Analysing and thickening a narrative: on Archibald Henderson's history of the twenty-seven lines

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Abstract

The aim of this chapter is to show non-specialists in the history of mathematics how a given historical account can be challenged and revisited in order to thicken its descriptions. The example chosen for this is the history of the twenty-seven lines of cubic surfaces published in 1911 by Archibald Henderson, which has since become the source of the usual history of the subject. After describing the context in which this history was written and highlighting its weaknesses, I return to the description of the 1849 paper of Arthur Cayley where the existence of the twenty-seven lines is proved to show what Henderson missed: the technical description of this work, its place in diverse frameworks of the time and its circulation in the 1850s and 1860s.

A theorem well known to modern algebraic geometers states that any non-singular cubic surface contains exactly twenty-seven straight lines, provided the surface is seen as an object of the complex projective space $\mathbf{P}^3(\mathbf{C})$.

Let me first explain the mathematical terms involved in this statement. A cubic surface, also called a surface of order 3 or a surface of the third order or third degree, is a surface which can be defined by a polynomial equation of degree 3, such as $x^3 - 2xy^2 + z^3 + 3y - 1 = 0$, where x, y, z are coordinates of space. The surface is said to be non-singular if a tangent plane can be defined at any of its points, which roughly amounts to the fact that it does not contain any point that looks like a peak or a pinch. That a cubic surface is an object in $\mathbf{P}^3(\mathbf{C})$ means that it is made of points with complex coordinates, as well as points situated "at infinity". From the point of view of coordinates and equations, points at infinity are best handled by the introduction of homogeneous coordinates x, y, z, w and homogeneous equations. The homogeneous form of the above equation is $x^3 - 2xy^2 + z^3 + 3yw^2 - w^3 = 0$,

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Figure 1: A model of a non-singular cubic surface with twenty-seven real lines not entirely made of points at infinity. The model is dated between 1881 and 1899. Credits: Cu-Dro-002 / Collections de l'Institut Henri Poincaré, Paris (CC BY-NC-SA).

and the points at infinity of the corresponding surface are those for which $w = 0$.

Such a complex projective framework allows to state intersection theorems nicely. For instance, Bezout's celebrated theorem states that a curve of order n and a curve of order m in the complex projective plane $\mathbf{P}^2(\mathbf{C})$ intersect in exactly mn points.[1] Similarly, in the projective complex space $\mathbf{P}^3(\mathbf{C})$, the intersection of a plane (which is a surface of order 1) and a surface of order n is a curve of order n. And it is in this framework that the twenty-seven-lines theorem makes best sense: for instance, a real cubic surface being given, some of the twenty-seven lines it contains may be made up entirely of points at infinity, while others may not be made entirely of real points (see figure 1).

To get a sense of how the history of this theorem has been presented by

[1] More precisely, this result is true if the curves have no common component and if the counting of the intersection points is made by taking multiplicities into account. A (plane) curve of order n is a curve which can be defined by an equation of degree n.

mathematicians since the middle of the twentieth century, let me look at the publications whose title or review in *Mathematical Reviews* contains the expression "27 lines". According to the *MathSciNet* on-line database, as of April 2025, there are 172 such publications, dated between 1940 and 2023. About a quarter of them include historical material about the twenty-seven lines.

Although the scope and focus of this historical material varies greatly from one publication to another, there is a consensus in attributing the first statement and the first proof of the twenty-seven-lines theorem to the British mathematicians Arthur Cayley and George Salmon, in two articles published in 1849 [Cayley 1849a; Salmon 1849]. In two thirds of the cases, the historical facts, be they on Cayley and Salmon or on other points, are stated without reference to any source. However, the cases of the remaining third have in common that they cite (sometimes indirectly) a specific reference from which the historical content is taken unchanged, namely a 1911 book by Archibald Henderson entitled *The Twenty-Seven Lines upon the Cubic Surface* [Henderson 1911].[2] This book, or, more specifically, its preliminary "Historical Summary", hence appears as the source of the usual history of the twenty-seven-lines theorem.[3]

But this status does not mean that Henderson's narrative is without defects. I will show in this chapter that this narrative, although not containing any obviously false statement, suffers from many methodological weaknesses and consists in a succession of facts that are listed without being investigated in depth, either in terms of their own description or their contextualisation. The result is a history of the twenty-seven-lines theorem which is poor in my view, but which I will show how to make more complex, at least in its first elements. Moreover, since the type of Henderson's writing can be found in many current mathematical texts that include historical remarks (be they on the twenty-seven lines or on any other theorem, object or concept), the reflections developed on Henderson's case can be seen as applying to such texts as well – it goes without saying that I am thus sympathetically expressing my point of view as a historian on how to approach the history of mathematics, and that this is in no way intended to disparage mathematicians who include historical content in their publications.

In the first section of the chapter, I will begin by explaining the background of Henderson's writing of his book, examining his sources and bibliographic

[2] By indirect citation, I mean that a reference cites a text of which the historical material is taken from Henderson's book. For instance, [Shioda 2015] cites [Hunt 1994], of which the historical material comes from Henderson.

[3] There are texts, such as [Dolgachev 2012, pp. 562–565], which escape my *MathSciNet* research and which, in addition to Henderson's book, use other sources. Nevertheless, such works seem to me to display the same characteristics of historical writing as Henderson, even if they cite more references than the latter does. In this sense, the remarks I will make in the course of this chapter about the very shape of Henderson's narrative may also apply to them.

references, and then presenting the historical summary, organised by Henderson in a sequence of topical, apparently independent paragraphs. To evidence the poorness of this summary's content and to propose simultaneously avenues to enrich it, I will offer, in the second section, my own analysis of Cayley's 1849 paper: this will highlight what Henderson misses, which includes the technical description of this paper and its links with Cayley's other research of the time. The circulation of this paper between 1849 and 1870 will be dealt with in the third section. In the conclusion, after briefly revisiting another part of Henderson's historical summary to show the limits of its topical division, I will summarise my main criticisms of this summary, which I will connect to the evolution of the history of mathematics as a research discipline.

1. Henderson's twenty-seven lines

1.1 A second Ph.D., under Dickson's supervision

Archibald Henderson (1877–1963) was an American mathematician, literary critic, biographer and historian.[4] In 1894 he entered the University of North Carolina as a student, and he received a Ph.D. in 1902 with a dissertation entitled *The Cone of the Normals and an Allied Cone for Surfaces of the Second Degree*. Although Henderson was then hired as an associate professor of mathematics at the same university, he chose to continue his mathematical training. Having already spent a semester at the University of Chicago in 1901, he chose to continue his studies there in 1902–1903. In particular, in 1902, he began a work on the twenty-seven lines of the cubic surfaces in view of obtaining a Ph.D. of the University of Chicago, under the supervision of Leonard Eugene Dickson.[5]

Between 1903 and 1905, Henderson published three mathematical papers involving the twenty-seven lines of cubic surfaces [Henderson 1903, 1904, 1905].[6] In 1910–1911 he travelled to Europe, where he met Henry Frederick Baker and Bertrand Russell in Cambridge, Issai Schur and Hermann Schwarz in Berlin, and Émile Picard and Édouard Goursat in Paris. A first version of the book *The Twenty-Seven Lines upon the Cubic Surface* was published at the term of the trip, as one of the *Cambridge Tracts in Mathematics and Mathematical Physics*, the three aforementioned papers being included almost verbatim in

[4] The biographic information of this paragraph come from [Putzel 1988] and the short vita in [Henderson 1915].

[5] According to Della Dumbaugh, it was very unusual to prepare a second doctorate in the United States of America at that time. With the aim of expanding his mathematical knowledge, Henderson was certainly attracted by the then very active University of Chicago: between 1862 and 1934, this University awarded 237 doctorates in mathematics while the University of North Carolina awarded only 2, among which Henderson's one [Richardson 1936, p. 203]. On the University of Chicago, see for instance [Parshall and Rowe 1994, ch. 9].

[6] In this paragraph and the following one, I rely on the list of Henderson's publications provided by the *Jahrbuch über die Fortschritte der Mathematik* and the *Mathematical Reviews*.

this book [Henderson 1911]. After World War I it was republished in Chicago, as the Ph.D. dissertation as such, the only modification compared to the 1911 version being the addition of a short biographical notice [Henderson 1915].

The works of Henderson published after 1915, however, do not concern the twenty-seven lines any more, which seems to indicate that this topic was closely linked to Dickson's personal agenda. This is corroborated by the fact that at the beginning and the end of the time period during which Henderson worked on his dissertation, Dickson published several papers and books involving at least in part the twenty-seven lines [Dickson 1901a,b, 1902, 1915; Miller, Blichfeld, and Dickson 1916]. Dickson, of course, is also very well known for his *History of the Theory of Numbers*, which he began to design in 1911, and which he saw as a way to familiarise himself with arithmetical topics.[7] It is not clear to me whether the historical part of Henderson's dissertation was intended to have such a role in the practice of mathematics but, as will be seen below, the way of writing the history of mathematics was certainly the same.

Henderson's dissertation is first and foremost of a mathematical nature, the "Historical Summary" being only seven pages long out of the hundred of the whole. Apart from a note of thanks, a table of content, a general introduction, a bibliography, a table and some figures, the text is made up of seven mathematical chapters which mainly appear as a survey of the twenty-seven lines, in the sense that they present various results on the topic, most of them being already known.[8] That the aim was above all to take stock of what was known about the twenty-seven lines is in fact indicated in the introduction, where Henderson also warned the reader that the topic was so large that he could not treat every aspect of it:

> The problem of the twenty-seven lines upon the cubic surface is of such scope and extent, and is allied to so many other problems of importance, that to give a *résumé* of all that has been done upon the subject would enlarge the present memoir into an extensive book. It has not proved feasible to attempt to cover even the geometrical phases of the problem, in their extension in particular to the cognate problem of the forty-five triple tangent planes, although the two subjects go hand in hand. In this memoir, however, is given a general survey of the problem of the twenty-seven lines, from the geometric standpoint. [Henderson 1911, p. 8]

Since this warning was placed after the historical summary, it is very likely that it applied mostly to the seven mathematical chapters rather than to this summary. Nevertheless I will show that the latter also suffers from serious

[7] See [Dumbaugh Fenster 1999a,b]. As described in these references, Dickson saw other purposes in the history of mathematics. Moreover, Dickson stressed the importance of travelling to Europe to broaden one's knowledge on a subject, which echoes Henderson's own trajectory.

[8] Two chapters include short sections of a historical nature, which repeat and develop certain points from the historical summary [Henderson 1911, pp. 14, 54].

shortcomings. To get a first idea of this, let me now analyse Henderson's bibliographic references and possible sources.

1.2 Sources and bibliographic references of Henderson

Henderson did not make explicit how he gathered the sources that served him to write his book and, in particular, his historical summary. However, a clue can be found in the first paragraph of this summary:[9]

> The literature of the subject is very extensive. In a bibliography on curves and surfaces compiled by J. E. Hill, of Columbia University, New York, the section on cubic surfaces contained two hundred and five titles [*Bull. Am. Math. Soc.* Vol. III (1897), pp. 136–146]. The Royal Society of London Catalogue of Scientific Papers, 1800–1900, volume for *Pure Mathematics* (1908), contains very many more. [Henderson 1911, p. 1]

The two references given here to evidence the profusion of the mathematical works on cubic surfaces appear as potential bibliographical sources. Let me examine them both.

The paper cited by Henderson was authored by John Ethan Hill (1864–1941), then a tutor at Columbia University who had earned his doctorate in 1895 at Clark University with a dissertation called *On Quintic Surfaces*.[10] In this paper, Hill gave "a brief sketch of a bibliography of curves and twisted curves, prepared by [him]" [Hill 1897, p. 133]. He did not explain how he formed this bibliography, and contented himself with shortly presenting the diverse sections he created there. In particular, Hill evoked that among the 3,715 references of the bibliography, 205 deal with cubic surfaces, which corresponds to Henderson's number given above. That said, Hill provided bibliographic data only for three of these 205 references: a paper by Leopold Mossbrugger [1841], presented as the first one to have ever dealt with cubic surfaces specifically, and the two 1849 papers by Cayley and Salmon where the existence of the twenty-seven lines is proved [Cayley 1849a; Salmon 1849].

Henderson included this information in his historical summary, where he devoted a paragraph to Mossbrugger's paper before turning to the works of Cayley and Salmon. In fact, not only was the historical content taken over from Hill, but so was the wording of the very beginning of the historical summary and of a part of the paragraph on Mossbrugger. Indeed, compare the following lines:

> It is remarkable that the first paper that I can find that deals specifically with the cubic surface is one by L. Mossbrugger [...]. Although one may say that the classification of cubic surfaces is practically complete, the

[9] In this quote and the following ones by Henderson, what has been put into brackets appears in footnotes in the original.

[10] See the *Obituary Record of Graduates of Yale University Deceased During the Year 1941–1942*, on p. 170.

> study of these surfaces appears, still to-day, to have the same fascination as was exhibited when the discovery of the existence and the relations of the 27 lines of the general cubic surface was first announced. [Hill 1897, p. 137]

with:

> While it is doubtless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year to year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of the discovery of the twenty-seven lines upon the cubic surface. [...] The first paper that deals specifically with the cubic surface was by L. Mossbrugger. [Henderson 1911, p. 1]

This form of plagiarism, as well as the fact that Henderson seems to have taken Hill's narrative as it is, prompts us to read Henderson's historical summary with caution and to reflect on the seriousness with which he wrote it. For instance, to accept Mossbrugger's 1841 paper as the beginning of the history of cubic surfaces is by no means self-evident: according to the *Encyklopädie der mathematischen Wissentschaften mit Einschluß ihrer Anwendungen*, some 1829 research of Julius Plücker already involved a general cubic surface [Meyer 1928, p. 1439].

If there is no doubt that Henderson used Hill's paper, nothing allows me to know whether he had access to Hill's bibliography. This bibliography is recorded neither in the *Jahrbuch* nor in the *Mathematical Reviews*, and I could not find any trace of it. Therefore it is impossible to me to compare the 205 papers on cubic surfaces that are supposed to be listed in it with the bibliographic items of Henderson.

However, it is certain that Henderson used the *Catalogue of Scientific Papers* to gather sources on the twenty-seven lines.[11] Henderson cited one of the three volumes of the subject index, which classifies the listed papers in disciplinary parts. These parts are divided into several sections, themselves made of diverse subsections. Bibliographical references are listed under these subsections, and are, according to the cases, grouped by keywords. For instance, in the part related to geometry, a section on "Algebraic curves and surfaces of degree higher than the second" contains a subsection on "Algebraic surfaces of degree higher than the second". A header called "Configurations" exists in this subsection, under which the twenty-seven lines finally appear as keywords (see Table 1).

While the authors of the *Catalogue* did not explain how they created and attributed the keywords, the examination of the entries shows that these keywords often (but not systematically) reflect words from the titles of the

[11] Let me recall that the *Catalogue* is the outcome of a project of bibliographic review of articles published in scientific journals in the nineteenth century [Wagner-Döbler and Berg 1996].

papers that are associated with the them. For instance, a note by Camille Jordan entitled “Sur une nouvelle combinaison des vingt-sept droites d’une surface du troisième ordre” is associated with the keywords “27 straight lines, new combination” [Jordan 1870a]. Cayley’s 1849 paper on the twenty-seven lines, on its part, is called “On the Triple Tangent Planes of Surface of Third Order” and has “Triple tangent planes” as keywords.

The existence of such keywords makes the twenty-seven lines an explicit, well-identifiable topic in the *Catalogue*, and the example of Cayley’s paper proves that comparing Henderson’s bibliographic references with the *Catalogue* requires to consider the keywords “Triple tangent planes” as relevant as “27 straight lines”. It turns out that every item having these keywords in the *Catalogue* are bibliographical references in Henderson. The converse is not true, as exemplifies the paper [Maschke 1888], which appears in the *Catalogue* section devoted to discrete groups with the keywords “Quaternary group of substitutions with ternary Hessian for sub-group”.[12]

At this point, I should make it clear that by “bibliographic references” I mean the set of the references that appear somewhere in Henderson’s book. This does not coincide with the bibliography *per se*, understood as the list of 75 references given at the end of the book, since there are 24 publications cited in the historical summary all the while being absent from the bibliography.[13]

Conversely, some references of the bibliography are not cited anywhere in the book. Now, most of these references have erroneous titles in the bibliography, and these titles seem to have been forged from the corresponding *Catalogue* keywords. For instance, the real title of [Affolter 1874] is “Zur Theorie der Flächen dritter Ordung”, while it is presented as “The Twenty-Seven Lines on the Cubic Surface” in Henderson, and has “27 straight lines” as keywords in the *Catalogue*. In addition, the pages of these litigious references are never given fully: in the exact same way as in the *Catalogue*, only their first page is indicated in the bibliography. Hence Henderson obviously completed his bibliography with the help of the *Catalogue*, but did not care to have the right titles and the complete page numbers – in particular, it is most likely that he did not even read these articles.

All this confusion, once again, prompts us to take a cautious view on Henderson’s historical work. Further, to assess now the extent of Henderson’s bibliographic references, and thus get a better idea of the (in)completeness of his account, a possibility is to use the *Jahrbuch über die Fortschritte der Mathematik*.

Contrary to the index of the *Catalogue*, the *Jahrbuch* does not possess any

[12]Other references that exist in Henderson cannot appear in the *Catalogue* because they are books or Ph.D. dissertations, or because they were published after 1900. It is possible that Henderson simply collected such references without using any reviewing tool.

[13]The other parts of the book do not contain references that do not appear in the bibliography.

Algebraic Curves and Surfaces of degree higher than the second

7600 General

7610 Metrical and projective properties of algebraic plane curves of degree higher than the second

7630 Special plane algebraic curves

7640 Algebraic surfaces of degree higher than the second

— Contacts, and Tangent Lines and Planes

— Surfaces, 2nd and 3rd, 2nd, 3rd and 4th, degrees

— Surfaces, 3rd degree

— Configurations

– 27 real straight lines, forms of surfaces containing.
–— straight lines.
–— — —, delineation.
–— — —, determination and classification of surfaces with respect to.
–— — —, equation.
–— — —, — determining, resolution.
–— — —, groups of substitutions connected with.
–— — —, new combination.
–— — —, and parabolic curve.
–— — —, representation on plane.
–— — — and 45 triple tangent planes.
–— — — — — — — — — and 36 double-sixers, construction of models showing lines.
– Triple tangent planes.
–— — —, Cayley's theorem, proof.
–— — —, polyhedral configurations.
–— — —, property.

— Surfaces, 3rd and 4th, 4th, 4th and 5th, 5th, $(m+n)$th, nth degrees

7650 Special algebraic surfaces

7660 Skew algebraic curves

Table 1: The twenty-seven lines in the *Catalogue of Scientific Papers*. Note that apart from "27 straight lines" and "Triple tangent planes", which gather 7 and 3 papers, respectively, the other keywords are associated with only one paper. Moreover, the header "Configurations" contains other keywords that those written in this table.

header corresponding to the topics of cubic surfaces or of the twenty-seven lines. The search for mathematical texts whose title contains the phrase “twenty-seven lines” or its French, German or Italian equivalents and which were published between 1868 and 1911 yields 34 results (the lower bound of this time interval corresponds to the year when the *Jahrbuch* begins its reviews, while the upper bound is the year of the first publication of Henderson’s monograph). Among these results, only 6 are not bibliographic references in Henderson.[14] This search in the titles thus brings only a handful of new publications – which confirms that the *Catalogue* keywords were often created from the titles of the papers and that Henderson used the *Catalogue*.

Now, if one searches for publications whose title or review contain “twenty-seven lines”, 101 results are obtained, among which 66 are not in Henderson’s bibliographic references. A large part of these new references are continuations of works that appear in Henderson: this is the case, for instance, of a series of papers by Dickson and by William Burnside, which are connected to the study of groups associated with the twenty-seven lines, a topic which does appear in the historical summary.

Other references, on the contrary, reflect some research involving the twenty-seven lines that is not evoked at all by Henderson, either in the historical summary or the mathematical chapters. A part of them is connected with some research by Alfred Clebsch who, in a 1866 paper, established the representation of any cubic surface on a plane, which means, in modern terms, that he proved the existence of a birational transformation between any cubic surface and the projective plane [Clebsch 1866]. Because they are crucial to understand this birational transformation, the twenty-seven lines are present in this article, an article which is important in the development of the issue of surface representations at the end of the nineteenth century [Brigaglia, Ciliberto, and Pedrini 2004, pp. 303–304; Rowe 2021; Lê 2022, pp. 33–35].

These examples show that Henderson’s historical account is far from exhausting the topic of the twenty-seven lines. Of course, this is not problematic in itself since exhaustiveness is probably nothing else than a pipe dream in the history of mathematics. But Henderson’s reader must bear in mind the limitations his account of the twenty-seven straight lines may present, limitations that the author unfortunately did not make explicit.

Leaving aside the issue of oversights of this nature, let me now turn to the content of the historical summary.

[14] For this comparison, I do not count some short notes, such as [Jordan 1869b], that announce a larger publication, such as [Jordan 1870b]. Henderson, indeed, seemed to have systematically cited the latter kind of texts only.

1.3 A topical historical summary

This summary is divided into several paragraphs. The first two relate to the high number of publications dealing with cubic surfaces and to Mossbrugger's paper: I accounted for them earlier. With the exception of one of them, all the other paragraphs begin with a sentence which clearly attributes to them a given topic related to the twenty-seven lines – the exceptional paragraph consists in the listing of bibliographical references on several of these topics. In their order of appearance in Henderson's narrative, the topics are the following:

- The existence of the twenty-seven lines with Arthur Cayley and George Salmon.
- The basis for "a purely geometric theory" of cubic surfaces with Jacob Steiner.
- The issue of the notation of the lines, and the concept of a double-six.
- The works of Rudolf Sturm and Luigi Cremona on cubic surfaces "from the synthetic standpoint".
- The classifications of cubic surfaces in regards with the reality of the lines they contain or their singularities.
- The models of cubic surfaces or their lines.
- The shapes and models of cubic surfaces.
- The links between the twenty-seven lines and the twenty-eight double tangents of plane quartic curves.[15]
- The links between the twenty-seven lines and the Pascal configuration.[16]
- References on the concept of double-six, and on the link between the twenty-seven lines and the twenty-eight double tangents.
- The research of Corrado Segre on cubic varieties in a four-dimensional space.
- The group-theoretic standpoint on the twenty-seven lines.

In each paragraph, Henderson briefly describes the topics and provides bibliographical references of related mathematical works. His descriptions consist mainly in successions of facts, which are sometimes augmented with anecdotes.

[15] A quartic curve is a curve of order 4. A non-singular quartic curve being given in $\mathbf{P}^2(\mathbf{C})$, there are exactly 28 lines that are tangent to the curve in two distinct points.

[16] The Pascal configuration is the configuration obtained from the six lines that join opposite vertices of a hexagon inscribed in a conic. Pascal's theorem states that the mutual intersections of these lines consist in three aligned points.

For instance, the paragraphs dealing with the existence of the twenty-seven lines by Cayley and Salmon, on the one hand, and with Steiner's research, on the other hand, are the following:

> The theory of straight lines upon a cubic surface was first studied in a correspondence between the British mathematicians Salmon and Cayley; and the results were published, *Camb. and Dublin Math. Journal*, Vol. IV. (1849), pp. 118–132 (Cayley), pp. 252–260 (Salmon). The observation that a definite number of straight lines must lie on the surface is initially due to Cayley, whereas the determination of that number was first made by Salmon. [Salmon, *Geometry of Three Dimensions*, 4th edition, §530, note. Cf. also Cayley, *Coll. Math. Papers*, Vol. I. note, p. 589.]
>
> The basis for a purely geometric theory of cubic surfaces was laid by Steiner ["*Ueber die Flächen dritten Grades*," read to the Berlin Academy, 31st January, 1856; *Crelle's Journ.*, Vol. LIII.] in a short but extremely fruitful and suggestive memoir. This paper contained many theorems, given either wholly without proof, or with at most the barest indication of the method of derivation—a habit of "*ce célèbre sphinx*," as he has been styled by Cremona. [Henderson 1911, pp. 1–2]

Before interrogating the content of these paragraphs, let me first note that here again Henderson is plagiarising, this time by repeating almost verbatim one of the cited texts. Indeed, the story of the discovery of the twenty-seven lines is told in Salmon's *Geometry of Three Dimensions* as follows:

> The theory of right lines on a cubical surface was first studied in the year 1849, in a correspondence between Prof. Cayley and me, the results of which were published, *Cambridge and Dublin Mathematical Journal*, vol. IV., pp. 118, 252. Prof. Cayley first observed that a definite number of right lines must lie on the surface; the determination of that number as above, and the discussions in Art. 533 were supplied by me. [Salmon 1882, p. 496]

There is no reason to doubt the veracity of the events reported by Salmon (and thus by Henderson), especially as they were also reported by Cayley. But it is fair to question the value of a narrative of which a non-negligible part consists in an almost exact repetition of extracts written by mathematicians of the past. In fact, this way of writing the history of mathematics, as well as the division of the twenty-seven-lines theme in several topics, somehow echoes Dickson's *History of the Theory of Numbers*, which was explicitly meant to be a list of "facts" that were not supposed to be commented on by the historian [Dumbaugh Fenster 1999a, pp. 163–164].

Moreover, the two quoted paragraphs showcase other kinds of problems and questions that are to be found throughout the historical summary.

First, Henderson never provided any precise clue on the mathematical proofs of the reported theorems or on the technical or heuristic sources used by past mathematicians. For instance, he only alluded to Steiner's geometric theory of cubic surfaces, without explaining what this refers to, while nothing was said at all on how Cayley and Salmon proved the existence of the twenty-seven lines. What kind of objects and techniques did the proofs of these mathematicians involve? Were these proofs self-contained? Were they inspired by past research? Moreover, Henderson never commented on the status of the results about the twenty-seven lines within the cited papers. Did the twenty-seven lines represent the actual core of these papers, or were they only a side issue for Cayley, Salmon and Steiner? More generally, what was the status of the twenty-seven lines in their mathematical research of the time?

Alongside these questions about the content of each paragraph, others exist about the relations between these paragraphs. In the case of the above examples, Steiner's research is presented as having launched the study of cubic surfaces by purely geometric methods. Does it mean that Cayley and Salmon never used such methods? Whether this is the case or not, did Steiner know about the British contributions? Was his purely geometric theory at least in part inspired by them? Did his "many theorems" include results, such as the very existence of the twenty-seven lines, that were already known by Cayley and Salmon? Similarly, Henderson did not explain if the mathematicians involved in the other paragraphs learned of the existence of the twenty-seven lines from the publications of the British duo, nor if they made use of some of their technical results.

Finally, the very division in topical paragraphs of the historical summary is also questionable. How did Henderson choose and define the topics to be accounted for? Why, as seen above, did he left out some themes, such as the plane representations of cubic surfaces? How did he operate the distribution a given contribution in a given topic? To what extent are these topics related to each other, from the point of view of the mathematical content? Furthermore, does the topical division correspond to an actual partition of the mathematical papers cited, or could these papers be associated with several topics? Why are some papers given in the bibliography not dealt with in the historical summary? Is it because they do not fit to the chosen topics, or because they were seen as insignificant?

This litany of interrogations could be continued indefinitely and there is no question of addressing all of them, my point being just to suggest how Henderson's history of the twenty-seven lines could be enriched. That said, I will now consider Cayley's 1849 paper and describe it in detail, which will lead me to answer to some of the previous questions.

2. A focus on Cayley's 1849 paper

Henderson reported that Cayley and Salmon developed their results on the twenty-seven lines in a correspondence between them, the former having found that any cubic surface contains a certain number of lines while the latter proved that this number is 27. This story was already accounted for in the references cited by Henderson: in Cayley's 1849 paper [Cayley 1849a, p. 132] and, as noted above, in texts published afterwards, namely in Cayley's *Collected Papers*, which he edited himself [Cayley 1889, p. 589], and in Salmon's famous *Treatise on the Analytic Geometry of Three Dimensions* – while Henderson's only cited the fourth edition of the treatise of 1882, the information was actually already included in the first edition, published twenty years before [Salmon 1862, p. 386].

Unfortunately the letters between Cayley and Salmon seem to be lost. As explained by Tony Crilly, in 1923, the letters in Cayley's Nachlass were returned to their original author if he or she was still alive, which was not the case for Salmon, while "[m]any letters written by others have not survived" [Crilly 2006, p. 559]. I will thus leave aside everything concerning the epistolary circumstances of the proof of the twenty-seven-lines theorem, and focus on Cayley's original paper.

2.1 The existence of the twenty-seven lines

This paper was published in the *Cambridge and Dulin Mathematical Journal*. Of its fifteen pages, the first two are devoted to the existence of the twenty-seven lines, of which two proofs are actually proposed by Cayley.

The first one presupposes the existence of lines upon a given cubic surface: "A surface of the third order contains in general a certain number of straight lines", asserted Cayley in opening his paper [Cayley 1849a, p. 118]. Cayley then considered the planes that pass through such a line. Since the intersection of a plane and a cubic surface is a cubic curve, in this case, the cubic curve must be composed of the line and a conic. Further, since the two points where this line and this conic intersect are double points of the cubic curve, they are both counted with multiplicity 2 when considered as intersection points of the plane and the surface.[17] In other words, they are points where the plane is tangent to the surface.

Cayley's idea, then, was to look for the planes for which the residual conic degenerates into two lines. If this happens, the intersection of such a plane with the surface is made of three lines and, as just explained, their three mutual intersection points are points of tangency: this is why the plane is called a "triple tangent plane". Cayley continued:

> The number of lines and treble tangent planes is determined by means of a theorem easily demonstrated, *viz.* that through each line there may be drawn five (and only five) treble tangent planes. [Cayley 1849a, p. 118]

[17]If the considered point is $(0, 0, 0)$, which can be assumed without any loss of generality, this is easily seen when thinking of the multiplicity orders as the lowest degrees in the equations of these objects.

This theorem, which was not proved by Cayley,[18] was used as follows. Consider one of the five triple tangent planes given by the theorem. It contains three lines and, according to the same theorem, there are four new triple tangent planes associated with each of these three lines. This gives 12 triple tangent planes and thus 24 new lines on the surface. Together with the three original ones, the process thus yields 27 lines on the surface.

Reciprocally, Cayley proved that these lines are the only ones that can be found on the surface. To do this, he remarked that since the three lines of a triple tangent plane form the exact intersection of this plane with the surface, any line L included in the surface intersects the plane in a point that belongs to one of the three lines, say L'. Now, the plane defined by L and L' intersects the surface in a cubic curve which contains L and L'. This curve thus necessarily consists in three lines, and the plane that contains them is one of the triple tangent planes in which L is included. This proves that L does correspond to a line found by the process presented above. Cayley concluded:

> Hence the whole number of lines upon the surface is twenty-seven; and it immediately follows that the number of treble tangent planes is forty-five.[19] [Cayley 1849a, p. 119]

Let me stop momentarily the description of Cayley's paper and comment a bit on what precedes.

First, as has been seen, Cayley started his paper *in medias res*, by asserting that cubic surfaces contain lines. In particular, he did not explain at all what was his incentive for investigating this topic. It should be noted, however, that in previous research he had already dealt with cubic surfaces that contain lines and possess triple tangent planes. Indeed, in a 1844 paper devoted to cubic curves, Cayley considered a cubic surface defined by the condition that it contains the six edges of a given tetrahedron – this is the surface that would later be called "Cayley's surface" [Cayley 1844]. He remarked that this surface is touched along each edge by exactly one plane, and he proved that the six tangent planes thus obtained intersect two by two in three lines that are coplanar and included in the cubic surface. Although Cayley did not make it explicit, this provides a (singular) cubic surface with nine lines and eleven triple tangent planes. Other publications of the time show that Cayley also worked with quadric surfaces and their infinitely many lines, as well as with the "wave surface", a special quartic surface of which he studied special plane intersections [Cayley 1846a, 1848]. Therefore, if the specific reasons why Cayley studied cubic surfaces, their lines and triple tangent planes remain unclear, these questions were in line with his earlier investigations on algebraic surfaces

[18] Ernest De Jonquières proved it in a paper where he exposed Cayley's and Salmon's 1849 works [Jonquières 1859, pp. 135–136].

[19] Each of the 27 lines belongs to 5 planes and, reciprocally, each plane contains 3 lines. Thus the number of planes is $\frac{27\times5}{3} = 45$.

of low order.[20]

Second, the proof of the existence of the twenty-seven lines (and the forty-five triple tangent planes) was incomplete for two reasons. The first one is that it presupposed the existence of at least one line upon the surface. This point can be seen as unproblematic, since Cayley provided another proof where this hypothesis is not needed: in a way, this merely makes the first proof dependent on the second one. The second reason is that Cayley explicitly omitted the demonstration of a theorem that he used crucially to enumerate the lines and the triple tangent planes. Apart from these two points, the proof appears as quite solid and clear. In particular, it necessitates only to know that a cubic surface is intersected by a plane in a cubic curve, and that such a curve may degenerate into one line and one conic, or into three lines. From this point of view, it is possible that Cayley presented this incomplete proof – and, indeed, as the first one in his paper – because he saw it as being the most intuitive and accessible for his readers. Finally, it is noteworthy that this proof did not involve any equation between coordinates, and could therefore be qualified as purely geometric: this counters the idea that such a viewpoint was characteristic of Steiner, as Henderson's narrative might suggest.

Let me now turn to the second proof, which Cayley presented as having "the advantage of not assuming *a priori* the existence of a line upon the surface" [Cayley 1849a, p. 119]. The key idea here was to consider a tangent cone to the surface, that is, a cone made of the lines in space that are tangent to the surface and pass through an arbitrary vertex outside the surface. Cayley asserted that any double tangent plane of the cone is a double tangent plane of the cubic surface, and thus intersects this surface in a line and a conic.[21] Reciprocally, a line in the surface being given, the plane containing this line and passing through the vertex of the cone is a double tangent plane of the cone. As Cayley wrote, this proves that there are as many lines in the surface as double tangent planes of the cone.

Cayley then invoked two texts to complete his proof. He first cited a recent paper by Salmon [1847] to state that the tangent cone is a surface of order 6 which has no double line and 6 cuspidal lines – Salmon's paper contained a more general result, giving the order, number of double lines and number of cuspidal lines of a cone tangent to a surface of any order m. Cayley finally cited Julius Plücker's famous book *Theorie der algebraischen Curven* [Plücker 1839] to conclude:

> [B]y the formula in Plücker's "Theorie der Algebraischen Curven," p. 211,

[20] See [Crilly 2006, part 2] for a rich account of Cayley's activities between 1844 and 1849.

[21] Let O be the vertex of the cone. By definition, this cone is made of the lines OM, where M belongs to the surface and OM is tangent to it. A double tangent plane of the cone is a plane that contains two edges of the cone, say OM and OM'. If T is such a double tangent plane, it is tangent to the surface in M and M'. Hence the cubic curve that is its intersection with the surface has M and M' as double points: it is necessarily a degenerate cubic, made of the line MM' and a conic.

stated so as to apply to cones instead of plane curves, (viz. n being the order, x the number of double lines, y that of the cuspidal lines, u that of the double tangents planes, then

$$u = \frac{1}{2}n(n-2)(n^2-9)-(2x+3y)(n^2-n-6)+2x(x-1)+6xy+\frac{9}{2}y(y-1)).$$

The number of double tangent planes is twenty-seven, which is therefore also the number of lines upon the surface. [Cayley 1849a, p. 119]

This formula, which is exactly the same as Plücker's, yields indeed $u = 27$ for the values $n = 6$, $x = 0$, $y = 6$.

This second proof of the existence of the twenty-seven lines suffered no gap, but, contrary to the first one, it appears as less adapted to understand the geometrical configuration obtained since it does not show how the twenty-seven lines are arranged into forty-five triple tangent planes. At any rate, this proof was also explicitly rooted in the research of other mathematicians, which is historically interesting since it displays more explicitly how Cayley's investigations was inscribed in collective frameworks of the time.

2.2 Equations and notations

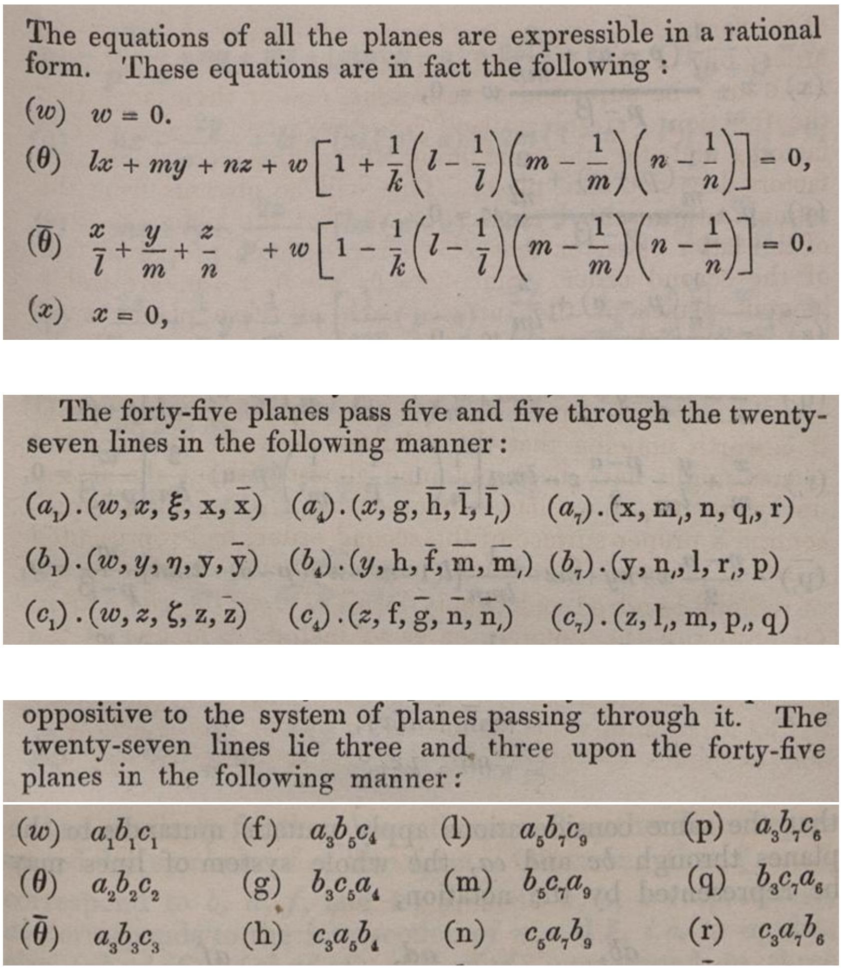
Although the two proofs of the existence of the twenty-seven lines were done, Cayley's paper was far from finished. The next move was to make explicit the equations of the forty-five triple tangent planes. For this, Cayley first proved that for an appropriate choice of space coordinates x, y, z, w, the equation of a cubic surface can be written as $wP + kxyz = 0$, where k is a numerical parameter and P is a quadratic polynomial.

In the case where P cannot be factorised into linear polynomials, Cayley showed that the previous equation of the cubic surface is of the form

$$w\left\{x^2+y^2+z^2+yz\left(mn+\frac{1}{mn}\right)+zx\left(n\ell+\frac{1}{n\ell}\right)+xy\left(\ell m+\frac{1}{\ell m}\right)+ xw\left(\ell+\frac{1}{\ell}\right)+yw\left(m+\frac{1}{m}\right)+zw\left(n+\frac{1}{n}\right)\right\}+kxyz=0,$$

where ℓ, m, n are parameters. From this equation, Cayley deduced the list of the forty-five equations of the triple tangent planes, each of which was associated with a symbol used to denote the plane: for example, the plane $w = 0$ was denoted by (w), the plane $x + \frac{1}{k}\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right)w = 0$ was denoted by (ξ), etc. (see figure 2).

Cayley then wrote down sixteen (out of 120) equations of the cubic surface, such as $w\mathrm{f}\bar{\mathrm{f}} + k\xi yz = 0$, where ξ, f and $\bar{\mathrm{f}}$ designate the left-hand sides of the equations of the planes (ξ), (f) and $(\bar{\mathrm{f}})$. He also listed the systems of five planes having a line in common – a line which is necessarily one of the twenty-seven –, on the basis of which he introduced a notation for these lines, consisting in



The equations of all the planes are expressible in a rational form. These equations are in fact the following :

$(w)\quad w=0.$

$(\theta)\quad lx+my+nz+w\left[1+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0,$

$(\bar{\theta})\quad \frac{x}{l}+\frac{y}{m}+\frac{z}{n}\quad +w\left[1-\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0.$

$(x)\quad x=0,$

The forty-five planes pass five and five through the twenty-seven lines in the following manner :

$(a_1).(w, x, \xi, \mathrm{x}, \bar{\mathrm{x}})$ $(a_4).(x, \mathrm{g}, \bar{\mathrm{h}}, \bar{\mathrm{l}}, \bar{\mathrm{l}}_{,})$ $(a_7).(\mathrm{x}, \mathrm{m}_{,}, \mathrm{n}, \mathrm{q}_{,}, \mathrm{r})$

$(b_1).(w, y, \eta, \mathrm{y}, \bar{\mathrm{y}})$ $(b_4).(y, \mathrm{h}, \bar{\mathrm{f}}, \bar{\mathrm{m}}, \bar{\mathrm{m}}_{,})$ $(b_7).(\mathrm{y}, \mathrm{n}_{,}, \mathrm{l}, \mathrm{r}_{,}, \mathrm{p})$

$(c_1).(w, z, \zeta, \mathrm{z}, \bar{\mathrm{z}})$ $(c_4).(z, \mathrm{f}, \bar{\mathrm{g}}, \bar{\mathrm{n}}, \bar{\mathrm{n}}_{,})$ $(c_7).(\mathrm{z}, \mathrm{l}_{,}, \mathrm{m}, \mathrm{p}_{,}, \mathrm{q})$

oppositive to the system of planes passing through it. The twenty-seven lines lie three and three upon the forty-five planes in the following manner :

(w)	$a_1b_1c_1$	(f)	$a_3b_5c_4$	(l)	$a_5b_7c_9$	(p)	$a_3b_7c_6$
(θ)	$a_2b_2c_2$	(g)	$b_3c_5a_4$	(m)	$b_5c_7a_9$	(q)	$b_3c_7a_6$
$(\bar{\theta})$	$a_3b_3c_3$	(h)	$c_3a_5b_4$	(n)	$c_5a_7b_9$	(r)	$c_3a_7b_6$

Figure 2: The beginning of the list of the equations of the forty-five triple tangents planes [Cayley 1849a, p. 121]. The letters p, α, β designate explicit functions of the parameters k, ℓ, m, n.

the symbols $a_1, \dots, a_9$, $b_1, \dots, b_9$, $c_1, \dots, c_9$. This notation did not contain in itself the trace of the incidence relations: the line a_1 was defined as the line common to the planes (w), (x), (ξ), (x), $(\overline{\mathrm{x}})$, the line b_1 was defined as the line common to the planes (w), (y), (η), (y), $(\overline{\mathrm{y}})$, and so on: it is Cayley's exhaustive listing that allowed understanding the association between lines and planes. Conversely, Cayley indicated which lines are contained in each plane (see figure 2).

In fact, Cayley was led to another notation of the twenty-seven lines, obtained this time by supposing that in the equation of the cubic surface $wP + kxyz = 0$, the quadratic polynomial P breaks up into two linear factors. In this case, this equation can be written as $ace - bdf = 0$, where $a, \dots, f$ are linear functions of the coordinates. Cayley added that a counting of the constants actually proved directly that the equation of any cubic surface can be written in this way. Without entering into the details here, let me just remark that the other notation of the twenty-seven lines used the letters $a, \dots, f$ and was established in close relation with the incidence properties of the lines.

After having shown the correspondence between this notation and the previous one, Cayley contented that:

> There is great difficulty in conceiving the complete figure formed by the twenty-seven lines, indeed this can hardly I think be accomplished until a more perfect notation is discovered. In the mean time it is easy to find theorems which partially exhibit the properties of the system. [Cayley 1849a, p. 127]

Moreover, at the end of the paper, Cayley explained that "the whole subject of this memoir was developed in a correspondence with Mr. Salmon, and in particular, that [he was] indebted to him for the determination of the number of the lines upon the surface and for the investigations connected with the representation of the twenty-seven lines by means of the letters a, c, e, b, d, f, as developed above." [Cayley 1849a, p. 132].

Cayley's own dissatisfaction of his notations was not mentioned by Henderson in the paragraph of the historical summary devoted to the issue of the notation:

> The notation first given by Cayley was obtained from some arrangement that was not unique, but one of a system of several like arrangements; but it was so complicated as scarcely to be considered as at all putting in evidence the relations of the lines and triple tangent planes. [Henderson 1911, p. 2]

As can be seen, Henderson overlooked that Cayley provided two notations, one due to Salmon. Nor did he explain what the mentioned notation is, or even that it was exposed in the 1849 paper, which certainly has the effect of breaking the coherence of this paper.

After having evoked a system of notation due to Andrew Hart, and published in [Salmon 1849], Henderson turned to the notation which was proposed by

Ludwig Schläfli and which "has remained unimproved upon up to the present time" [Henderson 1911, p. 2]. Nevertheless Henderson did not explain that Schläfli's and Cayley's approaches share common technical features, and the reader could not guess this since Henderson did not enter into the mathematical details. In fact, to develop his system of notation, Schläfli began by proving that the equation of any cubic surface can be brought to the form $uvw + xyz = 0$, where x, y, z, w are appropriate coordinates of space and u, v are linear functions of them [Schläfli 1858, p. 112]. Up to the + sign (which is here mathematically unimportant), this is the same form of the equation $ace - bdf = 0$ from which Cayley started for his second system of notation, and, just like Cayley, Schläfli made use of this special form of the equation to derive a new notation of the twenty-seven lines, a notation defined via the concept of a double-six (which refers to a set of twelve lines having special incidence relations). Schläfli did not cite Cayley's paper, and, as will be seen below, it is likely that he did not know its content. Still it is a pity that yet another form of coherence was missed in Henderson's account.

2.3 Incidence properties and configurations

As indicated in Cayley's previous quote, in spite of having an imperfect notation, Cayley used the one consisting in the symbols $a_1, \ldots, c_9$ to investigate some properties of the geometrical configuration formed by the twenty-seven lines.

For instance, he showed that any two non-intersecting lines, say a_1, b_1, are intersected by five lines a_2, b_1, a_5, a_7, a_9 having the following properties: they do not intersect each other, any four of them do not cut any line other than a_1, b_1, and any three of them are cut by exactly one other line.[22] Another result is that given three non-intersecting lines there exist six lines that do not intersect them: these six lines, then, form an hexagon,[23] of which each pair of opposite sides are cut by yet another pair of lines. Cayley stated several other such results and concluded: "The number of such theorems might be multiplied indefinitely, and the number of different combinations of lines or planes to which each theorem applies is also very considerable" [Cayley 1849a, p. 128].

The rest of Cayley's paper was devoted to other properties linked to the configuration of the twenty-seven lines and the forty-five triple tangent planes. While one of them is related to the anharmonic ratio of specific families of triple tangent planes, others concerned plane cubic curves, which Cayley realised as planar intersections of cubic surfaces: such a curve contains the twenty-seven points that are the intersections of the twenty-seven lines by the plane containing the curve, and Cayley investigated the incidence relations of these

[22] All the lines considered in this paragraph are some among the twenty-seven.

[23] Contrary to the planar case, six lines in space do not always form an hexagon. This is the case if adequate incidence relation between the lines exist.

points. For instance, he showed that there exist systems of fifteen points (among the twenty-seven) that are aligned three by three on fifteen lines, in such a way that, reciprocally, nine of these points are the mutual intersections of 2×3 lines, while the six remaining ones form an hexagon whose diagonals and sides intersect in a special way.

This type of results, which deal with the different incidence configurations of points, lines and planes, recalls strongly the activities that prefigure what would be called "tactics" by Cayley in 1864. As Caroline Ehrhardt [2015] explains, tactics was "a field of investigation at the crossroads of algebra, combinatorics and recreational mathematics" which, apart from Cayley, involved mathematicians such as Thomas Kirkman and James Joseph Sylvester. In particular, it was marked by investigations on the arrangement of things, and Cayley was engaged in such investigations already around 1850, typically with the famous fifteen-schoolgirls problems [Tahta 2006].

The series of papers that Cayley devoted to theorems related to *géométrie de position*, published between 1846 and 1851 but written between 1845 and 1849, are of the same vein [Cayley 1846b, 1847, 1849b, 1851]. In particular, Cayley investigated the combinatorics associated with configurations of points and lines in a plane. For instance, he studied the plane configuration made of fifteen points situated three by three on twenty lines, and proved that there exist ten systems of nine of these points which form, in two different ways, three triangles having special properties [Cayley 1846b, p. 216]. Cayley also researched thoroughly the Pascal configuration and listed many theorems of the same nature. To take but one example, Cayley recalled that Kirkman had proved that the 60 lines arising from Pascal's theorem[24] intersect each other three by three in 20 points that Steiner already found, and in 60 other points, labelled by the letter h. Other incidence properties of these points were recalled, after which Cayley added:

> I myself have since found that the sixty points h lie three by three on twenty lines X. All these theorems can be demonstrated quite easily when one knows how the points and lines are to be combined by constructing the points and lines h, J, &c. This is done in a very simple way, using a notation that I will explain.[25] [Cayley 1851, p. 550]

This geometric setting, the insistence on the combinatorics associated with configurations of points and lines, and the use of an adequate notation to

[24]Let me recall that Pascal's theorem states that if an hexagon is inscribed in a conic, its diagonals intersect two by two in three aligned points. Given six points in a plane, one can form 60 hexagons, which thus yields 60 lines. On Kirkman's research on the Pascal configuration, see [Tahta 2006, pp. 61–68].

[25]"Moi, j'ai depuis trouvé que les soixantes points h sont situés trois à trois sur vingt droites X. Tous ces théorèmes peuvent être démontrés assez facilement quand on connaît la manière suivant laquelle les points et les droites doivent être combinés en construisant les points et les droites h, J, &c. Cela se fait alors d'une manière très simple, au moyen d'une notation que je vais expliquer."

investigate this echo directly what was done in the paper on the twenty-seven lines. It is not the place here to discuss further Cayley's other works, my point being just to put into light the insertion of his research on the twenty-seven lines in a more general framework, both at the level of his own activities and at the level of collective investigations. Needless to say that all this is absent from Henderson's narrative.

3. Circulation of Cayley's paper

Another question that was not raised at all by Henderson is how Cayley's (and Salmon's) results circulated at their time. Looking at articles and books published after them already provides answers.

The item of Henderson's bibliography that just follows chronologically Cayley's article is a 1855 paper by Francesco Brioschi, who investigated diverse properties of the twenty-seven-lines configuration [Brioschi 1855]. Although he did not provide explicit bibliographic references, Brioschi referred to Cayley's and Salmon's works from the very outset of his paper:

> In a surface of the third order there are, in general, twenty-seven lines. Mr. Cayley proved that any three of these lines are situated in a plane (called triple tangent by that author), and that five triple tangent planes pass through any one of the straight lines.[26] [Brioschi 1855, p. 374]

Brioschi also knew of the content of Salmon's 1849 paper, since he recalled (and then used) Hart's notation of the twenty-seven lines, which was presented there. Such a knowledge of the papers of Cayley and Salmon is also obvious in a 1859 article of Ernest de Jonquières, which, as mentioned above, was devoted to present and complete the proofs of the two British mathematicians [Jonquières 1859]. Interestingly, De Jonquières' paper was published in *Nouvelles annales de mathématiques*, a journal aimed at students and teachers, which proves that the twenty-seven-lines theorem (and the associated names of Cayley and Salmon) circulated beyond the strict academic milieu already at the end of the 1850s.

On the other hand, Steiner published in 1856 a memoir which, as has been seen, was described by Henderson as the basis for a "purely geometric theory of cubic surfaces" [Steiner 1856]. Among others, Steiner presented several ways of generating cubic surfaces, avoiding any recourse to equations and coordinates. To take one example, the first way of generating a cubic surface lies in the theorem according to which, given any pair of trihedrons and any point in space, there exists exactly one cubic surface passing through this point and the nine lines that are the intersections of the planes of the trihedrons – from

[26]"In una superficie del terzo ordine esistono, in generale, ventisette rette. Il sig. Cayley ha dimostrato che tre qualsivogliano di esse rette sono situate in un piano, (da quell'autore chiamato triplo-tangente), e che per una qualunque delle rette stesse passano cinque piani tripli-tangenti."

this follows almost immediately that every cubic surface contains twenty-seven lines. Steiner's paper contained many other results, among which the existence of forty-five triangles that can be formed from the twenty-seven lines – Steiner thus rather talked about triangles while Cayley and Salmon dealt with triple tangent planes.

Although such results are clearly the same as Cayley's, it is noteworthy that Steiner did not cite him. In fact, Steiner did not cite any published work by himself or other mathematicians: only the name of Jean-Victor Poncelet appeared, associated with a result on pencils of quadric surfaces [Steiner 1856, p. 134].

Yet Steiner had been aware of Cayley's research on cubic surfaces at least since 1853 on, as is proved by one of his personal notes:

> Paris. July 1853. Notes.
>
> 1. *From Sylvester*. An Englishman (Cayley) is said to have found that a cubic surface, in general, contains 27 lines.[27]

This quote shows that Steiner did not know who Cayley was at this time, which is quite surprising since in 1853, Cayley (who was 32 year old) had already published more than a hundred papers, more than twenty of them (of which about eight dealt with geometry) in Crelle's *Journal für die reine und angewandte Mathematik*.[28]

In any case, the previous quote proves that the statement and proof of the existence of the twenty-seven lines by Cayley (and Salmon) did not circulate widely and uniformly just after 1849. Another testimony of this is the following extract of a letter from Schläfli to Cayley, dated 1856:

> I was prompted to do so by your discovery of the 27 straight lines on the surface of the 3rd degree, which Mr. Steiner communicated to me orally. However, I have not yet been able to get hold of your relevant treatise in the Cambridge and Dublin Mathematical Journal, and so I venture to present some of the results of my investigation without knowing to what extent they have been rendered superfluous by what has already been published.[29] [Graf 1905, p. 9]

One sees here that one issue for Continental mathematicians was to be able to find concretely the volumes of the *Cambridge and Dublin Mathematical*

[27]"Paris. Juli 1853. Notizen. 1. *Von Silvester* [sic]. Ein Engländer (Cayley) soll gefunden haben: dass f^3, im Allgemeinen, 27 G. enthält." [Graf 1896, p. 91]

[28]See Cayley's *Collected Papers*, volumes 1 and 2. On Cayley's will to be known on Continental Europe, including by the means of publishing in Continental research journals, see [Despeaux 2014, pp. 93–94].

[29]"Die Veranlassung dazu gab mir Ihre von Herrn Steiner mir mündlich mitgetheilte Entdeckung der 27 Geraden auf der Fläche 3ten Grades. Ihre betreffende Abhandlung im Cambridge und Dublin Mathematical Journal konnte ich freilich bis jetzt nicht zur Hand bringen, und so wage ich es einiges von den Resultaten meiner Untersuchung vorzulegen, ohne zu wissen, in wie weit dieselben durch bereits Erschienenes überflüssig gemacht sind."

Journal at the time.[30]

Let me take a final example, dated a bit later. In his 1870 *Traité des substitutions et des équations algébriques* and in the papers that preceded the publication of this book, Camille Jordan studied a certain algebraic equation associated with the twenty-seven lines [Jordan 1869a,c, 1870b]. In these publications, he attributed the result of the existence of the lines to Steiner: "Steiner has made known (*Journal de M. Borchardt*, t. LIII) the following [theorem]: *Any surface of the third degree contains twenty-seven lines*."[31] The *Traité*, however, also contains a note inserted at the end of the book, where Jordan pointed out that "Messrs. Cayley and Salmon had discovered and studied these lines before Steiner."[32] This correction seems to be the result of a remark that Luigi Cremona expressed to Jordan. Indeed, in a letter dated 19 December 1869 where he thanked Jordan for having sent him a first version of the corresponding part of the *Traité*, and where he manifested his interest for a question involving the twenty-seven lines and the so-called "hyperelliptic functions", he underlined in passing Cayley's and Salmon's priority over Steiner:

> Among other things, there is a question that arouses my curiosity to the highest degree: that of the connection between the search for the 27 lines of a cubic surface (which were discovered by Messrs. Cayley and Salmon, before Steiner) and the trisection of hyperelliptic functions.[33]

It is not clear to me why Jordan was not aware of the contributions of Cayley and Salmon. But I want to take this example and the previous ones as testimonies of the fact that the actual circulation of the results of the two British mathematicians is to be thought of with nuances: these results were neither totally ignored, nor broadly received on the Continent.[34]

As for Cayley himself, when commenting retrospectively in his *Collected Papers*, he made clear that: "As mentionned at the conclusion of the [1849] Memoir the whole subject was developped in a correspondance with Dr. Salmon. Steiner's researches upon Cubic Surfaces are of later date" [Cayley 1889, p. 589]. This remark, formulated about 40 years after the initial publication of the twenty-seven-lines theorem, evidences that Cayley had probably been annoyed,

[30] On this journal, see [Crilly 2004; Despeaux 2014].

[31] "Steiner a fait connaître (*Journal de M. Borchardt*, t. LIII) les théorèmes suivants : *Toute surface du troisième degré contient vingt-sept droites.* [...]" [Jordan 1869a, p. 147].

[32] "MM. Cayley et Salmon avaient découvert et étudié ces droites avant Steiner." [Jordan 1870b, p. 665].

[33] "Entre autres, il y a une question qui excite au plus haut degré ma curiosité : celle du rapprochement de la recherche des 27 droites d'une surface cubique (qui ont été découvertes par MM. Cayley et Salmon, avant Steiner) avec la trisection des fonctions hyperelliptiques." Extract of a letter from Cremona to Jordan, dated 19 December 1869 and kept at the Archive of the École polytechnique (ref. VI2A2(1855) 9).

[34] Other examples illustrate this: Sylvester [1861] only referred to Cayley and Salmon, while Friedrich August [1862] only referred to Steiner, and Heinrich Schröter [1863] cited the three mathematicians.

during this time interval, that too many people attributed the first proof of the theorem to Steiner instead of him and Salmon.[35]

4. Conclusion

My analysis of Cayley's article could obviously be deepened and widened in several ways, by seeking to understand its past and its future on a broader scale, by scrutinizing technical aspects that I have gone over quickly, by trying to interpret it in relation with Cayley's naturalist practice in mathematics [Crilly 2006, pp. 193–195] or with his mathematical and writing style [Lorenat 2023]. And the same kind of questionnaire could be applied to each of Henderson's bibliographic references to get a richer history of the twenty-seven-lines theorem.

To take an example that I have dealt with elsewhere, the paragraph of the historical summary devoted to group theory simply mentions that "Jordan first proved [in 1869] that the group of the problem of the trisection of the hyperelliptic functions of the first order is isomorphic with the group of the equation of the twenty-seventh degree, on which the twenty-seven lines of the general surface of the third degree depend" before moving on to Felix Klein's 1887 work on the same question [Henderson 1911, p. 6]. As the reader should now suspect, Henderson's sentence conceals a much more complex situation: this research by Jordan was actually part of a collective activity (around 1870) involving mathematicians such as Clebsch, Klein and Max Noether, and aimed at using the geometry of finite configurations – such as the twenty-seven lines, but also the twenty-eight double tangents to quartic curves, the nine inflection points of cubic curves, the sixteen lines of some quartic surfaces, etc. – to assimilate the (Galois) theory of substitution groups, which they found difficult and abstract [Lê 2013, 2015a, 2016]. And in this case too, the division into thematic paragraphs proposed by Henderson does not stand up to closer examination of the texts associated with the point of view of group theory: links exist with the works of Steiner, with the research on the twenty-eight double tangents and even with the research on the models and shapes of cubic surfaces.

I have not spared Henderson in this chapter, my criticisms being essentially of three kinds. First, the instances of almost verbatim copying of sources and the problems associated with the addition of bibliographical references that have obviously not been opened by Henderson can only make us suspicious of the seriousness of his work. Second, the absence of any explanation of the extent of the corpus relating to the twenty-seven lines and the choices made

[35] Contrary to what is written in [Lê 2015b, p. 50], Steiner did not designate the twenty-seven lines as "Cayley's lines" ("*Cayley'sche Geraden*"). Schläfli did so, in letters to Steiner [Graf 1896, p. 125 sqq.]. Although I want to be prudent on this point, the fact that Schläfli used this expression in his letters to his good friend Steiner may suggest that the latter was not offended by the attribution to Cayley of the discovery of the twenty-seven lines.

in writing the historical narrative, whether they concern the selection of the contributions described or the structuring into topical paragraphs, encourage us to question the scope of this account. And third, the very historical writing, consisting in a sequence of references and facts that are neither thoroughly described nor contextualised, cannot be seen as satisfactory.

To be fair, Henderson's history of the twenty-seven lines was explicitly presented as a historical *summary* preceding a mathematical work, and not as an extended historical account on its own right – which did not prevent it from becoming the source of the usual history of the subject, probably in the absence of an alternative produced since.

Moreover this historical summary was written at a time when the history of mathematics was not the research discipline of today.[36] Thanks to the efforts of their predecessors and contemporaries, historians of mathematics now have more studies at their disposal to enrich the description of a given episode. They are also aware of the need for rigour, whether in the formation and clarification of a corpus or in the degree of precision of descriptions, in order to construct a narrative and evaluate its historical scope and significance. And they have shown that, more often than not, the history of a theorem or a concept cannot be reduced to a linear story, neatly arranged chronologically or thematically, as is the case for Henderson's summary.

This chapter is therefore an invitation to non-historians not to be satisfied with such overly obvious accounts, and at the same time to reflect on the past of mathematical objects and theorems, and on the possibilities of accounting for this past in all its complexity.

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[36] Of course, not all histories of mathematics written in the nineteenth or early twentieth centuries are of the kind Henderson wrote. For an overview on the evolutions of the historiography of mathematics, see [Dauben and Scriba 2002; Remmert, Schneider, and Kragh Sørensen 2016].

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