“Are the *genre* and the *Geschlecht* one and the same number?” An inquiry into Alfred Clebsch’s *Geschlecht*

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This story has been told many times. Around 1880, while he was seeking to clarify and continue Lazarus Fuchs’s work on linear differential equations, Henri Poincaré recognized the importance of some special complex functions which he proposed to call Fuchsian functions. On June 12 1881, after having read Poincaré’s first papers on these functions, [Poincaré 1881c], Felix Klein wrote him a letter (the first of a long series) where he explained the similarity between these papers and the research on elliptic functions and algebraic equations that he had done shortly before, [Klein 1879b,a,d,e,c, 1880]. In a second letter, Klein expounded his objections regarding the appellation “Fuchsian functions,” which triggered the famous dispute about the correct way to call these functions.[1]

Apart from this argument, the first letters between the two mathematicians contain many questions that Poincaré asked to Klein about technical points that were unclear to him. One of these questions, formulated on June 27 1881, was about the identity of two numbers and the meaning of a phrase found in the 1880 paper of the previous list, which happens to be the first one that Poincaré read:

> In my memoir on Fuchsian functions, I divided the Fuchsian groups according to various classification principles, and, among others, according to a number that I call their *genre*. Likewise you divide the *Untergruppen* according to a number that you call their *Geschlecht*. Are the *genre* (as I understand it) and the *Geschlecht* one and the same number? I could not know, for I do not know what the *Geschlecht im Sinne der Analysis situs* is.[2]

Poincaré’s other letters frequently involved German words and cited German publications, which suggests that the issue, here, was not an inability to understand this language. Rather, what caused trouble to Poincaré was the very definition of the *Geschlecht* “in the sense of analysis situs,” a definition that Klein provided in his answer of July 2 1881:

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[1] On Poincaré, Klein, and Fuchsian functions (now called automorphic functions), see [Freudenthal 1955; Dieudonné 1982; Gray 2000; Saint-Gervais 2016] for instance.

[2] “Dans mon mémoire sur les fonctions fuchsiennes, j’ai partagé les groupes fuchsiens d’après divers principes de classification et entre autres d’après un nombre que j’appelle leur genre. De même vous partagez les *Untergruppen* d’après un nombre que vous appelez leur *Geschlecht*. Le *genre* (tel que je l’entends) et le *Geschlecht* sont-ils un seul et même nombre ? Je n’ai pu le savoir, car je ne sais pas ce que c’est que le *Geschlecht im Sinne der Analysis situs*.” The correspondence between Poincaré and Klein has been first published in *Acta mathematica*, [Nörlund 1923].

> "*Geschlecht im Sinne der Analysis situs*" is attached to any closed surface. It is equal to the maximum number of sections which can be made along loops without cutting the surface into pieces. Now, if the surface in question can be considered as the image of the systems of values w, z of an algebraic equation $f(w, z) = 0$, its *Geschlecht* is also the *Geschlecht* of the equation. Thus your *genre* and my *Geschlecht* are *physically the same numbers*; it is probably just that I see a Riemann surface and the associated definition of p more freely.[3]

Klein thus attested that the *genre* and the *Geschlecht*, which are numbers associated with surfaces and Riemann surfaces on the one hand, and algebraic equations with two unknowns on the other hand, are the same—their common value is what Klein designated by p.

Geschlecht and *genre* refer to what is currently well-known as the *genus*, which designates several (related) numbers linked to a variety of objects, among which those listed by Klein: it can be the number of holes in a compact, connected, orientable, boundaryless (real) surface, the complex dimension of the vector space of holomorphic 1-forms on a compact Riemann surface, or the quantity $p = \frac{(n-1)(n-2)}{2}$ attached to a non-singular complex algebraic curve defined by an equation $f(x, y) = 0$ of degree n.[4]

Now, the historical literature unanimously attributes to Bernhard Riemann (1826–1866) the paternity of the notion of genus, a notion which he brought to light in his celebrated works on complex analysis of 1851 and 1857 where, among others, he studied Abelian functions and introduced the surfaces that now bear his name [Riemann 1851, 1857a,c,b,d]. The same literature often credits Alfred Clebsch (1833–1872) with having proposed the name *Geschlecht* for this notion in 1865.[5] Therefore, because it pertains to a notion and an appellation which had appeared many years before, Poincaré's question could be seen as highlighting a mere mathematical shortcoming, a definition that he would have missed in his training or his early readings.[6]

In fact, one of the results of the present paper is that it was absolutely not common for mathematicians to associate the word *Geschlecht* with objects like surfaces and Riemann surfaces in 1881, and that it was Klein himself (followed by a handful of students and correspondents) who began to do so in his aforementioned publications of 1879 and 1880.

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[3]"‚Geschlecht im Sinne der Analysis situs' wird jeder geschlossenen Fläche beigelegt. Dasselbe ist gleich der Maximalzahl solcher in sich zurückkehrender Schnitte der Fläche, die man ausführen kann, ohne die Fläche zu zerstücken. Wenn jetzt die betreffende Fläche als Bild der Werthsysteme w, z einer algebraischen Gleichung $f(w, z) = 0$ betrachtet werden kann, so ist ihr Geschlecht eben auch das Geschlecht der Gleichung. Ihr ‚genre' und mein ‚Geschlecht' sind also *materiell dieselben Zahlen*, es liegt bei mir nur vermuthlich eine freiere Auffassung der Riemann'schen Fläche und der auf sie gegründeten Definition von p zu Grunde." I warmly thank Norbert Schappacher for having helped me translate this quotation. Unless otherwise stated, the other translations in this paper are mine.

[4]From a current point of view, these three numbers are indeed the same, provided the right links between surfaces, Riemann surfaces, and algebraic curves are established. See [Griffith and Harris 1994, Chap. 2] for instance. The German and French words *Geschlecht* and *genre* which appear in the previous quotations of Poincaré and Klein are still, nowadays, the equivalents of "genus." As will be seen, however, English-speaking mathematicians of the nineteenth century did not actually use this word.

[5]See, among others, [Scholz 1980; Gray 1998; Laugwitz 1999; Houzel 2002; Chorlay 2007; Eckes 2011; Bottazzini and Gray 2013; Popoescu-Pampu 2016; Saint-Gervais 2016]. Such historical works being primarily focused on Riemann's research, they only roughly (if at all) analyze Clebsch's act of naming. Further, they frequently make use of the words "genus" and *Geschlecht* to describe Riemann's work, although they do not appear in it. This issue will be discussed in my conclusion.

[6]Poincaré's youthful mathematical gaps have already been observed by historians, as in [Gray 2000, p. 200].

On the contrary, surfaces and Riemann surfaces were usually characterized with Riemann's original terminology of "connectivity," whereas the term *Geschlecht* was, from its very introduction by Clebsch in 1865, mainly linked to objects of projective geometry such as algebraic curves.[7]

More generally, the aim here is to ascertain to what extent this new association could be perceived as problematic at the beginning of the 1880s. Two lines of inquiry will be followed. The first one is to come back to Clebsch's 1865 introduction of the *Geschlecht*, and to understand in minute detail its ins and outs. As will be seen, this word first designated a taxon in a certain classification of algebraic curves, a classification grounded on a certain reinterpretation of Riemann's 1857 research on Abelian functions, which included a definition of the "connectivity" of surfaces and Riemann surfaces. This reinterpretation will be scrutinized to fathom both the motives of Clebsch and the related technical execution, which will allow us to fully apprehend the differences between his and Riemann's works, in particular for what concerns their use of geometry. Our second line of inquiry is an examination of how the notions of genus and connectivity circulated during the period 1857–1882. To do so, a relevant corpus of about 240 published papers will be analyzed in terms of clusters of citations, of disciplinary classification, and of distribution of words like "Clebsch," "Riemann," "genus," and "connectivity." Among other outcomes, this analysis will confirm what has been announced above, namely that surfaces and Riemann surfaces were almost exclusively characterized with their connectivity, while the genus was mostly linked to algebraic curves in the tradition of Clebsch. The words "genus" and "connectivity," as well as their foreign equivalents, will appear as disciplinary markers during the considered period of time, in the sense that in most cases, mathematicians purposely assigned them to specific objects coming from projective geometry or from analysis and analysis situs, respectively. After a short description of the situation in textbooks published at the turn of the century, concluding remarks will be devoted to some of the historiographic issues related to such disciplinary markers.

Before delving into Clebsch's research on the *Geschlecht*, we begin by presenting the parts of Riemann's work where the Riemann surfaces and the notion of connectivity appeared. Since these elements have already been abundantly described in the historical literature, our attention will be concentrated on the specific points that Clebsch later revisited.[8]

1 Riemann and the connectivity of surfaces

Riemann introduced the surfaces that are now called after him as a means to study functions of the complex variable, and especially multivalued complex functions. He first exposed these ideas in his 1851 dissertation [Riemann 1851], and presented them again (with slight differences) in three short papers which served as preliminaries to his celebrated memoir on Abelian functions [Riemann 1857a,c,b,d]. Contrary to the dissertation, which has not been published in any journal and thus hardly circulated at the time, the three short papers

[7]There are two 1865 papers of Clebsch where the *Geschlecht* of curves was introduced, both published in the same volume of Crelle's *Journal für die reine und angewandte Mathematik*. The first one is devoted to rational algebraic curves [Clebsch 1865c], while the second one tackles the issue of counting of singularities on algebraic curves [Clebsch 1865a].

[8]See the previously cited works, especially [Scholz 1980; Bottazzini and Gray 2013].

and the memoir on Abelian functions of 1857 were published in *Journal für die reine und angewandte Mathematik*.[9] Correspondingly, Clebsch later cited and used the memoir on Abelian functions, so that our focus will lie on this memoir and on the parts of the preliminary papers which are useful to understand it.

1.1 Defining the connectivity

In the first of these preliminary papers, after having recalled that "Gauss's well-known geometric representation" of complex quantities $z = x + yi$ consisted in representing them as points of a plane with coordinates x, y, Riemann indicated that the only functions w of $x + yi$ that he would consider satisfy $i\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$, and that such functions can be either single-valued, or multivalued.[10] Multivalued functions, he continued, possess in general several branches, which merge at the so-called *Verzweigungspunkte* (branch points, or ramification points). The algebraic functions, which are functions s of the complex variable z implicitly defined by an algebraic equation $F(s, z) = 0$, are, in general, multivalued functions, just as Abelian integrals, which are primitive functions $\int f(s, z)\,\mathrm{d}z$, where s is an algebraic function of z, and f is a rational function of two variables.[11]

For the sake of clarity, let us consider here one example, which is not to be found in Riemann's papers. The polynomial $F(s, z) = s^2 - z^2(z - 1)(z - 2)$ defines the algebraic function $s = z\sqrt{(z-1)(z-2)}$, which is multivalued because of the square roots: with each z are generally associated two opposite values of s. Further, the points $z = 1$ and $z = 2$ are two branch points, since the two corresponding values of s are equal in each case. The algebraic function (or, equivalently, the equation $F(s, z) = 0$) then allows to define Abelian functions, like $\int \frac{\mathrm{d}z}{z\sqrt{(z-1)(z-2)}}$ and $\int z^3\sqrt{(z-1)(z-2)}\,\mathrm{d}z$, which respectively correspond to the cases $f(s, z) = 1/s$ and $f(s, z) = sz^2$.

Riemann insisted on the fact that the surfaces which he was about to introduce, and which were later called Riemann surfaces, were a "geometric" way to represent the ramification of multivalued functions, in particular algebraic and Abelian functions: "In many investigations, notably in the study of algebraic and Abelian functions, it is advantageous to represent the branching of a multivalued function geometrically in the following way."[12] This manner consisted in defining, for a given multivalued function, a surface made of several infinitely thin sheets spread over the complex plane—one sheet for each branch of the function—and connected together at the branch points. As a result, Riemann added, functions which are

[9]See [Bottazzini and Gray 2013, pp. 277–279]. The dissertation was eventually included in Riemann's complete works, first published in 1876 under the supervision of Richard Dedekind and Heinrich Weber, [Riemann 1876]. In this edition, the four 1857 papers were gathered as a single article entitled "Theorie der Abel'schen Functionen."

[10]In modern parlance, the considered (single-valued) functions are holomorphic functions, their differentiability as real functions being implicitly assumed.

[11]In the image of Clebsch's usage (see below), mathematicians of the second half of the nineteenth century frequently used the phrases "Abelian integrals" and "Abelian functions" in an interchangeable way.

[12]"Für manche Untersuchungen, namentlich für die Untersuchung algebraischer und Abel'scher Functionen, ist es vortheilhaft, die Verzweigungsart einer mehrwerthigen Function in folgender Weise geometrisch darzustellen." [Riemann 1857a, p. 103]. The present translation is borrowed from [Riemann 2004, p. 81]. Moreover, the issues of visualization of branch points in the works of Riemann (among others) have been recently investigated in [Friedman 2019].

multivalued on the complex plane become single-valued when considered as defined on their associated surface.

These surfaces were then investigated from the point of view of analysis situs in the second preliminary paper, entitled "Theorems, from analysis situs, for the theory of the integrals of total differentials with two terms."[13] Starting from considerations about the values of integrals of functions defined on a "Riemann surface,"[14] Riemann proposed to distinguish the simply connected surfaces from the multiply connected ones:

> This leads to a differentiation of surfaces between the simply connected ones, in which each closed curve completely limits a part of the surface—such as a circle—, and the multiply connected ones, for which this does not happen—such as a ring limited by two concentric circles.[15] [Riemann 1857c, p. 106]

Riemann then refined this division by defining a surface to be $(n + 1)$-ply connected if there exist n closed curves on it which do not limit any part of the surface when considered either individually or collectively, but which do when taken together with any other closed curve on the surface.[16] The number $n + 1$ thus defined was called the order of connectivity of the surface.

Riemann showed that these notions could also be apprehended by considering cross-cuts, *i.e.* lines drawn on the surface and connecting two points of its boundary, or, in the case of a surface without boundary, lines starting from, and ending at a given point on the surface: from this viewpoint, he explained, a surface is simply connected if any cross-cut divides it into two pieces, and a $(n + 1)$-ply connected surface can be transformed into a simply connected one by operating n adequate cross-cuts.

Although he started from considerations on Riemann surfaces, Riemann made clear that the ideas and results related to the connectivity hold for "surfaces lying arbitrarily in space,"[17] which he exemplified, a few pages after (p. 108), with the triply connected surface of a torus. Four examples of Riemann surfaces (including one with a part consisting in two sheets) were also provided, together with drawings, to illustrate the notion of connectivity (see Figure 1).

[13]"Lehrsätze aus der analysis situs für die Theorie der Integrale von zweigliedrigen vollständigen Differentialen," [Riemann 1857c].

[14]Although Riemann obviously did not employ such an expression, I shall use it here to avoid conflating the surfaces with several sheets over the complex plane (the Riemann surfaces) and the usual surfaces in space, which were also considered by Riemann (see below). Such a form of anachronism will be discussed in the conclusion.

[15]"Dies veranlasst zu einer Unterscheidung der Flächen in einfach zusammenhangende, in welchen jede geschlossene Curve einen Theil der Fläche vollständig begrenzt — wie z. B. ein Kreis —, und mehrfach zusammenhangende, für welche dies nicht stattfindet, — wie z. B. eine durch zwei concentrische Kreise begrenzte Ringfläche." The "circle" given by Riemann as an example of a simply connected surface refers to what we would call a disk. Moreover, as Ralf Krömer pointed out to me, Riemann's systematic use of "zusammenhangend" instead of "zusammenhängend" is quite puzzling. As will be seen in the different following quotations, the other German-speaking mathematicians of the nineteenth century would commonly use the spelling "zusammenhängend."

[16]As is noted in [Bottazzini and Gray 2013, p. 287], Riemann's proof that this definition does not depend on the choice of the n closed curves was defective, and would be corrected by Alberto Tonelli in 1875.

[17]"[Diese Sätze] gelten für beliebig im Raume liegende Flächen." [Riemann 1857c, p. 106].

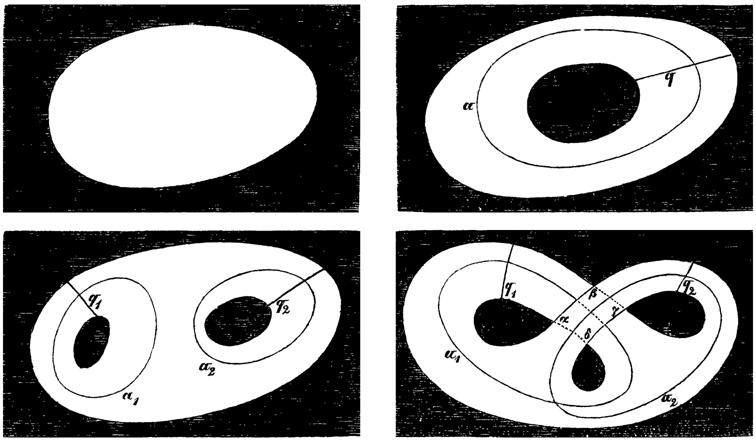

Figure 1 – The four (and only) drawings in [Riemann 1857c, p. 110]. On the first line, the surface on the left is simply connected, while that on the right is doubly connected. The two surfaces on the second line are triply connected.

1.2 Surfaces associated to algebraic functions

We now arrive to the memoir on Abelian functions [Riemann 1857d].[18] Once again, the reader is advised that the focus will be on the elements which would be later re-taken by Clebsch in the works that led to the notion of genus. Therefore we will not discuss the introduction of the θ-functions with p variables and their use in the solution of the so-called Jacobi inversion problem, to which the second part of Riemann's memoir was devoted; similarly, what relates to the so-called Riemann inequality (later completed into the Riemann-Roch theorem) or to the counting of the modules of classes of equations will not be mentioned here.[19]

In the first part of the memoir, Riemann remarked that a surface spread over the whole complex plane could be seen either as a surface with a boundary situated at infinity or as a closed surface, provided the point $z = \infty$ is added to it. He then proved that the order of connectivity of such a closed surface is necessarily an odd number $2p + 1$. In particular, this is the case for a Riemann surface associated to an algebraic function, and several sections of the memoir were devoted to prove a formula expressing the number p with the help of some of the characteristics of the considered algebraic function.

Specifically, Riemann took an algebraic function s defined by an equation $F(s, z) = 0$ of degree n in s, and he denoted by w the number of its branch points.[20] This number, Riemann stated, equals the number of pairs (s, z) such that $F(s, z) = \frac{\partial F}{\partial s}(s, z) = 0$ and $\frac{\partial F}{\partial z}(s, z) \neq 0$.[21]

[18]The third preliminary paper concerned the use of the Dirichlet principle to define functions on Riemann surfaces. It will not be discussed here, since Clebsch did not use it in his works on the genus.

[19]About the Riemann-Roch theorem, see [Gray 1998].

[20]An hypothesis that appeared in the middle of a proof is that the coefficients of F, seen as a polynomial in s, are polynomials of z with the same degree.

[21]The condition $F = \frac{\partial F}{\partial s} = 0$ detects the values of z_0 for which the polynomial $F(s, z_0)$ has multiple roots. This corresponds to the values z_0 where branches of the function s merge, which is the definition of a branch point. Riemann remarked that if, additionally, $\frac{\partial F}{\partial z} = 0$, the branches of the function actually do not become equal. In our example of $F(s, z) = s^2 - z^2(z-1)(z-2)$, the values of (s, z) satisfying $F = \frac{\partial F}{\partial s} = 0$ are $(0, 0)$, $(0, 1)$ and $(0, 2)$. The two latter are such that $\frac{\partial F}{\partial z} \neq 0$, contrary to $(0, 0)$, which, indeed, is not a branch point. Roughly speaking, this phenomenon is to be linked with the exponent 2 of z in F, which makes the function

Then, assuming that $\frac{\partial^2 F}{\partial s \partial z} - \frac{\partial^2 F}{\partial s^2}\frac{\partial^2 F}{\partial z^2} \neq 0$ for each couple (s, z) where $F = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial z} = 0$, he proved that

$$w - 2n = 2(p - 1). \tag{1}$$

This formula would be central Clebsch's later reinterpretation, but it was not Riemann's final one: denoting by m the degree of F in z, and by r the number of pairs (s, z) such that $F = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial z} = 0$, Riemann demonstrated that $w + 2r = 2(n - 1)m$, which, combined with the equation (1), yielded $p = (n - 1)(m - 1) - r$.

Another important result related to the number p involved the Abelian integrals of the first kind, which are the Abelian integrals (associated with a given equation $F = 0$) taking finite values everywhere on the corresponding Riemann surface. After having proved that these integrals can be written in the form

$$\int \frac{\varphi(s, z)\, \mathrm{d}z}{\frac{\partial F}{\partial s}},$$

where φ is a polynomial of degrees $n - 1$ in s and $m - 1$ in z which vanishes at the pairs (s, z) such that $F = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial z} = 0$, Riemann used the expression $p = (n - 1)(m - 1) - r$ to demonstrate that p equals the number of independent integrals of the first kind.[22]

Riemann then showed that if two equations $F(s, z) = 0$ and $F_1(s_1, z_1) = 0$ can be transformed one into the other by a rational transformation, the corresponding numbers p and p_1 are the same. This allowed him to propose a new classification of algebraic equations:

> One considers now *all irreducible algebraic equations between two variable quantities which can be transformed into one another by rational substitutions,* as belonging to one *class*[23] [Riemann 1857d, p. 133]

Interestingly, this notion of class of algebraic equations was one of the results which were explicitly stated and commented in the introduction of the memoir, which indicates that Riemann saw it as an important outcome of his research; he also suggested that it could be useful for "other investigations,"[24] yet without making explicit what he meant.

We finally observe that Riemann also interpreted in his own way the famous theorem of Abel on the addition of Abelian integrals; essentially, this theorem states that any sum of Abelian integrals (defined by the same grounding equation $F = 0$) can be expressed as (the opposite of) a sum of p integrals, whose upper bounds depend algebraically on the upper bounds of the integrals of the given sum.[25] However, although Clebsch would make an important use of this theorem in his research, he did not use Riemann's contribution, which will hence be left out here.

$\sqrt{z^2} = z$ single-valued.

[22]In modern terms, this result states that p is the dimension of the complex space of the holomorphic 1-forms on the given (compact) Riemann surface.

[23]"Man betrachte nun als zu einer *Klasse* gehörend *alle irreductiblen algebraischen Gleichungen zwischen veränderlichen Grössen, welche sich durch rationale Substitutionen in einander transformiren lassen* [...]." The proposed translation comes in part from [Riemann 2004, p. 111].

[24]"Der bei dieser Untersuchung sich darbietende Begriff einer *Klasse* von algebraische Gleichungen [...] dürfte auch für andere Untersuchungen wichtig [...] sein." [Riemann 1857d, p. 115].

[25]About Abel's theorem, see [Gray 1989, pp. 364–366] or [Houzel 2002, pp. 152–158].

1.3 Riemann's non-algebraic geometry

Our (sketchy) description of Riemann's research being done, it is useful for our purpose to discuss how geometry occurs in it, especially because a comparison with Clebsch's use of geometry will be made in the next section.

While the noun *Geometrie* is actually absent from Riemann's work on Abelian functions, the adjective *geometrisch* appears twice, in two passages which have been quoted above and which deal with specific "representations": that of the complex numbers as points of the plane on the one hand, and that of the mode of ramification of complex functions by means of Riemann surfaces on the other hand. These surfaces were thus seen as geometric by Riemann, but in a certain sense: as attests the title of the second preliminary note, the properties of Riemann surfaces developed therein—among which those related to the notion of connectivity—belonged to analysis situs.[26] In particular, this is reflected in the fact that the curves which were considered by Riemann are cross-cuts and loops, and that their relevant characteristics are (in modern parlance) topological.

By contrast, it is important to note that there is no algebraic curve in Riemann's memoir. When dealing with equations like $F(s, z) = 0$, Riemann only talked about algebraic equations, and never interpreted them as equations defining algebraic curves. Further, his proofs did not imply any other object related to algebraic curves, like tangents or cusps. As for the new classification stemming from the notion of birational equivalence, it concerned classes of equations (or of algebraic functions), and not of algebraic curves.

From this point of view, it seems clear that using the vocabulary of the theory of algebraic curves to describe Riemann's memoir would be dangerously misleading.[27] This observation, of course, rests on our analysis of what has been published, and not on what Riemann could have actually thought: for instance, in his history of mathematics, Klein asserted that "from the beginning, Riemann had a very good appreciation of the significance of his theories for algebraic geometry."[28] However, Detlef Laugwitz rightly highlighted that such an affirmation rested on "hindsight interpretation of [Riemann's] publications." [Laugwitz 1999, p. 140]. What should be added is that even if Riemann did think of algebraic curves when he developed his work on Abelian functions, the words that he chose for the publication prove that he did not want to present it in this perspective. Extracting notions and results from this memoir, importing them into other parts of mathematics, recognizing them as elements of something called "algebraic geometry," or interpreting them as related to algebraic curves would be done by other mathematicians, among whom Clebsch played a crucial role.

2 The *Geschlechter* of algebraic curves by Clebsch

As already written, Clebsch defined the notion of genus of algebraic curves in two 1865 papers. These two papers, which are mathematically independant from one another, belong to a series

[26]Let us recall that in the *Jahrbuch über die Fortschritte der Mathematik*, for instance, the section entitled "Reine, elementare und synthetische Geometrie" contained a subsection devoted to "Continuitätsbetrachtungen (Analysis situs)" from 1868 to 1916. The word "Topologie" was added in the parenthesis from 1890 on.

[27]The same remarks can actually be made when considering all the articles composing the collected works of Riemann, [Riemann 1876].

[28]"Riemann hat die Bedeutung seiner Theorien für algebraische Geometrie von Anfang sehr wohl erkannt." [Klein 1926, p. 296]. The translation is borrowed from [Laugwitz 1999, p. 139].

of publications of 1864–1866 pertaining to the theories of elliptic and Abelian functions and their possible applications to geometry, [Clebsch 1864b,a, 1865c,a,b; Clebsch and Gordan 1866a,b]. Among these texts, the 1864 memoir entitled "Ueber die Anwendung der Abelschen Functionen in der Geometrie" is of special importance for our investigation [Clebsch 1864a]. This memoir, which served as a technical basis for the 1865 papers where the notion of genus appeared, turns out to be the first of Clebsch's publications where Riemann is cited, and indeed for his 1857 memoir on Abelian functions.[29]

Accordingly, we begin by describing Clebsch's 1864 memoir on the application of Abelian functions to geometry, with a focus on what relates to his reappropriation of Riemann's research.

2.1 A homogenization of Riemann

In the introduction of this memoir, Clebsch first asserted that many applications of the theory of elliptic functions to geometry were known, and that his aim was to present analogous applications of the theory of Abelian functions, to which he immediately associated the name of Riemann.[30] Although Riemann's works had been known for several years, he continued, such applications were still lacking, probably because of the great difficulty of these works—Clebsch himself had struggled to understand Riemann's research on Abelian functions.[31] Furthermore, as will be seen, when Clebsch and Paul Gordan shortly later wrote their book on the theory of Abelian functions [Clebsch and Gordan 1866b], they described their approach as a simplification of that of Riemann, a simplification built in particular on the introduction of algebraic curves as Clebsch had precisely done in the 1864 memoir on Abelian functions. The approach developed in this memoir is thus to be seen both as a way to apply the theory of Abelian functions to geometry, and as the trace of how Clebsch used geometry to apprehend this theory.

Algebraic curves appeared in the very first technical step of Clebsch, which consisted in establishing a certain link between curves of order n and classes of Abelian functions. To do so, he recalled that "each algebraic equation $F(s, z) = 0$, thanks to which s is determined as a function of z, founds, according to *Herr* Riemann, a class of Abelian integrals,"[32] with which the number p of independent integrals of the first kind is associated. Citing explicitly Riemann's 1857 paper, he then recalled that this number p can be computed thanks to the formula

$$p = \frac{w}{2} - (n - 1), \tag{2}$$

[29]More generally, all the mentions of Riemann in Clebsch's published papers refer, either explicitly or implicitly, to this memoir of 1857.

[30]Clebsch did not cite any specific paper where elliptic functions would have been used to prove geometric theorems, but he vaguely mentioned the fact that Steiner had foreseen such theorems without the help of these functions. This allusion is probably to be linked with [Clebsch 1864b], a paper which was somewhat presented as a prequel of the memoir on Abelian functions (see [Clebsch 1864b, p. 105]), and where Clebsch demonstrated a theorem about cubic curves stated by Steiner in 1846 by first proving that such (non-singular) curves can be parameterized by elliptic functions. This paper of Clebsch is analyzed in [Lê 2018a].

[31]See [Brill and Noether 1894, p. 320], where the authors mention a letter of 1864 to Gustav Roch, where Clebsch would have expressed his difficulties to become acquainted with this research. See also [Laugwitz 1999, p. 140].

[32]"Jede algebraische Gleichung $F(s, z) = 0$, vermöge deren s als Function von z bestimmt ist, begründet nach Herrn Riemann eine Classe von Abelschen Integralen." [Clebsch 1864a, p. 190].

where n is the degree in s in the equation $F(s,z)=0$, and w is the number of pairs (s,z) such that $F(z,s)=\frac{\partial F}{\partial s}(z,s)=0$ and $\frac{\partial F}{\partial z}(z,s)\neq 0$. Further, following Riemann, he supposed that if a pair (s,z) makes F, $\frac{\partial F}{\partial s}$, and $\frac{\partial F}{\partial z}$ vanish, then it does not make $\frac{\partial^2 F}{\partial s\partial z}-\frac{\partial^2 F}{\partial s^2}\frac{\partial^2 F}{\partial z^2}$ vanish.

Clebsch then performed a first homogenization, as he explained that Riemann's equation $F(s,z)=0$ could be replaced by another kind of equation, which would be more "convenient" for the geometric applications:

> The form of the equation $F(s,z)=0$ which is taken as the basis by *Herr* Riemann is such that if the equation is ordered by the powers of s, each of its coefficients is of the same order in z. For the geometric application, it is convenient to consider another form as the general basic form, namely that which can be transformed into a homogeneous function of the nth order of the three variables x_1, x_2, x_3 by substituting $-\dfrac{a_1x_1+a_2x_2+a_3x_3}{b_1x_1+b_2x_2+b_3x_3}$ for z and $-\dfrac{\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}{\beta_1x_1+\beta_2x_2+\beta_3x_3}$ for s, the a, b, α, β being arbitrary constants.[33] [Clebsch 1864a, p. 191]

Clebsch denoted $f(x_1,x_2,x_3)$ the homogeneous function in question, and, after having explicitly interpreted $f(x_1,x_2,x_3)=0$ as the equation of a curve of the nth order, he asserted that its reduction to the form $F(s,z)=0$ with the help of

$$\begin{cases}(a_1x_1+a_2x_2+a_3x_3)+s(b_1x_1+b_2x_2+b_3x_3)=0\\(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)+z(\beta_1x_1+\beta_2x_2+\beta_3x_3)=0\end{cases}\tag{3}$$

corresponds to "the representation of the curve as the intersection of corresponding rays of the pencils (3), where s, z are the variable parameters."[34]

Since Clebsch neither provided any detail on what he meant here, nor gave any bibliographic reference, we propose the following explanations (which include the drawings of Fig. 2) to help our reader understand what is at stake.[35] For a given complex number s, the first equation of the system (3) represents a line in the plane, say D_s; thus, when considering every possible value of s, this equation represents a pencil of lines (or of rays, to borrow Clebsch's wording) parameterized by s, that is, a family made of all the lines passing through one given point. Similarly, the second equation also represents a pencil of lines Δ_z, with parameter z. The idea, then, is to define a correspondence between the two pencils thanks to an equation $F(s,z)=0$. Specifically, to each given s correspond a number of z's, namely the solutions of $F(s,z)=0$, so that to each line D_s of the first pencil correspond several lines Δ_z

[33]"Herr Riemann legt die Gleichung $F(s,z)=0$ zu Grunde in der Form, dass, wenn man die Gleichung nach Potenzen von s ordnet, jeder Coefficient von gleich hoher Ordnung für z sei. Für die geometrische Anwendung ist es zweckmässig, eine andere Form als allgemeine Grundform zu betrachten, nämlich diejenige, welche durch Substitution von $-\frac{a_1x_1+a_2x_2+a_3x_3}{b_1x_1+b_2x_2+b_3x_3}$ für z, $-\frac{\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}{\beta_1x_1+\beta_2x_2+\beta_3x_3}$ für s in eine homogene Function n^{ter} Ordnung der drei Veränderlichen x_1, x_2, x_3 übergeführt werden kann, wo die a, b, α, β beliebige Constante bezeichnen."

[34]"[...] entspricht dann der Darstellung der Curve als Durchschnitt entsprechender Strahlen der Büschel (3.), in denen s, z die veränderlichen Parameter bilden." [Clebsch 1864a, p. 191].

[35]It is difficult to guess what could have been the references of Clebsch here. In the section of the *Encyklopädie der mathematischen Wissenschaften* devoted to the generations of curves, Luigi Berzolari [1906, pp. 353–358] made the story of this topic begin with some works of Isaac Newton and Colin Maclaurin. The cited nineteenth-century predecessors of Clebsch (principally Steiner, Hermann Grassmann, Michel Chasles, and Ernest De Jonquières) seem to have considered the problem of generating curves as the intersection of two pencils of (higher) curves in a single-valued correspondence, whereas Clebsch proposed to generate a curve as the intersection of two pencils of lines in a multivalued correspondence.

of the second pencil. Reciprocally, to each line of the second pencil correspond several lines of the first pencil. Such a correspondence between the pencils having thus been defined, one considers the intersection points of every corresponding lines. What Clebsch meant in the previous quotation is that these points of intersection form an algebraic curve, and that it is possible to find an adequate equation $F(s, z) = 0$ such that this algebraic curve is the given one, defined by $f(x_1, x_2, x_3) = 0$. Furthermore, an important point (which remained quite vague in Clebsch's explanations) is that it is possible to choose an equation $F(s, z) = 0$ of degree n in s.[36]

Clebsch then sought for an expression of the number w depending on the order n of the curve $f(x_1, x_2, x_3) = 0$. He first remarked that if a pair (s, z) satisfies $F(s, z) = \frac{\partial F}{\partial s}(s, z) = 0$ and $\frac{\partial F}{\partial z}(s, z) \neq 0$, the line D_s of the first pencil is to be counted as double. This, Clebsch claimed, meant that D_s is a tangent of the curve $f = 0$. The case where (s, z) annihilates F, $\frac{\partial F}{\partial s}$, and $\frac{\partial F}{\partial z}$ corresponds to the case where the curve $f = 0$ has a double point (which is the intersection of D_s and Δ_z). Finally, the condition $\frac{\partial^2 F}{\partial s \partial z} - \frac{\partial^2 F}{\partial s^2}\frac{\partial^2 F}{\partial z^2} \neq 0$ imply that all the possible double points are ordinary, *i.e.* they are points of the curve where there are two distinct tangents.

Denoting by d the number of double points of the curve, Clebsch concluded that "according to a known formula for the number of tangents" [Clebsch 1864a, p. 192], one has

$$w = n(n-1) - 2d. \tag{4}$$

Again, no bibliographic reference was given by Clebsch. This number w, called the "class" of the curve $f = 0$, counts the number of tangents to the curve which can be drawn from a given point situated outside the curve, and the formula (4) had been established in some works of Julius Plücker published in the mid-1830s, [Plücker 1834, 1835].[37] It is actually a special case of the first of the four so-called Plücker formulas, which relate the order, the class, and the numbers of inflection points, cusps, double tangents, and double points of an algebraic curve: in its complete form, the first Plücker formula is $w = n(n-1) - 2d - 3r$, where r is the number of cusps. In the case treated by Clebsch, the hypotheses made on F imply that $r = 0$, which gives the formula (4). That it was well-known for Clebsch and the readers of his time can be measured by the fact that it was included in George Salmon's influential book on higher curves, [Salmon 1852, p. 63]—Clebsch actually cited this book as a reference for the Plücker formulas in the paper [Clebsch 1865a], which will be discussed in the next subsection.

Comparing the formulas (1) and (4) eventually yielded

$$p = \frac{(n-1)(n-2)}{2} - d.$$

[36]This point is highlighted in [Brill and Noether 1894, pp. 320–321]. As is made clear in this reference, starting from a general equation $F(s, z) = 0$ of degree n in s and then considering the intersection of the pencils (3) leads to an algebraic curve of degree greater than n and with higher singularities. Clebsch's approach goes the other way around: starting from a general curve of order n defined by $f(x_1, x_2, x_3) = 0$, it is possible to describe it as the intersection of two pencils of lines, the correspondence of which is defined by an adequate equation $F(s, z) = 0$ of degree n in s.

[37]At the beginning of the nineteenth century, Jean-Victor Poncelet had stated that the number of tangents that can be drawn from a point outside a given curve of order n was at most $n(n-1)$ [Poncelet 1817/1818]. The word "class" is due to Joseph-Diez Gergonne, who, a decade later, defined a curve to be of the mth class if m tangents can be traced through a given point, [Gergonne 1827]. These works of Poncelet and Gergonne are linked to the duality controversy. See [Lorenat 2015] and the references given therein.

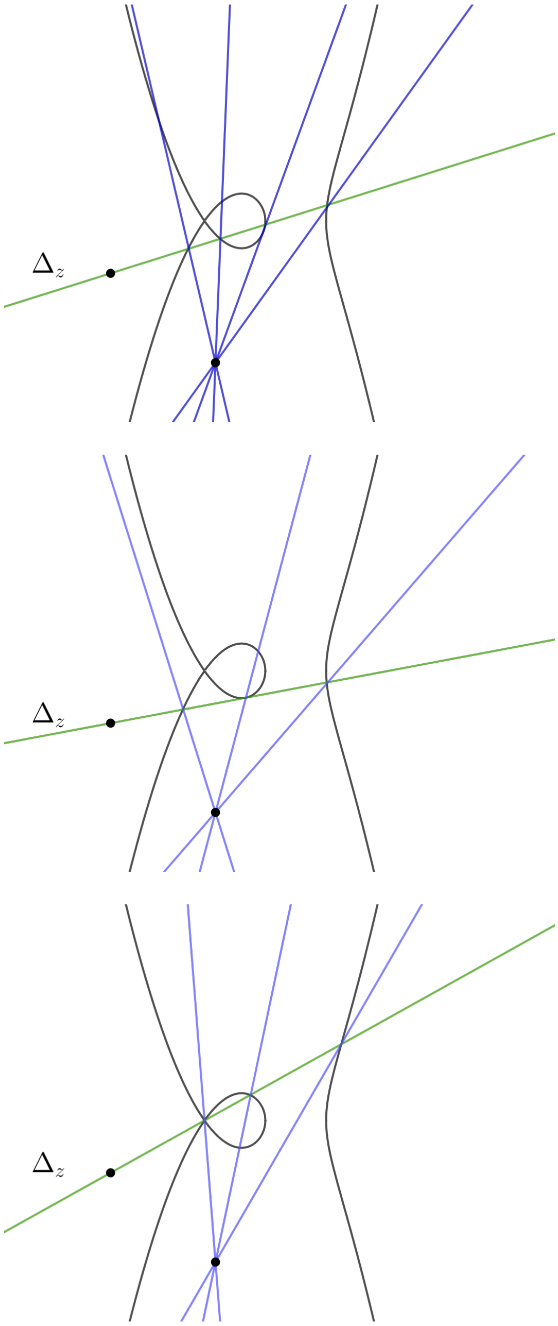

Figure 2 – Generation of an algebraic curve (in black) as the intersection of two pencils of lines in a multivalued correspondence. On the top, four blue lines D correspond to the green one Δ_z, and the intersection points are points of the generated curve. In the middle, two of the blue lines are coincident, which corresponds to the case where Δ_z is a tangent. On the bottom, two of the blue lines are also coincident, but correspond to a double point of the curve. These drawings are mine (apart from [Clebsch 1873], the publications of Clebsch considered in the present paper do not contain any diagram).

This number p, which would later be called the genus of the curve $f = 0$, was here qualified as "the number which gives the class of the Abelian functions associated with a curve of the nth order" [Clebsch 1864a, p. 192]. Clebsch then presented examples of curves corresponding to the first positive values of p:

> The elliptic functions ($p = 1$) thus correspond to the general curves of the third order, to the curves of the fourth order with two double points, etc.; the first class of Abelian transcendents corresponds to the curves of the fourth order with one double point, or to those of the fifth order with four double points, etc.; the following class ($p = 3$) corresponds to the general curves of the fourth order, etc.[38] [Clebsch 1864a, p. 192]

It is precisely this link between Abelian functions and algebraic curves, expressed through the number p, which would later serve as the basis for the definition of the notion of genus. In a footnote, Clebsch added allusively that what he had done can be extended to the case where the considered curve has cusps. He did not make it more precise here, but in [Clebsch 1865a], he stated (without proof) that in this case, one has

$$p = \frac{(n-1)(n-2)}{2} - d - r,$$

where r is the number of cusps; a demonstration was provided in the book written with Gordan, [Clebsch and Gordan 1866b, p. 15].

The next theorem proved by Clebsch was of the utmost importance in the memoir, being the one which allowed him to derive new results on algebraic curves. Clebsch presented it as an application of Abel's theorem about the sum of Abelian integrals to the study of the intersection of curves: a non-singular curve C of order n and nm points on it being given, Clebsch's theorem gives necessary and sufficient conditions for these points to belong to one curve of order m; more precisely, these conditions bear on the values taken by the p integrals of the first kind associated to C in each of the mn points. Interestingly, before beginning the proof of this theorem, Clebsch emphasized once more the importance of considering homogeneous objects: "By taking as a basis homogeneous equations instead of those used by Abel, the Abelian theorem (volume 4, p. 200, of this journal) can be expressed in a way which leads to important consequences"[39] [Clebsch 1864a, p. 193].

The proof actually implied yet another kind of homogeneous objects. Indeed, Clebsch insisted on the fact that "according to Riemann" and thanks to the equalities

$$\frac{x_2 \mathrm{d}x_3 - x_3 \mathrm{d}x_2}{\frac{\partial f}{\partial x_1}} = \frac{x_3 \mathrm{d}x_1 - x_1 \mathrm{d}x_3}{\frac{\partial f}{\partial x_2}} = \frac{x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1}{\frac{\partial f}{\partial x_3}},$$

the Abelian integrals of the first kind that are associated with the curve $f(x_1, x_2, x_3) = 0$ can be written in the form

$$\int \Theta \cdot \frac{\sum \pm c_1 x_2 \mathrm{d}x_3}{c_1 \frac{\partial f}{\partial x_1} + c_2 \frac{\partial f}{\partial x_2} + c_3 \frac{\partial f}{\partial x_3}},$$

[38] "So entsprechen die elliptischen Functionen ($p = 1$) den allgemeinen Curven dritter Ordnung, den Curven vierter Ordnung mit zwei Doppelpunkten etc.; die erste Classe der Abelschen Transcendenten den Curven vierter Ordnung mit einem Doppelpunkt, oder denen fünfter Ordnung mit vier Doppelpunkten etc.: die folgende Classe ($p = 3$) den allgemeinen Curven vierter Ordnung etc."

[39] "Der Abelsche Satz (Band 4 pag. 200 dieses Journals) kann nun, mit Zugrundelegung homogener Gleichungen statt der von Abel benutzten, in einer Weise ausgesprochen werden, welche wichtige Folgerungen gestattet."

where c_1, c_2, c_3 are arbitrary complex numbers, and Θ is a homogeneous polynomial of order $n-3$ in x_1, x_2, x_3 that vanishes at every double point of the curve—as Clebsch proved in [Clebsch 1864a, pp. 195–196] and [Clebsch and Gordan 1866b, pp. 14–15], such a polynomial is determined by $\frac{(n-1)(n-2)}{2} - d = p$ coefficients. The homogeneous form of the integrals of the first kind was deduced from the form given in 1857 by Riemann, namely

$$\int \frac{\varphi(s, z)\,\mathrm{d}z}{\frac{\partial F}{\partial s}}.$$

Just as equations, Abelian integrals thus underwent a special process of homogenization before being used in Clebsch's proofs.[40]

The geometric applications of Clebsch's version of the addition theorem consisted in a variety of situations, which included many theorems on the enumeration of curves satisfying given conditions. For instance, Clebsch proved that if a curve of order n is given, and if one considers positive integers m, r such that $mn - pr \geqslant 0$ and $m \geqslant n-2$, then there exists r^{2p} curves of order m having a contact of order r with the given curve, in p given points.

Let us finally observe that Clebsch also showed how the theory of Abelian functions could be used to study space curves:[41] similarly to the planar case, Clebsch began by bridging the consideration of such a curve to that of two corresponding pencils of planes. This helped him prove that the number p can be expressed by $p = \frac{(k-1)(k-2)}{2} - d$, where k is the order of the space curve and d is the number of lines drawn from a given point and meeting the curve in two points. Clebsch also demonstrated a theorem giving conditions for points on a space curve to lie on one and the same algebraic surface—here again, Abelian integrals were written homogeneously written before being used in the proofs.

The number p thus played a central role in the 1864 memoir on the application of Abelian functions to geometry, and this role possessed several interrelated facets. If he first presented it as the number of Abelian integrals of the first kind associated with a given curve, thus preserving Riemann's point of view, Clebsch then proved the fundamental formula $p = \frac{(n-1)(n-2)}{2} - d$, which can be seen as a way to compute of p within the new framework of algebraic curves. This number then repeatedly occurred in Clebsch's version of the theorem of Abel and its applications to the enumeration of special curves, which emphasized its Abelian character. However, even if this Abelian aspect of p reappeared here and there in the subsequent publications of Clebsch (including in one of the two 1865 articles where the genus was defined), it somewhat withdrew to the profit of the direct link with the singularities of a curve and the property invariance by birational transformations, which was emphasized in the other article of 1865.

2.2 Introducing the *Geschlechter*

The genus, as defined in 1865, was introduced as a new way to classify algebraic curves: as attest the next quotations, the word *Geschlecht* employed by Clebsch did not designate

[40]Homogeneous integrals had already been considered in a 1862 paper of Siegfried Aronhold, [Aronhold 1862], in the case of elliptic integrals. This paper, which was cited by Clebsch in the 1864 memoir we are studying here, was central in the latter's research on elliptic parameterization of cubic curves published in [Clebsch 1864b]. See [Lê 2018a, pp. 12–18].

[41]Clebsch only considered space curves which are the complete intersections of two algebraic surfaces.

the number p as it is the case nowadays, but referred to the collection of all the curves having the same number p, and thus associated with the same class of Abelian functions.[42] This employment echoes other classifications of curves that had been proposed before Clebsch, where the same vocabulary of *Geschlecht*, together with *Ordnung* and *Classe*, designated the different taxa.[43] To take but one (German) example from the nineteenth century, Plücker [1835, p. v] used the word *Geschlechter* to refer to taxa of curves of the third order that had been called *genera* (in Latin) and *genres* (in French) by Leonhard Euler and Gabriel Cramer in their celebrated books of the mid-eighteenth century, [Euler 1748; Cramer 1750].[44] Clebsch thus took an existing term attached to the classificatory terminology, and attributed to it a new technical meaning to it; moreover, he proposed to subvert the old hierarchy where the orders were the top divisions.

Indeed, in the article [Clebsch 1865c], devoted to the study of curves admitting a rational parameterization, Clebsch began by recalling that "the class of the Abelian functions that are linked to an algebraic plane curve of the nth order is determined by the number $p = \frac{n-1.n-2}{2}$" if the curve has no singular point, and that this number has to be diminished by the number of double points and cusps, if there are any. The new classification was then described as follows:

> Instead of classifying the algebraic curves in orders, and making subdivisions in them according to the number of double points and cusps that they contain, one can classify them into *genera* [*Geschlechter*] according to the number p; in the first genus are thus the curves for which $p = 0$, in the second one those for which $p = 1$, etc. Hence the different orders appear reciprocally as subsections of the genera; and any order appears in the genera [from $p = 0$] to $p = \frac{n-1.n-2}{2}$, where the most general curve of the nth order—that is, free of any double point or cusp—finds its place.[45] [Clebsch 1865c, p. 43]

Clebsch added that the homogeneous coordinates of a curve belonging to a given genus can be expressed as rational functions of two parameters s, z connected by an equation $F(s, z) = 0$ founding the class of the corresponding Abelian functions. In particular, in the case of $p = 0$, he explicitly stated that s could be expressed rationally as a function of z, which implied that the curve could itself be parameterized rationally.

The rest of the paper was devoted to the study of rational curves, a study consisting in listing and counting their possible singular points, in stating the geometric version of

[42] That said, Clebsch himself designated p as the genus of a given curve shorlty after. See [Clebsch 1868a, p. 1238] for instance.

[43] This vocabulary, which is that of the naturalist classifications, was also used in classifications of other objects: it was for instance the case for the classification of quadratic forms that had been proposed by Carl Friedrich Gauss in his 1801 *Disquisitiones Arithmericae*. See [Lemmermeyer 2007; Goldstein 2016].

[44] Euler first called these taxa *species,* but, after having enumerated them, he added that he should have called them *genera* [Euler 1748, p. 126].

[45] "Statt die algebraischen Curven nach Ordnungen einzutheilen, und in diesen Unterabtheilugen zu machen nach der Anzahl der Doppel- und Rückkehrpunkte, welche dieselben aufweisen, kann man dieselben in *Geschlechter* eintheilen nach der Zahl p; zu dem ersten Geschlecht also alle diejenigen für welche $p = 0$, zum zweiten diejeniegen, für welche $p = 1$, u.s.w. Dann erscheinen umgekehrt die verschiedenen Ordnungen als Unterabtheilungen in den Geschlechtern; und zwar kommt jede Ordnung in allen Geschlechtern vor bis zu $p = \frac{n-1.n-2}{2}$, wo dann die allgemeinste, d.h. von Doppel- und Rückkehrpunkten völlig freie Curve $n$ter Ordnung ihre Stelle findet."

Abel's theorem in the case $p = 0$, and in applying it to enumerate curves satisfying given conditions.[46]

In the other 1865 paper where the notion of genus was defined, [Clebsch 1865a], Clebsch did not insist on the possible parameterizations of curves. Instead, he emphasized the fact that curves which can be transformed into one another by a birational transformation[47] belong to the same genus:

> According to the principles given [in my memoir on Abelian functions [Clebsch 1864a]], one can divide the curves into *genera*, according to the class of Abelian functions to which they lead, or according to the value of the number p to which they correspond. Now, when a curve is deduced from another in such a way that to each point or each tangent of one curve generally corresponds one unique point or one unique tangent, *both curves lead to the same Abelian integrals,* and hence belong to the same *genus* and have the same p. This theorem, indeed, is just another clothing of the one given by *Herr* Riemann in this journal [Riemann 1857d, p. 133].[48] [Clebsch 1865a, p. 98]

The theorem of Riemann to which Clebsch referred in this quotation is the one stating that two algebraic equations belong to the same class if they can be transformed into one another by a rational substitution. It is interesting to remark that Clebsch himself saw the invariance of the number p as "just another clothing" of Riemann's theorem. Yet, even if such a proximity was thus brought to light, Clebsch still distinguished between the two theorems: while his bore on algebraic curves, that of Riemann pertained on algebraic equations.

Clebsch then showed how the invariance of p could help obtain information on what he called the "characteristic numbers" of an algebraic curve, that is, its order n, its class m, and the numbers i, t, d, r of its inflection points, double tangents, double points, and cusps, respectively.[49] Citing the book by Salmon on algebraic curves, [Salmon 1852, p. 91], he first recalled that these numbers are related by the Plücker formulas:

$$\begin{aligned} m &= n^2 - n - 2d - 3r \\ n &= m^2 - m - 2t - 3i \\ i &= 3n^2 - 6n - 6d - 8r \\ r &= 3m^2 - 6m - 6t - 8i, \end{aligned}$$

[46]The exact same questions—which are clearly reminiscent of those of the 1864 memoir—were tackled in a twin paper dedicated to the curves with $p = 1$ (which can thus be parameterized with the help of elliptic functions), [Clebsch 1865b].

[47]Clebsch repeatedly talked about "univocal transformations" (*eindeutige Transformationen*) or about curves which "correspond to each other point by point" without specifying the meaning of these expressions. If such appellations seem to insist only on the (almost) bijective character of the considered applications, all the examples that Clebsch treated are indeed what we now see as birational transformations.

[48]"Nach den dort gegebenen Principien kann man die Curven in *Geschlechter* eintheilen nach der Classe Abelscher Functionen, auf welche sie führen, oder nach dem ihnen entsprechenden Werthe der Zahl p. Wenn nun aus der gegebenen Curve eine andere so abgeleitet wird, dass jedem Punkte oder jeder Tangente der einen Curve im Allgemeinen immer nur ein einziger Punkt oder eine einzige Tangente der andern entspricht, *so führen beide Curven auf dieselben Abelschen Integrale,* gehören also demselben *Geschlechte* an, und besitzen dasselbe p. In der That ist dieser Satz nur eine andere Einkleidung desjenigen, welchen Herr Riemann dieses Journal Band 54 pag. 133 gegeben hat."

[49]Clebsch's notation in the paper which we are studying here was explicitly borrowed from Salmon's book on higher plane curves, [Salmon 1852, p. 91]: m for the order, n for the class, and ι, τ, δ, $\varkappa$ for the other numbers. For the sake of clarity, I choose here to change this notation in order to make them (at least partially) correspond to those of Clebsch's previous papers.

and he added that one also has

$$\frac{(n-1)(n-2)}{2} - d - r = \frac{(m-1)(m-2)}{2} - t - i.$$

This equality, he explained, can be directly deduced from the Plücker formulas, but can also be seen as a consequence of the invariance of p. Indeed, Clebsch observed that a curve and its dual have the same number p because each point of the former is associated with a tangent to the latter, and reciprocally.[50] Since the order, number of double tangents and number of inflection points of the dual are respectively equal to the class, number of double points and number of cusps of the original curve, one has $p = \frac{(m-1)(m-2)}{2} - t - i$, which yields the previous formula.

Clebsch also pointed out that the invariance of p provides other interesting identities, such as

$$\frac{(n-1)(n-2)}{2} - d - r = \frac{(m-1)(m-2)}{2} - t - i = \frac{(n'-1)(n'-2)}{2} - d' - r' = \frac{(m'-1)(m'-2)}{2} - t' - i',$$

where the letters $n', \dots, i'$ designate the characteristic numbers of any curve which is rationally equivalent to a given curve with characteristic numbers $n, \dots, i$. As an example, he proposed to compute the characteristic numbers of the evolute of a given curve[51]: first he asserted that, in this case, it is possible to prove directly that $i' = 0$ and $m' = n^2 - 2d - 3r$. Then, by using the Plücker formulas and those stemming from the invariance of p, he deduced that the order of the evolute is $n' = 3n(n-1) - 6d - 8r$, and he provided similar expressions for r', t', and d' in function of the characteristic numbers of the original curves.[52]

Both of the papers where the genus was defined were thus devoted to the study of algebraic curves. However, while the classificatory aspect of this notion was somewhat inert in the paper on rational curves (the genus being chosen once and for all), it appeared as more dynamic in the other one, where the invariance of the genera by birational transformations was at the core of Clebsch's approach. More generally, this invariance was the property which is most used in Clebsch's other publications.

2.3 Usages of the genus in Clebsch's works

Indeed, if one looks at how the notion of genus intervened in the works of Clebsch published after 1865, it appears that the classificatory facet of the notion was expressed only discretely as a new way to organize the research on curves from a global point of view: apart from the 1865 papers where he studied the curves with $p = 0$ and $p = 1$, Clebsch only wrote one short paper devoted to the curves of genus $p = 2$, [Clebsch 1869b]. He also occasionally used the genus to classify specific curves: for example, in [Clebsch 1869a], the genus was the principal way to divide the curves drawn on a ruled surface of the third order; for each p, the different possibilities for the order of the curves, together with other characteristic numbers, were then listed.

[50] Just like in the previous quotation, the rational character of this correspondence remained tacit.

[51] The evolute of a curve is the locus of all the centers of curvature of the curve, or, equivalently, the envelope of all the normals.

[52] Clebsch actually erroneously started from $i' = r$ in his original paper. In a supplement to a paper published in the same volume of Crelle's *Journal* [Clebsch 1865d], he indicated that Arthur Cayley had pointed to him that this equality, which came from an earlier article of Steiner, was wrong. He then gave the correct formulas for the characteristic numbers of the evolute of a curve, as we reported here.

The direct link between curves and Abelian functions (expressed through the possible parameterizations of curves, for instance) also remained rarely invoked in Clebsch's publications, to the profit of a combination of the expression $p = \frac{(n-1)(n-2)}{2} - d$ and its invariance by birational transformations, a combination used to gain knowledge on algebraic curves and algebraic surfaces. A typical example of this has been depicted in the previous subsection, with the evolute of a curve. Clebsch also repeatedly used this property for what he called the study of the geometry on surfaces. For instance, in [Clebsch 1866], he proved that every cubic surface can be represented on the plane, which means that there exists a birational transformation between any cubic surface and the plane. This representation was a means to understand cubic surfaces, notably *via* the study of the algebraic curves that are drawn on them: Clebsch knew how the curves included in the plane of representation are transformed, by the representation, into curves drawn on the surface; as this representation is birational, the genus of such curves remains invariant, which allowed Clebsch to find formulas for the characteristic numbers of the curves upon the surface in function of those included in the plane of representation. The genus also helped Clebsch study the very possibility for an algebraic surface to be represented on the plane: in [Clebsch 1868b], he proved that if a quartic surface has this property, it necessarily contains a conic section made of double points. His proof consisted in starting from the fact that every curve obtained as the intersection of the surface by a plane is a quartic curve, and he showed that such a curve corresponds, in the representation, to a (plane) cubic curve. Since such cubics are "in general" of genus 1, so are the quartics drawn on the surface. Hence, because of the formula $p = \frac{(n-1)(n-2)}{2} - d$, these quartics necessarily possess two double points, from which Clebsch deduced that the surface contains a double conic section.

The initial classificatory framework was occasionally reactivated in the papers where Clebsch used the notion of genus of algebraic curves to define other notions of genera by analogy. For instance, in 1868, as he proposed a definition of a notion of genus for algebraic surfaces,[53] Clebsch began by recalling that:

> The theorems of M. Riemann on algebraic functions of two variables gave a principle of classification of algebraic curves. I proposed to name *genus* of a curve the number $p = \frac{(n-1)(n-2)}{2} - d$ [...], n being the order of the curve and d the number of double points or cusps.[54] [Clebsch 1868a, p. 1238]

After having recalled that two curves must have the same genus to be transformed into one another rationally, he wrote:

> Now, I found that for algebraic surfaces, there are theorems which are completely analogous [... and which] allow us to classify these surfaces according to their *genus* p. Two surfaces must belong to the same genus to be transformed into one another in a rational way.[55] [Clebsch 1868a, pp. 1238-1239]

[53]Clebsch only considered complex algebraic surfaces. In particular, the introduced notion of genus for these surfaces was not linked to the notion of connectivity of real surfaces, or of Riemann surfaces (the latter being, in modern parlance, one-dimensional complex manifolds).

[54]"Les théorèmes de M. Riemann sur les fonctions algébriques à deux variables ont donné un principe pour classifier les courbes algébriques. J'ai proposé de nommer *genre* d'une courbe le nombre $p = \frac{(n-1)(n-2)}{2} - d$ [...], n étant l'ordre de la courbe, d le nombre de ses points doubles ou de rebroussement."

[55]"Maintenant j'ai trouvé que, pour les surfaces algébriques, il y a des théorèmes tout à fait analogues [... qui nous permettent] de classifier ces surfaces eu égard à leur *genre p*. Deux surfaces devront appartenir au même genre pour être transformées l'une en l'autre d'une manière rationelle."

The genus defined here by Clebsch is equal to the number of arbitrary coefficients in the equation of a surface of order $n-4$ passing through the singular curve of the given surface—this echoes the equality between the genus of a curve of order n and the number of coefficients in the equation $\Theta = 0$ of a curve passing through its double points. Clebsch then stated that in the case of a surface having no singular curve, the genus is equal to $\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}$: the analogy of the definition of the genus of a surface was thus sustained by an analogy of related theorems and formulas.[56]

Another notion of genus was defined by Clebsch for objects which he called "connexes," and, consequently, for algebraic differential equations of the first order, [Clebsch 1872, pp. 442-446].[57] Here again, both the idea of classification and the analogy with Riemann's theorem on Abelian integrals were emphasized:

> The ideas developed [in the present paper] lead to an interesting application of the indicated theory, since they provide a classification of the algebraic differential equations of the first order which is absolutely analogous to the Riemannian classification of the Abelian integrals.
>
> These investigations are connected with my extension of the Riemannian theorem on the conservation of the number p by univocal transformations, or, geometrically expressed, on the conservation of the curve-genus.[58] [Clebsch 1872, p. 443]

Significantly, if Clebsch presented once more his new classification as rooted in the research of Riemann on Abelian integrals, he clearly expressed a distinction between this research and its geometrical expression connected with the genus of algebraic curves.

2.4 A few simple and elementary considerations

As already explained, all the mentions to Riemann in Clebsch's publications refer to the paper on Abelian functions of 1857, and they first occurred in the memoir on the application of Abelian functions to geometry, so that what we have depicted above represents the way that Clebsch first assimilated the research of Riemann. This research, as Clebsch admitted, was difficult for him to grasp, and his modification of Riemann's theory grounded on the introduction of objects and techniques from projective geometry is to be seen as a way to tame what Riemann had written on Abelian functions. When they presented their book on these functions before the French *Académie des sciences*, Clebsch and Gordan insisted on the simplification which was supposedly provided by the new geometric framework:

> The theory of Abelian functions, of which [...] the full development [has been given] in 1857 by M. Riemann, can nevertheless be greatly simplified regarding the fundamental principles on which it is based. We manage to use, instead of M. Riemann's profound and

[56]The notion of the genus of algebraic surfaces was later re-worked by other mathematicians (including Arthur Cayley, Max Noether, and, later, Italian geometers), who actually proposed different such notions. See [Houzel 2002, p. 205; Brigaglia, Ciliberto, and Pedrini 2004, pp. 312-318].

[57]Clebsch defined a connex with the help of an equation $f(x_1, x_2, x_3; u_1, u_2, u_3) = 0$ between the homogeneous coordinates of points x and of lines u in the plane. It is thus an object made of pairs of points and lines.

[58]"Die hier entwickelten Vorstellungen führen auf eine interessante Anwendung der angedeuteten Theorie, indem sie eine Classification der algebraischen Differentialgleichungen erster Ordnung liefer, welche der Riemannschen Classification der Abelschen Integrale durchaus analog ist. Diese Untersuchungen knüpfen sich an die Erweiterung, welche ich dem Riemannschen Satze über die Erhaltung der Zahl p bei eindeutigen Transformationen, oder, geometrisch ausgedrückt, über die Erhaltung des Curvengeschlechtes [...] gegeben habe."

> transcendental principles, a few simple and elementary considerations. [...] We make use of a double geometric representation.[59] [Clebsch and Gordan 1866a, pp. 183–184]

As was then made clear, the double geometric representation consisted, on the one hand, to consider the bounds and the paths of integration of Abelian integrals as made of points of the complex plane, and, on the other hand, to start from a homogeneous equation $f(x_1, x_2, x_3) = 0$ representing a plane curve.

Correspondingly, the very first sentence after the introductory pages of Clebsch and Gordan's book set up the framework very clearly,[60] for it took as a working basis the homogeneous equation $f(x_1, x_2, x_3) = 0$ of an algebraic curve; the next step was to introduce homogeneous integrals. The first three chapters of the book contained approximately the theorems that Clebsch had proved in his previous papers, such as the geometric version of Abel's addition theorem, the equality $p = \frac{(n-1)(n-2)}{2} - d - r$, and the invariance of p under birational transformation. But the book addressed many other questions, such as the so-called inversion problem, the monodromy of Abelian integrals, the introduction of Θ-functions, and the problem of division of these functions, which are all topics which had been tackled by Riemann in 1857.

Yet one central feature of Riemann's research was completely obliterated in Clebsch and Gordan's book: that of Riemann surfaces. The two mathematicians explained that their use of geometric considerations allowed to "avoid every consideration of functions in general, which is always awkward, because they tie in with a concept which is completely undetermined and contains unknown possibilities."[61] Riemann surfaces were not explicitly pointed out as problematic objects, but they appeared nowhere in the book; significantly, in the paragraph devoted to branch points, Riemann's memoir on Abelian functions was cited for the terminology "*Verzweigungspunkte*," but what cited and effectively used was Victor Puiseux' research, which studied algebraic functions with the help of paths in the complex plane.[62]

The surfaces that Riemann had introduced as a geometric way to represent the ramification of algebraic functions were thus avoided, in Clebsch's research, to the profit of other geometric devices which allowed him to tackle the theory of Abelian functions. Algebraic curves, in particular, played a prominent role, and their introduction was accompanied by several processes of homogenization, and by techniques and results coming from projective geometry, like the generation of curves by pencils of lines or the formula giving the class of a curve—and reciprocally, the obtained results on Abelian functions offered new theorems on algebraic curves. By contrast, one sees how remote the research of Riemann on Abelian functions was

[59]"La théorie des fonctions abéliennes, dont [...] le développement complet [a été donné] en 1857 par M. Riemann, est néanmoins susceptible d'une grande simplification en ce qui concerne les principes fondamentaux sur lesquels elle est basée. Nous sommes parvenus à mettre, au lieu des principes transcendants et profonds de M. Riemann, quelques considérations simples et élémentaires. [...] Nous nous servons d'une double représentation géométrique."

[60]The introduction of the book also mentioned the double geometric representation. Here Clebsch and Gordan alluded to "the work of Messrs. Briot and Bouquet." This certainly referred to the book by two mathematicians on doubly periodic functions, [Briot and Bouquet 1859], which made an important use of integration along complex paths *à la* Cauchy.

[61]"Man vermied auf diese Weise insbesondere alle Betrachtungen über Functionen im Allgemeinen, welche immer misslich sind, weil sie an einem völlig unbestimmten und unbekannten Möglichkeiten enthaltenden Begriff anknüpfen." [Clebsch and Gordan 1866b, p. vi].

[62]See [Clebsch and Gordan 1866b, p. 80 sqq.]. The given reference is [Puiseux 1850].

from such considerations. Riemann's geometry (close, if not equal, to analysis situs) and Clebsch's geometry were therefore different, both in the objects that they involved and in their mode of intervention in the theory of Abelian functions.

Nevertheless, the avoidance of Riemann surfaces for the handling of Abelian functions does not mean that Clebsch rejected these objects once and for all (for ontological reasons for instance), as proves the existence of a paper precisely devoted to the "theory of Riemann surfaces" [Clebsch 1873].[63] This paper was a continuation of some research of Jacob Lüroth, [Lüroth 1871], and was aimed at finding a standard way to arrange the cuts and the branch points linking together the several sheets of a Riemann surface. Algebraic curves, however, are totally absent from these investigations, which indicates that Clebsch did not mix what was supposed to remain separated: the theory of algebraic curves (and their genera) on the one hand, and, on the other hand, that of Riemann surfaces (and their connectivity).

3 Genus vs. connectivity, 1857–1882 (and beyond)

To follow and compare how other mathematicians employed the notions of genus and of connectivity, and associated them with algebraic curves or (Riemann) surfaces, and with the names of our two authors, I gathered a corpus of papers with the help of a textual search of the keywords "Clebsch," "Riemann," "genus," "connectivity," and slightly different versions of them, in English, French, German, and Italian[64] in the volumes of seven major research journals published between 1857 and 1882: *Journal für die reine und angewandte Mathematik*, *Mathematische Annalen* (published from 1869 on), *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, *Journal de mathématiques pures et appliquées*, *The Quarterly Journal of Pure and Applied Mathematics*, *Proceedings of the London Mathematical Society* (from 1865 on), and *Annali di matematica pura ed applicata* (from 1858 on). This textual search has been conducted on the `txt` files of these volumes that are available on different online archive websites, such as `archive.org`, `hathitrust.org`, or `gallica.bnf.fr`. It should be stressed that the quality of the available `txt` files is sometimes imperfect and that a few of the volumes of these journals are not available online, which implies that some papers which actually contain the mentioned keywords have not been detected by the textual search. Nevertheless, the results that have been obtained by analyzing the resulting corpus appear to be sufficiently coherent to assure the robustness of my conclusions.

Among the papers thus spotted, I only kept those which do correspond to our subject: first, I removed the papers where the words "genus" or "connected" do not have a technical sense—for instance, the French word *genre* is a synonym of "kind," or "type," and is frequently used in sentences such as "*ce genre de problème*" ("this kind of problem"). I also neglected the articles where the word "genus" does have a technical term, but which is not connected to the genus of curves or surfaces, etc., like in the case of the genus of quadratic forms, but also of the genus in the botanical sense (let us recall that the *Comptes rendus* included many

[63] This paper, which was published posthumously, is the only one among Clebsch's publications to tackle the notion of Riemann surface.

[64] For instance, the search also included that of words such as "connected," in order to detect phrases like "$(2p + 1)$-ply connected." Moreover, for reasons that will be explained in the next subsection, I also searched for the English word "deficiency."

Journal	Articles
Journal für die reine und angewandte Mathematik	62
Mathematische Annalen	98
Comptes rendus hebdomadaires des séances de l'Académie des sciences	38
Journal de mathématiques pures et appliquées	3
The Quarterly Journal of Pure and Applied Mathematics	6
Proceedings of the London Mathematical Society	17
Annali di matematica pura ed applicata	19

Table 1 – Number of papers within each journal found by our textual search.

non-mathematical papers). Finally, I did not retain the papers where Clebsch or Riemann are invoked for features that are not related to our topic: this was mostly the case for papers where Riemann is cited only for his works on trigonometric series or on prime numbers.

This selection produced a corpus of 243 articles (including those of Clebsch which have been previously described), non-homogeneously distributed among the considered journals, as is shown in Table 1. The papers of this corpus have been studied by identifying their general topic, examining their references, and analyzing in which way the genus or the connectivity intervened in them. Before turning to the general results of this investigation, a few words must be devoted to the specificity of the papers written by British mathematicians.

3.1 The British case

In a paper dated October 1865, Arthur Cayley [1865] proposed to define the "deficiency" of an algebraic curve as follows. He first recalled that a curve of order n has at most $\frac{(n-1)(n-2)}{2}$ double points, and that curves having this maximal number of double points can be parameterized rationally; this result, he wrote, is a particular case of a theorem of Riemann (at this point Cayley only alluded to the year 1857, but he explicitly cited the memoir on Abelian functions a few lines after) which had been explicitly stated and proved by Clebsch in his paper on rational curves, [Clebsch 1865c]. Cayley then proposed another proof of this theorem, following which he added:

> Before going further, it will be convenient to introduce the term "Deficiency," viz. a curve of the order n with $\frac{1}{2}(n-1)(n-2) - D$ dps [double points], is said to have a deficiency = D: the foregoing theorem is that for curves with a deficiency = 0, the coordinates are expressible rationally in terms of a parameter θ. [Cayley 1865, p. 2]

Cayley then acknowledged that Clebsch [1865a] had already proved that two curves which can be transformed into one another by a birational transformation have the same deficiency. The problem that he tackled in the rest of the paper was to find, for a given curve of deficiency D, an equivalent curve of minimal order. Cayley's proof will not be discussed here, but some points linked to the introduction of the "deficiency" are worth a comment.

First, even if Cayley's deficiency and Clebsch's *Geschlecht* were recognized to be the same numbers quickly after 1865,[65] it is interesting to highlight the differences between the frameworks within which the two mathematicians introduced their respective notions. As we

[65] See for instance [Chasles 1867, p. 825], or the other references given below in this section.

saw, the term *Geschlecht* initially referred to classes of curves, and not to the number p itself, which underlined Clebsch's classificatory viewpoint. On the contrary, the deficiency was, by definition, a number, which measures the amount of double points that are missing to a curve to be rational. If the deficiency could evidently offer a way to classify algebraic curves, the word itself and Cayley's way to introduce it show that this was not his primary aim.

Actually, after the word *Geschlecht* switched from its definition as a class of curves to the number p itself, Cayley persistently continued to use the term "deficiency," and so did all the British mathematicians of the corpus.[66] This specificity, however, did not hamper communication. For instance, when Cayley published English papers in *Mathematische Annalen*, he used the word "deficiency" and added the German equivalent *Geschlecht* in parentheses, like in [Cayley 1871, p. 472]. Cayley also published one note in French in the *Comptes rendus* where he indicated that he had called the number D the *défaut* of the curve; there, he wrote the English "deficiency" in parentheses but he did not make appear the French *genre*, [Cayley 1866, p. 587]. Reciprocally, when Michel Chalses described the works of Cayley and Clebsch in [Chasles 1867, p. 825], he used the French words *défaut* and *déficience* for the former, and *genre* for the latter. Finally, although Klein repeatedly used the word *Geschlecht* in his German publications, one paper published in the *Proceedings of the London Mathematical Society* and translated by Olaus Henrici, [Klein 1877], talked about the deficiency of curves.[67]

I found no trace of discussion on the right choice of words to be adopted, or of disagreement between English-speaking mathematicians and others.[68] Hence it is difficult to know whether the British mathematicians kept using the word "deficiency" for a precise reason or not: because of the initial difference between the introduction of this term and that of *Geschlecht*, or because of something which would be related to a sort of British identity, for instance.[69]

Regarding our general questioning in this paper, it is however important to stress that this asymmetry of vocabulary did not seem to impact the technical organization of the mathematical work, and that just like the German *Geschlecht*, the French *genre*, and Italian *genere*, the word "deficiency" was used only for algebraic curves and algebraic surfaces, and was not related to Riemann surfaces. The following description of our corpus, written with the current word "genus," thus encompasses all the English contributions.

[66]Five mathematicians of the corpus were born in Britain: Arthur Cayley (19 papers in the corpus), Samuel Roberts (5 papers), Henry John Stephen Smith (2 papers), William Kingdon Clifford, and Joseph Wolstenholme (1 paper each). Olaus Henrici, who was born in Germany but had lived in England from 1865 on, is also to be counted as one of the Britisch contributors of the corpus, with one paper written in English.

[67]The only other English paper in our corpus signed by a non-British mathematician is a note of Charles Hermite published in the *Proceedings*, [Hermite 1871]. There (p. 344) one reads the word "deficiency," whereas Hermite talked about the *genre* in another paper of the corpus, [Hermite 1878, p. 298]. Hermite's note in English is indicated to be an extract from a letter to Cayley, and has been presented to the *London Mathematical Society* by the latter. Although there is no explicit evidence of it, a possibility is that Hermite's original letter was in French and that Cayley translated it into English.

[68]This is the case in the corpus, but also outside. Clebsch and Cayley did write to one another, as is indicated in some of their publications (see for instance footnote 52) but their letters have not been retrieved. The other known letters of Clebsch do not evoke this subject, nor do the letters from Cayley to Cremona for instance, [Israel 2017].

[69]I did spot uses of "genus" with the textual search, but such uses referred to quadratic forms only. Another case of such a persistent asymmetric use of words in English, German, and French publications is detailed in [Lê 2018b, pp. 250–252]; it concerns what have been called *Schliessungsprobleme* in German.

3.2 Clusters, classifications, and keywords

The analysis of the corpus which is proposed here follows the methods used in [Goldstein 1999], which consist in looking at all the references made in the texts of the corpus and searching for clusters of texts, that is, sets of texts which cite themselves a great deal or share common references, but rarely cite texts of other clusters. Four main clusters thus appear.[70] They will be described in regards to their main themes and references, their disciplinary classification in the *Jahrbuch über die Fortschritte der Mathematik* (for the papers published after 1868), and the way of how our keywords (genus, connectivity, etc.) appear and are used.

The first cluster is by far the biggest, as it gathers about 120 papers (almost half of the corpus), published between 1865 and 1882 in all the journals from which we started, although a large part of the total mass of *Mathematische Annalen* appears here. These papers relate to several topics bearing for the greatest part on algebraic curves and surfaces. Among these topics, the principal ones are the study of algebraic surfaces from the point of view of birational transformations (with works of Clebsch, Cayley, Cremona, and Max Noether among others), and the so-called principle of correspondence and its applications (a topic researched by Cayley, Hieronymous Zeuthen, and Alexander Brill above all).[71] A group of texts, of which many were written by Klein, and which contain investigations at the junction of projective geometry and topology such as the research on the notion of connectivity for a projective real surface [Klein 1874a], or on the possible shapes of real cubic surfaces, also belong to this first cluster, [Schläfli 1871; Klein 1873; Rodenberg 1879].[72] The papers of Clebsch where the genus is employed (and which we described above) are part of this cluster, and appear to be abundantly cited by the other authors (which include some of Clebsch's students: Noether, Brill, and Klein), which shows their importance for the research represented here. This first cluster gathers almost all the papers of the corpus which are classified under the sections "analytic geometry" and "pure, elementary, and synthetic geometry" of the *Jahrbuch*—let us recall that the latter section includes a subsection devoted to "continuity considerations (analysis situs)," represented by the aforementioned works. As for the apparition of our keywords, the cluster contain in a vast majority the notion of genus (of algebraic curves and algebraic surfaces), while the works classified under "analysis situs" contain both the notions of genus (of curves) and of connectivity of (Riemann) surfaces. We finally observe that the notion of the genus of algebraic surfaces exclusively appears in this first cluster.

The second cluster is made of about 25 papers which all directly cite Riemann's 1857 memoir on Abelian functions. Published from the middle of the 1860s on by mathematicians like Johannes Thomae, Heinrich Weber, Gustav Roch, or Lazarus Fuchs, these papers are classified in the section "Function theory" of the *Jahrbuch* and deal with diverse questions related to Abelian functions and Θ-functions of several variables—the articles studying Θ-functions, in addition of frequently citing themselves and citing Riemann's 1857 memoir, also share as common sources one paper of Riemann on these functions [Riemann 1866]

[70]Taken together, these four clusters represent about 76% of the corpus. The remaining texts are either isolated elements or parts of (very) small clusters. Our analysis has been conducted with the help of the tools offered by the collective online database `Thamous`, initiated and maintained by Alain Herreman.

[71]The texts linked to this topic could be characterized as a sub-cluster, in the sense that they cite themselves a lot and cite the other texts of the cluster less, but still more than the texts outside the cluster.

[72]About such works of Klein (and Schläfli), see [Scholz 1980, pp. 164–167].

and Clebsch and Gordan's book on Abelian functions. In terms of journals, Crelle's *Journal für die reine und angewandte Mathematik* is over-represented here, although *Mathematische Annalen* also appears in the cluster, yet more modestly. In any case, these texts rarely invoke the notion of genus; on the contrary, they almost systematically involve in one way or another the order of connectivity of a Riemann surface. A paper of Thomae [1881] appears as an exception, since it mentions the genus of a Riemann surface—it will be discussed below, together with seven other such exceptional cases.

The third cluster is approximately of the same size as the second one, but is concentrated between 1879 and 1881. Almost all of its texts are written by Klein. They bear on the link between the solution of algebraic equations (of degree 5, 7, and 11) and the transformation of elliptic functions, and have all been published in *Mathematische Annalen*. One paper appears to be at the core of this highly coherent cluster, namely that where Klein presented his solution of the quintic equation [Klein 1879b]; the common references which do not belong to our corpus are the works of Charles Hermite, Francesco Brioschi, Leopold Kronecker, and Ludwig Kiepert on the same subject.[73] The cluster also includes papers of students of Klein, in the image of Walther Dyck's article on the theory of groups and its application to the groups of transformations of hyperbolic tessellations [Dyck 1882], or Adolf Hurwitz's paper on modular functions [Hurwitz 1882]; Poincaré's series of notes devoted to Fuchsian functions also appears to be attached to this cluster [Poincaré 1881c,d, 1882]. In the image of the second cluster, the vast majority of the texts in the third one relate to the theory of functions, according to the *Jarhbuch*. The situation in view of our keywords, however, is different, as there are many co-occurrences of the genus of curves and of the connectivity of (Riemann) surfaces. Moreover, this cluster is the one containing six of the eight exceptional cases talking about the genus of surfaces and Riemann surfaces, among which Klein's 1880 paper which caused trouble to Poincaré, as seen in our introduction, [Klein 1880].

Finally, the fourth cluster is almost entirely made of notes (about 15 in number) published by Paul Appell, Émile Picard, and Henri Poincaré in the French *Comptes rendus*, between 1880 and 1882. These notes have two books that do not belong to the corpus as common references, namely Briot and Bouquet's book on elliptic functions in its second edition [Briot and Bouquet 1875] and Briot's one on Abelian functions [Briot 1879]. They deal with the theory of linear differential equations (in the continuation of Fuch's research), and with questions linked with Abelian functions—although these topics seem to be close to those appearing in the second and third clusters, no citation link between them and the fourth one is to be observed. The classification of the *Jahrbuch* situates the papers of this fourth cluster in the sections of function theory, and of differential and integral calculus. Here, it is the genus which systematically appears to the detriment of the connectivity; it is principally related to algebraic curves, sometimes to algebraic equations. In one paper of Poincaré (the last of our eight exceptional cases), it is used to characterize an usual surface in space, [Poincaré 1881a].

There is, therefore, a certain correlation between the use of the notions of genus and of connectivity, the disciplinary classification offered by the *Jahrbuch*, and the clusters that we brought to light. To be more specific, whereas the second and the fourth clusters are chiefly characterized by the presence of only one of the two notions—the connectivity (of

[73] About Hermite's solution of the quintic with the help of elliptic functions, see [Goldstein 2011]. For Kronecker, see [Petri and Schappacher 2004]. See also [Gray 2000], notably for the case of the seventh-degree equation.

Riemann surfaces) and the genus (of algebraic curves and equations), respectively—, the others make use of both of them. But the first cluster, mainly classified in geometry, reserves the connectivity for the papers which come under the subsection of analysis situs, while the third one, situated in function theory, is marked by concurrent employments of the two notions. It is thus remarkable that the clusters, which have been detected by the examination of the citations, can be retrieved by such a study.[74]

From the viewpoint of the *Jahrbuch* classification, the previous description also evidences that the connectivity is, generally speaking, and (at least) until 1882, associated with analysis (in particular for its link with Riemann surfaces) and analysis situs. Similarly, the genus is principally connected to geometry (either pure or analytic), and appears in analysis situs only in conjunction with the connectivity. In this sense, and even if there exist some papers which are classified under one unique section of the *Jahrbuch* and which contain both of the notions, the words "genus" and "connectivity" thus appear as disciplinary markers.

3.3 Objects and names associated with the genus and the connectivity

Finer information can be obtained, of course, by taking a closer look at the papers of the corpus. Here the focus will be on how the notions of genus and of connectivity were linked to mathematical objects, and how these notions (and some related results) were attributed to Clebsch or Riemann.

First of all, it must be emphasized that many papers used one notion or the other as technical elements which appeared in the proofs, without being defined, commented on, or attached to any name or reference. This is particularly true for the genus of algebraic curves, which thus appears to be completely (and quite rapidly) integrated in the research on such curves as an usual notion. In such cases, although the link with Abelian functions is activated from time to time, the genus mostly appears through the formula $p = \frac{(n-1)(n-2)}{2} - d$ and its birational invariance, in the image of what we wrote about Clebsch's publications.

It is also instructing to look at the (fewer) papers which involved both the notions of genus and of connectivity. Let us for instance consider a 1882 paper of Ferdinand Lindemann devoted to the study of particular complex functions—it is classified among the function-theoretic papers, and belonging to the second cluster [Lindemann 1882]. In this paper, Lindemann was led to consider the function

$$\zeta = \frac{\sqrt{z^2 - b^2} - \sqrt{z^2 - c^2}}{\sqrt{c^2 - b^2}}$$

of the complex variable z. Lindemann explained that one can "think of the values of ζ as spread over the z-plane in a four-sheeted Riemann surface Z,"[75] and listed the branch points of this surface. Then he transformed the previous equation rationally into

$$\frac{1}{4}\zeta^4(c^2 - b^2) - z^2\zeta^2 + \frac{1}{2}\zeta^2(c^2 + b^2) + \frac{1}{4}(c^2 - b^2) = 0,$$

and explained:

[74]That such clusters can represent mathematical themes, for instance, is a result of [Goldstein 1999, p. 205].

[75]"Die Werthe von ζ können wir uns über der z-Ebene in einer vierblättrigen Riemann'schen Fläche Z ausgebreitet denken." [Lindemann 1882, p. 329].

> This is the equation (in the unknowns ζ, z) of a curve of the fourth order with one self-touching point (two infinitely neighboring double points) which belongs to the axis at infinity $\zeta = 0$. Its genus is equal to $\frac{1}{2}3 \cdot 2 - 2 = 1$, so that the connectivity of the four-sheeted surface Z is equal to 3.[76] [Lindemann 1882, p. 329]

The algebraic curve introduced by Lindemann was thus a means to compute the connectivity of the Riemann surface representing ζ, and the previous quotation clearly shows that the notion of genus was associated with the algebraic curve, while that of connectivity was linked to the Riemann surface, even if the two objects were technically tied together. Let us additionally remark that if the equation of the curve was not homogeneized by Lindemann, its projective character appeared through the consideration of points at infinity for the computation of its genus.

Lindemann's paper is representative of many papers which were classified under one unique section of the *Jahrbuch*, but which strictly separated the genus and of connectivity in regards to the objects to which each of these notions were related, thus reflecting a sort of disciplinary distinction made on the objects themselves.

Sometimes, such a distinction was reinforced by attributing these notions to Clebsch or Riemann. To take one example, in a paper devoted to Jacobi's inversion problem, Hermann Stahl first advised his reader that "the Riemannian theory and the geometric expression by Clebsch and Gordan are assumed to be known," and then fixed the notation:

> Let $F(s, z) = 0$ be the grounding algebraic equation, and let the ramification of s as function of z be represented by the $(2p+1)$-ply connected surface T [...]. For the geometric interpretation, we write the equation homogeneously in the form $f(x_1, x_2, x_3) = 0$ and we consider it as a curve of order n and genus p.[77] [Stahl 1880, p. 171]

Hence the differentiation was double in Stahl's text: while Riemann surfaces and their connectivity appeared on the side of Riemann and his theory, the geometric interpretation, with homogeneous equations, algebraic curves and their genus, was attached to Clebsch and Gordan.

In some cases, the name of Clebsch was tied to the notion of genus, although the latter was not associated with a curve, but with an algebraic equation or an algebraic function:

> If $2p + 1$ is the order of connectivity of this [Riemann] surface and w is the number of simple branch points (a branch point of higher order being considered as an accumulation of several simple ones), one has, according to Riemann, the relation
>
> $$p = \frac{1}{2}w - n + 1.$$

[76] "[D]ies ist (in Veränderlichen ζ, z) die Gleichung einer Curve 4. Ordnung mit einem Selbstberührungspunkte (zwei unendlich benachbarten Doppelpunkten) im unendlich fernen Punkte der Axe $\zeta = 0$. Ihr Geschlecht ist gleich $\frac{1}{2}3 \cdot 2 - 2 = 1$, und daher der Zusammenhang der vierblättrigen Fläche Z ist gleich 3."

[77] "Die Riemannsche Theorie und die geometrische Ausdruckweise von Clebsch und Gordan sind als bekannt vorausgesetzt. [...] Es sei $F(s, z) = 0$ die zu Grunde liegende, algebraische Gleichung und die Verzweigung von s als Function von z sei dargestellt durch die $(2p + 1)$-fach zusammenhängende Fläche T [...]. Für die geometrische Interpretation schreiben wir die Gleichung homogen in der Form $f(x_1, x_2, x_3) = 0$ und betrachten dieselbe als Curve von der Ordnung n und vom Geschlecht p."

> The number p is called, after Clebsch, the genus of the equation $F = 0$.[78] [Weber 1873, p. 345]

The association of the names of Clebsch and Riemann to the notions of genus and of connectivity could also be expressed more implicitly, as mathematicians did not cite them for the notions themselves, but for related results: this can be seen for Riemann in the previous quotation of Weber. Another example is a 1881 note of Poincaré, where one reads: "A relation of genus p, $f(x, y) = 0$, can be supposed to be of degree $p + 1$ (Clebsch and Gordan, *Theorie der Abelschen Functionen*)."[79] This extract represents the first occurrence of the notion of genus in this note of Poincaré, and the citation to the Clebsch and Gordan's book, which is given for the result on the degree of f, and not for the definition of the genus, surreptitiously attach these names to the notion of genus.

Other papers, even if they implied only one of the two notions, presented ambiguities which came from other kinds of formulations, and associated the name of Riemann with curves, or with the genus, or with both. For instance, in 1878, Weber defined, "with Riemann, an algebraic function of genus p by an irreducible algebraic equation between the two variables, $F(s, z) = 0$, such that the ramification of s can be represented by a $(2p + 1)$-ply connected Riemann surface."[80] No curve intervened here, but the sentence of Weber tied Riemann's name to the notion of genus, *via* algebraic functions.

Another telling example is that of Brill and Noether, who, in the beginning of their famous paper on the application of algebraic functions to geometry, recalled the existence, for any given curve f, of a certain number "which [will be] denote[d] by the letter p after the process of Riemann and Clebsch among others, and name[d], after Clebsch, the *genus* of the curve." The name of Clebsch was clearly related to the appellation itself, but, a few lines after, Brill and Noether added that "Riemann has ordered the algebraic curves of genus p in classes,"[81] and thus explicitly attributed some research on algebraic curves to Riemann. These authors were not the only ones to make such a shift in the attributions: to take a final example, we mention a 1871 paper on the principle of correspondence by Zeuthen, who recalled that he had previously proposed a "geometric" proof ot the invariance of the genus by birational transformation, a result that he called "Riemann's theorem, on which Mr. Clebsch grounded the division of curves in *genera*"; a little bit later, he added: "Thus the curves are of the same genus (theorem of Riemann)".[82]

The numerous slight variations which can be seen in these examples show the complexity

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[78]"Ist die Ordnung des Zusammenhangs dieser Fläche $2p+1$, die Anzahl der einfachen Verzweigungspunkte w, wobei ein Verzweigungspunkt höherer Ordnung als ein Anhäufung von mehreren einfachen Verzweigungspunkten betrachtet wird, so besteht nach Riemann die Relation: $p = \frac{1}{2}w - n + 1$. Die Zahl p heisst nach Clebsch das Geschlecht der Gleichung $F = 0$."

[79]"Or une relation de genre p, $f(x, y) = 0$, peut être supposée de degré $p + 1$ (Clebsch et Gordan, *Theorie der Abelschen Functionen*)." [Poincaré 1881b, p. 958].

[80]"Wir definiren nach Riemann eine algebraische Function s von z vom Geschlecht p durch eine irreductible algebraische Gleichung zwischen beiden Variabeln $F(s, z) = 0$, welche die Eigenschaft hat, dass die Verzweigung von s sich durch eine $(2p + 1)$-fach zusammenhängende Riemann'sche Fläche T darstellen lässt." [Weber 1878, p. 35].

[81]"[... eine Zahl,] die wir nach dem Vorgang von Riemann, Clebsch u. A. mit dem Buchstaben p bezeichnet haben und in der Folge mit Clebsch das *Geschlecht* der Curve f nennen wollen. [...] Riemann hat die algebraischen Curven vom Geschlechte p in Classen geordnet. [Brill and Noether 1874, p. 300]."

[82]"[Le] théorème de Riemann sur lequel Mr. Clebsch a fondé la division des courbes en *genres*." [Zeuthen 1871, p. 150] Later, on p. 153: "Les courbes sont donc du même genre (Théorème de Riemann)."

and the blurring of the attribution process to Riemann and to Clebsch of what relates to the genus and the connectivity.[83] Nevertheless, even if the names of the two mathematicians were possibly mixed up, the association of the genus and the connectivity with objects was not: the latter was always related to (Riemann) surfaces, while the former characterized algebraic curves or surfaces, and sometimes algebraic functions or equations. As stated above, the genus happened to be tied to surfaces and Riemann surfaces only in a handful of papers, which happen to be all directly linked with Klein.

3.4 The genus of (Riemann) surfaces

Chronologically speaking, the first three papers of the eight excpetional ones were written by Klein himself, [Klein 1879b,c, 1880]; theypapers belong to our third cluster and deal with the problem of solving algebraic equations with the help of elliptic functions. Three other papers have been written by mathematicians who were students of Klein, namely Walther Dyck [1880, 1882] and Adolf Hurwitz [1882].[84] One of the two remaining papers is an extract from a letter from Carl Johannes Thomae to Klein which tackles the quesiton of constructing algebraic functions associated with a given Riemann surface, [Thomae 1881]. As for the last one, it is a note of Poincaré on curves attached to a differential equation $F(x, y, \frac{dy}{dx}) = 0$, [Poincaré 1881a], which has been presented to the *Académie des sciences* on December 5 1881, that is, a few month after the first letters between him and Klein that have been mentioned in our introduction.

The two 1879 papers of Klein were respectively devoted to the solution of the fifth-degree and the eleventh-degree equations. If they involved the "genus of Riemann surfaces," they also abundantly involved the genus of algebraic equations and curves. The first occurrence of the genus of a Riemann surface happened in the flux of the first paper, [Klein 1879b, p. 127], as Klein simply evoked "the genus p of the Riemann surface" associated with an algebraic function defined by an equation $\varphi(s, z) = 0$, without specifying what he meant: the terminology seemed to be straightforwardly transferred from the algebraic equation (or function) to the corresponding Riemann surface, no word of caution being expressed. The genus was then computed with the help of what we now call the Riemann-Hurwitz formula, which links together the genus, the number of sheets, and the ramification indexes of a Riemann surface—this formula was also central in the paper on the equation of degree 11, [Klein 1879c]. A few pages later, however, Klein operated another shift, consisting in passing from the conception of a Riemann surface as being spread over the plane to that of an usual surface in space (obtained by gluing together edges of a hyperbolic polygon).[85] Interestingly, this shift was accompanied by a terminological shift, which attached the word "genus" to a surface "thought of as lying freely in space":

[83] As indicate the previous examples, the replacement of Clebsch by Riemann in their association with algebraic curves does not seem to be correlated to the chronology.

[84] Let us emphasize that Klein, Dyck, and Hurwitz are mathematicians who continued to research the topic of Riemann surfaces after 1882. See [Friedman 2019, pp. 125–140].

[85] Klein had already tackled the question of the link between a real algebraic curve $f(x, y) = 0$ and the Riemann surface that represents the algebraic function y, [Klein 1874b, 1876]. In these papers, Klein clearly distinguished the genus, which he connected to curves, and the connectivity, which was reserved for Riemann surfaces. Another paper of the corpus was devoted to present a way to conceive a Riemann surface of connectivity $2p + 1$ as an usual surface in space with p holes, [Clifford 1876]—Clifford did not use any specific name for the number p, there. About this contribution, see [Scholz 1980, pp. 161–163].

> [This] is just the Riemann surface that we are looking for; but, instead of being spread over the J-plane in $n+1$ sheets, it is thought of as lying freely in space. [...] The rule, according to which the genus of the surface is computed from the number of sheets and the ramification points, is transformed into the so-called generalized polyhedron theorem of Euler: $e + s = k - 2p + 2$.[86] [Klein 1879b, p. 134]

Thus, contrary to the case of the "genus of a Riemann surface," the association of the word "genus" with a surface in space was accompanied by some comments, which highlight the correlated issue of identifying these two kinds of surfaces.

This process of considering a surface in space obtained from a fundamental polygon (and representing a Riemann surface) was exactly what Klein alluded to in the third paper, devoted to modular functions: "The closed surface, which is constituted by the reunion of corresponding edges of the fundamental polygon has, in the sense of analysis situs, a certain genus p."[87] As the reader may remember, this is exactly the passage which left Poincaré puzzled in his letter to Klein of June 1881 which we quoted in our introduction.

Poincaré's note of December 1881, [Poincaré 1881a], is not directly linked to the previous works of Klein, but its publication date and a part of its content strongly invites to recognize the latter's influence. A passage of this text, indeed, is strikingly reminiscent of what Klein expounded in his publications of 1879 and in his letter on the meaning of the *Geschlecht im Sinne der Analysis situs*: as Poincaré explained the interest of studying curves drawn on a certain surface, he explained:

> This surface is composed with a certain number of closed sheets [*nappes*]. Let S be one of these sheets and let p be its genus, that is, the number of separated cycles that can be drawn on this sheet without separating it into two different areas (thus a sphere, and, in general, a convex surface will be of genus 0, a torus will be of genus 1, a surface which was convex at first and in which p holes would have been pierced will be of genus p).[88] [Poincaré 1881a, p. 952]

Poincaré then also recalled what Klein called Euler's generalized polyhedron theorem of Euler. If he did not provide any reference about this theorem, or about the notion of the genus of a surface, the fact that he recalled the definition of the genus (in the exact same wording as Klein's) and gave elementary examples suggests that he thought that it would be useful for his readers, who could be as unsettled as he had been when reading Klein.

The other papers, where Dyck, Hurwitz, and Thomae talked about "the genus of a Riemann surface," were not explicit about the origin and the meaning of this expression, and their authors used it like it was completely natural. In Thomae's paper, the genus was apparently taken as the number of independent of the first kind on a Riemann surface, and

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[86]"[Dies] ist eben die Riemann'schen Fläche, welche wir suchen; nun ist sie, statt $(n+1)$-blättrig über der J-Ebene ausgebreitet zu sein, frei im Raume gelegen gedacht. [...] Die Regel, vermöge deren man das Geschlecht der Fläche aus Blätterzahl und Verzweigungspunkten berechnet, verwandelt sich für sie in den sogenannten verallgemeinerten Euler'schen Polyedersatz: $e + s = k - 2p + 2$." Here the letters e, s, and k respectively designate the numbers of vertices, edges, and faces of the considered polyhedron.

[87]"Die geschlossene Fläche, welche durch Vereinigung der zusammengehörigen Kanten des Fundamentalpolygons entsteht, besitzt, im Sinne der Analysis situs, ein gewisses Geschlecht, p." [Klein 1880, p. 64].

[88]"Cette surface se compose alors d'un certain nombre de nappes fermées. Soit S une de ces nappes, p son genre, c'est-à-dire le nombre de cycles fermés que l'on peut tracer sur cette nappe sans la séparer en deux régions différentes (ainsi une sphère, et en général une surface convexe sera de genre 0, un tore sera de genre 1, une surface primitivement convexe dans laquelle on aurait percé p trous sera de genre p)."

was used in a reasoning related to algebraic and Abelian functions. As for the articles of Klein's students, the genus was computed with the Riemann-Hurwitz formula, and was attached either to a Riemann surface or to a surface in space obtained by gluing the edges of a hyperbolic polygon, and supposed to represent a Riemann surface: this perfectly echoes the research of Klein which we described above, and which was abundantly cited by Hurwitz and Dyck.

On June 27 1881, when Poincaré asked Klein about the meaning of the "genus in the sense of analysis situs," he had certainly not read those of the exceptional papers which had been published at the time: apart from the papers of Klein which he did not know, the only possible one (when considering the dates) is the Dyck's article of 1880, which appeared in the same volume of *Mathematische Annalen* as Klein's 1880 one.[89] Further, and even if the way of how our corpus has been built invites to be cautious, it is likely that very few papers which do not belong to this corpus actually contained the problematic phrase. In other words, even if Poincaré had read at length the mathematical literature of the time, he would have certainly not encountered the "genus of a Riemann surface." Hence, rather than highlighting a youthful shortcoming, his question to Klein reflects the originality of the latter's use of the word *Geschlecht* to characterize surfaces and Riemann surfaces instead of objects of projective geometry.[90]

3.5 Epilogue: a view at the turn of the century

One may naturally wonder what happened to the phrase "genus of a Riemann surface" after 1882, and, in particular, if it quickly replaced the expressions involving the connectivity, as it is now the case. Since it would exceed the scope of the present paper, I will not try to answer this question in the same way that I did for the period 1857–1882; an interesting view, however, can be gained by looking at some of the textbooks on complex analysis or algebraic functions which were published at the turn of the century.[91] A clear evolution is to be observed in these textbooks, as can be seen in the following examples.

A first, yet exceptional, case it the 1882 book by Klein on "Riemann's theory of algebraic functions" [Klein 1882].[92] It appears that Klein used none of the words "genus" and "connectivity": he just defined "Riemann's p" as the maximum number of loops that can be drawn on a closed surface without dividing it (pp. 25–26), and then repeatedly used phrases like "a surface with $p = 1$" (p. 26 sqq.). Throughout the book, Klein insisted on the number p itself, and not on $2p + 1$, which would be the corresponding order of connectivity; that he did not name it *Geschlecht* might indicate that he knew that this terminology could be seen as problematic by his readers, when applied to surfaces and Riemann surfaces.

[89] The article of Thomae, dated April 1881, was probably not in circulation yet: it was published in the third sub-issue of the 18th volume of *Mathematische Annalen*, which contained a paper of Guiseppe Veronese dated June 1881.

[90] As already stated, genera of other objects (such as quadratic forms) existed at the time, although they were directly connected to analysis situs: the existence of several, independent notions of genera could have encouraged Poincaré to think that the "genus in the sense of analysis situs" was yet another one.

[91] I considered the books listed in the general bibliographies of the chapters of the *Encyklopädie der mathematischen Wissenschaften* devoted to these subjects [Osgood 1901; Wirtinger 1901].

[92] This book is the one where Klein presented his description of Riemann surfaces with the help of considerations coming from physics. See [Scholz 1980, pp. 182–188].

To take another example of a book written by a mathematician who contributed to our corpus, Camille Jordan's celebrated *Cours d'analyse de l'École polytechnique* included (in its second edition) a chapter on Abelian integrals [Jordan 1894]. The distinction, there, was clear, as Jordan defined the connectivity of Riemann surfaces on the one hand, and the genus of algebraic curves on the other hand, without ever mixing the terminology.[93]

Such mixes occurred in textbooks written by mathematicians from younger generations.[94] Thus Appell (1855–1930) and Édouard Goursat (1858–1936), in their book on algebraic functions, [Appell and Goursat 1895], first defined the order of connectivity of a Riemann surface, then proved that it is of the form $2p + 1$ in the case of a closed surface, and called p the genus of the surface (p. 229)—the exact same process occurred in a book by Luigi Bianchi (1856–1928) on complex analysis, [Bianchi 1901, p. 244]. As for Heinrich Burkhardt (1861–1914), he directly defined the *Geschlechtzahl* of a closed Riemann surface, and then recurrently employed expressions like: "the sphere has the genus $p = 0$, the torus has the genus $p = 1$," etc. [Burkhardt 1899, p. 9].

The English case is also particularly telling. For instance, Henry Frederick Baker (1866–1950) talked about the deficiency of a Riemann surface, "as defined by Riemann by means of connectivity" [Baker 1897, p. 6]—this was the only occurrence of the word "connectivity" in this book. Like his continental colleagues, Baker thus described Riemann surfaces with the word that had been kept for algebraic curves, but he did it by following the English usage of "deficiency." Another phenomenon happened in the *Theory of Functions of a Complex Variable* of Andrew Forsyth (1858–1942), [Forsyth 1893]. The author first defined and presented the properties of the connectivity of surfaces, and, in the case of a closed surface, he proved that the connectivity is an odd number $2p + 1$. He then explained that "the surface is often said to be of *class* p," and added in a footnote that "the German word is *Geschlecht*; French writers use the word *genre*, and Italians *genere*." [Forsyth 1893, p. 324]. Hence, if Forsyth reported the continental terminology usually linked to algebraic curves, he did not use the appellation "deficiency" for a Riemann surface. But the situation changed in the third edition of the book of 1918,[95] where the exact same explanation was provided, up to an eventual replacement of "class" by "genus" [Forsyth 1918, p. 372].

The terminology thus progressively evolved towards what is now customary, but the few examples given here indicate that during a long period of time, the connectivity and the genus of surfaces and Riemann surfaces coexisted. To provide a last point of comparison, we finally observe that in Hermann Weyl's highly influential book on Riemann surfaces, [Weyl 1913], these surfaces were characterized by their *Geschlecht*, while the notion of connectivity remained apparent only in the phrase "simply connected surfaces."

[93] Jordan's contribution to our corpus is a paper of 1866 dealing with polyhedrons [Jordan 1866]. In this paper, Jordan explained that "a surface is of species (m, n) if it is limited by m closed contours, and if n closed contours which do not intersect themselves can be drawn on it, without separating it into two different areas," to which he added that, if $m \neq 0$, "a surface of species (m, n) is $(m + 2n)$-ply continuous (*zusammenhængend), when one gives to this term the same definition than M. Riemann's.*" [Jordan 1866, pp. 1339–1441].

[94] Jordan and Klein were born in 1838 and 1849, respectively.

[95] I could not get access to the second edition.

4 Of words and meanings

If associating the word “genus” to a surface or a Riemann surface is nowadays absolutely ordinary, I have tried, throughout this paper, to display the issues, both historic and historiographic, that are hidden behind such an apparent banality.

Clebsch’s 1865 introduction of the *Geschlechter* did not consist in merely christening a notion that was already in its definitive form, without yet having a name, in Riemann’s research on Abelian functions. Their definition was rooted in a veritable revisit of this research, a homogeneous, geometric revisit which placed algebraic curves at the core of both its objectives and its own technical functioning. Thus the *Geschlechter* were tied to projective curves, their singular points, their tangents, their generation by pencils of lines, their parameterizations, their diverse enumerations, etc.—all of which were absent from Riemann’s memoir of 1857. After 1865, Clebsch and the other mathematicians continued to link the genus to such objects, even when they extended the notion to algebraic surfaces for instance; this is particularly true for the numerous works of our first cluster, where the genus mostly appeared through its expression $p = \frac{(n-1)(n-2)}{2} - d$ and its invariance by birational transformations, that is, through fundamental features that Clebsch had first brought to light. The order of connectivity $2p + 1$, for its part, was kept for the description of surfaces and Riemann surfaces, where the quantity p occurred under the garment of the number of independent integrals of the first kind or of the number of sections that are necessary to disconnect a surface.

If the terms “genus” and “connectivity” thus marked specific objects, they also marked specific dynamics of research, in the sense that our clusters, which have been detected by inspecting the citation links between the texts of the corpus, could also, in part, be characterized by the distribution of these terms in the corpus, a complete distinction being obtained by the additional consideration of the disciplinary classification of the *Jahrbuch*. In particular, the papers of the first cluster, which are those belonging to the geometric sections, never involved the connectivity, at the exception of the ones included in the subsection of analysis situs. On the other hand, the papers classified under “Function theory” were constituents of the three other clusters, which could be discriminated by the presence or absence of our keywords. Thus, the third cluster, organized around Klein’s works on elliptic functions and algebraic equations, was marked by a relative abundance of uses of both the genus and the connectivity, and by the new direct association of the genus with surfaces and Riemann surfaces.

This semantic association, as we saw, reflected and incarnated two dependent shifts made by Klein. At first, the name *Geschlecht* was tied to a Riemann surface through the intermediary of an equation $\varphi(s, z) = 0$, seen as defining both an algebraic curve and an algebraic function with its corresponding Riemann surface. These objects, however, remained clearly separated: the new phrase “genus of a Riemann surface” was not linked to an ontological merger of curves and Riemann surfaces. On the contrary, Klein’s way to view “more freely” a Riemann surface as an usual surface of space was the mathematical act which led to transport the genus to yet another object, and so to talk about the “genus of a surface.” These new expressions were then adopted by some of Klein’s colleagues, and seem to have progressively circulated and, at some point, to have replaced Riemann’s initial “connectivity.”

These movements, which operated on mathematical objects themselves, thus led to the forging of original phrases. But with original phrases come changes in the representations of the words that they involve.

Just as words of the natural language, mathematical words, beyond their sole technical definition(s), carry connotations. They are intertwined in networks of meanings linked to the personal knowledge of the people who employ or read them, and to the collective representations which are made of them, and which include genealogies of (groups of) authors, objects and domains of mathematics that may be naturally connected to them, as well as advocated methods and values for instance—meanings coming from outside mathematics could certainly be added to this list. Such networks of meanings, further, are not invariable with time; the transformations that mathematics undergo make them evolve, by making some of their facets change, emerge, or withdraw.

Before the beginning of the 1880s, mathematicians had no reason to spontaneously associate the "genus" with Riemann surfaces, since this word was tied to objects and works of projective geometry above all. In this respect, it is particularly telling that the few mathematicians of the time who made use of the word "genus" to account for Riemann's research were led to do so only because they also ascribed to him results on algebraic curves; as was then common usage, no one would have characterized Riemann's past achievements related to his surfaces with a word which referred to other kind of objects.

This phenomenon, of course, is not just to be linked with the use or the avoidance of words that do not appear in Riemann's research to describe it. For instance, nineteenth-century mathematicians rapidly employed the phrase "Riemann surface" in such narratives—and so did I in the present investigation. The difference is that "surface" was Riemann's word, and that adjoining his name was precisely a way to refer to the type of surfaces which he had introduced: "Riemann surface" had no meaning attached to anterior works of others, or stemming from different parts of mathematics. Unlike "genus," it did not carry a network of meanings which would not be conform to the representations of the time.

As for "the genus of a Riemann surface," its circulation and progressive adoption surely contributed to make the networks of meanings of the word "genus" evolve, by giving more room to Riemann and, at the same time, weakening the position of Clebsch. Conversely, because this expression has now become standard in mathematics, historians may tend to use it spontaneously to describe Riemann's research, thus perpetuating the custom of inscribing the latter into their readers' network of "genus" in quite a distorted way.

But networks of meanings are not invariable. The present paper, I hope, may contribute to change that of *Geschlecht*, by restoring some elements of its history that have been obliterated with time, and by offering a glimpse at two notions which had, for a moment, their own trajectories before being amalgamated under the same name.

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