well-posedness results. The only exception to this will be when constructing statistical solutions for the barotropic Navier-Stokes equations: as a matter of fact, as already mentioned above, that construction is based on the notion of finite energy weak solutions, which are known to exist globally in time [177, 123]. Thus, apart from the latter case, questions about the lifespan of solutions arise. For instance, we will often be interested in providing lower bounds for the time of existence of solutions in terms of the norms of the initial data, as well as in deriving blow-up/continuation criteria similar in spirit to the Beale-Kato-Majda continuation criterion for the incompressible Euler equations. Sometimes, instead, we will be able to formulate assumptions on the initial data to guarantee that the corresponding solutions must blow up in finite time.

To conclude this introductory chapter, we mention also that an underlying common theme of Part II is the study of some questions related to turbulence theory. First of all, the notion of statistical solution is strictly linked to turbulence theory, as the rough idea is to take averages over flows and look at how the fluid behaves statistically; besides, we will use statistical solutions to investigate the validity of the so-called *ergodic hypothesis* from physics and engineering in the context of energetically open systems. In addition, we will be interested in studying well-posedness of systems of equations used to model a turbulent motion of a fluid: in the next chapter we will consider the 1-D counterpart of a special k-ε model, also known as Kolmogorov's two-equation model of turbulence, whereas in Chapter 6 we will deal with a system for non-homogeneous fluids having odd viscosity; such an odd viscosity term is sometimes used in the description of a turbulent flow as a coherent collection of various systems of vortices at different scales.

Part II unfolds in the following way. In Chapter 5 we will consider systems of viscous flows, where viscosity here has to be interpreted in the classical sense, as a dissipative mechanism acting during the motion. In that chapter, we will mainly work in the context of compressible flows. In Chapter 6, instead, we will study systems describing the dynamics of inviscid fluids, or better of fluids whose motion does not present any dissipative effect. In that part, the flows will mainly be (quasi-)incompressible.

Chapter 5

Well-posedness for some viscous models

This chapter is devoted to the study of well-posedness questions related to various viscous models for non-homogeneous fluids.

The main system we consider is the compressible barotropic Navier-Stokes system. In Section 5.2 we will revise Hoff theory of weak solutions for shock data and see how to generalise those results by using a tangential regularity approach. In Section 5.3, instead, we will use the classical weak solutions theory by Lions-Feireisl in order to build up a theory of statistical solutions for energetically open systems. Besides, this approach will enable us to investigate the validity of the so-called ergodic hypothesis from turbulence theory in this context.

We point out that, in this part, we avoid any discussion on strictly related models, like the Navier-Stokes-Fourier system (see *e.g.* [131] for a comprehensive study) or systems presenting density-dependent degenerate viscosity coefficients (see [24, 22, 190, 216] and references therein, for instance). As a matter of fact those systems present their own specificities, thus driving us very far from the scopes of our presentation.

In the last part of the chapter, see Section 5.4, we will consider instead a one-dimensional reduction of the Kolmogorov two-equation model of turbulence. We will show a local in time well-posedness result, together with finite time blow-up for special classes of initial data.

Because of the importance played by the notion of weak solutions to the compressible Navier-Stokes system in this chapter, let us start with an introductory section about it.

Works presented in the chapter

(P.17) R. Danchin, F. Fanelli, M. Paicu: *A well-posedness result for viscous compressible fluids with only bounded density.* **Anal. PDE**, 13 (2020), n. 1, 275-316.

(P.19) F. Fanelli, E. Feireisl: *Statistical solutions to the barotropic Navier–Stokes system.* **J. Stat. Phys.**, 181 (2020), n. 1, 212-245.

(P.24) F. Fanelli, E. Feireisl, M. Hofmanová: *Ergodic theory for energetically open compressible fluid flows.* **Phys. D**, 423 (2021), Paper n. 132914.

(P.28) F. Fanelli, R. Granero-Belinchón: *Finite time blow-up for some parabolic systems arising in turbulence theory.* **Z. Angew. Math. Phys.**, 73 (2022), n. 5, Paper n. 180.

(S.2) F. Fanelli, R. Granero-Belinchón: *Well-posedness and singularity formation for the Kolmogorov two-equation model of turbulence in 1-D.* Submitted (2021).

Not mentioned, but in this context

(P.14) F. De Anna, F. Fanelli: *Global well-posedness and long-time dynamics for a higher order Quasi-Geostrophic type equation.* **J. Funct. Anal.**, 274 (2018), n. 8, 2291-2355.

5.1 Compressible Navier-Stokes: weak solutions theories

Let Ω be a smooth domain of $\mathbb{R}^d$, with $d \geq 2$. We left aside the case $d = 1$, as it is in some sense special and, by now, quite well understood [155]. Througout this chapter, we will consider either the case $\Omega = \mathbb{R}^d$ or the case in which Ω is bounded, supplemented with suitable (non-homogeneous, Dirichlet-type) boundary conditions.

For $(t,x) \in \mathbb{R}_+ \times \Omega$, we describe the motion of a viscous fluid of density $\rho = \rho(t,x) \geq 0$ and velocity field $u = u(t,x) \in \mathbb{R}^d$ by the barotropic Navier-Stokes sytem

$$\begin{cases} \partial_t \rho + \mathrm{div}\,\big(\rho\, u\big) \,=\, 0 \\ \partial_t\big(\rho\, u\big) \,+\, \mathrm{div}\,\big(\rho\, u \otimes u\big) \,+\, \nabla P(\rho) \,-\, \mu\,\Delta u \,-\, \lambda\,\nabla u \,=\, 0\,, \end{cases} \tag{5.1}$$

where $P = P(\rho)$ is the pressure field, while μ and λ are, respectively, the shear viscosity and the bulk viscosity coefficients, which we assume to be strictly positive constants here.

Throughout this part, we assume that the pressure function satisfies

$$P \in \mathcal{C}^1\big([0,+\infty[\big)\,, \qquad P'(z) > 0 \quad \text{for } \ z > 0\,, \qquad P'(z) \approx z^{\gamma-1} \quad \text{for } \ z \to +\infty\,, \tag{5.2}$$

for a certain $\gamma > 1$. The prototypical example of pressure function is given by the Boyle law $P(\rho) = A\,\rho^\gamma$, for some constant $A > 0$, but more general functions can be considered, according to the previous hypothesis. More restrictions about the value of γ will appear in a while. For the time being, let us remark that the monotonicity assumption $P'(z) > 0$ on the pressure is fundamental to set down a weak solution theory, although some results exist in the case when this assumption does not hold, see *e.g.* [122, 25].

5.1.1 Energy inequality

The barotropic Navier-Stokes system (5.1) possesses an energy balance law. Before presenting it, a few definitions are in order. First of all, we introduce the *pressure potential* $H = H(z)$, defined through the ODE

$$\forall\, z > 0\,, \qquad z\, H'(z) \,-\, H(z) \,=\, P(z)\,.$$

This in particular implies that $H''(z) = P'(z)/z$, so H is a convex function on $[0,+\infty[$ owing to the assumptions formulated on the pressure law. From now on, we fix the choice

$$H(z) \,=\, z \int_1^z \frac{P(s)}{s^2}\,\mathrm{d}s\,. \tag{5.3}$$

Next, we define, for $z \geq 0$ and $\overline{z} > 0$, the functional

$$\mathcal{H}\big(z\,\big|\,\overline{z}\big) \,:=\, H(z) \,-\, H(\overline{z}) \,-\, H'(\overline{z})\,\big(z \,-\, \overline{z}\big) \tag{5.4}$$

as the Bregman divergence associated to the convex function H.

Finally, we define the energy associated to a couple of functions $\big(\rho, u\big)$, representing the density and the velocity field of a fluid respectively, as the sum of the kinetic and internal energies. More concretely, in the case of Ω bounded, we define the function

$$\mathcal{E}\big(\rho, u\big) \,:=\, \int_\Omega \left(\frac{1}{2}\,\rho\,|u|^2 \,+\, H(\rho)\right)\,\mathrm{d}x\,. \tag{5.5}$$

In the case in which Ω is unbounded, instead, it is better to measure the variations of the density with respect to a reference state, which, without loss of generality, we can assume to be 1[1]. Then, we set

$$\mathcal{E}\big(\rho, u\big) \,:=\, \int_\Omega \left(\frac{1}{2}\,\rho\,|u|^2 \,+\, \mathcal{H}\big(\rho\,\big|\,1\big)\right)\,\mathrm{d}x\,. \tag{5.6}$$

[1] Notice that the couple $(1, 0)$ is a static state of equations (5.1). More general static states can be considered, in different situations; see Chapter 8 below, for instance.

Then, performing an energy estimate on the momentum equation, for any solution $\big(\rho,u\big)$ to system (5.1), we (formally) obtain the following energy inequality: for any $t\geq0$, one has

$$\mathcal{E}\big(\rho,u\big)(t)\,+\,\int_0^t\int_\Omega\Big(\mu\,|\nabla u|^2\,+\,\lambda\,|\mathrm{div}\,u|^2\Big)\,\,\mathrm{d}x\,\,\mathrm{d}t\,\leq\,\mathcal{E}\big(\rho_0,u_0\big)\,, \tag{5.7}$$

where we have set $\mathcal{E}\big(\rho,u\big)(t)\,=\,\mathcal{E}\big(\rho(t),u(t)\big)$ and where we have assumed that we have fixed the initial datum $\big(\rho_0,u_0\big)$ for the problem (5.1). Of course, we assume that the initial datum has been chosen of finite energy, *i.e.* so that $\mathcal{E}\big(\rho_0,u_0\big)<+\infty$.

Notice that, proceeding formally, one would rather get an equality, instead of the previous relation; however, at the level of the theory of weak solutions (where an approximation procedure is needed to solve the equations), one can rigorously prove that only the inequality (5.7) is satisfied.

We also remark that the previous estimate is valid as soon as there is no contribution from the boundary of the domain Ω. Thus, the energy inequality (5.7) holds in the whole space case $\Omega=\mathbb{R}^d$ (or even in the d-dimensional torus $\Omega=\mathbb{T}^d$), or whenever Ω has smooth boundary $\partial\Omega$, provided the equations are supplemented with homogeneous (no-slip or complete-slip) boundary conditions. We will see in Section 5.3 how the previous relation has to be modified in case some contribution from the boundary terms would appear.

5.1.2 Lions-Feireisl theory of weak solutions

Based on the energy inequality (5.7), in the late 90's Lions [177] (with subsequent improvements by Feireisl and collaborators [135, 123]) was able to set down a theory of weak (distributional) solutions at the level of regularity imposed by that relation. These are called *finite energy weak solutions*, and they are known to be globally defined in time. This theory plays the same role of the Leray solutions for the incompressible (homogeneous) Navier-Stokes system.

In this subsection, let us briefly highlight some delicate points of the analysis.

First of all, we remark that, from the energy inequality (5.7), one can only deduce that ρ (or $\rho-1$, in the case we consider an unbounded domain Ω) belongs to $L^\infty\big(\mathbb{R}_+;L^2(\Omega)+L^\gamma(\Omega)\big)$. The L^2 part comes from the region in which $\rho\approx1$ (the so-called "essential part"), whereas the growth of the pressure function (5.2) at $+\infty$ yields only a bound in L^γ in the complementary region (the so-called "residual part" of the domain). In particular, one does not dispose of any L^∞ bound for the density at the level of regularity of the finite energy weak solutions. Nonetheless, one can deduce bounds for the velocity field u alone in $L^2_{\rm loc}\big(\mathbb{R}_+;H^1(\Omega)\big)$ from viscosity and Sobolev embeddings.

As a consequence of the previous bound on the density function, the pressure $P(\rho)\approx\rho^\gamma$ is seen to be only $L^\infty\big(\mathbb{R}_+;L^1(\Omega)\big)$, and this is of course a problem in the proof of existence of weak solutions, which consists, as previously hinted to, in constructing smooth approximate solutions and passing to the limit in the approximation parameters by a compactness argument. Then, it is not clear that limit of the pressure function applied to the approximate densities is in fact the pressure function computed at the limit density. The first point to solve this issue is to get additional integrability for the density functions: this follows from a delicate and intricate argument, based on the use of harmonic analysis tools (in the whole space case) and of Bogovskii operator (for bounded Ω). Next, in order to guarantee that the limit of the pressure functions is indeed the pressure function related to the limit density, one needs some strong convergence properties of the approximate densities. For this, one uses properties related to the *effective viscous flux* (see what said in Chapter 4; more details will be given in the next subsection), together with a convexity argument: with these ingredients at disposal, one can thus control the oscillations in the density functions and prove in fact that, if there are no oscillations at initial time, then they must be zero also at later times.

We remark that carrying out the previous argument for obtaining the strong convergence of the densities requires the fundamental restriction $\gamma>d/2$ on the index appearing in (5.2). In addition, we stress the fact that the use of the effective viscous flux is based on the *isotropy* of the viscous stress tensor, meaning that the viscosity coefficients μ and λ are uniform in all the three space directions. Some improvements in this direction have recently been obtained in [25], but the general picture is still unclear. Isotropy of the viscous stress tensor will play a role in the study of singular limit problems for rotating fluids, see in particular the paragraph related to Ekman boundary layers in Section 8.5. Finally, we point out that the Lions-Feireisl theory of global in time finite energy weak solutions suffers of the same pathologies as Leray's theory of weak solutions for the incompressible Navier-Stokes sytem: their uniqueness and regularity, together with the validity of the energy equality, are outstanding open questions.

5.1.3 Hoff theory for shock data

It is clear that the weak solutions theory exposed above lacks of enough properties to give any kind of qualitative information on the solutions. Strong solutions theory does allow for a more precise description of the dynamics, but it imposes limitations on the kind of phenomena one can describe: for instance, considering densities which have discontinuities at an interface (a situation which frequently occurs in nature, and so is highly desirable to capture theoretically) is out of reach in that framework.

It turns out that, in parallel to the weak solutions theory by Lions-Feireisl, Hoff [155, 156] developed a theory for the so-called "shock data" (namely, initial data for which the initial density is discontinuous) and constructed global in time finite energy weak solutions emanating from finite energy initial data having sufficiently small energy. The smallness of the initial energy here is crucial; this assumption marks the difference with what exposed in Subsection 5.1.2 about the other class of finite energy weak solutions to system (5.1). Let us immediately point out that, in work [156] devoted to the multi-dimensional case, Hoff missed the compactness arguments of Lions-Feireisl (and in fact, the existence result in that paper is stated only for linear pressure laws $P(\rho) = A\rho$), but the existence theory for large data [177, 123] uses in a fundamental way ideas which go back to Hoff theory (and in turn rely on some observations of Serre [212] in 1-D).

The key observation of Hoff was that, although the density ρ and $\operatorname{div} u$ may present discontinuities, the quantity

$$F := \operatorname{div} u - \frac{1}{\nu}\Big(P(\rho) - P(1)\Big) \qquad\qquad (\text{with } \nu := \mu + \lambda),$$

called *effective viscous flux*, remains continuous in the dynamics. This property, together with the smallness of the initial energy, allowed Hoff to derive, under suitable assumptions, additional regularity also for other quantities, namely the vorticity $\omega := \nabla \times u$ of the fluid and the advective derivative $\dot{u} := \partial_t u + u \cdot \nabla u$. Thus, for almost any time $t > 0$, $\omega(t)$ and $F(t)$ are proved to be $H^1(\mathbb{R}^d)$, with moreover $F(t)$ Hölder continuous over $\mathbb{R}^d$, although they are not so at time $t = 0$; as a matter of fact, the proof relies on energy estimates with time weights. In addition, this argument yields a global L^∞ bound for the density variations $\rho(t) - 1$.

In a subsequent paper [157], Hoff gave a description of the propagation of discontinuities of the density in 2-D, for all times. Thus, if the initial density is piecewise α-Hölder continuous for some $\alpha \in]0, 1[$, but it has a sufficiently small jump across a Jordan curve $\mathcal{K}_0$ of class $\mathcal{C}^{1+\alpha}$, then (under a smallness condition on the initial datum and under a slightly better regularity assumption on the initial velocity) one can prove that the gradient of the velocity field u satisfies $\nabla u \in L^1_{\rm loc}\big(\mathbb{R}_+; L^\infty(\mathbb{R}^d)\big)$ and that both $\rho(t) - 1$ and $\operatorname{div} u(t)$ are piecewise α-Hölder continuous, presenting a jump discontinuity across the $\mathcal{C}^{1+\alpha}$ Jordan curve $\mathcal{K}_t$, which is simply the initial curve $\mathcal{K}_0$ transported by the flow of the solution. In particular, discontinuities on the density function are not smoothed out by the effect of the viscosity, but they instead persist for all times (although the size of the jump decreases exponentially in time).

The advantage of the previous result is that it gives a rather precise description of the evolution of the density discontinuities for all times. On the other hand, it only holds true in the two-dimensional case and under the very severe assumption of linear pressure laws $P(\rho) = A\rho$, *i.e.* $\gamma = 1$. Another limitation is that both results of [156, 157] only give existence of solutions, but they are not able to guarantee uniqueness.

Uniqueness of solutions in this framework was tackled by Hoff afterwards, in paper [158]. The result was obtained through stability estimates in $L^2\big([0,T] \times \mathbb{R}^d\big)$ for the velocity u and in $L^\infty\big([0,T]; H^{-1}(\mathbb{R}^d)\big)$ for the density ρ. In order to carry out the stability estimates, Hoff formulated a series of assumptions which had to be verified by the solutions $\big(\rho_j, u_j\big)_{j=1,2}$, and then he checked that almost all those assumptions were verified by the solutions he constructed in his previous works. As a matter of fact, a couple of assumptions were elusive and remained as true hypotheses in the uniqueness result. The first one reads $\nabla u_j \in L^1\big([0,T]; L^\infty(\mathbb{R}^d)\big)$; it was necessary in order to pass to Lagrangian coordinates and avoid, in this way, the loss of derivatives caused from the (hyperbolic) equation satisfied by the density function. Of course, the solutions constructed in paper [157] satisfied that assumption, but (as declared by Hoff in his paper, see page 1743 of [158]) a precise characterisation of initial data which generate solutions possessing that property was unclear. The second condition was imposed on the pressure function: for passing from a H^{-1} control on the pressure $P(\rho)$ to a H^{-1} control on ρ, he asked either for (roughly) a L^p bound for the gradient of the density (an assumption which is not well-suited in order to describe densities having jump discontinuities) or for the pressure law to be linear, hence again $P(\rho) = A\rho$. This allowed him to close the stability estimates.

5.2 Compressible Navier-Stokes: beyond Hoff's theory

In paper [84], we generalised Hoff's theory, obtaining a description of the evolution of the density discontinuities together with uniqueness for general pressure laws (in fact, they do not even need to be monotonically increasing) and in any space dimension $d \geq 1$. In order to explain the result in this direction, some preliminary notation is needed.

First of all, we assume that the pressure function satisfied

$$P(\rho) \,\in\, W^{1,\infty}_{\rm loc}(\mathbb{R}_+)\,, \qquad \text{with} \qquad P(1) = 0\,.$$

Observe that the last condition is certainly not restrictive. In addition, we define the two quantities

$$\sigma \,:=\, \rho - 1 \qquad \text{and} \qquad v \,:=\, -\frac{1}{\nu}\,\nabla(-\Delta)^{-1}P(\rho)\,. \tag{5.8}$$

Recall that we have set $\nu \,:=\, \mu + \lambda$ to be the sum of the two viscosity coefficients. As a matter of fact, the vector field v is defined in [84] as $v \,:=\, -\,\nabla(\mathrm{Id} - \nu\Delta)^{-1}P(\rho)$, as the operator $(\mathrm{Id} - \nu\Delta)^{-1}$ is better behaved for low frequencies than $(-\Delta)^{-1}$. However, for the scopes of this presentation it is enough to consider the previous definition. Then, we introduce the *effective velocity*

$$w \,:=\, u \,-\, v\,.$$

Observe that $\mathrm{div}\, w \,=\, F$ is the effective viscous flux of Hoff. The point is that, from the momentum equation, we find that w satisfies the following equation:

$$\partial_t w \,+\, \mathcal{L}w \,=\, G\,, \tag{5.9}$$

where we have defined the Lamé operator $\mathcal{L}$ by the formula $\mathcal{L}w \,=\, -\,\mu\,\Delta w \,-\, \lambda\,\nabla\mathrm{div}\, w$ and the forcing term G by

$$G \,=\, -\,(1+\sigma)(w+v)\cdot\nabla(w+v) \,-\, \sigma\,\partial_t w \,+\, \frac{1}{\nu}\,(1+\sigma)\,\nabla(-\Delta)^{-1}\partial_t P(1+\sigma)\,. \tag{5.10}$$

The point is that $\mathcal{L}v + \nabla P(\rho) = 0$, and this cancellation allows us to avoid the presence of the term $\nabla P(\rho)$ (which would demand a control on one derivative of the density) in the right-hand side of equation (5.9). Observe that, owing to the low frequency corrector $(\mathrm{Id} - \nu\Delta)^{-1}$, in [84] this relation is not exactly zero, but the result is however more regular than $P(\rho)$.

Now, the idea is to apply *maximal regularity* estimates to the parabolic problem (5.9), an approach which only requires integrability of the right-hand side G, but no regularity. First of all, we observe that dealing with the convective term, namely the first term appearing in (5.10), requires to develop a maximal regularity type estimates also for first order and zero-th order derivatives of w. We point out that, from the maximal regularity approach, we are able to retrieve the information $\operatorname{div} w \in L^1_T(L^\infty)$. Thus, if initially σ_0 is small, say $\|\sigma_0\|_{L^\infty} \leq \varepsilon$ for a small enough $\varepsilon > 0$, then we can propagate this information by the mass equation and say that σ remains small, at least locally in time. Thus, the $\partial_t w$ term appearing in the definition of G becomes just a remainder, to be absorbed into the left-hand side in the maximal regularity estimates. Finally, one more use of the mass equation guarantees us that the last term on the right of (5.10) is of lower order.

In the end, we are able to close the estimates in some finite time interval $[0,T]$. The statement can be roughly stated as follows.

Theorem 5.1. *Let $d < p < +\infty$ and $1 < r < r^* = r^*(d,p)$. Assume that $\sigma_0 \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies $\|\sigma_0\|_{L^\infty} \leq \varepsilon$, for some small $\varepsilon > 0$. Assume also that the initial velocity u_0 is such that $w_0 := u_0 - v_0$, with $v_0 = -\frac{1}{\nu}\nabla(-\Delta)^{-1}P(1+\sigma_0)$, belongs to the homogeneous Besov space $\dot{B}^{2-2/r}_{p,r}(\mathbb{R}^d)$.*

Then there exist a time $T > 0$ and a solution $\big(\rho, u\big)$ of the barotropic Navier-Stokes system (5.1) on $[0,T] \times \mathbb{R}^d$ such that:

- *$\sigma := \rho - 1$ verifies $\|\sigma\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq 4\varepsilon$, together with $\sigma \in \mathcal{C}^0\big([0,T]; L^q\big)$ for all $p \leq q < +\infty$;*
- *$u = v + w$, where v is defined as in (5.8), so in particular $\sigma \in \mathcal{C}^0\big([0,T]; W^{1,q}\big)$ and w satisfies*

$$
\begin{aligned}
& w \in \mathcal{C}^0\big([0,T]; \dot{B}^{2-2/r}_{p,r}\big) \cap L^{r_0}\big([0,T]; L^\infty\big)\,, \\
& \nabla w \in L^{r_1}\big([0,T]; L^p\big)\,, \qquad \partial_t w\,,\ \nabla^2 w\ \in L^r\big([0,T]; L^p\big)\,,
\end{aligned}
$$

for two suitable indices r_0 and r_1.

Remark that the functional framework is slightly subcritical here. As a matter of fact, in order to be critical one sould have $2 - \frac{2}{r} = -1 + \frac{d}{p}$, whereas here there holds $\frac{2}{r} + \frac{d}{p} < 3$ under our assumptions.

Next, let us comment on uniqueness. Instructed by the work of Hoff [158] and more recent works by Danchin [82, 44], the key point in order to avoid the hyperbolicity of the mass equation (which would cause a loss of derivatives, thus requiring a higher order regularity condition on the density function) is to pass to Lagrangian coordinates. For doing so, having $\nabla u \in L^1_T(L^\infty) = L^1\big([0,T]; L^\infty(\mathbb{R}^d)\big)$ seems to be a minimal requirement. Now, owing to maximal regularity estimates and Sobolev embeddings, the condition $p > d$ ensures that $\nabla w \in L^1_T(L^\infty)$. On the other hand, one has

$$
\nabla v \;=\; -\,\frac{1}{\nu}\,\nabla^2(-\Delta)^{-1}P(1+\sigma)\,,
$$

with $P(1+\sigma) \in L^\infty([0,T] \times \mathbb{R}^d)$. However, except in the case $d = 1$ (for which then uniqueness holds true), the operator $\nabla^2(-\Delta)^{-1}$ is a singular integral operator, so it does not map L^∞ into itself. On the other hand, one can easily see that $\operatorname{div} v \,=\, P(1+\sigma) \in L^1_T(L^\infty)$ and $\operatorname{curl} v = 0$ owing to the gradient structure of v. Thus, we are in a quite similar (dual, to some extent) situation as when dealing with the regularity of vortex patches solutions for the 2-D incompressible Euler equations.

Based on this analogy, we follow the pioneering ideas by Chemin [37, 38, 39] and resort to the notion of *tangential regularity* (also often referred to as “striated regularity”). In practice, one fixes a non-degenerate family $\mathcal{X}_0$ of vector fields which are tangent to the hypersurface of discontinuity (say) Σ_0 of the density function. In this context, "non-degenerate" means that, at any point $x \in \Sigma_0$ of the hypersurface, one has $d-1$ vector fields of the family which generate the whole tangent space $T_x\Sigma_0$ of Σ_0 at x, so that one can control all the directions on the tangent space *via* the vector fields of the family. Then, one looks at the derivatives $\partial_X f$ of a function $f \in L^\infty$ along those directions $X \in \mathcal{X}_0$, and at their time evolution.

As it was the case for the 2-D incompressible Euler equations, having striated regularity for σ is enough to deduce that ∇v belongs to $L^1_T(L^\infty)$, so in particular ∇u also belongs to the same space and then one can prove uniqueness (still, the proof is not direct and requires some efforts, see more details below). Roughly speaking, the uniqueness statement can be formulated in the following way.

Theorem 5.2. *Let the assumptions of Theorem 5.1 be in force. Let $\mathcal{X}_0$ be a non-degenerate family of vector fields belonging to L^∞, with gradient in L^p. Assume that, for any $X_0 \in \mathcal{X}_0$, one has $\partial_{X_0}\sigma \in L^p$.*

Then there exists a unique (local in time) solution $\big(\rho, u\big)$ to system (5.1) satisfying the same properties as in Theorem 5.1. In addition, $\nabla u \in L^1_T(L^\infty)$ and one can transport the family $\mathcal{X}_0$ through the flow of u, getting a new family $\mathcal{X}_t$. Then the vector fields of the family $\mathcal{X}_t$ still belong to L^∞ and have gradient in L^p, and the tangential regularity of σ is propagated in time: for any $t \in [0,T]$ and any vector field $X_t \in \mathcal{X}_t$, one has $\partial_{X_t}\sigma(t) \in L^p$.

Notice however that, despite the similarity with the Euler case, the proof of the propagation of the striated regularity here is much more involved. One of the main reasons is that the velocity field u is no more of free divergence, and this affects very much the transport estimates for propagating the integrabilities of the vector fields $X \in \mathcal{X}_t$ and the one of $\partial_X\sigma$. In particular, at some point of the proof one has to establish the property $\partial_X\nabla u \in L^1_T(L^p)$: getting it is based on a very delicate commutator process, combined with the use of the Hardy-Littlewood maximal function.

Last but not least, we point out that, despite the property $\nabla u \in L^1_T(L^\infty)$ holds in this framework, the proof of uniqueness also required some extra work. The main reason for this is that passing in Lagrangian coordinates introduces some variable coefficients in the Lamé operator, making the use of the previous approach (based on maximal regularity estimates) inconclusive; on the other hand, a perturbative argument around the constant coefficients case would involve a dangerous loss of derivatives, which would in turn require more smoothness for the density functions. Thus, the idea is rather to work with the true velocity field u of the fluid and proceed to stability estimates in L^2 norms.

5.3 Compressible flows: statistics and turbulence

In [111, 112], we built up a theory of *statistical solutions* for the barotropic Navier-Stokes system (5.1). In this section, we briefly report on those studies.

Statistical solutions were first introduced in the pioneering works of Foiaş and Prodi [139] on the incompressible Navier-Stokes equations. More works in that context are [218, 65, 140] and, more recently, [141, 142]. The basic idea behind statistical solutions is to fix a measure on the set of all initial data for the system, and to look at how this measure evolves under the flow[2] of the equations. This can be seen as a new attempt to restore uniqueness of solutions, based on the principle that one does not know whether the problem admits or not a unique solution for any reasonable (finite energy) initial datum, but what one really sees is expected to be, "in average", always the same.

[2]Of course, here the word "flow" is not really appropriate, as the problem is not know to be well-posed for low regularity initial data; however, we allow ourselves such a terminology, which we consider quite illustrative.

5.3.1 A semiflow selection approach

In [111], we implemented the previous idea for the problem (5.1), set on a smooth bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and supplemented with non-homogeneous boundary conditions:

$$u_{|\partial\Omega} = u_B \qquad \text{and} \qquad \rho_{|\Gamma_{\rm in}} = \rho_B\,, \tag{5.11}$$

where we have defined

$$\Gamma_{\rm in} := \Big\{ x \in \partial\Omega \;\Big|\; u_B(x)\cdot n(x) < 0 \Big\} \qquad \text{and} \qquad \Gamma_{\rm out} := \Big\{ x \in \partial\Omega \;\Big|\; u_B(x)\cdot n(x) \geq 0 \Big\},$$

where $n(x)$ is the exterior normal to $\partial\Omega$ at x, so that $\partial\Omega = \Gamma_{\rm in}\cup\Gamma_{\rm out}$. Here above, u_B and ρ_B are given profiles; without loss of generality, we can assume them to be defined on the whole closure $\overline{\Omega}$. We assume in addition that $\rho_B \geq \rho_* > 0$.

The notion of statistical solutions developed in [111] is based on the theory of global in time finite energy weak solutions, as presented in Section 5.1 above, and in fact it can be seen as an extension of it, see more details below. We immediately point out that, in this context, the energy inequality (5.7) becomes

$$\begin{aligned}
&\frac{\rm d}{{\rm d}t}\int_\Omega\left(\frac{1}{2}\,\rho\,|u-u_B|^2 + H(\rho)\right)\,{\rm d}x + \int_\Omega\Big(\mu\,|\nabla u|^2 + \lambda\,|{\rm div}\,u|^2\Big)\,{\rm d}x \\
&\qquad + \int_{\Gamma_{\rm out}} H(\rho)\,u_B\cdot n\,{\rm d}S_x + \int_{\Gamma_{\rm in}} H(\rho_B)\,u_B\cdot n\,{\rm d}S_x \\
&\leq -\int_\Omega\Big(\rho u\otimes u + P(\rho)\,{\rm Id}\Big):\nabla u\,{\rm d}x + \frac{1}{2}\int_\Omega \rho\,u\cdot\nabla\,|u_B|^2\,{\rm d}x \\
&\qquad + \int_\Omega\big(\mu\,\nabla u + \lambda\,{\rm div}\,u\,{\rm Id}\,\big):\big(\nabla u_B + {\rm div}\,u_B\,{\rm Id}\,\big)\,{\rm d}x + \int_\Omega \rho\,g\cdot(u-u_B)\,{\rm d}x\,,
\end{aligned} \tag{5.12}$$

where we assume that an external force $\rho\,g$ appears now in the right-hand side of the momentum equation in (5.1). We refer *e.g.* to [36] for an adaptation of the theory exposed in Subsection 5.1.2 to the case of open systems (see also [134]).

As a matter of fact, weak solutions can be seen as a special class of the statistical solutions constructed in [111], for initial measures which are Dirac masses sitting in the data space. This is possible by implementing a semiflow selection procedure in the spirit of Krylov [166], see also [33, 20, 21, 19, 13] for more recent applications. More precisely, define the data space

$$\mathcal{D} := \Big\{ \big[\rho_0, u_0, d\big] \;\Big|\; d = \big[\rho_B, u_B, g\big]\,,\ \rho_0 \in L^\gamma(\Omega)\,,\ u_0 \in L^2(\Omega) \Big\},$$

and look at it as a Borel subset of the Polish space

$$X_{\mathcal{D}} := W^{-k,2}(\Omega) \times W^{-k,2}(\Omega;\mathbb{R}^d) \times \mathcal{C}^0(\partial\Omega) \times \mathcal{C}^1(\overline{\Omega};\mathbb{R}^d) \times \mathcal{C}^0(\overline{\Omega};\mathbb{R}^d)\,.$$

Here, we fix some $k \geq 1 + d/2$. As a matter of fact, one should rather work with the momentum $m = \rho\,u$ rather than with the velocity field u; in addition, one should include the energy functional

$$\mathcal{E}\big(\rho, u \,\big|\, u_B\big) := \int_\Omega\left(\frac{1}{2}\,\rho\,|u - u_B|^2 + H(\rho)\right)\,{\rm d}x \tag{5.13}$$

in the definition of the data space $\mathcal{D}$. However, for the sake of clarity, we skip those details in this (rather informal) discussion. Using the above mentioned semiflow selection, for any $t \geq 0$ one is able to define a Borel-measurable map $\mathbf{U}(t)$ on $\mathcal{D}$ such that

$$\mathbf{U}(t)\,:\,\big[\rho_0, u_0, d\big] \mapsto \big[\rho(t,\cdot), u(t,\cdot), d\big]$$

where the couple $\big(\rho, u\big)$ is a finite energy weak solution to the Navier-Stokes system (5.1) with non-homogeneous boundary conditions (5.11) and external force $\rho\,g$, associated to the initial datum

$\big(\rho_0, u_0\big)$. The map $\mathbf{U}$ selects a unique finite energy weak solution emanating from the initial datum: it turns out that, for any $t \geq 0$ and almost any $\tau > 0$, one has

$$\mathbf{U}(t+\tau) \,=\, \mathbf{U}(t) \,\circ\, \mathbf{U}(\tau)\,,$$

where the "almost any" property depends on the triplet of data $\big[\rho_0, u_0, d\big] \in \mathcal{D}$ fixed.

Thus, given an initial measure $\mathcal{V}_0$ on the data space $\mathcal{D}$, one can use the family of operators $\big(\mathbf{U}(t)\big)_{t\geq 0}$ to define the evolution of $\mathcal{V}_0$ at time t as the push-forward measure by the operator $\mathbf{U}(t)$. In particular, one can construct a family of (Markov) operators $\big\{M_t\big\}_{t\geq 0}$ on the space of positive probability measures on $\mathcal{D}$, which exactly represents the new notion of statistical solutions introduced in [111].

To be more precise and formulate some statement in a more rigorous way, let us introduce the set

$$\mathfrak{P}[\mathcal{D}] \,:=\, \Big\{\text{complete Borel probability measure } \nu \text{ on } X_{\mathcal{D}} \text{ such that } \operatorname{Supp}(\nu) \,\subset\, \mathcal{D}\Big\}\,.$$

Then one has the following definition.

Definition 5.3. A *statistical solution* to the problem (5.1)-(5.11) is a family of (Markov) operators $\big(M_t\big)_{t\geq 0}$ on $\mathfrak{P}[\mathcal{D}]$ which enjoys the following properties:

- $M_0 = \mathrm{Id}$, namely $M_0(\nu) = \nu$ for any $\nu \in \mathfrak{P}[\mathcal{D}]$;
- each M_t is linear on convex combinations: for any $t \geq 0$, for any $\big(\nu_j\big)_{j\in\{1\ldots N\}} \subset \mathfrak{P}[\mathcal{D}]$ and any $\alpha \in \mathbb{R}^N$, with $\alpha_j \geq 0$ for all j and $\sum_j \alpha_j = 1$, one has
$$M_t\left(\sum_{j=1}^{N} \alpha_j\, \nu_j\right) \,=\, \sum_{j=1}^{N} \alpha_j\, M_t(\nu_j)\,;$$
- $\big(M_t\big)_{t\geq 0}$ possesses the following "almost semigroup" property: for any $\nu \in \mathfrak{P}[\mathcal{D}]$, there exists a set of full measure $\mathcal{R}_\nu \subset [0, +\infty[$, with $0 \in \mathcal{R}_\nu$, such that, for all $t \geq 0$ and all $\tau \in \mathcal{R}_\nu$, one has
$$M_{t+\tau} \,=\, M_t \,\circ\, M_\tau\,;$$
- for any $\nu \in \mathfrak{P}[\mathcal{D}]$ and any $t \geq 0$, one has
$$M_t(\nu) \,=\, \int_{\mathcal{D}} \delta_{\big[\rho(t,\cdot),u(t,\cdot),d\big]}\, \mathrm{d}\nu(\rho_0, u_0, d)\,,$$
where $\big(\rho, u\big)$ is a global in time finite energy weak solution to (5.1)-(5.11) supplemented with initial datum $\big(\rho_0, u_0\big)$.

In particular, the last property of the previous definition implies that, if ν_0 is "deterministic" (namely, a Dirac delta sitting on some initial datum), then its evolution $M_t(\nu_0)$ is also deterministic (namely, it is a Dirac delta sitting in the corresponding finite energy weak solution). In this sense, we can claim that the notion of statistical solution introduced in Definition 5.3 represents a suitable extension of the theory of weak solutions.

Now, using the semiflow selection operators $\big(\mathbf{U}(t)\big)_{t\geq 0}$ introduced above, one has the following theorem

Theorem 5.4. *Let $\gamma > d/2$ in* (5.2)*. Then, there exists a statistical solution $\big(M_t\big)_{t\geq 0}$ to problem* (5.1)-(5.11)*, defined by the following procedure.*

Let $\mathcal{V}_0 \in \mathfrak{P}[\mathcal{D}]$. For any $t \geq 0$, defined the probability measure $\mathcal{V}_t$ as the push-forward measure of $\mathcal{V}_0$ by the operator $\mathbf{U}(t)$ introduced above: for any $t \geq 0$ and any bounded continuous function Φ on $X_{\mathcal{D}}$, define

$$\int_{X_{\mathcal{D}}} \Phi(\rho, u, d) \, \mathrm{d}\mathcal{V}_t(\rho, u, d) \, := \, \int_{\mathcal{D}} \Phi \circ \mathbf{U}(t)[\rho_0, u_0, d] \, \mathrm{d}\mathcal{V}_0(\rho_0, u_0, d) \, .$$

Then, for any $t \geq 0$ define $M_t(\mathcal{V}_0) \, := \, \mathcal{V}_t$.

We conclude this part with a series of comments. First of all, we want to point out that, as an immediate consequence of our construction, we have that the measures $\mathcal{V}_t$ constructed in Theorem 5.4 satisfy an averaged version of the equations of conservation of mass and momentum and of the energy inequality. In addition, by use of the *relative energy inequality*, one can establish some continuity properties for statistical solutions which are supported on the set of regular data. On the other hand, we notice that, as the weak solutions associated to some datum are not know to be unique and our construction strongly depends on the performed semiflow selection $\big(\mathbf{U}(t)\big)_{t\geq 0}$, statistical solutions are (not known to be) unique.

Finally, the improvement of our approach, based on the the use of the semiflow selection principle, with respect to previous results on statistical solutions for the incompressible Navier-Stokes equations, consists essentially of two main factors: the "almost semigroup" property and the consistency with the notion of finite energy weak solutions, namely the fact that the image of a Dirac delta remains a Dirac delta at any later time.

5.3.2 Stationary statistical solutions

In [112], we used a statistical solutions approach to investigate the validity of the so-called *ergodic hypothesis* from turbulence theory in the context of open fluid systems, namely of fluid systems verifying the non-homogeneous boundary conditions (5.11). The ergodic hypothesis postulates that, for large enough times the system approaches a statistical equilibrium, hence the statistics of a (turbulent) flow can be completely described by means of a single invariant measure. To be a little bit more precise, given a dynamical system on some state space X, say

$$\mathbf{U} : \, \mathbb{R}_+ \times X \, \longrightarrow \, X \, ,$$

the ergodic hypothesis postulates the existence of a $\mu \in \mathfrak{P}[X] := \big\{\text{probability measures on } X\big\}$ such that, for μ-almost any $x_0 \in X$ and for any (say smooth) function F on X, one has

$$\text{(5.14)} \qquad \text{when } \; T \to +\infty \, , \qquad \frac{1}{T} \int_0^T F\big(\mathbf{U}(t, x_0)\big) \, \mathrm{d}t \, \longrightarrow \, \int_X F \, \mathrm{d}\mu \, .$$

It turns out that such probability measure μ must be invariant for the dynamics. Such a μ, if it exists, completely describes the long-time behaviour of the system, as time-averages, for large enough times, approach the ensemble average with respect to that measure.

As the fluid equations are not known to be well-posed in general, the notion of invariant measure has to be replaced in this context by the one of *stationary statistical solution*. Before going on, let us immediately stress the importance of considering non-homogeneous boundary conditions (5.11) from the modeling point of view. As a matter of fact, according to the Clausius principle (*i.e.* the second law of thermodynamics), for energetically closed systems the dynamics must approach the state with maximal entropy, which is necessarily an equilibrium (static state) of the system; we refer to *e.g.* [136] for results in this spirit for compressible fluid flows. In this case, the ergodic hypothesis is satisfied, the invariant measure μ being the Dirac mass sitting on that equilibrium; however, the support of μ is trivial, so the piece of information encoded by the ergodic hypothesis is quite poor. Accordingly, genuine turbulence can persist in the long run only for energetically open systems, in which the effects of energy dissipation are counterbalanced by the injection of mass and energy through interaction with the exterior at the boundary.

For the sake of coinciseness of the presentation, we avoid to enter into details of the study of [112] and keep the discussion quite unformal, limiting ourselves to convey the basic ideas, without nonetheless making them completely rigorous.

Following [211, 181], we adopted a dynamical system approach, which consists in considering a whole (entire, defined for all $t \in \mathbb{R}$) trajectory as initial datum and in looking at the dynamics generated by time-shift operators: given an entire trajectory $[\rho, u]$ which is a (weak) solution of the Navier-Stokes system, we define the shifted trajectory $S_\tau[\rho, u]$ by the formula

$$\forall\, t \in \mathbb{R}\,, \qquad\qquad S_\tau[\rho,u](t,\cdot)\;:=\;[\rho,u](t+\tau,\cdot)\,.$$

In this context, the ω-limit sets $\omega[\rho, u]$ associated to entire trajectories $\big(\rho, u\big)$ are the right objects to study the long-time behaviour of the system.

Next, pushing forward the point of view of [111], we defined a statistical solution as a stochastic process with continuous paths ranging in the space of entire trajectories and supported by solutions of the problem. This corresponds to lifting the arguments exposed in Subsection 5.3.1 to the space of (entire) trajectories, something which is always possible to do (notice however that, in this new approach, one misses the possibility of performing a semiflow selection). In this sense, statistical solutions of our deterministic problem can be seen as special solutions of the corresponding stochastic PDE with stochastic forcing term and random initial data, in which the stochastic forcing term vanishes. A stationary statistical solution is then a stationary process which is supported by entire (weak) solutions of the problem.

The first problem to face in order to study the validity of the ergodic hypothesis was to show existence of suitable conditions on the boundary data $\big(\rho_B, u_B\big)$ and the external force g able to guarantee that the total energy of the solutions remains bounded for all times $t \in \mathbb{R}$. As a matter of fact, owing to the non-homogeneous boundary conditions (5.11), some energy is injected in the system; however, from a statistical point of view very few can be said on the long-time behaviour of the system, if the total energy becomes unbounded. One has the following result.

Proposition 5.5. *Let the external force $g = g(x)$ be potential, i.e. $g = \nabla G$ for some $G \in \mathcal{C}^1(\overline{\Omega})$, and assume that*

$$\mathbb{D}u_B\;\geq\;0\,, \qquad\qquad \text{with} \qquad \operatorname{div} u_b\;\geq\;\alpha\;>\;0\,.$$

Let $\big(\rho, u\big)$ be a finite energy weak solutions of (5.1)-(5.11) *on $]\tau, +\infty[$, for some $\tau \geq -\infty$. Then, there exists a constant $\overline{E}$ depending only on the data of the problem, such that*

$$\limsup_{t\to+\infty}\;\mathcal{E}\big(\rho(t),u(t)\,\big|\,u_B\big)\;\leq\;\overline{E}\,,$$

where the function $\mathcal{E}\big(\rho, u\,\big|\,u_B\big)$ has been defined in (5.13).

We observe that the assumption on the external force g could be relaxed, so that g needs not be potential, at the price of imposing some more stringent condition on the velocity profile u_B.

Therefore, given some energy level $\overline{E} > 0$, one can define the set

$$\mathcal{U}\big[\overline{E}\big]\;:=\;\left\{\big(\rho,u\big)\quad \text{entire finite energy weak solutions}\;\;\Big|\;\;\sup_{t\in\mathbb{R}}\mathcal{E}\big(\rho(t),u(t)\,\big|\,u_B\big)\;\leq\;\overline{E}\right\}.$$

Owing to Proposition 5.5, one has that $\mathcal{U}\big[\overline{E}\big] \neq \emptyset$. Moreover, one can establish the following fundamental property.

Proposition 5.6. *The set $\mathcal{U}\big[\overline{E}\big]$ is a compact shift-invariant subset of the trajectory space $\mathcal{T} := \mathcal{C}^0\big(\mathbb{R};\widetilde{\mathcal{D}}\big)$, where we have defined $\widetilde{\mathcal{D}}\;:=\;L^1(\Omega)\times W^{-k,2}(\Omega)$.*

The proof of Proposition 5.6 relies on the property of *asymptotic compactness* of bounded trajectories. This fundamental property is derived from the study of a differential inequality for the density oscillation defect $D = D(t)$, defined as in the classical theory [177, 123] by the formula

$$D(t) \;:=\; \int_\Omega \Big(\overline{\rho \log \rho} \;-\; \rho \log \rho \Big)\, \mathrm{d}x \;\geq\; 0\,.$$

The main point is that the differential inequality, which reads

$$\frac{\mathrm{d}}{\mathrm{d}t} D \,+\, \Psi(D) \,\leq\, 0\,, \qquad\qquad \text{with} \qquad \Psi \in \mathcal{C}^0(\mathbb{R}), \quad \Psi(z)z \geq 0 \;\; \forall\, z \neq 0\,,$$

is the only information at one's disposal (as the notion of initial datum, in this approach, is apparently missing). It turns out that the previous relation is however enough to establish that $D \equiv 0$ must vanish identically, providing the sought asymptotic compactness of entire bounded trajectories.

Finally, one can focus on the study of the structure of the ω-limit set $\omega[\widetilde{\rho}, \widetilde{u}]$ associated to an entire bounded trajectory $\big(\widetilde{\rho}, \widetilde{u}\big)$, which is also a compact shift-invariant subset of the trajectory space $\mathcal{T}$. An implementation of the Krylov-Bogoliubov argument thus yields the existence of a stationary statistical solution $[\rho, u]$ ranging on $\omega[\widetilde{\rho}, \widetilde{u}]$, or in other words a measure μ on $\mathcal{T}$ which is shift-invariant and whose support is contained in $\omega[\widetilde{\rho}, \widetilde{u}]$.

Theorem 5.7. *For any* $\big(\widetilde{\rho}, \widetilde{u}\big) \in \mathcal{U}\big[\overline{E}\big]$*, there exists a stationary statistical solution* $[\rho, u]$ *such that* $[\rho, u] \in \omega[\widetilde{\rho}, \widetilde{u}]$ *almost surely.*

At this point, the Birkhoff-Khinchin ergodic theorem implies that, μ-a.s. on $\omega[\widetilde{\rho}, \widetilde{u}]$, one has convergence of the time-averages

$$\frac{1}{T} \int_0^T F\big(\rho(t), u(t)\big)\, \mathrm{d}t \;\longrightarrow\; \overline{F} \qquad\qquad \text{for} \qquad T \to +\infty\,, \tag{5.15}$$

for any (say bounded continuous) function F defined on the space of data. It would be tempting to compare this convergence result with the validity of the ergodic hypothesis (5.14). The problem is that the limit quantity $\overline{F}$ is only an observable, and not necessarily an ensemble average as predicted by the ergodic hypothesis.

In fact, it turns out that the function $\overline{F}$ appearing in (5.15) is a conditional expectation with respect to the σ-algebra of shift invariant sets in $\mathcal{T}$. Thus, $\overline{F}$ is a true espectation (ensemble average) if the stationary statistical solution μ is *ergodic*, that is, for any shift invariant borel set $B \subset \mathcal{T}$, one has $\mu(B) = 0$ or 1. By Krein-Milman theorem, such an ergodic stationary statistical solution always exists, but there may exist more than one statistical solution sitting on a given ω-limit set, so there is no reason, in general, for the measure μ given by Krylov-Bogoliubov to be ergodic.

All in all, the conclusion of our study about the validity of the ergodic hypothesis for energetically open systems can be stated in the following form.

Theorem 5.8. *Let* $\big(\rho, u\big) \in \mathcal{U}\big[\overline{E}\big]$*. If there exists a unique invariant measure* μ *sitting on* $\omega[\rho, u]$*, then the ergodic hypothesis holds true for that trajectory* $\big(\rho, u\big)$ *and for any other trajectory* $\big(r, v\big)$ *in the support of* μ*: for any bounded continuous function* F *on* $\widetilde{\mathcal{D}}$*, one has*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T F\big(\rho(t), u(t)\big)\, \mathrm{d}t \;=\; \int_{\widetilde{\mathcal{D}}} F\, \mathrm{d}\nu_0\,,$$

where $\nu_0 \,:=\, \nu \circ \pi_0^{-1}$ *is the push-forward measure on* $\widetilde{\mathcal{D}}$ *related to* $\pi_0 : \mathcal{T} \longrightarrow \widetilde{\mathcal{D}}$*, the projection of any trajectory at time* $t = 0$*.*

We conclude this section by mentioning that further developments in the theory of statistical solutions for compressible fluid systems can be found in [124, 137, 130], for instance.

5.4 Kolmogorov two-equation model of turbulence

In this section, we are still interested in some questions related to turbulence theory. This time, we will study directly a system of equations derived to describe a fully developed turbulent flow. This is the so-called Kolmogorov two-equation model of turbulence [213], which was introduced by Kolmogorov in 1942 to describe a fluid in a fully developed turbulent regime. The equations read as

$$
\begin{cases}
\partial_t u \,+\, (u\cdot\nabla)u \,+\, \nabla\pi \,-\, \nu \,\mathrm{div}\left(\dfrac{k}{\omega}\,\mathbb{D}u\right) \,=\, 0 \\
\partial_t \omega \,+\, u\cdot\nabla\omega \,-\, \alpha_1 \,\mathrm{div}\left(\dfrac{k}{\omega}\,\nabla\omega\right) \,=\, -\,\alpha_2\,\omega^2 \\
\partial_t k \,+\, u\cdot\nabla k \,-\, \alpha_3 \,\mathrm{div}\left(\dfrac{k}{\omega}\,\nabla k\right) \,=\, -\,k\,\omega \,+\, \alpha_4\,\dfrac{k}{\omega}\,\left|\mathbb{D}u\right|^2 \\
\mathrm{div}\, u \,=\, 0\,.
\end{cases}
\tag{5.16}
$$

Here, as usual, u is the velocity field of the fluid and π its pressure, while ω and k are microscopic quantities representing, respectively, the mean frequency of the turbulent fluctuations and the mean turbulent kinetic energy. In particular, one must have $\omega \geq 0$ and $k \geq 0$. The symbol $\mathbb{D}u$ represents the symmetric part of the Jacobian matrix of u. Finally, the various parameters ν and $\alpha_{1,2,3,4}$ are positive constants; they can be set all equal to 1 for the sake of the present discussion.

Equations (5.16) were introduced by Kolmogorov with not so many explanations. As it appears for other k-ε models in turbulence theory, the equation for k may be justified after derinving the equations for the so-called Reynolds stress and applying the eddy viscosity assumption by Boussinesq and Prandtl; on the contrary, the equation for ω is rather phenomenological and no rigorous theoreticla grounds are given for it, see *e.g.* [93]. However, it is clear that system (5.16) captures very well the mechanism of turbulence by which the energy which is dissipated at large scales by viscosity (the ν-term in the first equation) is transferred, through an energy cascade, to small scales and feeds up (*via* the α_4-term) the turbulent motion.

5.4.1 Previous results on the Kolmogorov two-equation model

Despite the Kolmogorov two-equation model of turbulence (5.16) belongs, to some extent, to the class of k-ε models, broadly studied in the literature (see for instance [199]), its analysis is quite recent. We summarise here the main results obtained about its well-posedness.

In [198], Mielke and Naumann established the existence of global in time "finite energy" weak solutions to (5.16) in the three-dimensional torus $\mathbb{T}^3$. These are solutions analogous, in spirit, to the ones discussed in Section 5.1 for the barotropic Navier-Stokes sytem: they are based on a natural energy inequality for equations (5.16), which in particular implies

$$
u\,,\,\omega \,\in\, L^\infty\big(\mathbb{R}_+;L^2(\mathbb{T}^3)\big) \,\cap\, L^2\big(\mathbb{R}_+;\dot H^1(\mathbb{T}^3)\big)\,.
$$

However, the basic observation is that, at the level of the energy estimates, the function k looks to be not better than $L^\infty\big(\mathbb{R}_+;L^1(\mathbb{T}^3)\big)$. In fact, some gain of integrability for both k and its gradient can be deduced by fine parabolic estimates with source term in L^1, see [198], but we will not enter into the details of that. Here, we want only to comment that the low integrability of k is the main reason for writing "finite energy" between quotation marks above. Related to this issue, one notices in [198] the appearing of a defect measure in the weak formulation of the equation for the mean turbulent kinetic energy k.

Skipping the details, we point out that the analysis of [198] was strongly based on the parabolic effect of the equations, which allows to gain some suitable information on the space gradients ∇u, $\nabla\omega$ and ∇k. For this, it was fundamental for the analysis to find conditions which ensured the non-vanishing of the "viscosity coefficient" k/ω. Thus, the authors formulated the following assumption: there exists positive constants $0 < \omega_* \leq \omega^*$ and $k_* > 0$ such that

$$\omega_* \leq \omega_0 \leq \omega^* \qquad \text{and} \qquad k_0 \geq k_* . \tag{5.17}$$

Now, an ODE structure (or, in other words, the parabolic maximum principle) hidden in system (5.16) allows to propagate those L^∞ bounds also at later times, even at the level of weak solutions: for any $t \geq 0$, one (formally) has

$$\omega_*(t) \leq \omega(t) \leq \omega^*(t) \qquad \text{and} \qquad k_0 \geq k_*(t) ,$$

for suitable functions $\omega_*(t)$, $\omega^*(t)$ and $k_*(t)$ which are decreasing in time, but are, for any $t \geq 0$ fixed, strictly positive. Thus, parabolicity of the system is ensured, together with space compactness for the unknowns u, ω and k.

The same assumption (5.17) played a key role in the strong solutions theory, as developed by Kosewski and Kubica (see [165] for a local in time result, [164] for a global in time result under a smallness condition of the initial datum), still in the domain $\mathbb{T}^3$. However, it should be noticed that, from a physical standpoint, while the bounds appearing in (5.17) for ω_0 look reasonable, the lower bound for k_0 seems to be highly questionable.

The only work dealing with the possible vanishing of k_0 was paper [27] by Bulíček and Málek. There, the authors constructed global in time finite energy weak solutions to system (5.16), with however some important differences with respect to the study of [198]. First of all, Bulíček and Málek introduced a reformulation of the Kolmogorov system, obtained by looking at the total energy function $E := \frac{1}{2}|u|^2 + k$ as a new unknown, to be used in place of k. As a matter of fact, the key remark was that E satisfies a better equation than k. When coming back to the original unknowns $\big(u, \omega, k\big)$, however, they only recovered suitable weak solutions of (5.16), in accordance with the result of [198]. Secondly, the solved the equations in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, supplemented with suitable non-trivial boundary conditions; thus, their study was able to capture boundary-induced turbulent phenomena. Finally, another novelty of the analysis of [27] was the relaxation of the assumption (5.17) concerning the initial turbulent kinetic energy k_0, which was supposed to satisfy $k_0 > 0$ in Ω, together with $\log k_0 \in L^1(\Omega)$.

5.4.2 A one-dimensional reduction

In [114], we focused on a one-dimensional reduction of the Kolmogorov system (5.16): we dismessed the divergence-free condition over u and, correspondingly, we suppressed the pressure term appearing in the momentum equation. The equations reduced then to

$$\begin{cases} \partial_t u + u\,\partial_x u - \nu\,\partial_x\left(\dfrac{k}{\omega}\,\partial_x u\right) = 0 \\ \partial_t \omega + u\,\partial_x \omega - \alpha_1\,\partial_x\left(\dfrac{k}{\omega}\,\partial_x \omega\right) = -\,\alpha_2\,\omega^2 \\ \partial_t k + u\,\partial_x k - \alpha_3\,\partial_x\left(\dfrac{k}{\omega}\,\partial_x k\right) = -\,k\,\omega + \alpha_4\,\dfrac{k}{\omega}\,\left|\partial_x u\right|^2 . \end{cases} \tag{5.18}$$

Our goal was to investigate more in depth well-posedness questions in the case of vanishing mean turbulent kinetic energy k_0. As our focus was not on boundary phenomena, but rather on effects which are produced in the bulk, we set the previous equations on the simple one-dimensional domain $\mathbb{T} = \mathbb{T}^1$.

Our main result can be stated as follows.

Theorem 5.9. *System* (5.18) *is well-posed, locally in time, in the space $H^m(\mathbb{T})$, for any $m \geq 2$. However, there exist smooth initial data such that the corresponding solutions blow up in finite time.*

Concerning the well-posedness statement, we point out that, as the system is now degenerate parabolic, it is absolutely natural to work at Sobolev regularities which are well-suited for hyperbolic equations (namely, $m \geq 2 > 1 + d/2$ for $d = 1$). On the other hand, the proof of the well-posedness contains some subtle points, linked with the degeneracy of the system for $k \approx 0$, which we would like to highlight in what follows. We want also to give more details on the blow-up phenomenon.

It turns out that all the essential points of the analysis can be seen on the following toy-model:

$$
\begin{cases} \partial_t u \,+\, u\,\partial_x u \,-\, \partial_x\big(\xi\,\partial_x u\big) \,=\, 0 \\ \partial_t \xi \,+\, u\,\partial_x \xi \,-\, \alpha\,\partial_x\big(\xi\,\partial_x k\big) \,=\, \xi\,\big|\partial_x u\big|^2 . \end{cases} \tag{5.19}
$$

In the previous system, the unknown u plays the same role as u in (5.18), while the unknown ξ plays the role of the quantity k/ω. In the equation for ξ, we have kept the main terms appearing in the third equation of (5.18), namely the transport, the diffusion with degenerate diffusion coefficient for $\xi \approx 0$ and the energy transfer term $\xi\,|\partial_x u|^2$ on the right-hand side. We have also kept the presence of the viscosity coefficient $\alpha > 0$ in the second equation of (5.19), as it will play a role later on.

In passing, we notice that the toy-model (5.19) shares similarities with the Prandtl model of turbulence and other one-equation models of turbulence, which have been studied *e.g.* in [35, 171] (see also references therein).

We also remark that system (5.19) behaves quite closely to the original 1-D model (5.18) only for higher order norms, or in other terms for high frequencies; if one would like to preserve also the basic energy estimates, one should add a term $-\xi$ on the right-hand side of the equation for ξ.

This having been said, here let us focus on propagation of higher norms only, and more precisely of H^2 regularity; higher regularity can be treated by the same token. We start by differentiating twice the equation for u, getting an evolution equation for $\partial_x^2 u$:

$$
\partial_t \partial_x^2 u \,+\, u\,\partial_x \partial_x^2 u \,-\, \partial_x\big(\xi\,\partial_x \partial_x^2 u\big) \,=\, \big[u, \partial_x^2\big]\,\partial_x u \,+\, \partial_x\left(\big[\xi, \partial_x^2\big]\,\partial_x u\right) ,
$$

where the first commutator term on the right is the classical commutator coming fom the transport term, whereas the second one is produced by the presence of the variable viscosity coefficient ξ. Now, we compute

$$
\partial_x\left(\big[\xi, \partial_x^2\big]\,\partial_x u\right) \,=\, \partial_x\Big(\partial_x^2 \xi\,\partial_x u \,+\, 2\,\partial_x \xi\,\partial_x^2 u\Big) .
$$

Then, we observe that, in an energy estimate (simply test the relation for $\partial_x^2 u$ by $\partial_x^2 u$ itself), there is no chance to control the terms coming from this commutator without using the viscosity, which is however degenerate here. On the other hand, we remark that we can write

$$
\int_{\mathbb{T}} \partial_x\Big(\partial_x \xi\,\partial_x^2 u\Big)\,\partial_x^2 u\,\,\mathrm{d}x \,=\, -\int_{\mathbb{T}} \partial_x \xi\,\partial_x^2 u\,\partial_x^3 u\,\,\mathrm{d}x \,=\, -\,2\int_{\mathbb{T}} \partial_x \sqrt{\xi}\,\partial_x^2 u\,\sqrt{\xi}\,\partial_x^3 u\,\,\mathrm{d}x\,,
$$

which now can be controlled by using the (yet degenerate) parabolic effect of the equations. This simple computation suggests to work with the unknown $\sqrt{\xi}$ rather than ξ. As a matter of fact, it can be seen that formulating the system in terms of $\big(u, \sqrt{\xi}\big)$ yields better properties on the commutator term $\partial_x\left(\big[\xi, \partial_x^2\big]\,\partial_x u\right)$, which becomes now under control.

Therefore, propagation of higher order regularity norms is obtained by working with $\sqrt{\xi}$ instead of ξ. The well-posedness part of Theorem 5.9 has to be ment in this sense: H^m regularity is formulated on $\sqrt{\xi}$ (which implies, in particular, that also ξ belongs to H^m).

Next, let us formulate a more precise blow-up result.

Theorem 5.10. *Let* $\big(u_0, \sqrt{\xi_0}\big) \in H^m(\mathbb{T}) \times H^m(\mathbb{T})$, *with* $m \geq 3$, *be such that:*

- u_0 *is odd with respect to the origin and* ξ_0 *even;*

- $\xi_0(0) = 0$;
- $\partial_x u_0 < 0$.

Then, there exists a time $T > 0$ such that, if the corresponding solution $\big(u, \xi\big)$ of system (5.19) *has not blown up before at a different place, then*

$$\lim_{t \to T^-} \partial_x u(t,0) \,=\, -\infty\,.$$

We observe that the previous blow-up result is very similar to the blow-up which occurs for the Burgers equation, and in fact one may object that, the equations (5.19) being local, the same phenomenon has to produce also in this context. Nonetheless, let us remark that here we are really at the boundary of Burgers theory. As a matter of fact, the function ξ_0 may vanish even only at one point, namely at $x = 0$, whereas it is easy to see that, if it is strictly positive everywhere, then global existence of smooth solutions holds. In addition, as ξ_0 may vanish at the origin only, there is no apparent reason for which the viscosity (which acts in any other point of the domain) should not work in order to smooth everything out and prevent the blow-up.

Inspired by the previous considerations, in [115] we were able to improve the previous blow-up result for the toy-model (5.19), and actually for a broader class of systems (see also [197] in this respect). More precisely, in a situation where the slope of u remains bounded (opposite then to the Burgers situation of Theorem 5.10), we proved blow-up of the curvature of the function ξ. The statement can be roughly formulated as follows.

Theorem 5.11. *Let $\big(u_0, \sqrt{\xi_0}\big) \in H^m(\mathbb{T}) \times H^m(\mathbb{T})$, with $m \geq 5$, be such that:*

- *u_0 is odd with respect to the origin and ξ_0 even;*
- *$\xi_0(0) = 0$, with $3\,\alpha\,\partial_x^2 \xi_0(0) > 1$;*
- $\partial_x u_0 \geq 0$.

Then, there exists a time $T > 0$ such that, if the corresponding solution $\big(u, \xi\big)$ of system (5.19) *has not blown up before at a different place, then*

$$\lim_{t \to T^-} \partial_x^2 \xi(t,0) \,=\, +\infty\,.$$

5.5 Some open questions and perspectives

We conclude the chapter with a list of some questions which drive our attention and which we plan to address in the near future.

Theory of compressible flows with shock data

Our first concern is to complete the well-posedness theory for viscous compressible fluids in presence of discontinuities of the density function.

To begin with, let us remark that the analysis of [84], explained in Section 5.2, holds only locally in time. This is in stark contrast with Hoff's theory [156, 157, 158], which instead holds true globally in time. Therefore, the first question which deserves attention in this context is whether or not it is possible to extend the results from [84] globally in time. The use of time-weighted energy estimates, like in Hoff, seems to be necessary. Yet, it is not clear how to reach a critical functional framework in our context, as the condition $\frac{2}{r} + \frac{d}{p} < 3$ seemed to be necessary in our approach.

A second main question concerns the case of heat-conducting fluids, whose dynamics is described by the full Navier-Stokes-Fourier system. In this direction, one would like to extend the result of [84] to fluids which present also temperature variations. Owing to the diffusive nature of the temperature equation, we expect that an approach based on maximal regularity would work also in this context, giving parabolic smoothing also for the temperature variable. However, it would be important to be able to get estimates independent of the diffusivity coefficient, in order to capture also the limiting case in which the temperature is simply advected by the velocity field. This limiting case seems particularly interesting because of its relations with two-fluid models and because it occurs in some approximated model for geophysical flows, see *e.g.* [128].

Well-posedness of turbulence models

The results exposed in Section 5.4, concerning well-posedness and singularity formation for the Kolmogorov two-equation model of turbulence, have some limitations from the theoretical viewpoint, inasmuch as the 1-D reduction (5.18) is, of course, not really physical. In particular, because of the absence of the incompressibility condition and of the pressure term, the model misses non-local effects, which encode the long-range interactions appearing in physics theories.

In this context, some questions then arise. First of all, one could extend the well-posedness and ill-posedness investigations to a non-local (yet one-dimensional) model, obtained by projecting the equation through the Hilbert transform $\mathbb{H}$ (which plays the same role as the Leray-Helmoltz projector in 1-D, as it is the only singular integral operator in one space dimension). One may hope for blow-up results in the same spirit of the celebrate work [200] by Montgomery-Smith about a 1-D reduction of the incompressible Navier-Stokes equations.

On the other hand, it would be highly desirable to extend the investigation to the full model (5.16) in any space dimension $d \geq 2$, and for sharp Sobolev (or critical Besov) regularities. The well-posedness results are like to hold, although the proof looks rather intricate, essentially because integration by parts have now to be replaced by a careful commutator process. Ill-posedness results are instead much less clear, because the arguments used in [114, 115] are essentially 1-D, even though the result of Theorem 5.11 looks much more flexible. In this respect, however, it is worth to point out that Theorem 5.11 holds true only for the toy-model (5.19) and it has not been proved yet for the Kolmogorov system, not even for its 1-D reduction.

Chapter 6

Well-posedness for inviscid fluid flows

In this chapter, we turn our attention to some inviscid fluid systems. We will discuss several models, putting in evidence common features and, at the same time, properties pertinent to the specific structure of each of them. We are mainly interested in developing a well-posedness theory in critical Besov spaces. As we will see, owing to the hyperbolic nature of the equations, the critical regularity for this kind of problems is dictated by the embedding

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\mathbb{R}^d) . \tag{6.1}$$

Because of the quasi-linear hyperbolic nature of the equations (with no additional structure given, for instance, by the linearisation around a special equilibrium [14, 15, 104]), all our results will be only local in time. However, sometimes we will investigate lower bounds for the lifespan of the solutions, especially in two space dimensions. In particular, by taking advantage of the (formal) proximity of the system under study to the incompressible Euler system, we will derive bounds able to say that, in the regime of small heterogeneity (the density, or the magnetic field...), say of size $\varepsilon > 0$, the lifespan T_ε of the corresponding solution tends to be larger and larger, namely $T_\varepsilon \longrightarrow +\infty$ when $\varepsilon \to 0^+$.

This chapter unfolds as follows. After a brief introduction, we will specialise on various models, namely a quasi-incompressible Euler system (see Section 6.2), the ideal magnetohydrodynamics (MHD in short, see Section 6.3) and finally a system for incompressible non-homogeneous fluids with odd viscosity (treated in Section 6.4).

We immediately point out that, strictly speaking, fluids with odd viscosity do present a viscous effect. However, only the skew-symmetric part of the viscous stress tensor appears, so this term does not dissipate energy (and in fact the odd viscosity term does not contribute to the energy balance at all). In particular, the system exhibits a hyperbolic-type behaviour, rather than parabolic. This is why we discuss this system in the present chapter.

Works presented in the chapter

(P.7) F. Fanelli, X. Liao: *The well-posedness issue for an inviscid zero-Mach number system in general Besov spaces.* **Asymptot. Anal.**, 93 (2015), n. 1-2, 115-140.

(P.8) F. Fanelli, X. Liao: *Analysis of an inviscid zero-Mach number system in endpoint Besov spaces for finite-energy initial data.* **J. Differential Equations**, 259 (2015), n. 10, 5074-5114.

(P.29) D. Cobb, F. Fanelli: *Elsässer formulation of the ideal MHD and improved lifespan in two space dimensions.* **J. Math. Pures Appl.** (9), 169 (2023), 189-236.

(S.4) F. Fanelli, R. Granero-Belinchón, S. Scrobogna: *Well-posedness theory for non-homogeneous incompressible fluids with odd viscosity.* Submitted (2022).

Not mentioned, but in this context

(P.20) D. Cobb, F. Fanelli: *Rigorous derivation and well-posedness of a quasi-homogeneous ideal MHD system.* **Nonlinear Anal. Real World Appl.**, 60 (2021), Paper n. 103284.

(P.21) F. Fanelli, E. Feireisl: *Some remarks on steady solutions to the Euler system in* $\mathbb{R}^d$. **Appl. Math. Lett.**, 116 (2021), Paper n. 107031.

(C.5) D. Cobb, F. Fanelli: *Symmetry breaking in ideal magnetohydrodynamics: the role of the velocity.* **J. Elliptic Parabol. Equ.**, 7 (2021), n. 2, 273-295.

6.1 Introduction

One of the main examples, if not the most emblematic example, of inviscid fluid equations is with no doubts the incompressible Euler equations

$$\begin{cases} \partial_t u \,+\, u\cdot\nabla u \,+\, \nabla\Pi \,=\, 0 \\ \operatorname{div} u \,=\, 0\,, \end{cases} \tag{6.2}$$

where, as usual, u represents the velocity field of the fluid and Π its (scalar) pressure field. Throughout this chapter, we assume that the fluid occupies the whole space $\mathbb{R}^d$, with $d \geq 2$, so that the previous system is set on $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$.

We refer to *e.g.* [182, 39, 180], or to Chapter 7 of [8] for an extensive study of the incompressible Euler equations. Here, we limit ourselves to recall that, owing to the quasi-linear hyperbolic structure given by the transport term $\big(\partial_t + u\cdot\nabla\big)u$, it is natural to solve equations (6.2) in spaces[1] $B^s_{p,r}$ for which the embedding property (6.1) holds true, namely for indices $(s,p,r)\in\mathbb{R}\times[1,+\infty]\times[1,+\infty]$ such that

$$s > 1+\frac{d}{p} \quad \text{and} \quad r\in[1,+\infty]\,, \qquad \text{or} \qquad s = 1+\frac{d}{p} \quad \text{and} \quad r=1\,. \tag{6.3}$$

As a matter of fact, the Euler system (6.2) is well-posed in these spaces, in general only locally in time. We point out that, in the endpoint case $p=+\infty$, an integrability condition (typically, L^2) for the velocity field is needed in the analysis.

As is well-known, the two-dimensional case plays a special role in the context of the incompressible Euler equations. As a matter of fact, one can introduce the *vorticity* of the fluid, defined as twice the skew-symmetric part of the Jacobian matrix of u. In 2-D, the vorticity can be simply identified with the scalar function

$$\omega \,:=\, \operatorname{curl} u \,=\, \partial_1 u^2 \,-\, \partial_2 u^1\,,$$

from which the velocity field u can be computed (at least formally) *via* the Biot-Savart law

$$u \,=\, -\nabla^\perp(-\Delta)^{-1}\omega\,.$$

Passing to the vorticity formulation of equations (6.2) erases the pressure term, which is a quadratic quantity[2], from the equations; however, in general it creates different bad (quadratic) terms, like the vortex stretching term. The advantage of working in 2-D is that those quadratic

[1] Here, we focus on the large class of non-homogeneous Besov spaces, which in particular includes the classical Sobolev and Hölder classes as special cases.

[2] We recall that, by taking the divergence of (6.2), one finds that Π satisfies

$$-\Delta\Pi \,=\, \operatorname{div}\big(u\cdot\nabla u\big)\,,$$

so that $\nabla\Pi \,=\, \nabla(-\Delta)^{-1}\operatorname{div}\big(u\cdot\nabla u\big)$.

terms identically vanish, thanks to the special structure of the convective term and the divergence-free condition on u. All in all, as already mentioned in Chapter 4, in 2-D the vorticity satisfies the simple transport equation

$$\partial_t\omega \,+\, u\cdot\nabla\omega \,=\, 0\,. \tag{6.4}$$

Owing to the fact that $\operatorname{div} u = 0$, this relation endows (at least formally) the system of an infinite number of conservation laws, namely

$$\forall\, p\in[1,+\infty]\,,\quad \forall\, t\geq 0\,,\qquad \|\omega(t)\|_{L^p} \,=\, \|\omega_0\|_{L^p}\,.$$

This property plays a key role in the theory and allows to prove that, in 2-D, the Euler system is globally in time well-posed. The simple equation (6.8) stands also at the basis of several low regularity theories for the incompressible Euler equations, like the Yudovich theory of vortex patches and the Delort theory of vortex sheets (see again [182, 39, 180]); we will not enter into the details of those topics here.

In this discussion, we want to insist rather on the global in time well-posedness of the Euler system in two space dimensions. As a matter of fact, this property seems to be lost, in general, whenever one perturbes the incompressible Euler equations with any kind of heterogeneity.

The probably simplest perturbation one may think of is the case when the fluid presents variations of density. In this case, according to the principle of conservation of mass and the Newton law, system (6.2) becomes

$$\begin{cases} \partial_t\rho \,+\, u\cdot\nabla\rho \,=\, 0 \\ \rho\Big(\partial_t u \,+\, u\cdot\nabla u\Big) \,+\, \nabla\Pi \,=\, 0 \\ \operatorname{div} u \,=\, 0\,. \end{cases} \tag{6.5}$$

We observe that, at least in absence of vacumm, *i.e.* for initial densities satisfying

$$0 \,<\, \rho_* \,\leq\, \rho_0 \,\leq \rho^*\,, \tag{6.6}$$

a property which is preserved by the flow (as u is incompressible), namely

$$\forall\, t\geq 0\,,\qquad \rho_* \,\leq\, \rho(t) \,\leq \rho^*\,,$$

at least for smooth enough solutions, we observe that we can divide the momentum equation in (6.5) by ρ and recast the system as, again, a coupling of transport equations. So, one can expect to solve (6.5) in the same functional classes for which system (6.2) is well-posed, namely in Besov spaces $B^s_{p,r}$ satisfying (6.1), or equivalently (6.3). This turns out to be indeed the case, see *e.g.* [81, 83] and references therein.

However, in stark contrast with the state-of-the-art mentioned above about the classical Euler equations, all the results known so far for system (6.5) are only *local in time*. This is an effect purely due to the non-homogeneity, namely the presence of a variable density ρ. As a matter of fact, the presence of the variable density entails two fundamental differences with respect to the homogeneous case. The first one concerns the analysis of the pressure, which now satisfies an elliptic equation with variable coefficients:

$$-\,\operatorname{div}\left(\frac{1}{\rho}\,\nabla\Pi\right) \,=\, \operatorname{div}\big(u\cdot\nabla u\big)\,. \tag{6.7}$$

This equation, however, implicates some difficulties only at the technical level, in the analysis of the regularity of the pressure gradient. The second main difference with the homogeneous case is instead much deeper and concerns the vorticity. Let us restrict to the case $d = 2$; dividing the momentum equation by ρ and computing the curl of the obtained relation, we find that, this time, the vorticity function ω satisfies

$$\partial_t\omega + u\cdot\nabla\omega + \nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\Pi = 0\,. \tag{6.8}$$

Namely, an additional term appears with respect to the case $\rho\equiv 1$; this additional term depends both on the density and on the pressure gradient and is responsible for production of vorticity in the dynamics. Keeping into account that $\nabla\Pi$ is a quadratic quantity and that the higher order norm of the density grows exponentially in the velocity field (owing to the transport equation for ρ), it is clear that the presence of this additional term is responsible for the local in time nature of the existence results available so far for the density-dependent Euler system (6.5).

Despite this, in [83] we were able to prove, in 2-D, an "asymptotically global" result in the regime of small non-homogeneities, in the following sense: if the size of the initial density variation ρ_0-1 is of size $\varepsilon>0$, then the lifespan $T_\varepsilon>0$ behaves like $\log\big|\log\varepsilon\big|$. More precisely, the lifespan T of a solution corresponding to an initial datum $\big(\rho_0,u_0\big)\in B^1_{\infty,1}\times\big(B^1_{\infty,1}\cap L^2\big)$ can be bounded from below as

$$T \geq \frac{C}{\|u_0\|_{L^2\cap B^1_{\infty,1}}}\log\left(1+C\log\left(1+\frac{1}{\|\nabla\rho_0\|_{B^0_{\infty,1}}}\right)\right). \tag{6.9}$$

It is worth to observe that the classical bound coming from quasi-linear hyperbolic theory would instead be of the form

$$T \geq \frac{C}{\|u_0\|_{L^2\cap B^1_{\infty,1}}+\|\rho_0-1\|_{B^1_{\infty,1}}}\,,$$

which does not imply at all the "asymptotically global" well-posedness (in the sense specified above) of system (6.5). We also point out here the importance of working at critical regularity $B^1_{\infty,1}$. Indeed, at that level of regularity, one can estimate the vorticity ω in the space $B^0_{\infty,1}$: now, for Besov spaces having regularity index $s=0$, one can dispose of improved transport estimates by Vishik [217] and, later, Hmidi and Keraani [153], which provides a growth of the Besov norm of the solution which is *linear* in the Lipschitz norm of the transport field. This linear growth is crucial in order to prove a lower bound like (6.9). On the other hand, a continuation criterion, analogous in spirit to the Beale-Kato-Majda criterion for the 3-D (homogeneous) Euler equations, guarantees us that the lifespan of a smooth solution does not depend on the regularity: the lifespan of a $B^s_{p,r}$ solution, with (s,p,r) satisfying (6.3) and $s\gg 1$, is the same as the lifespan of the solution considered at regularity $B^1_{\infty,1}$.

In fact, we will see in Section 6.3 that, for the previous argument to work, is not really necessary to work at critical regularity. All what one really needs in order to implement the previous procedure is a solid well-posedness theory in some high regularity space, supplemented with a continuation criterion in terms of low regularity norms, and more precisely of norms which can be controlled by the critical $B^0_{p,r}$ regularity.

6.2 A quasi-incompressible Euler system

In papers [118, 117], we extended the analysis of [81, 83] to the case of the following quasi-incompressible Euler system:

$$\begin{cases} \partial_t\rho + \mathrm{div}\,\big(\rho\,v\big) = 0 \\ \partial_t\big(\rho\,v\big) + \mathrm{div}\,\big(\rho\,v\otimes v\big) + \nabla\Pi = 0 \\ \mathrm{div}\,v + \mathrm{div}\,\left(\dfrac{\kappa(\rho)}{\rho}\,\nabla\rho\right) = 0\,, \end{cases}$$

where $\kappa(\rho)$ is a smooth scalar function of the density ρ. This system was rigorously derived by Alazard [1, 2] from the full compressible heat-conducting Euler system, in the regime of low Mach numbers, for large entropy variations.

We refer to this system as "quasi-incompressible", because the divergence of the velocity field v, although different from 0, is prescribed; in addition, when $\kappa \equiv 0$ one exactly recovers the density-dependent incompressible Euler system (6.5).

By the change of unknown

$$u = v - \nabla b, \qquad b = b(\rho) \qquad \text{such that} \quad b'(\rho) := -\frac{\kappa(\rho)}{\rho},$$

and assuming absence of vacuum as in (6.6), the previous equations can be recasted in the form of an incompressible fluid system:

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho - \operatorname{div}\big(\kappa(\rho)\,\nabla\rho\big) = 0 \\ \partial_t u + (u+\nabla b)\cdot\nabla u + \lambda(\rho)\,\nabla\pi = h(\rho,u) \\ \operatorname{div} u = 0, \end{cases} \tag{6.10}$$

where we have set $\lambda(\rho) = \rho^{-1}$, the function π is a new pressure function (different from the hydrodynamic pressure Π) and the forcing term h is defined by

$$h(\rho,u) := -u\cdot\nabla\nabla b + \rho\big(u\cdot\nabla\lambda\big)\nabla b + \rho\big(\nabla b\cdot\nabla\lambda\big)\nabla b - \operatorname{div}\big(\nabla b\otimes\nabla b\big).$$

About system (6.10), we remark the parabolic effect on the density functions (see the first equation) and the fact that the velocity field u is transported by an effective velocity $u+\nabla b$, which is not divergence-free; however, we expect its divergence to be smooth enough, owing to the parabolic smoothing for the density (recall that $b = b(\rho)$). On the other hand, we also notice that the forcing term consumes two derivatives of the density, hence the global well-posedness of this system is not clear and in fact remains as an open problem.

In [118] we proved local in time well-posedness of system (6.10) in Besov spaces verifying the Lipschitz embedding (6.1). However, owing to the analysis of the pressure, which satisfies an equation very similar to (6.7), we needed to restrict the integrability index p to the interval $p\in[2,4]$. As a matter of fact, this condition allows for a L^2 control of the quantity inside the divergence in the right-hand side of (6.7), so for a L^2 bound (hence, low frequency bound) for the pressure $\nabla\pi$ *independently* of the coefficient $\lambda(\rho) = \rho^{-1}$. In addition, we point out that, because of the parabolic nature of the density equation, the use of Chemin-Lerner spaces [42] was needed, making the analysis somehow more involved.

In the subsequent work [117], instead, we considered the endpoint functional framework $B^s_{\infty,r}$, corresponding to the choice $p=+\infty$, under conditions (6.3) on the indices $s\in\mathbb{R}$ and $r\in[1,+\infty]$. For reasons linked with the analysis of the pressure again, we supplemented this assumption with a finite energy (*i.e.* L^2) condition on both $\rho_0 - 1$ and u_0. Besides a local in time well-posedness result also in this new setting, we were also able to prove a lower bound for the lifespan of the solution in dimension $d=2$, implying the same "asymptotically global" well-posedness result mentioned in the previous section. The bound looks more complicated than (6.9), essentially due to the presence of $h(\rho,u)$ on the right-hand side of the momentum equation; however, it is still enough to deduce that, when $\rho_0 - 1$ is of size $\varepsilon > 0$ small (in a suitable norm), then the lifespan T_ε of the corresponding solution diverges to $+\infty$ when $\varepsilon\to 0^+$.

Theorem 6.1. *Let $d=2$. Fix an initial datum $\big(\rho_0,u_0\big)$ such that (6.6) is verified, $\operatorname{div} u_0 = 0$ and both $\rho_0 - 1$ and u_0 belong to the space $B^1_{\infty,1}\cap L^2$. Assume moreover that $\|\rho_0 - 1\|_{B^1_{\infty,1}} \leq 1$.*

Then, there exists a constant $c>0$, depending only on ρ_* and ρ^*, such that the lifespan of the corresponding solution $\big(\rho,u\big)$ to system (6.10) satisfies the lower bound

$$T \geq \frac{c}{\Gamma_0}\log\left(1+\frac{c}{\Gamma_0^2}\log\left(1+\frac{c}{\|\rho_0-1\|_{B^1_{\infty,1}}}\right)\right),$$

where we have defined $\Gamma_0 := 1 + \|\rho_0 - 1\|_{L^2}^2 + \|u_0\|_{L^2\cap B^1_{\infty,1}}$.

The key points of the analysis of [117] are essentially two. The first main ingredient is the proof of new *a priori* estimates in endpoint Chemin-Lerner spaces $\widetilde{L}^q_T(B^s_{\infty,r})$ for parabolic equations with variable coefficients in divergence form, where the variable coefficients may have large oscillations (namely, they are not assumed to be close to a constant).

Proposition 6.2. *Consider the parabolic problem*

$$\partial_t\rho - \mathrm{div}\left(\kappa\,\nabla\rho\right) = f\,, \qquad\qquad \rho_{|t=0} = \rho_0\,,$$

with $\kappa = \kappa(t,x)$ *such that* $0 < \kappa_* \leq \kappa \leq \kappa^*$ *and* ρ_0 *verifying* (6.6). *Let* $s > 0$ *and* $r \in [1,+\infty]$.

For any $\varepsilon > 0$ *fixed, there exists a positive constant* C*, depending only on the set of data* $\big(d, s, r, \rho_*, \rho^*, \kappa_*, \kappa^*, \varepsilon\big)$*, such that the following* a priori *estimate holds true:*

$$\|\rho\|_{\widetilde{L}^\infty_T(B^s_{\infty,r})\cap\widetilde{L}^1_T(B^{s+2}_{\infty,r})} \leq C\left(1 + \|\kappa\|^{\frac{2}{\varepsilon(1-\varepsilon)}}_{L^\infty_T(\mathcal{C}^\varepsilon)}\right)\left(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}^1_T(B^s_{\infty,r})} + \int_0^T\zeta(t)\,\mathrm{d}t\right),$$

where we have defined

$$\zeta(t) := \left(1 + \|\kappa\|^{\frac{2}{1+\varepsilon}}_{L^\infty_T(\mathcal{C}^{1+\varepsilon})}\right)\|\rho(t)\|_{B^s_{\infty,r}} + \|\nabla\kappa(t)\|_{L^\infty}\,\|\nabla\rho(t)\|_{B^s_{\infty,r}} + \|\nabla\kappa(t)\|_{B^s_{\infty,r}}\,\|\nabla\rho(t)\|_{L^\infty}\,.$$

Those estimates are proved by a microlocal analysis argument, consisting of a decomposition of the solution ρ both in the physical space and in the frequency space. They come into play in an essential way first of all in the analysis of the forcing term $h(\rho,u)$, which requires a control on $\nabla^2\rho$, and also in the proof of the improved (with respect to hyperbolic theory) lower bound on the lifespan of the solutions, providing enough regularity for the divergence of the transport field $u + \nabla b$, where $b = b(\rho)$.

As a matter of fact, the second important point of the analysis is the extension of the improved transport estimates by Vishik [217] and Hmidi and Keraani [153] to the case of transport fields which are not necessarily of null divergence.

Proposition 6.3. *Consider the linear problem*

$$\partial_t f + w\cdot\nabla f = g\,, \qquad\qquad f_{|t=0} = f_0\,.$$

Then, for any $\theta > 0$*, there exists a constant* $C = C(d,\theta) > 0$ *such that the following* a priori *estimate holds true:*

$$\|f(t)\|_{B^0_{\infty,1}} \leq C\left(\|f_0\|_{B^0_{\infty,1}} + \|g(\tau)\|_{L^1_T(B^0_{\infty,1})}\right)\left(1 + \int_0^t\left(\|\nabla w\|_{L^\infty} + \|\mathrm{div}\,w\|_{B^\theta_{\infty,\infty}}\right)\,\mathrm{d}\tau\right).$$

Notice that the previous result now yields that the Besov norm $B^0_{\infty,1}$ of the transported quantity grows linearly with respect to the Lipschitz norm of the transport velocity w, *plus* a suitable Hölder norm of the divergence of w. At this point, as the vorticity ω associated to equations (6.10) is transported by $w = u + \nabla b(\rho)$ and $\mathrm{div}\,u = 0$, the suitable Hölder regularity for $\mathrm{div}\,w = \Delta b(\rho)$ is provided by the parabolic estimates of Proposition 6.2.

6.3 On the ideal magnetohydrodynamics

In a series of works [48, 50, 49], we devoted attention to a different system, which describes the evolution of a conducting fluid which is slightly non-homogeneous. This model was rigorously derived in [48] and turns out to be a slight variant of the well-known ideal MHD system, on which we focus from now on for the sake of simplicity.

Denote by u the velocity field of the fluid and by b the self-generated magnetic field[3]. The ideal MHD equations in dimension $d = 2, 3$ read

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u \\ \operatorname{div} u = \operatorname{div} b = 0 , \end{cases} \tag{6.11}$$

where the scalar function π is the MHD pressure, given as the sum

$$\pi = \Pi + \frac{1}{2} |b|^2$$

of the classical hydrodynamic pressure Π and the magnetic pressure $|b|^2/2$.

In the present section, after briefly reviewing some old and recent results on equations (6.11), we will specialise on the two-dimensional situation. As a matter of fact, remark that system (6.11) can be viewed as yet another non-homogeneous perturbation of the incompressible Euler equations (6.2) in the regime $b \to 0$ in some sense. Hence, our goal is to obtain results in the same spirit as the one mentioned above also for system (6.11).

6.3.1 Generalities about the ideal MHD

At a first glance, forgetting about the MHD pressure term $\nabla\pi$, we see that the terms on the right-hand side of the equations may be responsible for a loss of derivatives in the *a priori* estimates. On the other hand, performing a basic energy estimate on the system, we find out that those terms are of opposite sign, so they cancel out; this fact gives rise to the energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 \Big) = 0 .$$

Of course, the same cancellation occurs also for higher order estimates, with the price of lower order commutator terms to handle. Thus, still forgetting about the pressure term, we see that system (6.11) has the structure of a quasi-linear symmetric hyperbolic system, so it is natural to solve it, *locally in time*, in energy-based spaces $H^s(\mathbb{R}^d)$, with $s > 1 + d/2$ (in accordance with conditions (6.3) above). This was indeed done in [208].

However, as already observed in [208], it turns out that the structure of the non-linear terms in (6.11) is even richer. As a matter of fact, introducing the change of unknowns

$$\alpha := u + b \qquad \text{and} \qquad \beta := u - b ,$$

also known as *Elsässer variables*, system (6.11) can be recasted as a coupling of transport equations:

$$\begin{cases} \partial_t \alpha + \beta \cdot \nabla \alpha + \nabla \pi_1 = 0 \\ \partial_t \beta + \alpha \cdot \nabla \beta + \nabla \pi_2 = 0 \\ \operatorname{div} \alpha = \operatorname{div} \beta = 0 , \end{cases} \tag{6.12}$$

where the two pressure terms $\nabla\pi_1$ and $\nabla\pi_2$ can be viewed as the Lagrangian multipliers associated to the two (independent) divergence-free conditions on α and β. This transport structure allows, exactly as it happens for the Euler system, for propagation of L^p norms, thus for local in time well-posedness results in general spaces $B^s_{p,r}$, of course under condition (6.3). This was obtained in [210, 196]. The use of Elsässer variables revealed to also be fundamental in many other works related to the ideal MHD system: we refer *e.g.* to [32, 43, 16, 121] for further investigations around equations (6.11) in various directions.

[3]Pay attention, here b takes a different meaning than it had in the previous Section 6.2.

6.3.2 On the lifespan of planar solutions

We point out that previous studies focused, besides, on continuation criteria [32, 43] and sheding some light on the global well-posedness issue [10]. However, it seems that no better lower bounds on the lifespan of the solutions were obtained than the one given by hyperbolic theory, namely $T \gtrsim 1/\left\|(u_0,b_0)\right\|_X$, where we denote by X the favourite functional framework where one solves the equations.

Our goal in [50] was thus to take advantage on the one hand of the proximity of the ideal MHD equations to the incompressible Euler equations, on the other hand of the Elsässer formulation for solving the system in critical Besov spaces, in order to show, in the specific case of dimension $d=2$, an improved lower bound on the lifespan of the solutions, implying an "asymptotically global" well-posedness result in the same spirit of the ones mentioned in Sections 6.1, 6.2.

In particular, this required to work with data and solutions at $B^1_{\infty,1}$ level of regularity. Now, the question of the equivalence between the original MHD system (6.11) and the Elsässer formulation (6.12), as well as their equivalence with the respective projected systems obtained by using the Leray-Helmholtz projector $\mathbb{P}$, becomes subtle in the framework of solutions which are merely bounded[4]. As a matter of fact, the core of the problem is common to all those equivalence issues: in order to ensure the equivalence between the two formulations (either between original MHD and Elsässer, or between the original MHD and its projected version, or between the Elsässer formulation and its projected version), one has to solve a Laplace equation, so time-dependent constant-in-space solutions (which belong to the kernel of the operator $-\Delta$) may appear in the L^∞ framework. We refer to [46] and references therein for an in-depth analysis of this kind of problems in incompressible fluid mechanics.

It is evident that the problem can be removed by imposing suitable "boundary conditions" at infinity for the Laplace equations. In this sense, it is not surprising that, in [50], we rigorously established the equivalence of the two systems under an additional L^2 condition on the initial velocity field and magnetic field (in fact, our result holds for a large class of weak solutions satisfying a L^p integrability condition, for some $1 \leq p < +\infty$). The integrability condition exactly provides a good boundary condition at infinity, as it completely kills solutions which are constant in space.

Then, we showed the well-posedness of system (6.11) in $L^2 \cap B^s_{\infty,r}$ functional classes, under condition (6.3), together with a continuation criterion involving only the $L^1_T(L^\infty)$ norms of the gradients ∇u and ∇b.

Theorem 6.4. *Let $d \geq 2$ and $(s,r) \in \mathbb{R} \times [1,+\infty]$ satisfying conditions* (6.3) *with $p=+\infty$. Then system* (6.11) *is well-posed, locally in time, in the space*

$$\mathbb{X}^s_r := \left\{ \big(u,b\big) \in \big(L^2 \cap B^s_{\infty,r}\big)^2 \;\Big|\; \operatorname{div} u = \operatorname{div} b = 0 \right\}.$$

Moreover, let $\big(u,b\big)$ is a solution on $[0,T[\,\times\mathbb{R}^d$, with $\big(u(t),b(t)\big) \in \mathbb{X}^s_r$ for any $t \in [0,T[$. If $T<+\infty$ and

$$\int_0^T \Big(\|\nabla u(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^\infty} \Big)\, dt < +\infty\,,$$

then $\big(u,b\big)$ can be continued beyond T into a $\mathbb{X}^s_r$ solution.

The latterpart of Theorem 6.4, namely the continuation criterion, is particularly important, as it tells us that the lifespan of a solution does not depend on the level of regularity. In particular, this enabled us to prove an improved lower bound on the lifespan of planar solutions, implying the sought "asymptotically global" well-posedness result.

[4] In passing, in this respect we point out that the well-posedness result of [196] in $B^s_{\infty,r}$ spaces does not seem complete to us.

Theorem 6.5. *Let $d = 2$ and take an initial datum $\big(u_0, b_0\big) \in \mathbb{X}^2_1$. Then, the lifespan T of the corresponding $\mathbb{X}^2_1$ solution is bounded from below as follows:*

$$T \gtrsim \frac{1}{\left\|\big(u_0, b_0\big)\right\|_{L^2 \cap B^2_{\infty,1}}} \log^3\left(1 + C\,\frac{\left\|\big(u_0, b_0\big)\right\|_{L^2 \cap B^1_{\infty,1}}}{\|b_0\|_{B^1_{\infty,1}}}\right). \tag{6.13}$$

We point out here a delicate point of the analysis of [50]: while working with Elsässer variables in $B^1_{\infty,1}$, in order to get (6.13) one needs to come back to the original unknowns $\big(u, b\big)$ at some point, thus to bound b in $B^1_{\infty,1}$. This fact entails a loss of derivatives in the estimates, as it requires a $B^2_{\infty,1}$ bound for u. Now, one may immediately come back to the Elsässer variables in order to recover the symmetry and propagate the $B^2_{\infty,1}$ regularity, but the loss of derivatives does not disappear. In order to circumvent this problem and close the estimates leading to (6.13), one can use the basic principle behind the continuation criterion, telling us that having a bound for suitable lower order norms of the solution prevents the blow-up of the higher order norms. Making this principle quantitative allows one to estimate the $B^2_{\infty,1}$ norm in terms of the lower order $B^1_{\infty,1}$, thus getting the result (with the price of an additional logarithmic factor, with respect to the bounds obtained in [83, 117]).

Because of the reasons we have just expressed, Theorem 6.5 requires an additional $B^2_{\infty,1}$ regularity on both u_0 and b_0, in order to close the estimates and get (6.13). This can be however improved (as done in [49]), by using proximity of (6.11) to Euler in a better way, namely by passing directly to "modified" Elsässer variables: this approach allows one to avoid the higher regularity assumption on b_0, which thus needs to be $B^1_{\infty,1} \cap L^2$ only.

To conclude this part, we remark our argument shows in fact that, for getting an improved lower bound for the lifespan of the solutions, it is not really necessary to work in critical spaces. What one really needs is a well-posedness theory in some (high regularity) functional framework, complemented with a continuation criterion in terms of norms which can be controlled by $B^0_{\infty,1}$ regularity. For instance, in the case discussed here, we notice that $\|\nabla u\|_{L^\infty} \lesssim \|u\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}$ (and similarly for b), and one can use energy conservation and the transport-like equation for the vorticity in Elsässer variables in order to control the $B^0_{\infty,1}$ norm through the estimates of [217, 153].

6.4 Fluids with odd viscosity

In this section, we get interested in a model for fluids which display *non-dissipative* viscosity effects. Examples of such fluids arise both in quantum and classical hydrodynamics and there is an increasing amount of physical literature about them. We refer to the introduction of [116] for precise references. Many of those references focus on the case of incompressible homogeneous fluids and study questions linked to the free-surface problem; we refer to [150] for a mathematical investigation of that problem.

In this kind of studies, the two-dimensional setting occupies a special place, as *odd viscosity*, namely the non-dissipative response of the viscous tensor to stesses, is not incompatible with isotropy. Thus, we focused on fluids which occupy the whole plane $\Omega = \mathbb{R}^2$, although a similar analysis can be performed also in the periodic case $\Omega = \mathbb{T}^2$.

In [116], we assumed the fluid to be incompressible, but to present also density variations. The system of equations we studied writes as follows:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho\, \partial_t u + \rho\, u \cdot \nabla u + \nabla \Pi + \nu_0 \operatorname{div}\big(\rho\, \nabla u^\perp\big) = 0 \\ \operatorname{div} u = 0\,, \end{cases} \tag{6.14}$$

where the notation $v^\perp$ has been introduced in Chapter 4 and $\nu_0 \in \mathbb{R}$ is the (kinematic) odd viscosity coefficient. Actually, it turns out that nor its value, nor its sign are important in our analysis; for this reason, we immediately set $\nu_0 = 1$. Finally, as the odd viscosity tensor appearing in the previous equations is not symmetric, we point out that we have set

$$\operatorname{div}\left(\rho\,\nabla u^\perp\right) \;=\; \sum_{j=1,2} \partial_j\left(\rho\,\partial_j u^\perp\right).$$

As we are going to see in a while, this model behaves much more as a hyperbolic system rather than a parabolic one.

6.4.1 Well-posedness

In [116], we investigated questions linked to the well-posedness of equations (6.14) in a critical regularity framework. Inspired by [83, 105] about the Euler system (6.5), to which equations (6.14) reduce when $\nu_0 = 0$, we assumed absence of vacuum, represented by condition (6.6) on the initial density; this condition of course implies, for regular enough solutions, absence of vacuum also at any later time. Moreover, owing to the absence of any parabolic effect on the veclity field, we looked for well-posedness in Sobolev spaces $H^s \equiv B^s_{2,2}$ satisfying condition (6.3), namely $s > 2$ as $d = 2$ here.

To begin with, we observe that the previous system conserves the kinetic energy. As a matter of fact, the odd viscosity term $\rho\nabla u^\perp$ is ortoghonal to ∇u. Thus, by testing the momentum equation against u and using the mass conservation equation, after suitable integration by parts one formally obtains

$$\forall\, t \geq 0\,, \qquad\qquad \frac{1}{2}\int_\Omega \rho(t)\,\left|u(t)\right|^2\,dx \;=\; \frac{1}{2}\int_\Omega \rho_0\,\left|u_0\right|^2\,dx\,.$$

On the other hand, thanks to the fact that the density ρ is transported by the divergence-free vector field, one formally get that $\|\rho(t)-1\|_{L^2\cap L^\infty} = \|\rho_0 - 1\|_{L^2\cap L^\infty}$ for any time $t \geq 0$. However, it is clear that those bounds are not sufficient to develop a well-posedness theory for system (6.14). Therefore, we look for *a priori* estimates for $\rho - 1$ and u in H^s, for $s > 2$. In doing so, we see that two main problems arise, both linked, of course, with the presence of the odd viscosity term in the momentum equations. Let us explain those problems in detail.

Assume that $u \in H^s$. First of all, we see that the transport equation for ρ allows us to propagate the H^s regularity of the initial datum, thus getting $\rho - 1 \in H^s$. Next, we consider the equation for u: following [83, 105], we divide the momentum equation by ρ and perform H^s estimates. At this point, by writing

$$\frac{1}{\rho}\operatorname{div}\left(\rho\,\nabla u^\perp\right) \;=\; \Delta u^\perp \,+\, \nabla\log\rho\cdot\nabla u^\perp\,, \tag{6.15}$$

we see that the first term on the right-hand side is skew-symmetric (thus it vanishes when performing the H^s estimate), whereas no cancellations occur for the second term. The point is that this second term involves derivatives of ρ and u, in particular it only belongs to H^{s-1}; this precludes our hope of closing the H^s estimates.

The second issue, instead, is linked with the regularity of the pressure term, and it looks maybe even more serious than the previous one. In order to explain it, we restrict our attention for a while to the simpler case in which the fluid is assumed to be homogeneous, *i.e.* $\rho \equiv 1$. Then, equations (6.14) reduce to

$$\begin{cases} \partial_t u \,+\, u\cdot\nabla u \,+\, \nabla\Pi \,+\, \Delta u^\perp \;=\; 0 \\ \operatorname{div} u \;=\; 0\,. \end{cases}$$

Observing that, owing to the divergence-free condition, one has $\Delta u^\perp = \nabla\phi$, one is finally reconducted to solve an incompressible Euler equation

$$\partial_t u + u\cdot\nabla u + \nabla\widetilde{\Pi} = 0\,, \qquad \text{with} \qquad \widetilde{\Pi} := \Pi + \phi\,.$$

As the system is set in two space dimensions, one can thus solve globally the previous equation in H^s, for any $s>2$. Recall that the key for that is to get rid of the pressure term by passing in vorticity formulation. However, one can track the regularity of the pressure gradient and find that $\nabla\widetilde{\Pi}\in H^s$. This implies that the (original) hydrodynamic pressure $\nabla\Pi$ is only H^{s-2}. Now, we have seen in Section 6.1 that, for non-homogeneous fluids, there is no chance to get rid of the pressure gradient, exactly due to the variations of the density: for this reason, the property $\nabla\Pi\in H^{s-2}$ looks very dangerous in view of proving well-posedness for equations (6.14).

Despite this loss of derivatives, which occurs at two different levels, in [116] we could establish a local in time existence and uniqueness result.

Theorem 6.6. *Let $s>2$ and take an initial datum (ρ_0,u_0) such that $\rho_0-1\in H^{s+1}$ and $u_0\in H^s$. Assume in addition that condition* (6.6) *holds true.*

Then there exist a time $T>0$ and a unique solution $(\rho,u,\nabla\Pi)$ to system (6.14) *on $[0,T]\times\mathbb{R}^2$ such that:*

- *$\rho\in L^\infty\big([0,T]\times\mathbb{R}^2\big)$ verifies the same bounds as in* (6.6) *and $\rho-1\in\mathcal{C}^0\big([0,T];H^{s+1}\big)$;*
- *$u\in\mathcal{C}^0\big([0,T];H^s\big)$;*
- *$\nabla\Pi\in\mathcal{C}^0\big([0,T];H^{s-2}\big)$, but $\nabla\big(\pi-\rho\omega\big)\in\mathcal{C}^0\big([0,T];H^{s-1}\big)$, where $\omega=\operatorname{curl}u=\partial_1u^2-\partial_2u^1$ is the vorticity of the fluid.*

In additon to the previous statement, one can establish also some continuation criteria, similar in spirit (but more complicate in the statement) to the Beale-Kato-Majda criterion for the incompressible Euler equations. We skip the details about those continuation criteria in the present discussion.

Here, we want rather to comment on the regularity of the pressure function, and more precisely to highlight the fact that the difference $\nabla\big(\pi-\rho\omega\big)$, which belongs to $\mathcal{C}^0\big([0,T];H^{s-1}\big)$, is more regular than $\nabla\pi$ and $\nabla\omega$ alone, which are only $\mathcal{C}^0\big([0,T];H^{s-2}\big)$. To some extent, this fact is reminiscent of what happens for the Hoff effective viscous flux, see Chapter 5. We will see that this property plays a key role in our analysis.

6.4.2 A hidden hyperbolic structure

Let us briefly explain how the proof of Theorem 6.6 works. First of all, we want to explain how to solve the above mentioned problems concerning a *priori* estimates, focusing on propagation of higher regularity norms.

The crucial point of the analysis is the introduction of suitable *good unknowns*, which somehow symmetrise the system and make an underlying hyperbolic structure appear. We claim that the good unknowns for system (6.14) are

$$\omega := \operatorname{curl}u = \partial_1u^2 - \partial_2u^1 \qquad \text{and} \qquad \theta := \eta - \Delta\rho\,,$$

where we have defined

$$\eta := \operatorname{curl}(\rho u) := \partial_1\big(\rho\,u^2\big) - \partial_2\big(\rho\,u^1\big)\,.$$

Let us explain why looking at those quantities to control the high regularity norms of the solution (ρ,u) solves the issues mentioned in the previous subsection.

To begin with, we compute an equation for the vorticity ω. For doing so, we divide the momentum equation by ρ and take the curl of the resulting expression: similarly to (6.4), we find the equation

$$\partial_t\omega + u\cdot\nabla\omega + \nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\Pi + \mathcal{B}\big(\nabla u,\nabla^2\log\rho\big) = 0\,, \tag{6.16}$$

where the term $\mathcal{B}\big(\nabla u,\nabla^2\log\rho\big)$ is a bilinear term whose precise expression is not important at this level; it is enough to notice that it comes from the odd viscosity term and, because of the identity (6.15) and of the fundamental cancellation $\operatorname{curl} u^\perp = \operatorname{div} u = 0$, it presents only one derivative in u and two derivatives in ρ. In particular, if $u\in H^s$ and $\rho-1\in H^{s+1}$, then $\mathcal{B}\big(\nabla u,\nabla^2\log\rho\big)\in H^{s-1}$, which is the expected regularity for ω.

However, we notice the property $\rho-1\in H^{s+1}$ seems to be out of reach here: ρ is transported by the H^s vector field u, so at best one can hope to transport H^s norms only. Here, it comes into play the second good unknown θ: for the time being, we are able to guarantee that $\rho\,u$ belongs to H^s, hence $\eta\in H^{s-1}$; if, for any reason, we can prove that $\theta\in H^{s-1}$ as well, then we get $-\Delta\rho\in H^{s-1}$, which impies (thanks to the control on the L^2 norm for $\rho-1$) that $\rho-1\in H^{s+1}$.

As a matter of fact, we notice that the following two idendities hold true:

$$\begin{aligned}
\rho\,\partial_t u + \rho\,u\cdot\nabla u &= \partial_t\big(\rho\,u\big) + u\cdot\nabla\big(\rho\,u\big)\\
\operatorname{div}\big(\rho\,\nabla u^\perp\big) &= \Delta\big(\rho\,u^\perp\big) - \sum_j\partial_j\big(\partial_j\rho\,u^\perp\big)\\
&= \Delta\big(\rho\,u^\perp\big) - \nabla\rho\cdot\nabla u^\perp - \Delta\rho\,u^\perp\,.
\end{aligned}$$

Then, applying the curl operator to the momentum equation in (6.14) in order to find an equation for η, one naturally finds an equation for θ instead:

$$\partial_t\theta + u\cdot\nabla\theta = \frac{1}{2}\nabla^\perp\rho\cdot\nabla|u|^2 + \mathcal{B}\big(\nabla u,\nabla^2\rho\big)\,, \tag{6.17}$$

where we have again used the property $\operatorname{curl} u^\perp = \operatorname{div} u = 0$ and where the first term on the right corresponds to the "stretching term" arising from the application of the curl operator to the transport term $u\cdot\nabla\big(\rho u\big)$. At this point, it is clear that, under the conditions $\rho-1\in H^{s+1}$ and $u\in H^s$, the right-hand side of equation (6.17) belongs to H^{s-1}, therefore one can infer the property $\theta\in H^{s-1}$, which in turn implies, as already announced above, that $\rho-1\in H^{s+1}$.

Nonetheless, the problem for closing the argument comes from the pressure term, which still appears in equation (6.16). Notice that we have reduced the order of the problem, inasmuch as a H^{s-1} control is now sufficient (whereas we needed H^s regularity for $\nabla\Pi$ for propagating the same norm of the velocity), but hoping for this regularity seems still too much, in light of the discussion of Subsection 6.4.1. Nevertheless, let us give a closer look at the pressure term. On the one hand, arguing as for getting (6.7), we see that $\nabla\Pi$ solves the elliptic equation

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla\Pi\right) = \operatorname{div}\left(u\cdot\nabla u + \nabla\log\rho\cdot\nabla u^\perp\right) - \Delta\omega\,,$$

where we used identity (6.15) again. As the right-hand side of the previous relation is the divergnce of some $F\in L^2$, we can easily estimate the L^2 norm of $\nabla\Pi$ by Lax-Milgram theorem. On the other hand, expanding the derivatives on the left, we can compute an equation for $\Delta\Pi$, with the goal of bounding the high frequencies of the pressure gradient: we find

$$-\Delta\Pi = \widetilde{F} - \rho\,\Delta\omega\,,$$

for a suitable function $\widetilde{F}$ which belongs to H^{s-2}. This implies in particular that

$$-\Delta\big(\Pi-\rho\omega\big) = \widetilde{F} - \big[\rho,\Delta\big]\,\omega \qquad\Longrightarrow\qquad -\Delta\big(\Pi-\rho\omega\big)\in H^{s-2}\,,$$

which in turn yields $\nabla\big(\Pi - \rho\omega\big) \in H^{s-1}$. Now, we use this property in equation (6.16): we get

$$\begin{aligned}\nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\Pi \,&=\, \nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\big(\Pi-\rho\omega\big) \,+\, \nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\big(\rho\omega\big) \\ &=\, \nabla^\perp\left(\frac{1}{\rho}\right)\cdot\nabla\big(\Pi-\rho\omega\big) \,-\, \nabla^\perp\log\rho\cdot\nabla\omega\,.\end{aligned}$$

At this point, we observe that the first term on the right-hand side has H^{s-1} regularity, so it is a "good" forcing term in the equation for ω, and that the second term is instead a transport term for ω. Thus, the vorticity ω results to be transported by the H^s divergence-free vector field $u - \nabla^\perp \log\rho$, submitted to a H^{s-1} forcing term; hence, one can propagate H^{s-1} regularity for ω and finally close the *a priori* estimates.

As a conclusion, we have seen that it is the hyperbolic structure of equations (6.16) and (6.17), namely the coupling of those transport equations, which allows for the propagation of higher order norms of the solution and for the well-posedness result to hold.

Actually, it is worth to point out that further complications arise in the proof of Theorem 6.6, at the level of the proof of existence, the proof of uniqueness and the proof of the continuation criterion. For the sake of brevity, we avoid to enter into those details here. We only point out that, in our arguments, Littlewood-Paley theory enters into play in a crucial way, through sharp estimates for transport-diffusion equations in Chemin-Lerner spaces (as a matter of fact, we prove existence by viscous regularisation of the system) and paradifferential calculus (in order to avoid a new loss of derivatives in the stability estimates leading to uniqueness).

6.5 Some open questions and perspectives

We conclude this chapter by mentioning some problems which capture our attention and which we would like to consider in the future.

The first remark we want to make here is that all the results for models of non-homogeneous (density-dependent) fluids have been obtained so far under the crucial assumption of absence of vacuum, see *e.g.* [81, 83, 105, 118, 117, 116]. Then, we are interested in studying what happens in case vacuum regions appear initially (and then they are transported by the flow). We expect that an approach inspired by [114, 115], based on finding a suitable quantity, linked to the density, to control close to vacuum, may be fruitful. We also expect that it may be possible to exploit tangential regularity information in the vacuum regions.

On a different but related standpoint, we plan to address the question of well-posedness of some models encoding transitions from compressible to incompressible, like in models for crowd dynamics and granular media. To the best of our knowledge, existence for such models is known only by a singular limit approach [179, 203, 96] from a suitably penalised compressible problem. A direct well-posedness study on the target compressible-incompressible system seems to be missing, although some studies exist based on optimal transport theory [189, 191, 170]. The description of the dynamics of the saturated region (where the density reaches a maximal value) is also a very interesting problem in this context, although some results are available in the one-dimensional situation [73, 74]. Let us mention that similar questions arise also in the study of the interaction of waves with an immersed object [169, 161, 26] and in some models from biology [91, 94, 92].

The global well-posedness issue for the kind of systems discussed in this chapter seems to be a hard problem, difficult to get if one does not exploit any specific structure of the system around well-chosen equilibria.

On the other hand, we plan to investigate some *partially dissipative cases*, similar in spirit to recent results [67, 68] for symmetric hyperbolic systems.

Finally, we focus on the odd viscosity system (6.14), for which it seems that the mathematical literature is still poor. Several questions arise in this context.

As a direct continuation of work [116], the first one is to study the well-posedness of the system in general Besov spaces $B^s_{p,r}$ (still under condition (6.3), of course), for $p \neq 2$. In particular, we are especially interested in the case $p = +\infty$ and in recovering suitable lower bounds for the lifespan of solutions, implying an "asymptotically global" well-posedness result also in this context. In the same direction, we would like to investigate propagation of tangential regularity, in the same spirit of [105] for the density-dependent incompressible Euler equations (6.5).

In addition, we plan to consider the case in which a (traditional) diffusive viscosity term cohexists with the odd viscosity term. In this case, we are interested in obtaining global existence and uniqueness of solutions. In a first approach, we may work under the Boussinesq approximation.

Furthermore, it seems interesting to consider similar problems in higher dimension, namely for $d = 3$. The question of the well-posedness of the system arises even for homogeneous fluids (*i.e.* with $\rho \equiv 1$) in this case, as now the odd viscosity term is anisotropic. One may expect to find similar phenomena as the ones highlighted in the study of rotating fluids; yet, the loss of derivatives due to the odd term has to be taken into account.

Finally, we mention that the case of (density-dependent) compressible flows looks interesting and challanging. As a matter of fact, in the computations of Section 6.4 we have used several times the incompressibility condition $\operatorname{div} u = 0$ in order to get fundamental cancellations, which avoided the presence of higher order derivatives on the velocity field. In the compressible case, one cannot rely anymore on those cancellations, thus a further loss of derivatives appears, which depends on $\operatorname{div} u$. We may expect an interplay between potential part (which should disperse) and incompressible part of the solution, yet it seems to us that some additional information has to be extrapolated from the system, in order to avoid such a loss of derivatives.

Part III

Singular perturbation problems

Chapter 7

Overview of Part III

Part III is devoted to the study of a particular class of singular perturbation problems in Fluid Mechanics, which arise in the study of *geophysical flows*, like currents in the ocean and in the atmosphere.

For an accurate description of the dynamics of geophysical flows, it is important to take into account three main features:

(i) the almost incompressibility of the flow;

(ii) the effects due to stratification, namely variations in the density function caused by the action of gravity;

(iii) the action of a strong Coriolis force, due to the fast[1] rotation of the ambient system.

The relevance of these effects is measured by introducing, correspondingly, three physical adimensional parameters (having positive value): the Mach number Ma, linked with incompressibility, the Froude number Fr, linked with stratification, and the Rossby number Ro, related to fast rotation. Saying that the previous attributes are predominant in the dynamics corresponds to assuming that the values of those parameters are very small.

The complexity of the original system (often referred to as *primitive system* in the references which will be quoted in the sequel[2]) prompts physicists to derive reduced models, more easy to study and to handle for performing computations and numerical simulations. The *conditio sine qua non* of this reduction process is that the reduced models must retain, in some sense, all the main characteristics of the original system. Thus, advocating some sort of continuity of the solutions of the original equations with respect to the parameters appearing therein, the reduced models are obtained by formally setting the previously mentioned Ma, Fr and Ro to 0 in the original equations. It is apparent that this process is only formal; its precise justification requires a rigorous limit process, which nonetheless poses several mathematical difficulties. As a matter of fact, in this limit process one is usually faced to a singular perturbation problem, in the following sense. Firstly, some terms in the equations are penalised by the small parameters (in the sense that factors like $\frac{1}{Ma}$, $\frac{1}{Fr}$ and $\frac{1}{Ro}$ appear in the equations) and, then, tend to explose when performing the asymptotics for those parameters going to 0. In addition, in the limit process the nature of the equations often changes (at the level of the type of PDEs involved, order of the differential operators...), a fact which is a source of additional difficulties in the analysis.

Thus, on the mathematical side, there are at least two main problems when facing the rigorous study of the asymptotic limit $Ma,\ Fr,\ Ro\ \to 0$: first of all, one has to find the right functional

[1] Fast, here, refers to the fact that the speed of rotation of the ambient system, *e.g.* the Earth, is much larger than the characteristic velocity of the fluid. See for instance [70] for more details.

[2] This terminology has not to be confused with the system of *primitive equations*, used in geophysics to denote the Navier-Stokes system with hydrostatic approximation. In order to avoid any confusion, we will try to resort to a different name in this manuscript.

framework in which giving a rigorous justification to the limit argument; moreover, one has to deal with several technical issues related to the above mentioned change in the nature of the equations, like possible vanishing of suitable quantities and loss of uniform bounds for other quantities, for instance.

At this point, it is important to point out that those limit processes ($Ma \to 0$, $Fr \to 0$ and $Ro \to 0$) do not commute. Therefore, what one usually does is to focus on a *distinguished limit*, in which the relative orders of magnitude of Ma, Ro and Fr are fixed and interrelated, depending on the physical regime one is interested in. Besides, this setting gives the possibility of studying *multiscale limits*, in which precisely those physical parameters have different orders of magnitude, thus translating a hierarchy of importance that the various considered effects have on the dynamics. In our case, we will typically be in the situation in which the Rossby number Ro is taken equal to a small parameter $\varepsilon > 0$, namely $Ro = \varepsilon$, whereas we will set the Mach number $Ma = \varepsilon^m$, for some $m \geq 0$ (in practice, $m \geq 1$ almost all the time) and the Froude number $Fr = \varepsilon^n$, for $0 \leq n \leq m$ (but in fact more restrictive upper bounds for n in terms of m appear in the study).

In this part of the manuscript, we will see how to tackle the *fast rotation limit* $Ro = \varepsilon \to 0^+$ for various fluid models. In Chapter 8 we will focus on the case of compressible flows and on the multiscale analysis, while in Chapter 9 we will describe the situation for incompressibile fuids with variable density. In particular, we will always deal with *non-homogeneous flows*, meaning that we will take into account fluids having variable density $\rho \geq 0$. While those cases (the compressible and incompressible ones) retain some common features, deep differences arise in their study: let us briefly comment on them.

To begin with, let us give the precise form of the Coriolis operator $\mathfrak{C} = \mathfrak{C}(\rho, u)$ we adopt to model the effects of the fast rotation on the dynamics of the flow. We consider a very simple form of $\mathfrak{C}$, which is however well-justified, from the physical viewpoint, at mid-latitudes (see *e.g.* [41] for details): if we denote by $e_3 = (0, 0, 1)$ the unit vector directed along the vertical axis and by $\times$ the usual external product in $\mathbb{R}^3$, we have

$$\mathfrak{C}(\rho, u) := \frac{1}{Ro} e_3 \times \rho u = \frac{1}{Ro} \rho \big(-u^2, u^1, 0 \big) . \tag{7.1}$$

This corresponds to identifying the rotation axis with the vertical axis.

Notice that the vector $\mathfrak{C}(\rho, u)$ is pointwise orthogonal to the vector u, hence also with respect to the L^2 scalar product on $\mathbb{R}^3$. For later use, we remark that this remains true also in $\mathbb{R}^2$, if we take the projections onto the first two components of the two vectors $\mathfrak{C}(\rho, u)$ and u. Thus, the Coriolis term gives zero contribution whenever we perform an energy estimate on the momentum equation. Nonetheless, this is no more the case when, instead, we perform a H^s estimate, for $s > 0$, as derivatives of ρ now appear; then, one should make sure of their smallness in order to absorbe the prefactor $\frac{1}{\varepsilon}$. Observe that this is a remarkable effect due to the heterogeneity of the fluid. For this reason, we conduct the asymptotic study in the framework of global in time weak solutions *à la Leray* (namely, weak solutions possessing *finite energy*, as discussed in Section 5.1) for the various models under consideration.

Next, assume for a while that the fluid is compressible and barotropic, so the pressure function $P = P(\rho)$ is a known function of the density only (recall equation (5.1) above). In many physically relevant situations, the (small) Mach number Ma is of the same order of magnitude as the Rossby number Ro: in this instance, we can set $Ma = Ro = \varepsilon$ (which means $m = 1$ in the discussion above), for a small parameter $\varepsilon > 0$ which we want to send to 0. As a consequence of the physical scaling, the two predominant effects of the dynamics are the incompressibility (associated to the penalisation of the pressure term) and the fast rotation (coming from the penalisation of the Coriolis operator $\mathfrak{C}$), and these effects are kept in balance in the limit process. Now, if the densities are small perturbations of a constant state, say $\rho \approx 1 + \varepsilon r$, it is easy to see (at least formally) that, from that "equilibrium" between pressure and rotation terms, in the limit $\varepsilon \to 0^+$

one recovers the so-called *quasi-geostrophic balance* relation:

$$e_3 \times u + P'(1) \nabla r = 0 . \tag{7.2}$$

This relation is a key point of the study, as it entails deep consequences. Indeed, from (7.2) one recovers[3] that $r = r(t, x^h)$ and $u^h = u^h(t, x^h)$. Using the mass equation, one in turn discovers that the limit velocity field u only depends on (t, x^h). This is exactly the celebrated *Taylor-Proudman theorem* in Geophysics: in the limit of fast rotation, the fluid tends to have planar behaviour and the dynamics essentially takes place in planes orthogonal to the rotation axis, while remaining constant in the vertical direction (*i.e.* the direction parallel to the rotation axis). Hence, the motion (at least, the mean motion in the bulk) is fully described once we know $u^h = u^h(t, x^h)$. Now, relation (7.2) tells us that u^h is a 2-D divergence-free vector field which satisfies $u^h = P'(1) \nabla_h^\perp r$; in particular, r is a stream function for the 2-D limit velocity field u^h, so, in order to describe the limit dynamics, it is enough to exhibit an equation for r alone. As a matter of fact, relation $u^h = P'(1) \nabla_h^\perp r$ enters in the asymptotic study also at other levels, as it is used several times in the computations, when rigorously proving convergence of the original (ε-dependent) system to some reduced system.

Relation (7.2) marks a severe difference between the compressible fluid case and the incompressible fluid case. As a matter of fact, in the incompressible situation, equation (7.2) will be replaced by

$$e_3 \times \rho u + \nabla \Pi = 0 ,$$

where this time we do not make the assumption that the limit density is necessarily constant, $\rho_{\lim} \equiv 1$, and where we have denoted by Π the (unknown) pressure of the fluid. Notice that, for incompressible flows, it makes no more sense to speak about the Mach number. Indeed, in this situation the pressure in only a Lagrangian multiplier associated to the divergence-free constraint over u, and one disposes of no explicit formulas for it; in any case, Π cannot be expressed in terms of the density function alone. In particular, one misses the explicit relation between the limit velocity field and the limit density variation function, so also all the benefits which came with it. As we will see, in this instance the proof of the convergence becomes much more involved (in particular, for technical reasons we will have to restrict our attention to 2-D flows), and the limit dynamics will be (at least in the case when the reference density state is non-constant) underdetermined.

Before concluding this overview of Part III, let us make some additional comments, of more technical flavour.

First of all, we will always deal with general *ill-prepared* initial data. This means that we will assume only uniform bounds, in suitable norms, for the initial data, without requiring special relations on them or on their limit points.

In addition, we will always neglect boundary effects. It is well-known that the Taylor-Proudman theorem is somehow incompatible with the physical condition which wants the fluid to stick at the boundary of the domain (think for instance to the bottom of the ocean). This illusory contradiction finds its explanation in the presence of the so-called Ekman boundary layers, in which the fluid is slowed down until it results to be at rest at the boundary. Ekman boundary layers affect also the global dynamics (even in the interior of the domain), by what is known as the Ekman pumping phenomenon. In our study, we will mainly neglect those boundary issues, by imposing *complete-slip boundary conditions* in the compressible case and by working in two dimensions of space in the incompressible case (as already mentioned, the reduction to 2-D flows is actually imposed by other technical difficulties). We will make an exception to that in Section 9.3. In addition, we will also give more references about Ekman boundary layers and related phenomena.

[3] Here, for a 3-D vector field $v = (v^1, v^2, v^3)$, we have set $v^h = (v^1, v^2)$.

Finally, we point out that our study will be based on *weak compactness* techniques; in particular, our convergence results will be true, in general, only up to extraction of a suitable subsequence. Notice that, in order to pass to the limit in the equations, one needs strong convergence of suitable quantities to treat the non-linear terms. Nonetheless, this property cannot be simply derived as in classical arguments: owing to the presence of the singular parameters in the equations, the time derivatives of the solutions will not enjoy, in general, uniform boundedness of any type, precluding the possibility of application of classical arguments, like the Ascoli-Arzelà or Aubin-Lions theorems. Instead, we will adopt (for most of the problems) a *compensated compactness* approach first introduced by P.-L. Lions and Masmoudi [178] in the context of the incompressible limit of the compressible Navier-Stokes equations, and then adapted by Gallagher and Saint-Raymond [145] to the study of homogeneous incompressible fluids in fast rotation. This method is based on making use, by purely algebraic computations, of the wave system, namely the system which describes propagation of oscillations from the target configuration: by exploiting its structure, we are able to find special cancellations and relations in the non-linear terms, which in turn allow us to finally express them as a sum of terms which either are small, or converge in the limit $\varepsilon \to 0^+$. This is a very robust technique, which in principle requires almost no constraints on the orders of magnitude of the small parameters coming into play. The drawback of this approach is that it does not yield any quantitative convergence property. For this reason, sometimes (when possible) we will use instead different techniques, yielding strong convergence (we will get it by applying the celebrated RAGE theorm from scattering theory, see *e.g.* [71]), even with a precise rate (relative entropy methods, see *e.g.* [127] and references therein).

In Chapter 8 we are going to tackle the singular limit problem and the multiscale analysis for some compressible fluid models. In Chapter 9 we deal with a similar asymptotic study for density-dependent incompressible fluid systems.

Chapter 8

Fast rotation limit: compressible models

The present chapter is devoted to the study of the fast rotation asymptotics for models of *compressible* fluid flows.

As in Part II, we denote by the scalar function $\rho \geq 0$ the density of the fluid, by $u \in \mathbb{R}^3$ its velocity field and by $P = P(\rho)$ the pressure of the fluid; for simplicity of presentation, we will always assume $P(\rho) = A\,\rho^\gamma$, for some $A > 0$ and $\gamma > 3/2$. As in the previous chapters, we will denote by Du the Jacobian matrix of u and by $\nabla u = {}^t(Du)$ its transpose matrix; then, we adopt a shortened name to denote the viscous stress tensor

$$\mathbb{S}(Du) := \mu \left(Du + \nabla u - \frac{2}{3}\operatorname{div} u \operatorname{Id} \right) + \lambda \operatorname{div} u \operatorname{Id}\,, \tag{8.1}$$

where Id is the identity matrix and μ and λ are, respectively, the shear viscosity and the bulk viscosity coefficients, which we assume to be positive here for simplicity. For the time being, we also assume that μ and λ are fixed constant. Thus, this notation is consistent with the form of the viscosity terms appearing in equations (5.1), up to a change in the definition of the coefficients μ and λ there. The general form of the original system we will consider throughout this part is given by the following rescaled barotropic Navier-Stokes system:

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho\, u\big) = 0 \\ \partial_t\big(\rho\, u\big) + \operatorname{div}\big(\rho\, u \otimes u\big) + \dfrac{1}{\varepsilon^{2m}} \nabla P(\rho) + \dfrac{1}{\varepsilon} e_3 \times \rho\, u \\ \qquad\qquad\qquad\qquad - \operatorname{div} \mathbb{S}(Du) = \dfrac{1}{\varepsilon^{2n}} \rho \nabla G + \dfrac{1}{\varepsilon^2} \rho \nabla F\,, \end{cases} \tag{8.2}$$

where $m \geq 0$ and $0 \leq n \leq m$, and the two external forces

$$G(x) = -\,x^3 \qquad\qquad \text{and} \qquad\qquad F(x) = \left|x^h\right|^2 = (x^1)^2 + (x^2)^2$$

denote respectively the gravity and the centrifugal force. However, results concerning related systems (like the full Navier-Stokes-Fourier system, the Navier-Stokes-Korteweg equations...) will be discussed as well.

We set system (8.2) in $\mathbb{R}_+ \times \Omega$, where Ω is the infinite slab

$$\Omega := \mathbb{R}^2 \times \,]0,1[\,.$$

The "horizontal domain" $\mathbb{R}^2$ could be replaced by the periodic box $\mathbb{T}^2$ with almost no changes in the analysis. We supplement equations (8.2) with *complete-slip boundary conditions*

$$\big(u \cdot n\big)_{|\partial\Omega} = \big(u^3\big)_{|\partial\Omega} = 0 \qquad\qquad \text{and} \qquad\qquad \Big(\big(\mathbb{S}(Du)n\big) \times n\Big)_{|\partial\Omega} = 0\,, \tag{8.3}$$

where $n = \pm e_3$ denotes the exterior normal to the boundary $\partial\Omega$. Recall that the previous conditions allow one to avoid the appearing of Ekman boundary layers. We also fix the initial conditions

$$\rho_{|t=0} = \rho_{0,\varepsilon} \qquad \text{and} \qquad u_{|t=0} = u_{0,\varepsilon} .$$

More precise assumptions on the family of initial data $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}\big)_{\varepsilon\in\,]0,1]}$ will be specified later on. The important point, forced by the techniques we use to prove convergence (essentially, weak compactness methods), which in turn allow us to consider ill-prepared initial data, is that the initial densities need to be small perturbations around some *static state* $\widetilde{\rho}$ (in fact, the static states may themselves depend on $\varepsilon > 0$, and converge to some state $\widetilde{\rho}$ when $\varepsilon \to 0^+$: this is a typical situation in multiscale problems, see Subsection 8.1.2 below).

Works presented in the chapter

(P.10) F. Fanelli: *Highly rotating viscous compressible fluids in presence of capillarity effects.* **Math. Ann.**, 366 (2016), n. 3-4, 981-1033.

(P.13) F. Fanelli: *Corrigendum to "Highly rotating viscous compressible fluids in presence of capillarity effects".* **Math. Ann.**, 370 (2018), n. 3-4, 1789-1797.

(P.22) D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska: *A multi-scale problem for viscous heat-conducting fluids in fast rotation.* **J. Nonlinear Sci.**, 31 (2021), n. 1, Paper n. 21.

(P.26) F. Fanelli: *Incompressible and fast rotation limit for barotropic Navier-Stokes equations at large Mach numbers.* **Phys. D**, 428 (2021), Paper n. 133049.

(C.6) D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska: *On the influence of gravity in the dynamics of geophysical flows.* **Math. Eng.**, 5 (2023), n. 1, Paper n. 008.

Not mentioned, but in this context

(P.11) F. Fanelli: *A singular limit problem for rotating capillary fluids with variable rotation axis.* **J. Math. Fluid Mech.**, 18 (2016), n. 4, 625-658.

(P.27) E. Bocchi, F. Fanelli, C. Prange: *Anisotropy and stratification effects in the dynamics of fast rotating compressible fluids.* **Ann. Inst. H. Poincaré C Anal. Non Linéaire**, 39 (2022), n. 3, 647-704.

(P.30) F. Fanelli, E. Zatorska: *Low Mach number limit for degenerate Navier-Stokes equations in presence of strong stratification.* **Comm. Math. Phys.** accepted for publication (2022).

(S.1) F. Fanelli: *Geophysical flows and the effects of a strong surface tension.* Submitted for the Proceedings of a conference (2016), `http://arxiv.org/abs/1605.09210`.

(C.2) F. Fanelli: *A note on viscous capillary fluids in fast rotation.* Bruno Pini Math. Anal. Semin., Univ. Bologna, Bologna (2015), 86-102.

8.1 Introduction

As already anticipated in Chapter 7, the goal of our study is the following: given, for any $\varepsilon \in\,]0,1]$ fixed, a global in time finite energy weak solution $\big(\rho_\varepsilon, u_\varepsilon\big)$ to system (8.2) related to the datum $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}\big)$, recall the discussion in Section 5.1, we want to understand the asymptotic behaviour of the family $\big(\rho_\varepsilon, u_\varepsilon\big)_{\varepsilon\in\,]0,1]}$ for $\varepsilon \to 0^+$. This means first of all to prove convergence of $\big(\rho_\varepsilon, u_\varepsilon\big)_\varepsilon$

to some target state $(\rho_{\lim}, u_{\lim})$, and then to characterise the dynamics of that target state, *i.e.* to find the equations satisfied by $(\rho_{\lim}, u_{\lim})$.

Before entering into the details of our presentation, let us mention that the corresponding study for the classical incompressible Navier-Stokes (and Euler) equations, which somehow correspond to take $\rho \equiv 1$ in (8.2), has been widely performed in the past and is by now quite well-understood. We will avoid any discussion about it here; we refer the interested reader to [41] for a comprehensive treatement of the subject. Similarly, we avoid any discussion about the incompressible limit $Ma \to 0^+$ for compressible fluid systems, a topic which has been broadly studied so far and for which there exists an extended litterature. We refer for instance to [131] for an overview of the main results and for more references about the problem of the incompressible limit alone. In this chapter, we will only review results about the *combined effects* of the incompressible and fast rotation limits and focus on system (8.2).

8.1.1 Fluids in quasi-geostrophic balance

To begin with, let us focus on a simplified version of system (8.2), where we take $F = G = 0$ and we consider only the value $m = 1$: the equations of motion then read

$$(8.4) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) \,=\, 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \dfrac{1}{\varepsilon^2} \nabla P(\rho) + \dfrac{1}{\varepsilon} e_3 \times \rho u - \operatorname{div} \mathbb{S}(Du) \,=\, 0 . \end{cases}$$

First attempts to understand the asymptotic behaviour of system (8.4) for $\varepsilon \to 0^+$ were performed in [22, 23]. Those works mainly concerned 2-D flows and the case of well-prepared initial data. Moreover, the investigation of the convergence was performed by relative entropy methods. The general case of ill-prepared initial data and of 3-D domains was treated only afterwards by Feireisl, Gallagher and Novotný in [126]. The authors considered initial data of the following form[1]:

$$\rho_{0,\varepsilon} \,=\, 1 + \varepsilon\, r_{0,\varepsilon} \,, \qquad \text{with} \qquad \big(r_{0,\varepsilon}\big)_\varepsilon \,\Subset\, L^2(\Omega) \cap L^\infty(\Omega)$$

for the density functions and, for the velocity fields,

$$\big(u_{0,\varepsilon}\big) \,\Subset\, L^2(\Omega) \,.$$

Under these assumptions, Lions-Feireisl theory [177, 123] provides us with the existence of a global in time finite energy weak solution $(\rho_\varepsilon, u_\varepsilon)$, for any $\varepsilon > 0$ fixed. Next, consider the energy inequality (5.7), which in this context becomes

$$(8.5) \qquad \int_\Omega \left(\frac{1}{2} \rho_\varepsilon \, |u_\varepsilon|^2 + \frac{1}{\varepsilon^2} \mathcal{H}\big(\rho_\varepsilon \,|\, 1\big) \right) \mathrm{d}x + \int_0^t \int_\Omega \Big(\mu\, |\nabla u_\varepsilon|^2 + \lambda\, |\operatorname{div} u_\varepsilon|^2 \Big) \, \mathrm{d}x \, \mathrm{d}t \\ \leq \int_\Omega \left(\frac{1}{2} \rho_{0,\varepsilon} \, |u_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \mathcal{H}\big(\rho_{0,\varepsilon} \,|\, 1\big) \right)$$

for all $t \geq 0$, where the functional

$$\mathcal{H}\big(\rho \,|\, \widetilde{\rho}\big) \,:=\, H(\rho) - H(\widetilde{\rho}) - H'(\widetilde{\rho})\, \big(\rho - \widetilde{\rho}\big)$$

has been introduced in (5.4) and denotes the Bregman divergence of one density state ρ to the reference state $\widetilde{\rho}$, associated to the convex function H. Here, H stands for the classical pressure potential, defined as usual as a solution to the ODE

$$\rho\, H'(\rho) - H(\rho) \,=\, P(\rho) \,.$$

Recall the discussion of Subsection 5.1.1.

[1] Given a normed space X and a sequence $\big(f_\varepsilon\big)_\varepsilon \subset X$ of elements of it, we use the notation $\big(f_\varepsilon\big)_\varepsilon \Subset X$ to mean that the sequence is also *bounded* in X, *i.e.* one has $\|f_\varepsilon\|_X \leq C$, for an absolute constant $C > 0$ which is independent of $\varepsilon > 0$.

Properties of the mean motion

The energy inequality (8.5) immediately implies that

$$\rho_\varepsilon \,=\, 1 \,+\, \varepsilon\, r_\varepsilon\,, \qquad\qquad \text{with} \qquad \big(r_\varepsilon\big)_\varepsilon \,\Subset\, L^\infty\big(\mathbb{R}_+; L^2(\Omega) + L^\gamma(\Omega)\big)\,,$$

and that

$$\big(u_\varepsilon\big)_\varepsilon \,\Subset\, L^\infty\big(\mathbb{R}_+; L^2(\Omega)\big)\,, \qquad\qquad \big(\nabla u_\varepsilon\big)_\varepsilon \,\Subset\, L^2\big(\mathbb{R}_+; L^2(\Omega)\big)\,.$$

As a consequence, one can take weak limits (up to suitable extractions) in the respective functional spaces and obtain, for suitable limit-points r and u, the convergences

$$r_\varepsilon \overset{*}{\rightharpoonup} r \qquad\qquad \text{and} \qquad\qquad u_\varepsilon \overset{*}{\rightharpoonup} u \tag{8.6}$$

in the weak-$*$ topology of the respective spaces. In addition, one can rigorously justify the quasi-geostrophic balance

$$\frac{1}{\varepsilon^2}\,\nabla P(\rho_\varepsilon) \,\approx\, \frac{1}{\varepsilon}\, e_3 \times \rho_\varepsilon u_\varepsilon$$

mentioned in Chapter 7, as in fact $\frac{1}{\varepsilon^2}\,\nabla P(\rho_\varepsilon) \,\approx\, \frac{1}{\varepsilon}\, P'(1)\,\nabla r_\varepsilon$ and $\frac{1}{\varepsilon}\, e_3 \times \rho_\varepsilon u_\varepsilon \,\approx\, \frac{1}{\varepsilon}\, e_3 \times u_\varepsilon$ are of the same order.

In turn, one can rigorously prove that the target state (r,u) satisfies the quasi-geostrophic relation (7.2) of the previous chapter, together with all the consequences which derive from it, in particular the Taylor-Proudman theorem. Moreover, from the mass equation one deduces that the limit velocity profile u must be incompressible, namely

$$\mathrm{div}\, u \,=\, 0\,.$$

Combining all those properties with the complete-slip boundary conditions, one finally finds that

$$u \,=\, \Big(u^h, 0\Big)\,, \qquad\qquad \text{with} \qquad u^h \,=\, \nabla_h^\perp r\,, \qquad r \,=\, r(t,x^h)\,. \tag{8.7}$$

Convergence to the limit dynamics

The next goal is to compute a dynamical equation for u, or equivalently for r. For this, one wants to pass to the limit in the weak formulation of the momentum equation, when testing it against a test function which belongs to the kernel of the singular perturbation operator. Thus, from now on we fix some smooth

$$\psi \,=\, \Big(\nabla_h^\perp\varphi, 0\Big)\,, \qquad\qquad \text{with} \qquad \varphi = \varphi(t,x^h) \,\in\, \mathcal{D}\big(\mathbb{R}_+ \times \mathbb{R}^2\big)\,,$$

and we use it as a test function in the momentum equation. Notice that ψ is divergence-free, together with its horizontal component $\psi^h \,=\, \nabla_h^\perp\varphi$, and only depends on the horizontal variables.

As is well-known, the convergence properties (8.6) are not enough to pass to the limit $\varepsilon \to 0^+$, essentially because of the non-linearities arising in the original (ε-dependent) system. In the specific case of system (8.4), there are two main problems. The first one is the convergence of the convective term $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon$, which is non-linear in the velocity field, so the smallness $\rho_\varepsilon - 1 = O(\varepsilon)$ alone is not enough to compute its asymptotics. The second problem is the convergence of the Coriolis term $\frac{1}{\varepsilon}\, e_3 \times \rho_\varepsilon u_\varepsilon$, which is singular when $\varepsilon \to 0^+$. Of course, the story is different for the other singular term, namely the pressure term, because it is a gradient, thus it vanishes whenever one tests the momentum equation on ψ as fixed above.

Now, the issue linked with the Coriolis term is easily solved thanks to the use of the mass equation. Indeed, for ψ as above, one can write

$$\frac{1}{\varepsilon}\int_{\mathbb{R}_+\times\Omega} e_3 \times \rho_\varepsilon\, u_\varepsilon \cdot \psi \;\mathrm{d}x\,\mathrm{d}t \,=\, \frac{1}{\varepsilon}\int_{\mathbb{R}_+\times\mathbb{R}^2} \langle \rho_\varepsilon\, u_\varepsilon^h\rangle \cdot \nabla_h\varphi \;\mathrm{d}x^h\,\mathrm{d}t \tag{8.8}$$

$$= \int_{\mathbb{R}^2} \langle r_{0,\varepsilon} \rangle \, \varphi(0,\cdot) \, \mathrm{d}x^h \, + \, \int_{\mathbb{R}^2} \langle r_\varepsilon \rangle \, \partial_t \varphi \, \mathrm{d}x^h \, \mathrm{d}t \, ,$$

where we have denoted by

$$\langle f \rangle \, = \, \langle f \rangle (t, x^h) \, = \, \int_{]0,1[} f(t, x^h, x^3) \, \mathrm{d}x^3$$

the vertical average of a function f defined on $\mathbb{R}_+ \times \Omega$. In the previous relation (8.8), we notice that the singularity $\frac{1}{\varepsilon}$ has disappeared, thanks to the use of the mass equation and the fact that $\rho_\varepsilon - 1 \, = \, O(\varepsilon)$, a fact which in turn relies on the scaling assumption $Ma = O(\varepsilon)$, recall (8.5). On the other hand, this argument forces us to work with the quantity r, hence to take the curl of the momentum equation. As it is clear, in this instance this makes no worries, as the target velocity field u satisfies (8.7), so the limit dynamics can be completely described by an equation over r; this will *not* be the case for incompressible flows, see Chapter 9.

It remains us to deal with the convective term. Its analysis is more involved, because the presence of terms like $u_\varepsilon^j \, u_\varepsilon^k$ demands to find some strong convergence $u_\varepsilon \longrightarrow u$ of the velocity fields in (say) $L^2_T(L^2_{\rm loc})$. As already commented, in the above mentioned works [22, 23] this property was obtained by the use of the *relative entropy/relative energy* method, but required to have well-prepared initial data. In [126], the authors were able to prove dispersion properties linked with the fast rotation limit, relying on the study of the wave system

$$\tag{8.9} \begin{cases} \varepsilon \, \partial_t r_\varepsilon \, + \, \mathrm{div}\, V_\varepsilon \, = \, 0 \\ \varepsilon \, \partial_t V_\varepsilon \, + \, P'(1) \, \nabla r_\varepsilon \, + \, e_3 \times V_\varepsilon \, = \, \varepsilon \, f_\varepsilon \, , \end{cases}$$

where r_ε has been defined above and we have set $V_\varepsilon \, := \, \rho_\varepsilon u_\varepsilon$, and on the use of the well-known RAGE theorem from scattering theory [71]. The fact that fast rotation implies dispersion was already put in evidence for incompressible (homogeneous) fluids, see [41] for details; Strichartz estimates were also used for weakly compressible fluids, see [131] for an overview of this and related results. The point is that such properties had not been used before in the combined case of compressible fluids in fast rotation, because the more complicated structure of the wave system. It is worth to point out that, differently from Strichartz estimates, the use of the RAGE theorem gives only weak dispersion results, in the sense that it implies the desired strong convergence $u_\varepsilon \longrightarrow u$ in $L^2_T(L^2_{\rm loc})$, which is enough to pass to the limit in the original system (8.4), but without a precise rate of convergence.

All in all, one can rigorsouly prove the convergence of system (8.4) to a 2-D incompressible Navier-Stokes system for the velocity field $u^h \, = \, \nabla_h^\perp r$. As already said, the treatement of the Coriolis term imposes to work in vorticity formulation $\omega = \mathrm{curl}_h u^h \, = \, \Delta_h r$. One thus write an equation for the stream-function r, which is the so-called *quasi-geostrophic equation*:

$$\tag{8.10} \partial_t \big(r - \Delta_h r \big) \, - \, \nabla_h^\perp r \cdot \nabla_h \Delta_h r \, + \, \mu \, \Delta_h^2 r \, = \, 0 \, ,$$

set in $\mathbb{R}_+ \times \mathbb{R}^2$.

8.1.2 Multiscale analysis

After work [126], people has started to focus on the study of the more complex system (8.2). In particular, because of the possible choice of the values of the parameters m and n, the analysis of the multiscale limit problem has attracted a lot of interest.

In this subsection, we try to make a thorough panorama of the results available in the literature and, at the same time, to highlight the difficulties linked with this kind of problems.

A model case

The multiscale analysis of system (8.2) has begun with work [125] by Feireisl, Gallagher, Gérard-Varet and Novotný. There, the authors took $G = 0$ and studied the limit $\varepsilon \to 0^+$ in presence of the centrifugal force $F(x) = \left|x^h\right|^2$. They considered two different regimes: they assumed either $m > 10$ (anisotropic scaling, the incompressible limit is the predominant effect in the dynamics) or $m = 1$ (isotropic scaling, the incompressible and fast rotation limits are act with the same order of magnitude). Observe that, even when $m = 1$, the situation is different with respect to the one described in Subsection 8.1.1, as the pressure and Coriolis terms are in balance with the centrifugal force term now.

Before describing the corresponding results for the two cases, it is worth to point out that, F being unbounded in Ω, the presence of the centrifugal force in the system is a source of technical troubles and handling it requires a carefull process of localisation in space.

Let us now focus for a while on the anisotropic case $m \gg 1$. Notice that the energy inequality (8.5) immediately gives $\rho_\varepsilon - 1 = O(\varepsilon^m)$ in this case (notice that the factor ε^{-2m} appears in front of the term $\mathcal{H}\big(\rho_\varepsilon \,\big|\, 1\big)$, in place of the factor ε^{-2}). However, despite being a lower order effect compared to the small Mach number limit, the fast rotation (small Rossby number) gives also contributions to the target dynamics. Then, according to both the incompressible limit and the Taylor-Proudman theorem, one can prove convergence to a 2-D incompressible Navier-Stokes equation for the limit velocity field.

While, at the qualitative level, the target dynamics is essentially the same as the one derived in [126] (see Subsection 8.1.1), the proofs actually differ very much one from the other. This is related of course to the presence of a non-zero centrifugal force, but there are deeper complications than the ones mentioned above, deriving from the unbondedness of F in Ω. Needless to say, the main issue of the analysis was, once again, proving the convergence of the convective term $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon$. As already observed, when $m > 1$ the incompressible limit is predominant over the fast rotation limit; thus, the idea implemented in [125] was to resort to Strichartz estimates [100] for the low Mach number limit in order to prove local strong convergence of the velocity fields $\big(u_\varepsilon\big)_\varepsilon$. However, when writing the wave system, we see that

$$\text{(8.11)} \qquad \begin{cases} \varepsilon^m \,\partial_t r_\varepsilon \,+\, \operatorname{div} V_\varepsilon \,=\, 0 \\ \varepsilon^m \,\partial_t V_\varepsilon \,+\, P'(1)\,\nabla r_\varepsilon \,=\, \varepsilon\, f_\varepsilon \,-\, \varepsilon^{m-1}\, e_3 \times V_\varepsilon \end{cases}$$

(actually, the system reads a bit more complicated than that, because of the presence of the centrifugal force). In contrast with (8.9), we see that the Coriolis term gives rise to a forcing term which is *large*, namely, of order $O(\varepsilon^{m-1})$ instead of order $O(\varepsilon^m)$. This complication is the main reason for the constraint $m > 10$ appearing in [125].

In the case of isotropic scaling, *i.e.* $m = 1$, instead, we see that the fast rotation limit combines with an *anelastic limit*. As a matter of fact, because of the balance of the pressure and centrifugal force terms, one can consider densities which are small perturbations around a profile $\widetilde{\rho}(x)$ which is no more constant (whereas before, for $F = 0$, we had $\widetilde{\rho} \equiv 1$) and, in the specific case of $F = \left|x^h\right|^2$, unbounded in Ω. Thus, passing to the limit $\varepsilon \to 0^+$ in the mass equation yields the anelastic constraint $\operatorname{div}\big(\widetilde{\rho}\, u\big) = 0$, where u is the target velocity profile. On the other hand, we see that the three terms $\nabla P(\rho_\varepsilon)$, $e_3 \times \rho_\varepsilon u_\varepsilon$ and $\rho_\varepsilon \nabla F$ are in balance in the momentum equation, giving rise to the relation $u^h = \nabla_h^\perp\big(\frac{P'(\widetilde{\rho})}{\widetilde{\rho}}\, r\big)$ for the limit velocity field u and the limit r of the density oscillations. Thus, the velocity fields $\big(u_\varepsilon\big)_\varepsilon$ still convergence to a 2-D incompressible profile, according to the Taylor-Proudman theorem, which is expected to satisfy an incompressible Navier-Stokes equation with variable (static) density $\widetilde{\rho}$. However, as a consequence of the fact that the motion is more constrained than in the case $\widetilde{\rho} \equiv 1$ (the anelastic constraint appears as an additional condition the target velocity u has to satisfy, with respect to the previous cases), it turns out that passing to the limit in the convective term simply gives terms which stay in the kernel of the singular perturbation operator, thus the contribution of the convective term in the limit $\varepsilon \to 0+$ reduces to 0. Then, the target equation reads as a *linear* quasi-geostrophic type equation with variable coefficients.

We remark that the presence of the variable reference density state $\widetilde{\rho}$ appeared also in the definition of the singular perturbation operator, as a variable coefficient in front of the various differential operators involved. Hence, the study of propagation of waves became more involved, as spectral methods (which Strichartz estimates, RAGE theorem... are based on) were out of use in this context. The authors of [125] proved convergence (actually, vanishing) of the convective term in the limit $\varepsilon \to 0^+$ by use of a *compensated compactness* argument, following [178, 145].

More results on the multiscale limit: stratification effects

After [125], more papers appeared, dealing with the multiscale analysis of the ε-dependent system (8.2), see [133, 132] by Feireisl and Novotný, see also [167, 168] by Kwon, Novotný and collaborators for the case of the full Navier-Stokes-Fourier system. Without entering too much into the details, let us point out some important points of those studies.

First of all, those works considered the effects of gravity, namely the case in which $G(x) = -x^3$ is non-zero, with $F = 0$. However, the regime was always a (very) *low stratification* regime, meaning that either $n = 0$, so the gravity term was not penalised, or one imposed the constraint

$$1 \leq n < \frac{m}{2}\,.$$

In particular, the scaling $n = m/2$, which was somehow classical for the incompressible limit problem (see [131]), was out of reach here, as well as the strong stratification regime $n = m$ (and in particular thecase of the isotropic scaling $n = m = 1$). In addition, all the range of values $1 \leq m \leq 2$ could not be considered in those works.

On the other hand, the authors of those works were able to perform also a vanishing viscosity (and heat diffusivity) limit, deriving in this way inviscid limit equations, still remaining in the framework of ill-prepared initial data. This was possible thanks to a wise combination of the relative entropy inequality together with dispersive-type estimates (Van Der Corput lemma and similar arguments based on stationary phase) together with a decomposition of the initial data into a part which lives in the kernel of the singular perturbation operator plus an oscillating component.

As a last comment in this context, we want to mention that, as a consequence of the anisotropy (recall that, in particular, one has $m > 2$), the Coriolis term generated also in this context large forcing terms in the study of waves propagation, as in (8.11). Interestingly, in [132] the authors changed the approach to improve the range of values of possible m for which proving convergence (recall the previous constraint $m > 10$ appearing in [125]). The method consisted in seeing the *large* forcing term $\varepsilon^{m-1}\, e_3 \times V_\varepsilon$ as a *small perturbation* of the singular perturbation operator: one passes from the wave propagator

$$\mathcal{A}\,:\,\begin{pmatrix} r \\ V \end{pmatrix} \longmapsto \begin{pmatrix} \operatorname{div} V \\ \nabla r \end{pmatrix}\,,$$

pertinent in system (8.11), to the family of perturbed wave propagators

$$\mathcal{A}_\sigma\,:\,\begin{pmatrix} r \\ V \end{pmatrix} \longmapsto \begin{pmatrix} \operatorname{div} V \\ \nabla r \,+\, \sigma\, e_3 \,\times\, V \end{pmatrix} \,=\, \mathcal{A}\begin{pmatrix} r \\ V \end{pmatrix} \,+\, \sigma \begin{pmatrix} 0 \\ e_3 \,\times\, V \end{pmatrix}\,,$$

where $\sigma = \sigma(\varepsilon) = \varepsilon^{m-1}$. In terms of the family of operators $\mathcal{A}_\sigma$, with $\sigma = \varepsilon^{m-1}$, the wave system (8.11) can be recasted as

$$\varepsilon^m\, \partial_t \mathcal{U}_\varepsilon \,+\, \mathcal{A}_\sigma\big[\mathcal{U}_\varepsilon\big] \,=\, \varepsilon^m\, \mathcal{F}_\varepsilon\,,$$

where we have set $\mathcal{U}_\varepsilon = \big(r_\varepsilon, V_\varepsilon\big)$ and $\mathcal{F}_\varepsilon = \big(0, f_\varepsilon\big)$. The idea is then that, $\mathcal{A}_\sigma$ being a small perturbation of the singular operator $\mathcal{A}$ for σ small, the dispersive estimates which old true for the latter should hold also for the former operator, somehow independently of the perturbation parameter $\sigma = \varepsilon^{m-1}$.

We will find again this point of view in a while, when speaking of works [106, 109][2].

In the next sections, we present the main outcomes of our researches about the singular limit problem for fastly rotating fluids in presence of multiple scales. In the very last section of the chapter we present some open problems.

8.2 The case of capillary fluids

At first, in [106, 107] we focused our attention to a Navier-Stokes-Korteweg system discussed in [24] (see also more references therein), where we added a term $\mathfrak{C}(\rho, u)$ as in (7.1) accounting for the effects of a fast rotation of the ambient space.

As the Navier-Stokes-Korteweg system is often used for modelling capillary fluids, and more in general for diffuse interface models, to take into account large variations of the fluid density in small regions of space, the presence of the Coriolis force (which, on the contrary, is more relevant at large space scales) is certainly questionable from the physical standpoint. However, the mathematical problem (about which we are going to give more details in a while) looked interesting to us and was already considered in some previous works, see *e.g.* [22] by Bresch and Desjardins and [163] by Jüngel, Lin and Wu. We notice, however, that those works only treated the 2-D case, in the regime of vanishing capillarity (more details here below) and for well-prepared initial data (as a matter of fact, the convergence argument relied on a relative entropy method).

Let us now introduce the system of equations, which looks very similar to (8.4), with the exception of two main differences appearing in the momentum equation: the presence of a higher order capillary term $-\rho\nabla\Delta\rho$ and the choice $\mu = \mu(\rho) = \rho$ anbd $\lambda = 0$ in the definition of the tensor $\mathbb{S}$. In particular, notice that the viscosity coefficient $\mu = \mu(\rho)$ is degenerate close to vacuum. Specifically, the equations read as follows:

$$(8.12)\qquad \begin{cases} \partial_t\rho + \operatorname{div}\big(\rho u\big) = 0 \\ \partial_t\big(\rho u\big) + \operatorname{div}\big(\rho u\otimes u\big) + \dfrac{1}{\varepsilon^2}\nabla P(\rho) + \dfrac{1}{\varepsilon} e_3 \times \rho u \\ \qquad\qquad\qquad\qquad - \nu \operatorname{div}\big(\rho\,\mathbb{D}u\big) - \dfrac{1}{\varepsilon^{2(1-\alpha)}}\rho\nabla\Delta\rho = 0\,, \end{cases}$$

where $\nu > 0$ is a positive (viscosity) coefficient, the symbol $\mathbb{D}u := \big(Du + \nabla u\big)/2$ denotes the symmetric part of the Jacobian matrix Du of u and $\alpha \in [0,1]$ is a fixed parameter. Having $\alpha > 0$ means that we are in a *vanishing capillarity regime*, which then combines with the small Mach number and small Rossby number regimes; the value $\alpha = 0$ corresponds instead to the *constant capillarity* case, in which the capillary term is in balance with the pressure term and the Coriolis term. This terminology comes from the choice of the rescaling of the capillarity coefficient, previous to the adimensionalisation of the equations.

From a mathematical standpoint, the capillarity term provides one with a control on the higher order derivatives of the density, as it can be easily seen from an energy estimates. On the other hand, the specific choice of the viscosity coefficients $\mu(\rho) = \rho$ and $\lambda = 0$ in $\mathbb{S}$ endows system (8.12) with a very nice mathematical structure, which allows for a second energy inequality known under the name of *BD entropy estimate* (after Bresch and Desjardins): we refer *e.g.* to [24, 22], see also [190, 216, 201, 220] and references therein for more about the BD entropy and generalisations. Thus, if we forget the Coriolis term for a while, from the capillarity term

[2] We point out that we were not aware of work [132] at the time [106] was written.

$-\frac{1}{\varepsilon^{2(1-\alpha)}}\rho\nabla\Delta\rho$ one derives the property $\frac{1}{\varepsilon^{1-\alpha}}\nabla\rho \in L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)$ by energy estimates, and the property $\frac{1}{\varepsilon^{1-\alpha}}\Delta\rho \in L^2_{\mathrm{loc}}\big(\mathbb{R}_+;L^2(\Omega)\big)$ by the BD entropy structure.

Unfortunately, the presence of the singular Coriolis term seems to destroy the BD entropy structure of the system, meaning that the obtained estimates look to be not uniform in the small parameter $\varepsilon\in]0,1]$. The first contribution of [106] was to show that, in fact, the Coriolis term can be controlled *uniformly* with respect to ε in the BD entropy estimates. By using that fundamental property and the uniform buonds which derive from it, we were able to perform the asymptotic limit $\varepsilon\to 0^+$ for all the range of the parameter $\alpha\in[0,1]$, in the 3-D geometry $\Omega=\mathbb{R}^2\times]0,1[$ (supplemented with complete-slip boundary conditions (8.3) and Neumann boundary conditions for the density gradient) and for general *ill-prepared* initial data, thus extending works [22, 163] in all those directions.

Roughly speaking, assuming that the normalisation $P'(1)=1$ holds for the pressure function, the main results obtained in [106] can be summarised in the following way.

Theorem 8.1. *If* $\alpha\in]0,1]$*, the Navier-Stokes-Korteweg system* (8.12) *converges to the quasi-geopstrophic equation* (8.10).

If $\alpha=0$ *instead, then system* (8.12) *converges to the higher order quasi-geostrophic equation*

$$\partial_t\big(\mathrm{Id}-\Delta_h+\Delta_h^2\big)r-\nabla_h^\perp\big(\mathrm{Id}-\Delta_h\big)r\cdot\nabla_h\Delta_h^2 r+\frac{\nu}{2}\Delta_h^2\big(\mathrm{Id}-\Delta_h\big)r=0\,, \tag{8.13}$$

set in $\mathbb{R}_+\times\mathbb{R}^2$.

Let us briefly comment on the proof of the results of [106]. The three keywords here are:

(i) dispersion;

(ii) symmetrisation;

(iii) perturbation.

To begin with, one has to remark that the case $\alpha=1$ is exactly the same as the one treated in [126]: thus, an application of the RAGE theorem allows us to deduce the strong convergence properties which are needed in order to pass to the limit in the weak formulation of equations (8.12). This is *dispersion.*

The case $\alpha=0$ is also very similar. In this situation, the capillarity term is in balance with the pressure gradient and the Coriolis term, giving a more complicated structure of the quasi-geostrophic balance (7.2), namely

$$e_3\times u+\nabla\big(P'(1)\mathrm{Id}-\Delta\big)r=0\,.$$

Notice that now the singular perturbation operator, that is

$$\mathcal{B}\,:\,\begin{pmatrix} r\\ V\end{pmatrix}\longmapsto\begin{pmatrix}\operatorname{div}V\\ \nabla\big(P'(1)\mathrm{Id}-\Delta\big)r+e_3\times V\end{pmatrix},$$

is no more skew-adjoint with respect to the L^2 scalar product. However, a *microlocal symmetrization argument* in the spirit of the one discussed in Chapter 3 allows one to still apply the RAGE theorem. This is *symmetrisation.* In turn, one gets convergence to the higher order quasi-geostrophic type equation (8.13), whose mathematical study was performed in [95].

Finally, in the intermediate cases $0<\alpha<1$, the key idea is to look at the lower order capillarity term as a *perturbation* of the original singular perturbation operator $\mathcal{B}_0$ (which is the same as in [126] and which coincides, in the notation of Subsection 8.1.2, with the operator $\mathcal{A}_1$ obtained for $\sigma=1$). Then, we proved [106, 109] a version of the RAGE theorem for perturbations $\mathcal{B}_\sigma$ of a given symmetric operator $\mathcal{B}_0$, where again $\sigma=\sigma(\varepsilon)=\varepsilon^{2\alpha}$, in which however also the microlocal symmetrizers of the operators $\mathcal{B}_\sigma$ depend on σ. We stress the fact that, for obtaining this perturbed version of the RAGE theorem, the operators $\mathcal{B}_\sigma$ needed to satisfy a fundamental *spectral gap* condition, which allows to isolate the point spectrum $\sigma_p(\mathcal{B}_\sigma)$ from the continuous part $\sigma_c(\mathcal{B}_\sigma)$ of the spectrum of $\mathcal{B}_\sigma$ in a somehow uniform way (uniform with respect to σ). Such a spectral gap condition is naturally verified in the case of the penalised Navier-Stokes-Korteweg system (8.12). Using that generalisation of the RAGE theorem we were able, in turn, to take the limit $\varepsilon \to 0^+$ and prove the convergence to the classical quasi-geostrophic equation (8.10) also in the case $0 < \alpha < 1$.

8.3 Heat-conducting fluids

In [98], we devoted attention to the fast rotation limit of the full Navier-Stokes-Fourier system with Coriolis force, in presence also of gravity and of the centrifugal force. The Navier-Stokes-Fourier system describes the motion of a compressible fluid flow in which temperature variations are taken into account. At the mathematical level, an entropy balance is added to system (8.2), following the Feireisl-Novotný theory [131]; we avoid to give here the precise form of the original (ε-dependent) system, which would require to introduce further notation and constitutive relations.

As already mentioned in Subsection 8.1.2, the study of the fast rotation limit for the full Navier-Stokes-Fourier system was initiated in [167, 168], but under some restrictions on the various order of magnitues of the Mach number and Froude number: in those works one had either $n = 0$ or $1 \leq n < m/2$, moreover the centrifugal force F was set equal to 0. Our goal was to drop those restrictions, which looked of technical nature, rather than being natural contraints. In [98] we proved convergence to suitable target systems in all the regimes

$$n = \frac{m}{2} \qquad \text{and} \qquad m \geq 2 \qquad (m \geq 1 \text{ if } F = 0)\ .$$

For the sake of conciseness, we give a rough statement pertaining the anisotropic scaling $m > 1$ only, which already contains all the main ingredients of our analysis.

Theorem 8.2. *Let either $m \geq 2$ and $F(x) = \left|x^h\right|^2$, or $m > 1$ and $F = 0$. Define the density and temperature variations respectively as*

$$r_\varepsilon \,:=\, \frac{\rho_\varepsilon - 1}{\varepsilon^m} \qquad \textit{and} \qquad \Theta_\varepsilon \,:=\, \frac{\theta_\varepsilon - \overline{\theta}}{\varepsilon^m}\,,$$

where θ_ε are the temperature of the fluid at a given ε and $\overline{\theta}$ is a (constant) positive temperature reference state. Let r, u and Θ be weak limits (up to an extraction) of, respectively, the sequences $\left(r_\varepsilon\right)_\varepsilon$, $\left(u_\varepsilon\right)_\varepsilon$ and $\left(\Theta_\varepsilon\right)_\varepsilon$.

Then, according to the Taylor-Proudman theorem, one has $u = \left(u^h, 0\right)$, with $u^h = u^h(t, x^h)$ and $\mathrm{div}_h u_h = 0$. In addition, the triplet $\left(r, u^h, \Theta\right)$ solves the Oberbeck-Boussinesq system

$$\begin{cases} \partial_t u^h \,+\, u^h \cdot \nabla_h u^h \,+\, \nabla_h \Pi \,-\, \Delta_h u^h \,=\, \delta_2(m)\, \langle r \rangle\, \nabla_h F \\ \partial_t \Theta \,+\, u^h \cdot \nabla_h \Theta \,-\, \Delta \Theta \,=\, \delta_2(m)\, u^h \cdot \nabla_h F \\ \nabla\big(r \,+\, \Theta\big) \,=\, \nabla G \,+\, \delta_2(m)\, \nabla F\,, \end{cases}$$

supplemented with suitable initial conditions.

In the previous statement, all the physical constants appearing in the limit have been set equal to 1. We have also defined $\delta_2(m)$ to be the Kronecker delta, which has value 1 if $m = 2$, 0 otherwise. Moreover, following the notation introduced above, we have denoted by $\langle f \rangle$ the vertical average of a function f.

Let us also observe that, in the case $m = 1$, one gets a similar statement, up to the fact that one has to work with a suitable stream-function q of the velocity field u (which is now defined in terms of r and Θ) and a modified variable Υ in place of Θ for the temperature variations. Then, the target system is identified as a coupling of a quasi-geopstrophic equation for q with a suitable advection-diffusion equation for Υ.

Avoiding to enter further into the details, we want to highlight here a couple of facts about Theorem 8.2, which look interesting to us. The first remark is that, owing to the last relation in the target system, some (very mild) stratification effects remain in the limit: this is a consequence of the choice $n = m/2$ in the scaling. Of course, no vertical variations enter into play in the equation for u, but the density and temperature variations, R and Θ respectively, do present oscillations in the third variable. The second point we want to stress is the fact that the previous statement does not ask for any restriction about the values of the various scaling parameters, apart from the constraint $m \geq 2$ when $F \neq 0$; we notice however that this constraint looks now no more technical, but structural. We prefer not to explain this point further here; on the contrary, let us comment a bit more about the method of the proof of our results, which allows to drop (almost) all the restrictions on the scaling parameters.

The improvement with respect to the previous studies was possible thanks to the use of a *compensated compactness argument*, in the spirit of [145, 125]. For the time being, let us take $F = 0$. The simple but fundamental remark was to observe that, for any $m \geq 1$, the wave system (8.11) (which remains similar also in the case of the Navier-Stokes-Fourier system) already encodes all the compactness properties one needs to pass to the limit. Indeed, recalling that $\langle f \rangle$ denotes the vertical average of a function f, from (8.11) one easily gets

$$\begin{cases} \varepsilon^m\, \partial_t \langle r_\varepsilon \rangle \,+\, \mathrm{div}_h \langle V_\varepsilon^h \rangle \,=\, 0 \\ \varepsilon^m\, \partial_t \langle V_\varepsilon \rangle \,+\, P'(1)\, \nabla \langle r_\varepsilon \rangle \,+\, \varepsilon^{m-1}\, e_3 \times \langle V_\varepsilon \rangle \,=\, \varepsilon\, \langle f_\varepsilon \rangle\,. \end{cases}$$

Hence, applying the operator curl_h to the orizontal components of the momentum equation yields

$$\begin{cases} \varepsilon^m\, \partial_t \langle r_\varepsilon \rangle \,+\, \mathrm{div}_h \langle V_\varepsilon^h \rangle \,=\, 0 \\ \varepsilon^m\, \partial_t \mathrm{curl}_h \langle V_\varepsilon^h \rangle \,+\, \varepsilon^{m-1}\, \mathrm{div}_h \langle V_\varepsilon^h \rangle \,=\, \varepsilon\, \mathrm{curl}_h \langle f_\varepsilon^h \rangle\,, \end{cases}$$

and taking the difference of the two equations immediately gives the compactness of the quantity

$$\zeta_\varepsilon \,:=\, \mathrm{curl}_h \langle V_\varepsilon^h \rangle \,-\, \varepsilon^{m-1}\, \langle r_\varepsilon \rangle\,.$$

In turn, this is the only information one needs in order to prove convergence of the vertical averages appearing in the non-linear term (namely, theconvective term). As for the coupling of the terms encoding the vertical oscillations, the argument is very similar to [125] and shows that the interaction of the oscillating components is small and actually vanishes in the limit $\varepsilon \to 0^+$.

Thus, we see that the previous argument does not require any constraint on the values of the different parameters, at least for $F = 0$. As a matter of fact, if the centrifugal force F is not 0, its presence complicates things, whence the constraint $m \geq 2$ in order for the previous argument to still apply. At this point, let us make a comparison with the multiscale situation (namely, $0 \leq \alpha \leq 1$) treated in Section 8.2: the perturbative argument from [106] did not require any constraint on the values of the parameters either; however, we did not find a way to apply it in the context described in the present section, as now the (perturbed) family of singular perturbation operators do not satisfy the uniform spectral gap condition mentioned above.

We refer to paper [98] for the discussion about other technical difficulties which arise in this study (choice of the domain for the theory of weak solutions [131] to apply, localisation procedure owing to the presence of the centrifugal force, study of the static states. . .) and the way we dealt with them.

To conclude this part, let us mention the study performed in [99], where we restricted our attention to the barotropic system (8.4) with $F = 0$ for simplicity. In that paper, we were able to weaken the restriction over the parameter n and to prove convergence for the range of values

$$m < 2n \leq m+1 \qquad \text{if } \; m > 1\,, \qquad\qquad \frac{1}{2} < n < 1 \qquad \text{when } \; m = 1\,.$$

Observe that, under those assumptions, one still works in a regime of low stratification. The proof was still based on a compensated compactness argument and deeply used the specific form of the gravity, together with the constraints which the fast rotation imposes (in particular, the Taylor-Proudman theorem).

8.4 The singular limit for large Mach numbers

To conclude this panoramic view on multiscale analysis, we mention the problem treated in [110]. In that paper, we considered the simplest case of the barotropic Navier-Stokes system (8.2), where we took $F = G = 0$ and we devoted attention to the case in which the Mach number is large with respect to the Rossby number, namely to the regime

$$0 \leq m < 1 .$$

However, in order to obtain some non-trivial limit, we needed to compensate the strong Coriolis force by a gradient term: so, we decided to penalise (in the same vein as [85, 86]) the bulk viscosity coefficient λ which appears in the expression (8.1) of $\mathbb{S}(Du)$ and to take it equal to $\lambda(\varepsilon) = \varepsilon^{-2\beta}$, for some $\beta \geq 1$. It is worth to point out that the penalisation of the bulk viscosity already enforces the incompressibility constraint at the limit. Taking into account this fact, it would be natural to restrict the attention to the case $m = 0$: yet, our proof of convergence enables us to do that only in space dimension $d = 2$.

A rough formulation of the main results of [110] is contained in the following statement.

Theorem 8.3. *Let either $d = 3$ and $0 < m < 1$, or $d = 2$ and $0 \leq m < 1$. For any $\varepsilon \in]0, 1]$, define the quantity*

$$\sigma_\varepsilon := \frac{1}{\varepsilon} \left(\rho_\varepsilon - 1\right) .$$

Then the sequence of vertical averages $\big(\langle \sigma_\varepsilon \rangle\big)_\varepsilon$ is uniformly bounded in $L^2_{\rm loc}\big(\mathbb{R}_+; H^{-k}_{\rm loc}(\Omega)\big)$, for some $k \in \mathbb{N}$ large enough.

In addition, denote by σ a weak-$*$ limit of that sequence and by u a weak-$*$ limit of the sequence of the velocity fields. Then one has $u = \big(u^h, 0\big)$, with $u^h = u^h(t, x^h)$ and $\mathrm{div}_h u^h = 0$, and the couple $\big(\sigma, \omega\big)$, with $\omega := \mathrm{curl}_h u^h$, solves the equation

$$\partial_t\big(\omega - \sigma\big) + u^h \cdot \nabla_h \omega - \mu \, \Delta_h \omega = 0 ,$$

supplemented with suitable initial conditions.

The main ideas for proving the previous result go back to paper [113], which we will present in detail in the next chapter. We also refer to that discussion for an better explanation of Theorem 8.3. Here, we limit ourselves to highlight some delicate points of the statement and of the analysis.

First of all, we want to stress the similarity of the previous equation for ω and σ with the quasi-geostrophic equation (8.10). Nonetheless, here we miss a stream function relation linking ω and σ, so the target dynamics looks underdetermined. This is a consequence of the very low regularity space in which the uniform bounds for the family $\big(\langle \sigma_\varepsilon \rangle\big)_\varepsilon$ are found.

On the other hand, we observe that the uniform boundedness of $\big(\langle \sigma_\varepsilon \rangle\big)_\varepsilon$ is a key property of the analysis, which may look however surprising at a first sight. As a matter of fact, keeping in mind the energy inequality (8.5), one may expect to have $r_\varepsilon := \big(\rho_\varepsilon - 1\big)/\varepsilon^m$ to be uniformly bounded in some space (see also Theorem 8.2 in this respect), but here $0 \leq m < 1$. The improvement relies on the crucial use of the structure of the wave system, in a similar fashion as explained in the previous section when speaking about compactness of the functions $\big(\zeta_\varepsilon\big)_\varepsilon$. We refer to Chapter 9 for more details, where we will use a similar property in the incompressible framework.

Finally, let us comment a bit on the proof of Theorem 8.3. The main argument uses compensated compactness again (as presented in Section 8.3 above) and the structure of the new wave system at hand. However, when dealing with the convective term, in this argument one cannot avoid the presence of a bilinear expression, linking the averages of the third components of the vorticities $\langle \omega_\varepsilon^3 \rangle$ and of the velocity fields $\langle u_\varepsilon \rangle$. Some strong convergence property is thus needed to compute the limit of that expression. At this point, the key observation is that the potential part $\nabla \Phi_\varepsilon$ of the momenta $V_\varepsilon = \rho_\varepsilon u_\varepsilon \approx u_\varepsilon$ satisfies a heat equation with fast diffusion:

$$\partial_t \langle \Phi_\varepsilon \rangle \,-\, \frac{1}{\varepsilon^{2\beta}}\, \Delta_h \langle \Phi_\varepsilon \rangle \,=\, \langle G_\varepsilon \rangle \,,$$

where the suitable forcing term G_ε is *not* uniformly bounded in ε. However, steered byt the fact that space derivatives of the solution to the heat equation decay faster in time than the solution itself, one can prove sharp decay (in ε) estimates for the quantity $(-\Delta_h)^s \langle \Phi_\varepsilon \rangle$, for a large enough $s \geq s_0 = s_0(m, \beta)$. In turn, combining this property with the structure of the wave system again, one is able to prove compactness of the vorticity components $\big(\langle \omega_\varepsilon^3 \rangle\big)_\varepsilon$ in suitable norms, a fact which finally enables us to compute the limit of the convective term and to prove the convergence result.

8.5 Some open problems and perspectives

We list here a series of open questions which drive our attention and which we plan to consider in the near future.

Strongly stratified fluids

The first main problem in this context, which remains open even for the simplest case of barotropic fluids, consists in computing the fast rotation limit in presence of strong stratification effects. This corresponds to the scaling $n = m = 1$ in the original system (8.2).

So far, the only result in this direction is [129] by Feireisl, Lu and Novotný, but the convergence to the limit dynamics is proven only for well-prepared initial data, by mean of the relative entropy method. We refer also to [17] for related results, in connection with the study of Ekman boundary layers (more details below).

The main problem is that, in the strongly stratified case, the reference density state $\widetilde{\rho}$ depends on the vertical variable: one has $\widetilde{\rho} = \widetilde{\rho}(x^3)$. First of all, the validity of the Taylor-Proudman theorem is no more clear in this situation. Moreover, all the methods employed so far to deal with ill-prepared initial data and propagation of waves (spectral methods, compensated compactness. . .) seem to break down in such situation. New ideas are then required.

Let us mention that the problem really pertains to the coexistence of a fast rotation term and a strong stratification term. Indeed, the strong stratification limit for compressible fluids (even in presence of temperature variations) without Coriolis term has been computed in a number of situations, see *e.g.* [131] for an overview of the related literature (see also [184, 119] and references therein).

Anelastic limits

The previous observation prompts us to consider, more in general, singular limits in presence of an *anelastic constraint*. With this, we mean that we intend to devote attention to situations in which the reference density state $\widetilde{\rho}$ is non-constant; then, typically the mass equation enforces the anelastic constraint $\operatorname{div}\big(\widetilde{\rho}\, u\big) = 0$ in the limit.

Such limits are relevant not only in the framework of geophysical flows: they naturally arise also in the context multi-component fluids or in the one of capillariy fluids governed by Var der Waals (non-monotone) pressure laws, for instance. Let us focus on this latter case, namely on Van der Waals pressure laws. Looking at the Navier-Stokes-Korteweg system (8.12), we see that, in the constant capillarity regime (*i.e.* $\alpha = 0$ in those equations), the non-monotone pressure term is in balance with the capillary term. This balance allows to consider static density states $\widetilde{\rho}$ which are non-constant. Computing the asymptotics $\varepsilon \to 0^+$ in this setting, however, seems to be challanging, even in absence of the Coriolis term. As a matter of fact, complications come from some commutator terms appearing in the wave system owing to the presence of a variable $\widetilde{\rho}$ (notice that a regularisation-in-space procedure is often required), combined with the very poor controls one disposes of on the higher order derivatives of the density variations. This contrasts very much with the results which are available, for instance, for the incompressible limit (no Coriolis force) for strongly stratified barotropic flows (see again *e.g.* [184, 131], or [119] for a degenerate Navier-Stokes system). How to bypass that difficulty seems not clear at present.

Finally, in this context it is interesting to mention the study of [209] by Schochet and Xu. In that paper, the authors are able to consider singular limits in presence of three different time scales and to capture second order effects in the target dynamics. The study is performed on strong solutions to some hyperbolic system and makes use of revisited filtering techniques in order to deal with the multiple scales. Understanding this approach in the context of weak solutions for the models discussed in this chapter may open new perspectives in the study of the multiscale problem, which we would like to explore.

Ekman boundary layers and topography effects

The second main open problem in this context is to undestand the interactions of the acoustic-Poincaré waves (which drive oscillations around the target profiles) with the boundary. In this respect, there are two aspects to be considered.

The first question is to study *Ekman boundary layers* (which arise when imposing no-slip boundary conditions) for weakly compressible fast rotating fluids.

The case of incompressible homogeneous fluids is by now quite well-understood. We refer to *e.g.* [41] for an overview of the available results in that context. More recent studies have focused the effects of a resonant forcing at the surface of the domain [76] (see also [77] for a somehow related analysis) and on the stability of the Ekman boundary layers in the regime of vanishing viscosity [206, 185].

The first work addressing the study of Ekman boundary layers for weakly compressible fast rotating fluids is paper [23] by Bresch, Desjardins and Gérard-Varet. That study was recently generalised by [17] to the case of more general pressure laws and of strongly stratified fluids (following the analysis of [129]). However, those works only treated the framework of well-prepared initial data; this is linked with the techniques of the proof, which relied on the use of the relative entropy inequality. The description of the Ekman boundary layers for general ill-prepared initial data thus remains open in the case of compressible flows in fast rotation.

In addition, we point out that some limitations appear in works [23, 17], in the sense that the obtained convergence results are only *conditional* results. As a matter of fact, the presence of Ekman layers forces one to consider small viscosity coefficients in the vertical direction, to balance large vertical variations of the fluid in the small boundary layer. Now, the weak solutions theory by Lions-Feireisl [177, 135, 123] relies in an essential way on an isotropy condition on the viscous stress tensor $\mathbb{S}(Du)$, isotropy which enables one to derive a simple equation for the effective viscous flux (recall the discussion of Section 5.1). Notice that some recent improvements in the anisotropic case have been recently obtained, see [25] by Bresch and Jabin, but conditions on the adiabatic exponent γ appearing (5.2) are still too restrictive for applications to the study of geophysical flows. This observation shows that the investigation of existence of weak solutions to the compressible Navier-Stokes system in presence of anisotropic viscosity coefficients is an interesting and important open problem. However, strictly related to the question of the study of Ekman boundary layers for compressible fluid flows, we point out that paper [18] offers an alternative approach for a rigorous derivation of the target equations, even though it misses a precise description of the structure of the solutions inside the boundary layers. We will give more details about this in the next Chapter, see in particular Section 9.3.

We avoid any discussion here about *Munk boundary layers*, namely boundary layers which appear the the vertical boundaries of the domain, for which no works seem to be available for weakly compressible flows. As a matter of fact, the situation is poorly understood even for incompressible homogeneous flows, see [101, 23, 78] for studies in that direction.

The second question is related to *topography effects*. As a matter of fact, in this kind of studies one usually formulates a flatness assumption on the boundary of the domain $\Omega = \mathbb{R}^2 \times \,]0,1[\,$. In the homogeneous incompressible case, some generalisations have been made [183, 149] to the case of rough boundaries, where however the roughness is small in ε and the structure is periodic in space. Some recent studies [72, 75] have focused on the analysis of the well-posedness of the stationary Navier-Stokes-Coriolis system in presence of a non-flat bottom of size $O(1)$.

The general picture for domains with irregular boundary is still not understood, even for incompressible homogeneous fluids, although progresses have been recently made [40] by Chemin under a radial symmetry assumption, even in presence of emerging islands. Considering the analogous problem for non-homogeneous fluids seems to be somehow subordinated to advances in the incompressible case, together with progresses in the understanding of the strong stratification regime.

In this context, it would be interesting to take advantage of suitable conormal regularity of the flow at the boundary, following previous studies [186, 187] by Masmoudi and Rousset (see also [188] for an application in the context of the low Mach number limit of the barotropic Navier-Stokes system).

Chapter 9

Fast rotation limit: incompressible models

In this chapter, we deal with the fast rotation limit for models of fluids which are *incompressible* and, at the same time, present *density variations*. Despite the huge literature available in the (classical) incompressible homogeneous case, this study has been initiated only very recently.

For simplicity of exposition, we will focus on the fast rotation limit for the density-dependent incompressible Navier-Stokes system with Coriolis force, which reads

$$\left\{\begin{array}{l} \partial_t\rho \,+\, \mathrm{div}\,\big(\rho\, u\big) \,=\, 0 \\[1ex] \partial_t\big(\rho\, u\big) \,+\, \mathrm{div}\,\big(\rho\, u\otimes u\big) \,+\, \dfrac{1}{\varepsilon}\,\nabla\Pi \,+\, \dfrac{1}{\varepsilon}\,e_3\,\times\,\rho\, u \,-\, \mathrm{div}\,\big(\nu(\rho)\,\mathbb{D}u\big) \,=\, 0 \\[1ex] \mathrm{div}\, u \,=\, 0\,. \end{array}\right. \tag{9.1}$$

In equations (9.1), as in the previous chapter, we have denoted by $\mathbb{D}u \,=\, \big(Du + \nabla u\big)/2$ the symmetric part of the Jacobian matrix Du, where $\nabla u \,=\, {}^t(Du)$; on the contrary, we have now called Π the pressure function, to stress the fact that, now, the pressure is no more given, but is in fact an unknown of the problem. The viscosity coefficient $\nu(\rho)$ is assumed to depend continuously on the density function ρ and to be non-degenerate close to vacuum: we assume

$$\nu\in\mathcal{C}^0(\mathbb{R}_+) \qquad\qquad \text{such that} \qquad \forall\,\rho\geq 0\,,\quad \nu(\rho)\,\geq\,\nu_*\,>\,0\,,$$

for some positive constant $\nu_* > 0$. The previous assumption is particularly important, because we will work under assumptions which allow for existence of vacuum regions (namely, regions where $\rho = 0$).

However, notice that some variants of this model can be considered as well. We refer *e.g.* to [47, 48] for related studies in the case of an incompressible MHD system and to [207] for a similar investigation in the context of the density-dependent incompressible Euler system.

Works presented in the chapter

(P.16) F. Fanelli, I. Gallagher: *Asymptotics of fast rotating density-dependent incompressible fluids in two space dimensions.* **Rev. Mat. Iberoam.**, 35 (2019), n. 6, 1763-1807.

(P.23) D. Cobb, F. Fanelli: *On the fast rotation asymptotics of a non-homogeneous incompressible MHD system.* **Nonlinearity**, 34 (2021), n. 4, 2483-2526.

(S.4) M. Bravin, F. Fanelli: *Fast rotating non-homogeneous fluids in thin domains and the Ekman pumping effect.* Submitted (2022).

Not mentioned, but in this context

(P.20) D. Cobb, F. Fanelli: *Rigorous derivation and well-posedness of a quasi-homogeneous ideal MHD system.* **Nonlinear Anal. Real World Appl.**, 60 (2021), Paper n. 103284.

9.1 Introduction

Analogously to what done in Chapter 8, the goal of this chapter is to perform the limit $\varepsilon \to 0^+$ for a family $\big(\rho_\varepsilon, u_\varepsilon, \nabla\Pi_\varepsilon\big)_\varepsilon$ of finite energy weak solutions to the original system (9.1). Again, for suitable initial data (see more precise assumptions below), the existence of a finite energy weak solution to (9.1) at any $\varepsilon > 0$ fixed is available thanks to the theory of P.-L. Lions [176] (see also references therein for previous studies).

As, to the best of our knowledge, there are not so many results on this problem, this introduction will be mainly devoted to point out basic facts about system (9.1) which will appear in the study below, as well as to underline analogies and differences with respect to the corresponding study performed in Chapter 8 for compressible flows.

As already said, we will work in the framework of global in time finite energy weak solutions to system (9.1), whose theory was set up by P.-L. Lions in [176]. These are weak solutions $\big(\rho_\varepsilon, u_\varepsilon, \nabla\Pi_\varepsilon\big)$ which satisfy, for any $t \geq 0$, the energy inequality related to system (9.1), namely

$$\frac{1}{2}\int_\Omega \rho_\varepsilon\left|u_\varepsilon\right|^2\,\mathrm{d}x + \int_0^t\int_\Omega \nu(\rho_\varepsilon)\,\left|\mathbb{D}u_\varepsilon\right|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{2}\int_\Omega \rho_{0,\varepsilon}\left|u_{0,\varepsilon}\right|^2\,\mathrm{d}x\,, \tag{9.2}$$

where we have used the L^2 orthogonality the pressure and the Coriolis terms with u_ε. Notice that, owing to the possible presence of vacuum, which we will assume here (see comments below), the initial condition for the velocity field should rather be formulated in terms of the momentum $m_{0,\varepsilon}$ (roughly speaking, $m_{0,\varepsilon} \approx \rho_{0,\varepsilon}\,u_{0,\varepsilon}$), however, for the sake of simplicity of presentation, we omit here to enter into these technical details.

Next, we remark that, in Lions's theory, bounds for the density functions ρ_ε are derived from properties of pure transport equations driven by divergence-free velocity fields. In particular, those bounds consist on the (formal) preservation of all the L^p norms of the initial state $\rho_{0,\varepsilon}$ or (when convenient) of $\rho_{0,\varepsilon} - \overline{\rho}$, for some constant state $\overline{\rho}$ (typically, we take $\overline{\rho} = 1$ for simplicity). This will become more clear in the discussion below, when introducing our precise working assumptions on the initial density functions.

We also point out that, owing to the divergence-free constraint over $u = u_\varepsilon$ in (9.1), the momentum equation has to be tested on test functions ψ which are themselves divergence-free, namely such that $\operatorname{div}\psi = 0$. Notice that, in this way, the pressure term completely disappears from the weak form of the equations. As a matter of fact, in the incompressible framework the term $\nabla\Pi = \nabla\Pi_\varepsilon$ is simply a Lagrangian multiplier associated to the constraint $\operatorname{div} u_\varepsilon = 0$. In particular, in this context one does not speak anymore about the Mach number: the scaling appearing in (9.1) is justified by the observation that, with this weak formulation, in the limit $\varepsilon \to 0^+$ the singular Coriolis term may be compensated only by a gradient.

Despite the pressure gradient $\nabla\Pi_\varepsilon$ disappears from the weak formulation of the equations, its presence in the equations entails deep consequences at the level of our study. Notice that Π (or Π_ε) is no more a simple (known) function of the density[1]. There are at least three major consequences of this fact.

The first consequence has been pointed out in Chapter 7: in the fast rotation limit $\varepsilon \to 0^+$, one will miss the quasi-geostrophic balance equation (7.2) and all its consequences. We will see

[1] As is well-known, the term $\nabla\Pi$ can be recovered from (ρ, u) by solving an elliptic equation with variable coefficients, recall relation (6.7); in the viscous case, its regularity can be studied by resorting to properties of the Stokes operator, see [176] again.

later on that, as a by-product of the previous fact, the limit dynamics remains *underdetermined* in general, similarly to what happened in Section 8.4.

We immediately point out that, for somehow related reasons, we are unable at present to treat the asymptotic limit $\varepsilon \to 0^+$ for equations (9.1) in a 3-D domain (although something can be said in the case of thin domains, see Section 9.3 for more details in this respect). Therefore, unless otherwise specified, in all this chapter we will take a 2-D spatial domain with very simple geometry, namely

$$\Omega \, = \, \mathbb{R}^2 \qquad \text{or} \qquad \Omega \, = \, \mathbb{T}^2 \, .$$

Then, in system (9.1), we will replace the Coriolis term $e_3 \times \rho u$ with its horizontal projection

$$\rho\, u^\perp \, , \qquad \text{where we have set} \qquad u^\perp \, := \, \big(-u^2, u^1 \big) \, . \tag{9.3}$$

A second main effect of having $\nabla \Pi \, = \, \nabla \Pi_\varepsilon$ as an unknown of the problem appears at the level of the initial data. As a matter of fact, as explained at the beginning of Chapter 8, in this kind of problems one naturally considers initial data $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}\big)$ which are close to a static state $\big(\widetilde{\rho}_\varepsilon, 0\big)$, which may itself depend on $\varepsilon > 0$. Now, setting $u \equiv 0$ in system (9.1) yields that

$$\partial_t \widetilde{\rho}_\varepsilon \, \equiv \, 0 \qquad \Longrightarrow \qquad \widetilde{\rho}_\varepsilon(t,x) \, = \, \widetilde{\rho}(x) \, , \tag{9.4}$$

with $\widetilde{\rho}$ being independent of both time and $\varepsilon > 0$ (independence of ε also uses the momentum equation, in fact), whereas the momentum equation simply reduces to $\nabla \widetilde{\Pi} = 0$, which does not look of any practical use for our scopes. In particular, equation (9.4) allows us to take initial density states of the form

$$\rho_{0,\varepsilon} \, = \, \rho_0 \, + \, \varepsilon \, r_{0,\varepsilon} \, , \qquad \big(r_{0,\varepsilon}\big)_\varepsilon \, \Subset \, L^2(\Omega) \, \cap \, L^\infty(\Omega) \, . \tag{9.5}$$

Now, differently from the case of compressible flows, see *e.g.* system (8.4), there is no reason to take ρ_0 constant. Therefore, in our study we will focus on two different cases: either the *quasi-homogeneous case*, for which

$$\rho_0 \, \equiv \, 1$$

and then $\rho_\varepsilon - 1 = O(\varepsilon)$ for all later times, or the *fully non-homogeneous case*, in which we take a generic density profile ρ_0 which, for structural and technical reasons, we assume to satisfy

$$\rho_0 \, \in \, \mathcal{C}^2(\Omega) \, , \qquad \text{with} \qquad 0 \, \leq \, \rho_0(x) \, \leq \, \rho^* \, ,$$

for a suitable constant $\rho^* > 0$. Notice that we are able to consider the possible *presence of vacuum*, *i.e.* regions where ρ_0 may vanish (and so may do $\rho_{0,\varepsilon}$), proving the convergence of equations (9.1) (in 2-D or thin domains) for $\varepsilon \to 0^+$ in the same setting as Lions's weak solutions theory[2]. In fact, in order to perform the fast rotation limit $\varepsilon \to 0^+$ in (9.1), one needs an additional technical assumption on ρ_0: this is a sort of *non-degeneracy* condition of its critical points and reads

$$\forall \ \text{compact} \ \ K \, \subset \, \Omega \, , \qquad \lim_{\delta \to 0^+} \text{meas} \left\{ x \in K \ \Big| \ \ \big|\nabla \rho_0(x)\big| \, \leq \, \delta \right\} \, = \, 0 \, .$$

The previous condition is somehow a weakened form of a similar requirement already appearing in [145] (see also [107]). In the compressible case, this property can be often derived from the properties of the static state under study (see *e.g.* [125]).

Finally, the third major consequence of not having an explicit direct relation linking $\nabla \Pi = \nabla \Pi_\varepsilon$ to $\rho \, = \, \rho_\varepsilon$ appears at the level of energy estimates and is connected with the previous decomposition (9.5). Let us explain this issue a bit more in detail. Observe that, from (9.5) and

[2] In the case $\Omega = \mathbb{R}^2$, considering vacuum on the initial datum requires some conditions on the possible vanishing of the initial density. As those conditions are transported by the flow [176], they will be automatically verified *uniformly* in $\varepsilon \in \,]0,1]$ also for our problem. We do not enter into the details here and rather refer to [176, 113, 47].

the transport equation for ρ_ε, assuming all the smoothness which is needed, we may follow the flow $\psi_t^\varepsilon(x)$ of u_ε and get that $\rho_\varepsilon = R_\varepsilon + O(\varepsilon)$, where we have denoted by $R_\varepsilon = R_\varepsilon(t,x) = \rho_0\big((\psi_t^\varepsilon)^{-1}(x)\big)$ the transport of ρ_0 along the flow of u_ε. Now, one has $R^\varepsilon \to R$ in some sense (in the weak-$*$ topology of L^∞, for instance), for a suitable density profile R. However, in the fully non-homogeneous case it is not possible to say that the original densities $\rho_\varepsilon \approx R$ are quantitatively closed to the target profile, namely that $\rho_\varepsilon - R = O(\varepsilon)$. On the other side, such a quantitative bound is needed in the convergence argument (recall the discussion in Subsection 8.1.1, for instance). The problem is that, differently from the compressible case, recall the energy estimate (8.5) for instance, the pressure term $\nabla\Pi_\varepsilon$ does not give any bound for the densities ρ_ε, specifically it does not imply anymore the smallness of the density variations $\rho_\varepsilon(t) - R(t) = O(\varepsilon)$ at later times. In other words, despite the equation for ρ_ε looks very gentle, there is no reason for the density functions ρ_ε to be close to the target profile R, at least in the fully non-homogeneous case. Of course, in the quasi-homogeneous setting, for which $\rho_0 \equiv 1$, one has that $R_\varepsilon = R = 1$, so one does not see at all the problem we have just mentioned and the limit can be computed easily; we refer to Subsection 9.2.1 for details about this case.

In what follows, we are going to see how to bypass the previous obstacles and prove a convergence result for system (9.1), in the 2-D domain $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$ (in Section 9.2) and in 3-D thin domains (see Section 9.3, where we will also point out a new approach to the study of the Ekman pumping effect).

9.2 Asymptotics in two space dimensions

In this section, let us focus on the two-dimensional case $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$. In particular, we assume that the Coriolis term in system (9.1) is replaced by its "horizontal" counterpart (9.3). We are going to explain how to rigorously compute the limit $\varepsilon \to 0^+$ of system (9.1) in both the quasi-homogeneous and fully non-homogeneous cases. This was done in [113] in the case of the viscosity coefficient $\nu(\rho) \equiv 1$, and then extended in [47] to the case of general functions ν satisfying the assumptions fixed above (and in the context of an incompressible non-homogeneous MHD system).

9.2.1 The quasi-homogeneous case

To begin with, consider the quasi-homogeneous case $\rho_0 \equiv 1$ in equation (9.5), so that the initial density functions verify

$$\rho_{0,\varepsilon} = 1 + \varepsilon\, r_{0,\varepsilon}$$

for any $\varepsilon > 0$, where the family $\big(r_{0,\varepsilon}\big)_\varepsilon$ is bounded in $L^2(\Omega) \cap L^\infty(\Omega)$. This case is particularly favorable, as the simple transport equation verified by the densities allows us to deduce that

$$\forall\, \varepsilon \in \,]0,1]\,, \quad \forall\, t \geq 0\,, \qquad\qquad \rho_\varepsilon(t) = 1 + \varepsilon\, r_\varepsilon(t)\,,$$

together with the equation

$$\partial_t r_\varepsilon + \operatorname{div}\big(r_\varepsilon\, u_\varepsilon\big) = 0\,.$$

From this, one easily deduces the uniform bounds

$$\big(r_\varepsilon\big)_\varepsilon \Subset L^\infty\big(\mathbb{R}_+; L^2(\Omega)\big) \cap L^\infty\big(\mathbb{R}_+ \times \Omega\big)\,.$$

On the other hand, the energy inequality (9.2) implies

$$\big(u_\varepsilon\big)_\varepsilon \Subset L^\infty\big(\mathbb{R}_+; L^2(\Omega)\big) \cap L^2\big(\mathbb{R}_+; \dot H^1(\Omega)\big)\,. \tag{9.6}$$

Now, the space compactness of the velocity fields and the time compactness of the density variations $\big(r_\varepsilon\big)_\varepsilon$ allow one to prove that

$$r_\varepsilon\, u_\varepsilon \longrightarrow r\, u \qquad \text{in} \qquad \mathcal{D}'\big(\mathbb{R}_+ \times \Omega\big)\,,$$

where we have denoted $r := \lim r_\varepsilon$ and $u := \lim u_\varepsilon$. Of course, those limits are weak-$*$ limits in suitable topologies, and hold up to extraction of suitable subsequences.

Then, one can pass to the limit in all terms appearing in the equations, except the convective term. As a matter of fact, the Coriolis term, which is the other problematic term in the momentum equation in (9.1) owing to its singular nature, can be treated by writing

$$\frac{1}{\varepsilon}\,\rho_\varepsilon\, u_\varepsilon^\perp \;=\; \frac{1}{\varepsilon}\, u_\varepsilon^\perp \,+\, r_\varepsilon\, u_\varepsilon^\perp\,, \tag{9.7}$$

where we see that the first term on the right-hand side simply disappears from the weak formulation, because, owing to the divergence-free constraint over u_ε, the term $u_\varepsilon^\perp$ is a perfect gradient.

Thus, let us focus on the convective term. Assuming to have enough space regularity of all the quantities, following [145, 125] one writes

$$\mathrm{div}\,\big(\rho_\varepsilon\, u_\varepsilon \otimes u_\varepsilon\big) \;\approx\; \mathrm{div}\,\big(u_\varepsilon \otimes u_\varepsilon\big) \;=\; u_\varepsilon\cdot\nabla u_\varepsilon \;=\; \omega_\varepsilon\, u_\varepsilon^\perp \,+\, \frac{1}{2}\,\nabla\big|u_\varepsilon\big|^2\,, \tag{9.8}$$

where we have defined $\omega_\varepsilon := \mathrm{curl}\, u_\varepsilon = \partial_1 u_\varepsilon^2 - \partial_2 u_\varepsilon^1$ to be the vorticity of the velocity field u_ε. At this point, arguing similarly to Section 8.3 (recall that here we are already in two space dimensions), from the wave system

$$\begin{cases} \varepsilon\,\partial_t r_\varepsilon \,+\, \mathrm{div}\, V_\varepsilon \,=\, 0 \\ \varepsilon\,\partial_t V_\varepsilon \,+\, \nabla\Pi_\varepsilon \,+\, V_\varepsilon^\perp \,=\, \varepsilon\, f_\varepsilon \end{cases} \tag{9.9}$$

is it possible to get compactness of $\eta_\varepsilon := \mathrm{curl}\, V_\varepsilon$ in space-time, hence (as $\rho_\varepsilon = 1 + O(\varepsilon)$) compactness of ω_ε. In view of (9.8), this latter property enables us to pass to the limit in the convective term.

All in all, we have shown the following result.

Theorem 9.1. *In the quasi-homogeneous regime, i.e. for $\rho_0 \equiv 1$ in (9.5), in the limit $\varepsilon \to 0^+$ equations (9.1) converge to the following Navier-Stokes type system:*

$$\begin{cases} \partial_t r \,+\, \mathrm{div}\,\big(r\, u\big) \,=\, 0 \\ \partial_t u \,+\, \mathrm{div}\,\big(u\otimes u\big) \,+\, \nabla\Pi \,-\, \nu(1)\,\Delta u \,+\, r\, u^\perp \,=\, 0 \\ \mathrm{div}\, u \,=\, 0\,. \end{cases}$$

As a last comment, we point out that, in the quasi-homogeneous case, it is possible to prove a similar result even in a 3-D framework, even though, strictly speaking, this has not been done in [113, 47].

9.2.2 The fully non-homogeneous case

Let us now focus on the fully non-homogeneous case, when the densities are small perturbations around a generic (non-constant) reference state ρ_0. As already explained in Section 9.1, the fully non-homogeneous case is much more involved.

As a matter of fact, while the uniform bounds (9.6) for the velocity fields still hold true, up to some modifications because of the possible presence of vacuum regions. the analysis of the density functions changes completely. By transport theory, we are able to say that the density functions ρ_ε converge (weakly-$*$ in $L^\infty\big(\mathbb{R}_+\times\Omega\big)$, for instance) to some target profile ρ. However, obtaining quantitative bounds on the difference $\rho_\varepsilon - \rho$ (in other terms, saying that a decomposition similar to (9.5) holds also for later times) seems to be out of reach, at a first sight. On the other hand, those quantitative bounds seem to be really necessary, at least for treating the Coriolis term, keep in mind (9.7) for instance.

Nonetheless, we remark that the original (ε-dependent) system already encodes some smallness/compactness property for the density functions, in a sense that we are going to explain now. To begin with, we observe that, by balancing the Coriolis term by a gradient, in the limit $\varepsilon \to 0^+$ we must have

$$\rho\, u^\perp \,=\, \nabla\phi\,,$$

which immediately implies the property

$$\mathrm{div}\,\big(\rho\, u\big) \,=\, 0\,.$$

But it can be seen that $\rho_\varepsilon\, u_\varepsilon \,\longrightarrow\, \rho\, u$ in the sense of distributions (as it was the case in the previous subsection for the product $r_\varepsilon\, u_\varepsilon$). Then, one in turn deduces that $\partial_t\rho = 0$, which implies $\rho(t) \,=\, \rho_0$ is the initial (non-constant) reference density state for any later time. Thus, one can now write the *ansatz*

$$\rho_\varepsilon \,=\, \rho_0 \,+\, s_\varepsilon\,, \qquad\qquad \text{with} \qquad s_\varepsilon \,\stackrel{*}{\rightharpoonup}\, 0\,.$$

However, we are still missing quantitative smallness properties for the family of functions $\big(s_\varepsilon\big)_\varepsilon$. In order to deduce them, let us define $r_\varepsilon \,:=\, s_\varepsilon/\varepsilon$ and notice that the equations can be still recasted in the form of the wave system (9.9). At this point, by taking the curl of the momentum equation and then taking the difference of the obtained relation with the mass equation, one gets a bound (in very negative Sobolev spaces with respect to x) for the quantity $\eta_\varepsilon - r_\varepsilon$, so in turn for the funtions r_ε. To be more precise and fix ideas, one gets

$$\big(r_\varepsilon\big)_\varepsilon \,\Subset\, L^\infty_{\rm loc}\big(\mathbb{R}_+;H^{-2-}(\Omega)\big)\,. \tag{9.10}$$

This is a very bad piece of information (inasmuch as uniform bounds are available in spaces of very low regularity), but gives the sought quantitative smallness $\rho_\varepsilon - \rho_0 \,=\, O(\varepsilon)$. In addition, observe that an interpolation argument allows one to deduce the quantitative smallness $\rho_\varepsilon\, u_\varepsilon \,=\, \rho_0\, u_\varepsilon \,+\, O(\varepsilon^\kappa)$ in the sense of $\mathcal{D}'$, for a suitable $0<\kappa<1$. Of course, it is not possible to take $\kappa=1$ here, because of the very low space regularity of the functions r_ε.

With those properties at hand, in order to compute the limit dynamics we can resort to similar arguments as the ones used in Sections 8.1 and 8.3 for the compressible case. In particular, we have already seen that the products $\rho_\varepsilon\, u_\varepsilon$ converge to $\rho_0\, u$. Next, we can use the mass equation (thus passing to the vorticity formulation of the momentum equation) for treating the Coriolis term. Finally, we consider the viscosity term: we notice that either it is linear in u_ε (if $\nu\equiv 1$), or, when it is variable $\nu=\nu(\rho_\varepsilon)$, one uses DiPerna-Lions theory [102] for transport equations to deduce local strong convergence of the ρ_ε, whence convergence of $\nu(\rho_\varepsilon)\,\mathbb{D}u_\varepsilon$ to $\nu(\rho_0)\,\mathbb{D}u$.

In the end, once again the main difficulty lies in the study of the convergence of the convective term. For this, and in order to be able to treat the possible vanishing of ρ_0, similarly to (9.8) we can write

$$\mathrm{div}\,\big(\rho_\varepsilon\, u_\varepsilon\otimes u_\varepsilon\big) \,\approx\, \mathrm{div}\,\big(\rho_0\, u_\varepsilon\otimes u_\varepsilon\big) \,=\, \rho_0\,\nabla\phi_\varepsilon \,+\, \rho_0\,\omega_\varepsilon\, u_\varepsilon^\perp \,+\, u_\varepsilon\cdot\nabla\rho_0\, u_\varepsilon\,.$$

Now, the term $\rho_0\,\nabla\phi_\varepsilon$ can be shown to converge to some $\rho_0\,\nabla\phi$ in the sense of $\mathcal{D}'$. On the other hand, using the interpolation argument mentioned above, and in particular the decomposition $\rho_\varepsilon\, u_\varepsilon \,=\, \rho_0\, u_\varepsilon \,+\, O(\varepsilon^\kappa)$, one can compute

$$\rho_0\,\omega_\varepsilon \,\approx\, \eta_\varepsilon \,-\, u_\varepsilon\cdot\nabla^\perp\rho_0\,,$$

where, as above, we have set $\eta_\varepsilon \,=\, \mathrm{curl}\, V_\varepsilon$. The point is that

$$u_\varepsilon\cdot\nabla\rho_0\, u_\varepsilon \,-\, u_\varepsilon\cdot\nabla^\perp\rho_0\, u_\varepsilon^\perp \,=\, \widetilde{\phi}_\varepsilon\,\nabla\rho_0 \,=\, \nabla\big(\rho_0\,\widetilde{\phi}_\varepsilon\big) \,-\, \rho_0\,\nabla\widetilde{\phi}_\varepsilon\,,$$

owing to special cancellations which occur in the computations. From the previous relation, we finally deduce that

$$\mathrm{div}\,\big(\rho_\varepsilon\, u_\varepsilon\otimes u_\varepsilon\big) \,\approx\, \rho_0\,\nabla\Phi_\varepsilon \,+\, \nabla\widetilde{\Pi}_\varepsilon \,+\, \eta_\varepsilon\, u_\varepsilon^\perp\,,$$

for suitable quantities Φ_ε and $\widetilde{\Pi}_\varepsilon$. To conclude, one must treat the last term on the right. The key remark is that only the component along $\nabla^\perp\rho_0$ is important, as the other component gives a contribution of the same type as $\rho_0\,\nabla\Phi_\varepsilon$. But, when projecting onto the $\nabla^\perp\rho_0$ direction, we see that $u_\varepsilon^\perp\cdot\nabla^\perp\rho_0\,=\,u_\varepsilon\cdot\nabla\rho_0\,\approx\,\mathrm{div}\,V_\varepsilon$, up to small remainders, so one can use the wave system again (together with the space-time compactness of $\eta_\varepsilon - r_\varepsilon$) to deduce that the resulting term is also small, and in particular vanishes in the limit $\varepsilon\to 0^+$.

We remark that this argument works for all smooth test functions ψ which are divergence-free, without any other constraint; in particular, the proved convergence is a true convergence in $\mathcal{D}'$, without need to restricting to the subclass of test functions verifying in addition $\mathrm{div}\,\big(\rho_0\psi\big)=0$. Roughly speaking, those would be functions belonging to the kernel of the singular perturbation operator; using them would allow to erase the term $\rho_0\nabla\Phi$ appearing in the target equation (9.11) below.

To sum up, in the fully non-homogeneous case one gets the following result. For simplicity of presentation, assume $\nu\equiv 1$ here.

Theorem 9.2. *In the fully non-homogeneous regime, in the limit $\varepsilon\to 0^+$ equations* (9.1) *converge to the (underdetermined) linear scalar equation*

$$\partial_t\Big(\mathrm{curl}\,\big(\rho_0\,u\big)\,-\,r\Big)\,-\,\Delta\omega\,+\,\mathrm{curl}\,\big(\rho_0\,\nabla\Phi\big)\,=\,0\,, \tag{9.11}$$

where the vector field u satisfies in addition $\mathrm{div}\,u\,=\,\mathrm{div}\,\big(\rho_0\,u\big)\,=\,0$, $\omega\,=\,\mathrm{curl}\,u\,=\,\partial_1u^2-\partial_2u^1$ *is the vorticity of u and Φ is a suitable scalar distribution, which is an additional unknown of the system (together with u and r).*

We conclude this part by formulating some comments about the previous statement.

First of all, we notice that the term $\rho_0\,\nabla\Phi$ corresponds somehow to the Lagrangian multiplier asociated with the divergence-free constraint $\mathrm{div}\,\big(\rho_0\,u\big)\,=\,0$, but of course (contrarily to the classical pressure term) it does not vanish when taking the curl of the equations.

Secondly, we remark that, differently from the situation described in Chapter 8, one is no more able to derive a stream-function relation analogous to (8.7), hence one is no more able to prove that the vorticity ω equals Δr. So, the previous equation, although presenting a similar structure as the quasi-geostrophic equation (8.10), is indeed underdetermined, independently from the presence or not of the Lagrangian multiplier $\rho_0\,\nabla\Phi$.

Again forgetting about the term $\rho_0\,\nabla\Phi$, we notice that the main reason for the underdetermined nature of the limit system is the absence of an equation for the target density oscillation function r. In turn, this is due to the weak bounds (9.10) available for r_ε, so for r, in very low regularity spaces: such regularity does not allow us to get uniform bounds for the products $r_\varepsilon\,u_\varepsilon$ in spaces of distributions, so to pass to the limit in the mass equation and get a dynamical relation also for the target function r.

Finally, we point out that in [113] we gave a *conditional* convergence result, which states that, under suitable *uniform* bounds on the initial density perturbations $r_{0,\varepsilon}$ and, more importantly, on the (solution!) velocity fields $\big(u_\varepsilon\big)_\varepsilon$ both in time and space, one is able to recover a dynamical equation for r in the limit, thus getting a fully determined target system (up to the presence of "pressure terms" of the form $\rho_0\,\nabla\Phi$). The conditional convergence result can be roughly stated as follows (again, we take $\nu(\rho)\equiv 1$ here).

Theorem 9.3. *In addition to the hypotheses fixed in Section 9.1, assume moreover that $\rho_0\in W^{3,\infty}(\Omega)$ and that the following conditions hold true:*

(i) $\big(r_{0,\varepsilon}\big)_\varepsilon\,\subset\,H^{1+\beta}(\Omega)$, *for some $\beta\in\,]0,1[\,$;*

(ii) $\big(u_\varepsilon\big)_\varepsilon\,\subset\,L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+;H^1(\Omega)\big)\cap L^2_{\mathrm{loc}}\big(\mathbb{R}_+;H^2(\Omega)\big)$*;*

(iii) $\big(u_\varepsilon\big)_\varepsilon\,\subset\,\mathcal{C}^{0,\alpha}_{\mathrm{loc}}\big(\mathbb{R}_+;L^2(\Omega)\big)$, *for some $\alpha\in\,]0,1[\,$.*

Then there exist distributions Π, Γ_0 *and* Γ_1 *over* $\mathbb{R}_+ \times \Omega$ *such that the limit points* r *and* u *identified above satisfy the system*

$$\begin{cases} \partial_t r \,+\, u \cdot \nabla r \,=\, \operatorname{curl}\big(\rho_0 \,\nabla \Gamma_1\big) \\ \rho_0 \,\partial_t u \,+\, \nabla \Pi \,+\, \rho_0 \,\nabla \Gamma_0 \,+\, r\, u^\perp \,-\, \nu\, \Delta u \,=\, 0 \\ \operatorname{div} u \,=\, \operatorname{div}\big(\rho_0\, u\big) \,=\, 0\,. \end{cases}$$

Interestingly, we notice that, *in absence of vacuum*, the additional conditions (ii) and (iii) of the previous statement are met by weak solutions related to slightly more regular initial data (see *e.g.* [176, 79]). However, the corresponding bounds are not (known to be) verified uniformly with respect to the small Rossby number: this is the reason why the result of Theorem 9.3 remains only conditional.

9.3 The case of thin domains and the Ekman pumping effect

In this part, we briefly discuss the extension of the results of [113, 47] to the case of a thin 3-D domain. This corresponds to the study performed in [18].

Thus, in this section we consider the 3-D system of equations (9.1), where we set $\nu(\rho) \equiv 1$ for simplicity, in the thin infinite slab

$$\Omega_\varepsilon \,:=\, \mathbb{R}^2 \times \,]-\ell_\varepsilon, \ell_\varepsilon[\;,$$

for a suitable decreasing sequence $\big(\ell_\varepsilon\big)_\varepsilon$ satisfying

$$\forall\,\varepsilon \in]0,1]\,, \qquad \ell_\varepsilon \,>\, 0 \qquad\qquad \text{and} \qquad\qquad \ell_\varepsilon \,\searrow\, 0 \quad \text{for} \quad \varepsilon \to 0^+\,.$$

Notice that, analogously to the situation considered in Chapter 8, the exterior normal n_ε to the boundary $\partial\Omega_\varepsilon \,=\, \mathbb{R}^2 \times \{\pm \ell_\varepsilon\}$ is in fact independent of ε, as one has $n_\varepsilon \,=\, n \,=\, \pm e_3$. At the boundary $\partial\Omega_\varepsilon$, we impose *Navier-slip boundary conditions*, namely

$$(9.12) \qquad \big(u_\varepsilon \cdot n\big)_{|\partial\Omega_\varepsilon} \,=\, 0 \qquad \text{and} \qquad \big((\mathbb{D}u_\varepsilon) n \times n\big)_{|\partial\Omega_\varepsilon} \,=\, -\,\alpha_\varepsilon\,\big(u_\varepsilon \times n\big)_{|\partial\Omega_\varepsilon}\,,$$

for a suitable sequence $\big(\alpha_\varepsilon\big)_\varepsilon \subset \mathbb{R}_+$ of friction coefficients. We assume that

$$\exists\, \lim_{\varepsilon \to 0^+} \frac{\alpha_\varepsilon}{\ell_\varepsilon} \,=\, \lambda \,\in\, [0, +\infty[\,.$$

As a matter of fact, it turns out that, when $\lambda = +\infty$, the limit problem is only the trivial one, as one has $u \equiv 0$. When $\lambda = 0$, instead, we recover the two-dimensional result of Theorems 9.1 and 9.2 above: roughly speaking, the friction is so low that one is close to the complete slip setting, for which any boundary effect vanishes in the limit $\varepsilon \to 0^+$. Finally, whenever $\lambda > 0$, we are going to see that an additional term appears in the equations with respect to that situation; this is a consequence of the friction imposed at the boundary $\partial\Omega_\varepsilon$. This additional term encodes a well-known physical effect, which takes the name of *Ekman pumping phenomenon* (see more details below).

For simplicity of presentation, in what follows we will focus on the quasi-homogeneous regime, although corresponding results for the fully non-homogeneous case (in the same spirit of Theorem 9.2 above) can be obtained as well. We also take a constant viscosity coefficient $\nu(\rho) \equiv 1$. The result is the following one, where we reintroduce the indices h everywhere to stress the fact that, although the original system is 3-D, the target dynamics is only two-dimensional, as already seen in Chapter 8.

Theorem 9.4. *In the quasi-homogeneous setting $\rho_0 \equiv 1$, system (9.1), set on Ω_ε and supplemented with Navier-slip boundary conditions (9.12), converges to the 2-D damped Navier-Stokes type system*

$$\begin{cases} \partial_t r \,+\, \mathrm{div}_h\big(r\, u^h\big) \,=\, 0 \\ \partial_t u^h \,+\, \mathrm{div}_h\big(u^h \otimes u^h\big) \,+\, \nabla_h \Pi \,-\, \Delta_h u^h \,+\, r\,\big(u^h\big)^\perp \,+\, 2\,\lambda\, u^h \,=\, 0 \\ \mathrm{div}_h u^h \,=\, 0\,. \end{cases}$$

Some comments about the previous statement are in order.

First of all, we notice the presence of the term $2\lambda u^h$ in the limit system. This is an additional term with respect to the situation treated in Theorem 9.1, for instance; it originates from the friction condition (9.12) imposed at the boundary. This damping term (recall that, by our assumptions, one has $\lambda \geq 0$) exactly encodes the Ekman pumping phenomenon mentioned above. We recall that, at the physical level, the Ekman pumping effect consists in the dissipation of kinetic energy as a consequence of a global circulation phenomenon, called Ekman suction. The Ekman suction process originates from the presence of the Ekman boundary layers, so in turn from the friction of the fluid at the boundary (translated into the condition $\alpha_\varepsilon > 0$ in (9.12) above), but it involves also portions of the fluid in the interior of the domain (in this sense we called it "global"). We refer *e.g.* to Part I of [41] and to [70] for a more detailed explanation.

Secondly, we remark that, in a thin domain framework, the only reasonable quantities to work with are vertical averages. As a matter of fact, under our assumptions the energy balance for system (9.1) reads

$$\frac{1}{2\,\ell_\varepsilon}\int_{-\ell_\varepsilon}^{\ell_\varepsilon}\int_{\mathbb{R}^2} \rho_\varepsilon(t)\;|u_\varepsilon(t)|^2\;\mathrm{d}x_h\,\mathrm{d}x_3 \,+\, \frac{1}{2\,\ell_\varepsilon}\int_{-\ell_\varepsilon}^{\ell_\varepsilon}\int_0^t\int_{\mathbb{R}^2}|Du_\varepsilon|^2\;\mathrm{d}x_h\,\mathrm{d}\tau\,\mathrm{d}x_3$$
$$+\,\frac{\alpha_\varepsilon}{\ell_\varepsilon}\int_0^t\int_{\mathbb{R}^2\times\{-\ell_\varepsilon,\ell_\varepsilon\}}\big|u_\varepsilon\big|^2\;\mathrm{d}x_h\,\mathrm{d}\tau \,\leq\, \frac{1}{2\ell_\varepsilon}\int_{-\ell_\varepsilon}^{\ell_\varepsilon}\int_{\mathbb{R}^2}\rho_{0,\varepsilon}\;|u_{0,\varepsilon}|^2\;\mathrm{d}x_h\,\mathrm{d}x_3\,.$$

In particular, it is natural to formulate boundedness of suitable norms of vertical averages of the initial data and to prove convergence of the vertical averages of the solutions $\big(\rho_\varepsilon, u_\varepsilon\big)_\varepsilon$. We omit to inter into the details here and rather refer to [18].

Finally, we comment a bit on the proof of the previous result and highlight the main points of the analysis. To begin with, we observe that the previous energy inequality, combined with the boundary condition $\big(u_\varepsilon\cdot n\big)_{|\partial\Omega_\varepsilon} \,=\, 0$, gives that the vertical averages $\langle u_\varepsilon^3\rangle$ of the vertical component of u_ε must converge to 0. This is natural, if one thinks that considering the fast rotation limit for thin domains is equivalent to study the limit in a fixed domain, but with penalised vertical derivatives (see also [77] in this respect). Next, by using properties of the (averaged) mass equation, one can see that the vertical averages of the densities $\langle\rho_\varepsilon\rangle$ converge strongly to the target profile, and so do the vertical averages of the oscillation functions $\langle r_\varepsilon\rangle$ (as $\rho_0 = 1$ in the situation considered here). At this point, another fundamental ingredient of the proof is to derive Sobolev and Poincaré inequalities in thin domains. As a matter of fact, the direct application of those inequalities (and the control on the gradients of the velocity fields) allows us to "linearise" all the averages of some non-linear quantities, like the products $\rho_\varepsilon\, u_\varepsilon$ and the convective term, for instance. By speaking about "linearisation" we mean that, in order to understand the limit of the average of the non-linear quantities (typically products of densities and velocity fields), it is enough to prove convergence of the non-linear quantity (products) of the averages. After having obtained this reduction, similar arguments as the ones used for the 2-D problem apply, to show convergence of the vertical averages $\langle r_\varepsilon\rangle$ and $\langle u_\varepsilon^h\rangle$ towards suitable quantities r and u^h respectively, which satisfy the claimed target equations.

The proof of the correspoding result in the fully non-homogeneous case follows the same main lines. As a matter of fact, the principal difficulties for treating that case appear already in the

2-D problem and have already treated in Section 9.2. The statement can be roughly formulated as follows.

Theorem 9.5. *Assume to be in the fully non-homogeneous regime, with $\rho_0 = \rho_0(x^h)$ satisfying the assumptions fixed in Section 9.1. Then, system* (9.1)*, set on Ω_ε and supplemented with Navier-slip boundary conditions* (9.12)*, coverges to the equation*

$$\partial_t\Big(\mathrm{curl}_h\big(\rho_0\, u^h\big)\;-\;r\Big)\;-\;\Delta_h\omega\;+\;\mathrm{curl}_h\big(\rho_0\,\nabla_h\Phi\big)\;+\;2\,\lambda\,\omega\;=\;0\,,$$

where the vector field $u^h = u^h(t,x^h)$ satisfies in addition $\mathrm{div}_h u^h \;=\; \mathrm{div}_h\big(\rho_0\, u^h\big) \;=\; 0$ and where we have set $\omega \;=\; \mathrm{curl}_h u^h \;=\; \partial_1 u^2 - \partial_2 u^1$ to be the 2-D vorticity of u^h. In the previous equation, Φ is a suitable scalar distribution, which is an additional unknown of the system.

9.4 Some open questions and future perspectives

As usual, we conclude the present chapter with a list of open problems and questions which look interesting to us and which we would like to tackle in the future.

As a matter of fact, many problems which arise in the compressible case (see Section 8.5) are pertinent also in the context of incompressible non-homogeneous fluids in fast rotation. We think, for instance, to the study of Ekman and Munk boundary layers, of topography effects, of the interplay of a strong Coriolis force with gravity.

However, as it appears clear from the discussion above, the understanding of the fast rotation limit for density-dependent incompressible flows is poorer, if compared to the compressible instance. The main problem is that, in the general 3-D framework (9.1) and in the fully non-homogeneous case, it is not clear to us that the Taylor-Proudman theorem should remain true in the limit. In particular, we are not able to exclude that the target dynamics may present non-trivial vertical components. At the mathematical level, this represents a major obstacle when proving rigorous convergence results, as all the techniques (spectral methods, compensated compactness) used so far in this kind of problems strongly rely on the validity of the Taylor-Proudman theorem and on the fact of having a purely horizontal dynamics in the limit.

Therefore, it seems more important, actually essential for further developments on the subject, to focus rather on basic problems first. We plan to address two points in particular. The first one consists in finding some special classes of initial data, which would allow to spoil the target motion from any vertical component, thus to recover the validity of the Taylor-Proudman in the limit. The second direction is, in a first approximation, to focus on partial results, like proving convergence of only the vertical averages of the solution, for instance. Tackling the general case of the full convergence of the system seems to require the development of new approaches and/or techniques, which would allow to handle, in the limit process, the presence of non-trivial dependence on the vertical variable.

Bibliography

[1] T. Alazard. Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions. *Adv. Differential Equations*, 10(1):19–44, 2005.

[2] T. Alazard. Low Mach number limit of the full Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 180(1):1–73, 2006.

[3] G. Alberti, G. Crippa, and A. L. Mazzucato. Loss of regularity for the continuity equation with non-Lipschitz velocity field. *Ann. PDE*, 5(1):Paper No. 9, 19, 2019.

[4] S. Alinhac and P. Gérard. *Pseudo-differential operators and the Nash-Moser theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson.

[5] L. Ambrosio and G. Crippa. Continuity equations and ODE flows with non-smooth velocity. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(6):1191–1244, 2014.

[6] M. Avellaneda, C. Bardos, and J. Rauch. Contrôlabilité exacte, homogénéisation et localisation d'ondes dans un milieu non-homogène. *Asymptotic Anal.*, 5(6):481–494, 1992.

[7] H. Bahouri and J.-Y. Chemin. Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127(2):159–181, 1994.

[8] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[9] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[10] C. Bardos, C. Sulem, and P.-L. Sulem. Longtime dynamics of a conductive fluid in the presence of a strong magnetic field. *Trans. Amer. Math. Soc.*, 305(1):175–191, 1988.

[11] J. Bartsch, A. Borzí, F. Fanelli, and S. Roy. A theoretical investigation of Brockett's ensemble optimal control problems. *Calc. Var. Partial Differential Equations*, 58(5):Paper No. 162, 34, 2019.

[12] J. Bartsch, A. Borzí, F. Fanelli, and S. Roy. A numerical investigation of Brockett's ensemble optimal control problems. *Numer. Math.*, 149(1):1–42, 2021.

[13] D. Basarić. Semiflow selection for the compressible Navier-Stokes system. *J. Evol. Equ.*, 21(1):277–295, 2021.

[14] J. Bedrossian and N. Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.*, 122:195–300, 2015.

[15] J. Bedrossian, N. Masmoudi, and V. Vicol. Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow. *Arch. Ration. Mech. Anal.*, 219(3):1087–1159, 2016.

[16] R. Beekie, T. Buckmaster, and V. Vicol. Weak solutions of ideal MHD which do not conserve magnetic helicity. *Ann. PDE*, 6(1):Paper No. 1, 40, 2020.

[17] E. Bocchi, F. Fanelli, and C. Prange. Anisotropy and stratification effects in the dynamics of fast rotating compressible fluids. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 39(3):647–704, 2022.

[18] M. Bravin and F. Fanelli. Fast rotating non-homogeneous fluids in thin domains and the Ekman pumping effect. *Submitted*, 2022.

[19] D. Breit, E. Feireisl, and M. Hofmanová. Dissipative solutions and semiflow selection for the complete Euler system. *Comm. Math. Phys.*, 376(2):1471–1497, 2020.

[20] D. Breit, E. Feireisl, and M. Hofmanová. Markov selection for the stochastic compressible Navier-Stokes system. *Ann. Appl. Probab.*, 30(6):2547–2572, 2020.

[21] D. Breit, E. Feireisl, and M. Hofmanová. Solution semiflow to the isentropic Euler system. *Arch. Ration. Mech. Anal.*, 235(1):167–194, 2020.

[22] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2):211–223, 2003.

[23] D. Bresch, B. Desjardins, and D. Gérard-Varet. Rotating fluids in a cylinder. *Discrete Contin. Dyn. Syst.*, 11(1):47–82, 2004.

[24] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[25] D. Bresch and P.-E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math. (2)*, 188(2):577–684, 2018.

[26] D. Bresch, D. Lannes, and G. Métivier. Waves interacting with a partially immersed obstacle in the Boussinesq regime. *Anal. PDE*, 14(4):1085–1124, 2021.

[27] M. Bulíček and J. Málek. Large data analysis for Kolmogorov's two-equation model of turbulence. *Nonlinear Anal. Real World Appl.*, 50:104–143, 2019.

[28] N. Burq. Contrôlabilité exacte des ondes dans des ouverts peu réguliers. *Asymptot. Anal.*, 14(2):157–191, 1997.

[29] N. Burq, B. Dehman, and J. Le Rousseau. $\mathcal{C}^0$-vector field, measure propagation, and wave controllability on rough riemannian manifolds with boundary: part 1. *Submitted*, 2022.

[30] N. Burq, B. Dehman, and J. Le Rousseau. $\mathcal{C}^0$-vector field, measure propagation, and wave controllability on rough riemannian manifolds with boundary: part 2. *Submitted*, 2022.

[31] N. Burq, B. Dehman, and J. Le Rousseau. Measure propagation along $\mathcal{C}^0$-vector field and wave controllability on a rough compact manifold. *Submitted*, 2022.

[32] R. E. Caflisch, I. Klapper, and G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 184(2):443–455, 1997.

[33] J. E. Cardona and L. Kapitanski. Semiflow selection and Markov selection theorems. *Topol. Methods Nonlinear Anal.*, 56(1):197–227, 2020.

[34] C. Castro and E. Zuazua. Concentration and lack of observability of waves in highly heterogeneous media. *Arch. Ration. Mech. Anal.*, 164(1):39–72, 2002.

[35] T. Chacón Rebollo and R. Lewandowski. *Mathematical and numerical foundations of turbulence models and applications.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser/Springer, New York, 2014.

[36] T. Chang, B. J. Jin, and A. Novotný. Compressible Navier-Stokes system with general inflow-outflow boundary data. *SIAM J. Math. Anal.*, 51(2):1238–1278, 2019.

[37] J.-Y. Chemin. Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnel. *Invent. Math.*, 103(3):599–629, 1991.

[38] J.-Y. Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)*, 26(4):517–542, 1993.

[39] J.-Y. Chemin. Fluides parfaits incompressibles. *Astérisque*, (230):177, 1995.

[40] J.-Y. Chemin. Ekman boundary layers and topography (conference talk). In *Workshop on Analysis of PDEs.* Karlsruhe Institut of Technology (Karlsruhe, Germany), 2023.

[41] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier. *Mathematical geophysics*, volume 32 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, Oxford, 2006.

[42] J.-Y. Chemin and N. Lerner. Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations*, 121(2):314–328, 1995.

[43] Q. Chen, C. Miao, and Z. Zhang. On the well-posedness of the ideal MHD equations in the Triebel-Lizorkin spaces. *Arch. Ration. Mech. Anal.*, 195(2):561–578, 2010.

[44] N. Chikami and R. Danchin. On the well-posedness of the full compressible Navier-Stokes system in critical Besov spaces. *J. Differential Equations*, 258(10):3435–3467, 2015.

[45] M. Cicognani and F. Colombini. Modulus of continuity of the coefficients and loss of derivatives in the strictly hyperbolic Cauchy problem. *J. Differential Equations*, 221(1):143–157, 2006.

[46] D. Cobb. Bounded solutions in incompressible hydrodynamics. *Submitted*, 2021.

[47] D. Cobb and F. Fanelli. On the fast rotation asymptotics of a non-homogeneous incompressible MHD system. *Nonlinearity*, 34(4):2483–2526, 2021.

[48] D. Cobb and F. Fanelli. Rigorous derivation and well-posedness of a quasi-homogeneous ideal MHD system. *Nonlinear Anal. Real World Appl.*, 60:Paper No. 103284, 36, 2021.

[49] D. Cobb and F. Fanelli. Symmetry breaking in ideal magnetohydrodynamics: the role of the velocity. *J. Elliptic Parabol. Equ.*, 7(2):273–295, 2021.

[50] D. Cobb and F. Fanelli. Elsässer formulation of the ideal MHD and improved lifespan in two space dimensions. *J. Math. Pures Appl. (9)*, 169:189–236, 2023.

[51] F. Colombini, E. De Giorgi, and S. Spagnolo. Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 6(3):511–559, 1979.

[52] F. Colombini and D. Del Santo. A note on hyperbolic operators with log-Zygmund coefficients. *J. Math. Sci. Univ. Tokyo*, 16(1):95–111, 2009.

[53] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier. Time-dependent loss of derivatives for hyperbolic operators with non regular coefficients. *Comm. Partial Differential Equations*, 38(10):1791–1817, 2013.

[54] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier. A well-posedness result for hyperbolic operators with Zygmund coefficients. *J. Math. Pures Appl. (9)*, 100(4):455–475, 2013.

[55] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier. A note on complete hyperbolic operators with log-Zygmund coefficients. In *Fourier analysis*, Trends Math., pages 47–72. Birkhäuser/Springer, Cham, 2014.

[56] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier. The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients. *Comm. Partial Differential Equations*, 40(11):2082–2121, 2015.

[57] F. Colombini, D. Del Santo, F. Fanelli, and G. Métivier. On the Cauchy problem for microlocally symmetrizable hyperbolic systems with log-Lipschitz coefficients. *Indiana Univ. Math. J.*, 69(3):785–836, 2020.

[58] F. Colombini, D. Del Santo, and M. Reissig. On the optimal regularity of coefficients in hyperbolic Cauchy problems. *Bull. Sci. Math.*, 127(4):328–347, 2003.

[59] F. Colombini and F. Fanelli. A note on non-homogeneous hyperbolic operators with low regularity coefficients. *Rend. Istit. Mat. Univ. Trieste*, 42(suppl.):1–25, 2010.

[60] F. Colombini, E. Jannelli, and S. Spagnolo. Nonuniqueness in hyperbolic Cauchy problems. *Ann. of Math. (2)*, 126(3):495–524, 1987.

[61] F. Colombini and N. Lerner. Hyperbolic operators with non-Lipschitz coefficients. *Duke Math. J.*, 77(3):657–698, 1995.

[62] F. Colombini and G. Métivier. The Cauchy problem for wave equations with non Lipschitz coefficients; application to continuation of solutions of some nonlinear wave equations. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(2):177–220, 2008.

[63] F. Colombini and G. Métivier. Counterexamples to the well posedness of the Cauchy problem for hyperbolic systems. *Anal. PDE*, 8(2):499–511, 2015.

[64] F. Colombini and S. Spagnolo. Some examples of hyperbolic equations without local solvability. *Ann. Sci. École Norm. Sup. (4)*, 22(1):109–125, 1989.

[65] P. Constantin and J. Wu. Statistical solutions of the Navier-Stokes equations on the phase space of vorticity and the inviscid limits. *J. Math. Phys.*, 38(6):3031–3045, 1997.

[66] S. Cox and E. Zuazua. The rate at which energy decays in a string damped at one end. *Indiana Univ. Math. J.*, 44(2):545–573, 1995.

[67] T. Crin-Barat and R. Danchin. Partially dissipative hyperbolic systems in the critical regularity setting: the multi-dimensional case. *J. Math. Pures Appl. (9)*, 165:1–41, 2022.

[68] T. Crin-Barat and R. Danchin. Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit. *Math. Ann.*, to appear, 2023.

[69] G. Crippa, T. Elgindi, G. Iyer, and A. L. Mazzucato. Growth of Sobolev norms and loss of regularity in transport equations. *Philos. Trans. Roy. Soc. A*, 380(2225):Paper No. 24, 12, 2022.

[70] B. Cushman-Roisin and J.-M. Beckers. *Introduction to geophysical fluid dynamics*, volume 101 of *International Geophysics Series*. Elsevier/Academic Press, Amsterdam, second edition, 2011.

[71] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.

[72] A.-L. Dalibard and D. Gérard-Varet. Nonlinear boundary layers for rotating fluids. *Anal. PDE*, 10(1):1–42, 2017.

[73] A.-L. Dalibard and C. Perrin. Existence and stability of partially congested propagation fronts in a one-dimensional Navier-Stokes model. *Commun. Math. Sci.*, 18(7):1775–1813, 2020.

[74] A.-L. Dalibard and C. Perrin. Partially congested propagation fronts in one-dimensional Navier-Stokes equations. *J. Elliptic Parabol. Equ.*, 7(2):491–507, 2021.

[75] A.-L. Dalibard and C. Prange. Well-posedness of the Stokes-Coriolis system in the half-space over a rough surface. *Anal. PDE*, 7(6):1253–1315, 2014.

[76] A.-L. Dalibard and L. Saint-Raymond. Mathematical study of rotating fluids with resonant surface stress. *J. Differential Equations*, 246(6):2304–2354, 2009.

[77] A.-L. Dalibard and L. Saint-Raymond. Mathematical study of the β-plane model for rotating fluids in a thin layer. *J. Math. Pures Appl. (9)*, 94(2):131–169, 2010.

[78] A.-L. Dalibard and L. Saint-Raymond. Mathematical study of degenerate boundary layers: a large scale ocean circulation problem. *Mem. Amer. Math. Soc.*, 253(1206):vi+105, 2018.

[79] R. Danchin. Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Adv. Differential Equations*, 9(3-4):353–386, 2004.

[80] R. Danchin. Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients. *Rev. Mat. Iberoamericana*, 21(3):863–888, 2005.

[81] R. Danchin. On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework. *J. Differential Equations*, 248(8):2130–2170, 2010.

[82] R. Danchin. A Lagrangian approach for the compressible Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)*, 64(2):753–791, 2014.

[83] R. Danchin and F. Fanelli. The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces. *J. Math. Pures Appl. (9)*, 96(3):253–278, 2011.

[84] R. Danchin, F. Fanelli, and M. Paicu. A well-posedness result for viscous compressible fluids with only bounded density. *Anal. PDE*, 13(1):275–316, 2020.

[85] R. Danchin and P. B. Mucha. Compressible Navier-Stokes system: large solutions and incompressible limit. *Adv. Math.*, 320:904–925, 2017.

[86] R. Danchin and P. B. Mucha. From compressible to incompressible inhomogeneous flows in the case of large data. *Tunis. J. Math.*, 1(1):127–149, 2019.

[87] R. Danchin and P. B. Mucha. The incompressible Navier-Stokes equations in vacuum. *Comm. Pure Appl. Math.*, 72(7):1351–1385, 2019.

[88] R. Danchin and M. Paicu. Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux. *Bull. Soc. Math. France*, 136(2):261–309, 2008.

[89] R. Danchin and X. Zhang. Global persistence of geometrical structures for the Boussinesq equation with no diffusion. *Comm. Partial Differential Equations*, 42(1):68–99, 2017.

[90] R. Danchin and X. Zhang. On the persistence of Hölder regular patches of density for the inhomogeneous Navier-Stokes equations. *J. Éc. polytech. Math.*, 4:781–811, 2017.

[91] N. David and B. Perthame. Free boundary limit of a tumor growth model with nutrient. *J. Math. Pures Appl. (9)*, 155:62–82, 2021.

[92] N. David and M. Schmidtchen. On the incompressible limit of a tumour growth model with convective effects. *Comm. Pure Appl. Math.*, accepted for publication, 2022.

[93] P. A. Davidson. *Turbulence.* Oxford University Press, Oxford, second edition, 2015. An introduction for scientists and engineers.

[94] T. Dębiec, B. Perthame, M. Schmidtchen, and N. Vauchelet. Incompressible limit for a two-species model with coupling through Brinkman's law in any dimension. *J. Math. Pures Appl. (9)*, 145:204–239, 2021.

[95] F. De Anna and F. Fanelli. Global well-posedness and long-time dynamics for a higher order quasi-geostrophic type equation. *J. Funct. Anal.*, 274(8):2291–2355, 2018.

[96] P. Degond, P. Minakowski, and E. Zatorska. Transport of congestion in two-phase compressible/incompressible flows. *Nonlinear Anal. Real World Appl.*, 42:485–510, 2018.

[97] B. Dehman and S. Ervedoza. Observability estimates for the wave equation with rough coefficients. *C. R. Math. Acad. Sci. Paris*, 355(5):499–514, 2017.

[98] D. Del Santo, F. Fanelli, G. Sbaiz, and A. Wróblewska-Kamińska. A multiscale problem for viscous heat-conducting fluids in fast rotation. *J. Nonlinear Sci.*, 31(1):Paper No. 21, 63, 2021.

[99] D. Del Santo, F. Fanelli, G. Sbaiz, and A. Wróblewska-Kamińska. On the influence of gravity in the dynamics of geophysical flows. *Math. Eng.*, 5(1):Paper No. 008, 33, 2023.

[100] B. Desjardins and E. Grenier. Low Mach number limit of viscous compressible flows in the whole space. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1986):2271–2279, 1999.

[101] B. Desjardins and E. Grenier. On the homogeneous model of wind-driven ocean circulation. *SIAM J. Appl. Math.*, 60(1):43–60, 2000.

[102] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[103] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 25(1):1–41, 2008.

[104] T. M. Elgindi. On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation. *Arch. Ration. Mech. Anal.*, 225(2):573–599, 2017.

[105] F. Fanelli. Conservation of geometric structures for non-homogeneous inviscid incompressible fluids. *Comm. Partial Differential Equations*, 37(9):1553–1595, 2012.

[106] F. Fanelli. Highly rotating viscous compressible fluids in presence of capillarity effects. *Math. Ann.*, 366(3-4):981–1033, 2016.

[107] F. Fanelli. A singular limit problem for rotating capillary fluids with variable rotation axis. *J. Math. Fluid Mech.*, 18(4):625–658, 2016.

[108] F. Fanelli. Some local questions for hyperbolic systems with non-regular time dependent coefficients. *J. Hyperbolic Differ. Equ.*, 14(2):301–322, 2017.

[109] F. Fanelli. Corrigendum to "Highly rotating viscous compressible fluids in presence of capillarity effects" [MR3563230]. *Math. Ann.*, 370(3-4):1789–1797, 2018.

[110] F. Fanelli. Incompressible and fast rotation limit for barotropic Navier-Stokes equations at large Mach numbers. *Phys. D*, 428:Paper No. 133049, 20, 2021.

[111] F. Fanelli and E. Feireisl. Statistical solutions to the barotropic Navier-Stokes system. *J. Stat. Phys.*, 181(1):212–245, 2020.

[112] F. Fanelli, E. Feireisl, and M. Hofmanová. Ergodic theory for energetically open compressible fluid flows. *Phys. D*, 423:Paper No. 132914, 25, 2021.

[113] F. Fanelli and I. Gallagher. Asymptotics of fast rotating density-dependent incompressible fluids in two space dimensions. *Rev. Mat. Iberoam.*, 35(6):1763–1807, 2019.

[114] F. Fanelli and R. Granero-Belinchón. Well-posedness and singularity formation for the Kolmogorov two-equation model of turbulence in 1-d. *Submitted*, 2021.

[115] F. Fanelli and R. Granero-Belinchón. Finite time blow-up for some parabolic systems arising in turbulence theory. *Z. Angew. Math. Phys.*, 73(5):Paper No. 180, 19, 2022.

[116] F. Fanelli, R. Granero-Belinchón, and S. Scrobogna. Well-posedness theory for non-homogeneous incompressible fluids with odd viscosity. *Submitted*, 2022.

[117] F. Fanelli and X. Liao. Analysis of an inviscid zero-Mach number system in endpoint Besov spaces for finite-energy initial data. *J. Differential Equations*, 259(10):5074–5114, 2015.

[118] F. Fanelli and X. Liao. The well-posedness issue for an inviscid zero-Mach number system in general Besov spaces. *Asymptot. Anal.*, 93(1-2):115–140, 2015.

[119] F. Fanelli and E. Zatorska. Low Mach number limit for degenerate Navier-Stokes equations in presence of strong stratification. *Comm. Math. Phys.*, accepted for publication, 2022.

[120] F. Fanelli and E. Zuazua. Weak observability estimates for 1-D wave equations with rough coefficients. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 32(2):245–277, 2015.

[121] D. Faraco, S. Lindberg, and L. Székelyhidi, Jr. Bounded solutions of ideal MHD with compact support in space-time. *Arch. Ration. Mech. Anal.*, 239(1):51–93, 2021.

[122] E. Feireisl. Compressible Navier-Stokes equations with a non-monotone pressure law. *J. Differential Equations*, 184(1):97–108, 2002.

[123] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[124] E. Feireisl. Compressible fluid motion with uncertain data. *J. Math. Fluid Mech.*, 24(4):Paper No. 96, 17, 2022.

[125] E. Feireisl, I. Gallagher, D. Gérard-Varet, and A. Novotný. Multi-scale analysis of compressible viscous and rotating fluids. *Comm. Math. Phys.*, 314(3):641–670, 2012.

[126] E. Feireisl, I. Gallagher, and A. Novotný. A singular limit for compressible rotating fluids. *SIAM J. Math. Anal.*, 44(1):192–205, 2012.

[127] E. Feireisl, B. J. Jin, and A. Novotný. Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech.*, 14(4):717–730, 2012.

[128] E. Feireisl, R. Klein, A. Novotný, and E. Zatorska. On singular limits arising in the scale analysis of stratified fluid flows. *Math. Models Methods Appl. Sci.*, 26(3):419–443, 2016.

[129] E. Feireisl, Y. Lu, and A. Novotný. Rotating compressible fluids under strong stratification. *Nonlinear Anal. Real World Appl.*, 19:11–18, 2014.

[130] E. Feireisl and M. Lukáčová-Medvid'ová. Statistical solutions for the Navier–Stokes–Fourier system. *Submitted*, 2022.

[131] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Advances in Mathematical Fluid Mechanics. Birkhäuser Verlag, Basel, 2009.

[132] E. Feireisl and A. Novotný. Multiple scales and singular limits for compressible rotating fluids with general initial data. *Comm. Partial Differential Equations*, 39(6):1104–1127, 2014.

[133] E. Feireisl and A. Novotný. Scale interactions in compressible rotating fluids. *Ann. Mat. Pura Appl. (4)*, 193(6):1703–1725, 2014.

[134] E. Feireisl and A. Novotný. *Mathematics of open fluid systems.* Nečas Center Series. Birkhäuser/Springer, Cham, [2022] ©2022.

[135] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[136] E. Feireisl and H. Petzeltová. On the long-time behaviour of solutions to the Navier-Stokes-Fourier system with a time-dependent driving force. *J. Dynam. Differential Equations*, 19(3):685–707, 2007.

[137] E. Feireisl and A. Świerczewska Gwiazda. The Rayleigh–Bénard Problem for Compressible Fluid Flows. *Arch. Ration. Mech. Anal.*, 247(1):Paper No. 9, 2023.

[138] E. Fernández-Cara and E. Zuazua. On the null controllability of the one-dimensional heat equation with BV coefficients. *Comput. Appl. Math.*, 21(1):167–190, 2002. Special issue in memory of Jacques-Louis Lions.

[139] C. Foiaş. Statistical study of Navier-Stokes equations. I, II. *Rend. Sem. Mat. Univ. Padova*, 48:219–348 (1973); ibid. 49 (1973), 9–123, 1972.

[140] C. Foiaş, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[141] C. Foiaş, R. M. S. Rosa, and R. Temam. Properties of time-dependent statistical solutions of the three-dimensional Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)*, 63(6):2515–2573, 2013.

[142] C. Foiaş, R. M. S. Rosa, and R. Temam. Properties of stationary statistical solutions of the three-dimensional Navier-Stokes equations. *J. Dynam. Differential Equations*, 31(3):1689–1741, 2019.

[143] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.*, 7:345–392, 1954.

[144] K. O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11:333–418, 1958.

[145] I. Gallagher and L. Saint-Raymond. Weak convergence results for inhomogeneous rotating fluid equations. *J. Anal. Math.*, 99:1–34, 2006.

[146] F. Gancedo and E. García-Juárez. Global regularity for 2D Boussinesq temperature patches with no diffusion. *Ann. PDE*, 3(2):Paper No. 14, 34, 2017.

[147] F. Gancedo and E. García-Juárez. Global regularity of 2D density patches for inhomogeneous Navier-Stokes. *Arch. Ration. Mech. Anal.*, 229(1):339–360, 2018.

[148] F. Gancedo and E. García-Juárez. Regularity results for viscous 3D Boussinesq temperature fronts. *Comm. Math. Phys.*, 376(3):1705–1736, 2020.

[149] D. Gérard-Varet. Highly rotating fluids in rough domains. *J. Math. Pures Appl. (9)*, 82(11):1453–1498, 2003.

[150] R. Granero-Belinchón and A. Ortega. On the motion of gravity-capillary waves with odd viscosity. *J. Nonlinear Sci.*, 32(3):Paper No. 28, 33, 2022.

[151] Z. Hassainia and T. Hmidi. On the inviscid Boussinesq system with rough initial data. *J. Math. Anal. Appl.*, 430(2):777–809, 2015.

[152] T. Hmidi. On the Yudovich solutions for the ideal MHD equations. *Nonlinearity*, 27(12):3117–3158, 2014.

[153] T. Hmidi and S. Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Anal.*, 189(2):283–300, 2008.

[154] T. Hmidi and M. Zerguine. Vortex patch problem for stratified Euler equations. *Commun. Math. Sci.*, 12(8):1541–1563, 2014.

[155] D. Hoff. Discontinuous solutions of the Navier-Stokes equations for compressible flow. *Arch. Rational Mech. Anal.*, 114(1):15–46, 1991.

[156] D. Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1):215–254, 1995.

[157] D. Hoff. Dynamics of singularity surfaces for compressible, viscous flows in two space dimensions. *Comm. Pure Appl. Math.*, 55(11):1365–1407, 2002.

[158] D. Hoff. Uniqueness of weak solutions of the Navier-Stokes equations of multidimensional, compressible flow. *SIAM J. Math. Anal.*, 37(6):1742–1760, 2006.

[159] D. Hoff and M. Perepelitsa. Boundary tangency for density interfaces in compressible viscous fluid flows. *J. Differential Equations*, 253(12):3543–3567, 2012.

[160] A. E. Hurd and D. H. Sattinger. Questions of existence and uniqueness for hyperbolic equations with discontinuous coefficients. *Trans. Amer. Math. Soc.*, 132:159–174, 1968.

[161] T. Iguchi and D. Lannes. Hyperbolic free boundary problems and applications to wave-structure interactions. *Indiana Univ. Math. J.*, 70(1):353–464, 2021.

[162] V. J. Ivriĭ and V. M. Petkov. Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations. *Uspehi Mat. Nauk*, 29(5(179)):3–70, 1974. Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), III.

[163] A. Jüngel, C.-K. Lin, and K.-C. Wu. An asymptotic limit of a Navier-Stokes system with capillary effects. *Comm. Math. Phys.*, 329(2):725–744, 2014.

[164] P. Kosewski and A. Kubica. Global in time solution to Kolmogorov's two-equation model of turbulence with small initial data. *Results Math.*, 77(4):Paper No. 163, 31, 2022.

[165] P. Kosewski and A. Kubica. Local in time solution to Kolmogorov's two-equation model of turbulence. *Monatsh. Math.*, 198(2):345–369, 2022.

[166] N. V. Krylov. The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:691–708, 1973.

[167] Y.-S. Kwon, D. Maltese, and A. Novotný. Multiscale analysis in the compressible rotating and heat conducting fluids. *J. Math. Fluid Mech.*, 20(2):421–444, 2018.

[168] Y.-S. Kwon and A. Novotný. Derivation of geostrophic equations as a rigorous limit of compressible rotating and heat conducting fluids with the general initial data. *Discrete Contin. Dyn. Syst.*, 40(1):395–421, 2020.

[169] D. Lannes. On the dynamics of floating structures. *Ann. PDE*, 3(1):Paper No. 11, 81, 2017.

[170] H. Lavenant and F. Santambrogio. Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational mean field games. *Comm. Partial Differential Equations*, 43(12):1761–1802, 2018.

[171] J. Lederer and R. Lewandowski. A RANS 3D model with unbounded eddy viscosities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 24(3):413–441, 2007.

[172] X. Liao and P. Zhang. On the global regularity of the two-dimensional density patch for inhomogeneous incompressible viscous flow. *Arch. Ration. Mech. Anal.*, 220(3):937–981, 2016.

[173] X. Liao and P. Zhang. Global regularity of 2-D density patches for viscous inhomogeneous incompressible flow with general density: high regularity case. *Anal. Theory Appl.*, 35(2):163–191, 2019.

[174] X. Liao and P. Zhang. Global regularity of 2D density patches for viscous inhomogeneous incompressible flow with general density: low regularity case. *Comm. Pure Appl. Math.*, 72(4):835–884, 2019.

[175] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2*, volume 9 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Perturbations. [Perturbations].

[176] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[177] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[178] P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl. (9)*, 77(6):585–627, 1998.

[179] P.-L. Lions and N. Masmoudi. On a free boundary barotropic model. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 16(3):373–410, 1999.

[180] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[181] J. Málek and J. Nečas. A finite-dimensional attractor for three-dimensional flow of incompressible fluids. *J. Differential Equations*, 127(2):498–518, 1996.

[182] C. Marchioro and M. Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[183] N. Masmoudi. Ekman layers of rotating fluids: the case of general initial data. *Comm. Pure Appl. Math.*, 53(4):432–483, 2000.

[184] N. Masmoudi. Rigorous derivation of the anelastic approximation. *J. Math. Pures Appl. (9)*, 88(3):230–240, 2007.

[185] N. Masmoudi and F. Rousset. Stability of oscillating boundary layers in rotating fluids. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(6):955–1002, 2008.

[186] N. Masmoudi and F. Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.*, 203(2):529–575, 2012.

[187] N. Masmoudi and F. Rousset. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 223(1):301–417, 2017.

[188] N. Masmoudi, F. Rousset, and C. Sun. Uniform regularity for the compressible Navier-Stokes system with low Mach number in domains with boundaries. *J. Math. Pures Appl. (9)*, 161:166–215, 2022.

[189] B. Maury, A. Roudneff-Chupin, and F. Santambrogio. A macroscopic crowd motion model of gradient flow type. *Math. Models Methods Appl. Sci.*, 20(10):1787–1821, 2010.

[190] A. Mellet and A. F. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 32(1-3):431–452, 2007.

[191] A. R. Mészáros and F. Santambrogio. Advection-diffusion equations with density constraints. *Anal. PDE*, 9(3):615–644, 2016.

[192] G. Métivier. Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d'espace. *Trans. Amer. Math. Soc.*, 296(2):431–479, 1986.

[193] G. Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008.

[194] G. Métivier. L^2 well-posed Cauchy problems and symmetrizability of first order systems. *J. Éc. polytech. Math.*, 1:39–70, 2014.

[195] G. Métivier and K. Zumbrun. Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems. *Mem. Amer. Math. Soc.*, 175(826):vi+107, 2005.

[196] C. Miao and B. Yuan. Well-posedness of the ideal MHD system in critical Besov spaces. *Methods Appl. Anal.*, 13(1):89–106, 2006.

[197] A. Mielke. On two coupled degenerate parabolic equations motivated by thermodynamics. *Submitted*, 2021.

[198] A. Mielke and J. Naumann. On the existence of global-in-time weak solutions and scaling laws for Kolmogorov's two-equation model for turbulence. *ZAMM Z. Angew. Math. Mech.*, 102(9):Paper No. e202000019, 31, 2022.

[199] B. Mohammadi and O. Pironneau. *Analysis of the k-epsilon turbulence model.* RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[200] S. Montgomery-Smith. Finite time blow up for a Navier-Stokes like equation. *Proc. Amer. Math. Soc.*, 129(10):3025–3029, 2001.

[201] P. B. Mucha, M. Pokorný, and E. Zatorska. Approximate solutions to a model of two-component reactive flow. *Discrete Contin. Dyn. Syst. Ser. S*, 7(5):1079–1099, 2014.

[202] M. Perepelitsa. Weak solutions of the Navier-Stokes equations for compressible flows in a half-space with no-slip boundary conditions. *Arch. Ration. Mech. Anal.*, 212(3):709–726, 2014.

[203] C. Perrin and E. Zatorska. Free/congested two-phase model from weak solutions to multi-dimensional compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 40(8):1558–1589, 2015.

[204] J. Rauch. *Hyperbolic partial differential equations and geometric optics*, volume 133 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2012.

[205] M. Reissig and K. Yagdjian. Weakly hyperbolic equation with fast oscillating coefficients. *Osaka J. Math.*, 36(2):437–464, 1999.

[206] F. Rousset. Stability of large Ekman boundary layers in rotating fluids. *Arch. Ration. Mech. Anal.*, 172(2):213–245, 2004.

[207] G. Sbaiz. Fast rotation limit for the 2-D non-homogeneous incompressible Euler equations. *J. Math. Anal. Appl.*, 512(1):Paper No. 126140, 41, 2022.

[208] P. G. Schmidt. On a magnetohydrodynamic problem of Euler type. *J. Differential Equations*, 74(2):318–335, 1988.

[209] S. Schochet and X. Xu. Toward uniform existence and convergence theorems for three-scale systems of hyperbolic PDEs with general initial data. *Comm. Partial Differential Equations*, 47(12):2401 – 2443, 2022.

[210] P. Secchi. On the equations of ideal incompressible magnetohydrodynamics. *Rend. Sem. Mat. Univ. Padova*, 90:103–119, 1993.

[211] G. R. Sell. Global attractors for the three-dimensional Navier-Stokes equations. *J. Dynam. Differential Equations*, 8(1):1–33, 1996.

[212] D. Serre. Variations de grande amplitude pour la densité d'un fluide visqueux compressible. *Phys. D*, 48(1):113–128, 1991.

[213] D. B. Spalding. Kolmogorov's two-equation model of turbulence. *Proc. Roy. Soc. London Ser. A*, 434(1890):211–216, 1991. Turbulence and stochastic processes: Kolmogorov's ideas 50 years on.

[214] G. Strang. Necessary and insufficient conditions for well-posed Cauchy problems. *J. Differential Equations*, 2:107–114, 1966.

[215] S. Tarama. Energy estimate for wave equations with coefficients in some Besov type class. *Electron. J. Differential Equations*, pages No. 85, 12, 2007.

[216] A. F. Vasseur and C. Yu. Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations. *Invent. Math.*, 206(3):935–974, 2016.

[217] M. Vishik. Hydrodynamics in Besov spaces. *Arch. Ration. Mech. Anal.*, 145(3):197–214, 1998.

[218] M. J. Vishik and A. V. Fursikov. *Mathematical problems of statistical hydromechanics*, volume 9 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the 1980 Russian original [MR0591678] by D. A. Leites.

[219] T. Yamazaki. On the $L^2(\mathbf{R}^n)$ well-posedness of some singular or degenerate partial differential equations of hyperbolic type. *Comm. Partial Differential Equations*, 15(7):1029–1078, 1990.

[220] E. Zatorska. On the flow of chemically reacting gaseous mixture. *J. Differential Equations*, 253(12):3471–3500, 2012.

[221] E. Zuazua. Exact controllability for the semilinear wave equation. *J. Math. Pures Appl. (9)*, 69(1):1–31, 1990.

Heterogeneity and low regularity issues in PDEs arising from Fluid Mechanics

Abstract: This habilitation thesis is devoted to the study of some partial differential equations, with a special emphasis on models arising in fluid mechanics. The general question we address here is about how and to which extent the presence of some heterogeneity (variations in density or temperature of the fluid, interaction with the boundary, anisotropy. . .) affects the dynamics of the fluid. Motivated by physics, we perform our study in a low regularity framework.
The manuscript is composed of three main parts. In the first one, we study linear hyperbolic operators having variable, low regularity coefficients. We prove several well-posedness results with and without loss of derivatives, for coefficients having lower regularity than the Lipschitz one. In the second part, we focus on non-linear models related to fluid mechanics and address the question of their well-posedness. We deal with weak solutions, strong solutions at critical regularity and statistical solutions. We are interested in several questions, depending on the specific model under consideration: for instance, the description of the dynamics of interfaces, or questions linked with turbulence theory, or the attainment of suitable bounds on the lifespan of the solutions. In the third and last part, we continue the study of models from fluid mechanics, but from the angle of singular perturbation problems. We focus on systems describing the dynamics of geophysical flows: our aim is to rigorously derive, by tools from asymptotic analysis, reduced models in some physically relevant regimes. We perform the low Rossby number (fast rotation) limit for non-homogeneous (compressible and incompressible) fluid flows, for a wide range of the scaling parameters.

Keywords: heterogeneity; low regularity; fluid mechanics; variable density; well-posedness; singular limits.

Questions d’hétérogénéité et de faible régularité dans les EDP issues de la Mécanique des Fluides

Résumé: Ce mémoire d’habilitation est consacré à l’étude de quelques équations aux dérivées partielles, avec une attention particulière à des modèles issus de la mécanique des fluides. La question générale à laquelle l’on s’intéresse concerne l’influence de la présence d’une certaine hétérogénéité (des variations de la densité du fluide, l’interaction avec le bord, l’anisotropie…) sur la dynamique du fluide. Motivés par la physique, nous conduisons notre étude dans un cadre à faible régularité.
Le manuscrit se compose de trois parties principales. Dans la première partie, nous étudions des opérateurs hyperboliques linéaires qui présentent des coefficients ayant une faible régularité. On montre plusieurs résultats de caractère bien posé avec ou sans perte de dérivées, pour des coefficients qui satisfont une condition de régularité plus faible que celle de Lipschitz. Dans la deuxième partie, nous nous intéressons à des modèles non-linéaires en lien avec la mécanique des fluides et à la question de leur caractère bien posé. Nous considérons le cadre des solutions faibles, des solutions fortes à régularité critique et des solutions statistiques. Nous nous intéressons à plusieurs questions, selon le modèle spécifique traité: par exemple, la description de la dynamique des interfaces, ou des questions liées à la théorie de la turbulence, ou l’obtention de bornes inférieures sur le temps de vie des solutions. Dans la troisième et dernière partie, on continue l’étude de modèles de la mécanique des fluides, mais du point de vue de l’analyse de problèmes de perturbation singulière. Nous nous concentrons sur des systèmes décrivant la dynamique de flots géophysiques: le but est la dérivation rigoureuse, par des outils d’analyse asymptotique, de modèles réduits dans certains régimes physiquement pertinents. Nous étudions la limite à faible nombre de Rossby (c’est-à-dire à rotation rapide) pour des systèmes de fluides non-homogènes (compressibles ou incompressibles), pour un large spectre de valeurs des paramètres d’échelle.

Mots clés: hétérogénéité; faible régularité; mécanique des fluides; densité variable; caractère bien posé; limites singulières.

Image en couverture: The *Monte Conero* with the *Due Sorelle* cliff (Ancona, Italy). Crédit image: https://www.pinterest.it/pin/le-due-sorelle-sirolo-ancona-riviera-del-conero-italia--426716133423524007/.

UNIVERSITÉ DE LYON