Of fame and legacy: parametrising algebraic curves from Clebsch to Humbert

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Abstract

This article explores the legacy of Alfred Clebsch related to the topic of algebraic curve parametrisations, from his innovative methods using elliptic and Abelian functions (1864–1865) to their reworking by Georges Humbert in the framework of Fuchsian functions (1886). An in-depth analysis of the contributions of these two mathematicians, as well as others who contributed the topic during the two decades that separate their works, shows how Clebsch's methods and results were taken over and reinterpreted effectively. Additionally, reflections on how Clebsch's enduring fame was constructed and perpetuated at the end of the 19th century are provided.

In 1885, contextualising Georges Humbert's doctoral thesis *Sur les courbes de genre un*, Charles Hermite wrote in his report:

> It is to Clebsch, one of the illustrious geometers of Germany, whose discoveries occupy such a prominent place in the science of our time, that we owe these fertile and entirely new methods which are based on the use of doubly periodic functions and have shed the brightest light on many of the most difficult questions in geometry. All the analysts have read with awe the memoirs in which Clebsch gave the interpretation of Abel's theorem on the sums of integrals of algebraic differentials, those which he devoted to plane cubics, to the [quartic] which is the intersection of two surfaces of the second order, finally to the curves of any order which are of genus one.[1]

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[1] "C'est à Clebsch, l'un des illustres géomètres de l'Allemagne dont les découvertes occupent une si grande place dans la science de notre époque, que sont dues ces méthodes fécondes et entièrement nouvelles fondées sur l'emploi des fonctions doublement périodiques et qui ont jeté la plus vive lumière sur beaucoup de questions les plus difficiles de la géométrie. Tous les analystes ont lu avec admiration les mémoires dans lesquels Clebsch a donné l'interprétation du théorème d'Abel sur les sommes d'intégrales de différentielles algébriques, ceux qu'il a consacrés aux cubiques planes, à la quadrique [sic] intersection de deux surfaces du second ordre, enfin aux courbes d'un ordre quelconque qui sont de genre un." Cited from [Gispert 2015, p. 222]. In my translation, I corrected Hermite's mistaken use of "quadrique" instead of "quartic".

This statement on the enduring fame and influence of Alfred Clebsch's evoked works, published some twenty years earlier, is intriguing for at least three interrelated reasons.

First, despite Hermite's indisputable authoritative status, his characteristic hyperbolic writing style[2] and the circumstances in which he composed the preceding lines raise the question of the actual commonality of Clebsch's renown at the time. Second, beyond this renown, nothing is said on how Clebsch's methods and results might have effectively shaped and been integrated in later research. Third, Hermite's words suggest a certain interface between geometry and analysis[3] which calls for a more precise description of how it was implemented by Clebsch and how it possibly evolved in subsequent works.

This article seeks to investigate these issues, focussing on the topic in which Humbert's thesis was rooted: the global parametrisations of algebraic curves and their consequences. Specifically, Humbert introduced his work by recalling that Clebsch [1865b] had shown the possibility of parametrising any curve of genus one[4] by formulas such as

$$x = \varphi(t), \quad y = \psi(t),$$

where φ and ψ are elliptic functions, and the parameter t is a complex number.[5] In fact, at the beginning of the 1880s, Henri Poincaré had extended this result, proving that any curve of genus greater than one can be parametrised by Fuchsian functions.[6] Only one year after his thesis, Humbert [1886] published a broad paper where he mimicked the techniques developed in the thesis to study curves on the basis of their Fuchsian parametrisation, and where Clebsch was still cited, this time for his works on the applications of Abelian functions to geometry [Clebsch 1864a].

[2]This style is most apparent in Hermite's private letters, such as [Hermite 1984]. See also the commented examples given in [Goldstein 2011]. As for Hermite's style in his published papers, see [Lê 2024].

[3]At the end of the century, the word *géomètre* was still commonly employed to designate any mathematician, but Hermite's concurrent use of *analystes* in the second sentence offers an interesting contrast.

[4]An algebraic curve defined by an equation of degree n is said to be of order (or degree) n. If it has only double points as possible singularities, its genus is the number $p = \frac{(n-1)(n-2)}{2} - d$, where d is the number of double points. Thus the unit circle $x^2 + y^2 = 1$ is of order 2 and genus 0, the cubic $y^2 = x^3 - x$ is of order 3 and genus 1, etc.

[5]Elliptic functions are meromorphic functions with two independent periods, which echoes Hermite's qualifier "doubly periodic" seen above. In the second half of the 19th century, doubly periodic functions were seen as encompassing elliptic functions, the latter being understood as the reciprocal of elliptic integrals – see [Briot and Bouquet 1859], for instance. This distinction being unimportant for my purpose, I will neglect it in this paper. Let me also recall that parametrising a curve by the formulas $x = \varphi(t)$, $y = \psi(t)$ means that any of its points has coordinates $(\varphi(t), \psi(t))$ for some t. For instance, the unit circle $x^2 + y^2 = 1$ (minus the point $(-1, 0)$) is parametrised by $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$.

[6]Fuchsian functions can be seen as generalisations of elliptic functions. Technical explanations will be provided below, together with references on their history.

I gathered the corpus that forms the basis of the present investigations by beginning with Humbert's two mentioned texts [Humbert 1885, 1886], then moving backwards through their references until arriving to Clebsch's papers devoted to the applications of elliptic and Abelian functions to geometry [Clebsch 1864b,a, 1865b]. This provides a coherent set of articles and theses by Ferdinand Lindemann [1876; 1878], Hermite [1877a, 1878], Hermann Amandus Schwarz [1880], Robert D'Esclaibes [1880, 1881], Émile Picard [1880a, 1881] and Poincaré [1881a,b, 1882b,c], alongside those by Clebsch and Humbert.

My examination of these contributions will include a thorough analysis of the mathematical technique. By scrutinising how curves were parametrised and how this lead to specific theorems and proofs, my intention is not only to help the reader understand the mathematical content of this research and grasp at a fine-grained level the disciplinary articulations between functions and curves. I also aim at uncovering historical continuities which manifest at the core of the considered papers through designations of objects, statements of theorems and their applications, or proof techniques, and which remain invisible if one looks only at the introductions of these papers or at overall remarks made therein.

In addition to the results on Clebsch's legacy around curve parametrisations, three outcomes will emerge from the present investigations. First, it will illustrate Hermite's multifaceted role as a mathematician, who appears not only as a contributor of papers on the topic but also as the author of fundamental results on elliptic functions, as an instigator of research and as a social connector.[7] Second, it will shed light on the works and practice of Humbert, a mathematician who would become central in the mathematics of the time but is still poorly known by the historiography.[8] Third, it will contribute the history of Fuchsian functions by showing an example of how these functions circulated after their creation by Poincaré, a creation on which historical research has concentrated thus far.[9]

The sections of this paper follow an order which is a compromise between strict chronology and thematic groupings. The first one is devoted to Clebsch's 1864 and 1865 papers on the application of elliptic and Abelian functions to geometry, as well as his lectures on geometry, edited by Lindemann and published in 1876. The contributions described in the second section include published letters by Hermite and Lindemann (1877, 1878), a 1880 paper by Schwarz written "at Mr. Charles Hermite's invitation", and D'Esclaibes 1880 doctoral thesis, which was devoted to the applications of elliptic functions to curves of genus 1, and of which Hermite wrote the review; for thematic reasons,

[7] On Hermite's various mathematical commitments, see in particular [Goldstein 2012].

[8] Humbert's research on algebraic surfaces is described in [Houzel 2002, pp. 235–255; Brigaglia 2016]. See also [Goldstein 2009; Leloup 2009, pp. 152–157] for his role in the development of number theory. Biographical elements on Humbert will be given below.

[9] See, however, [Bergeron 2018; Goldstein 2023] for the links established by Poincaré himself between Fuchsian functions and arithmetic.

Humbert's 1885 thesis is also analysed in this section. Indeed, the third and fourth sections are dedicated to Fuchsian functions, first with the setting up of the Fuchsian parametrisation by Picard and Poincaré (1880, 1881), then to its applications to algebraic curves by Humbert (1886).

In the final, concluding section, I briefly mention the importance of the idea of parametrising geometric objects in Humbert's later research. Then I come back to the three issues stemming from Hermite's opening quote and examine them (in a reverse order) in synthesising the results obtained throughout the article. In particular, I propose some reflections on the fame and actual legacy of Clebsch's mathematical approach consisting in bridging together Abelian functions and geometry.

1. Clebsch

1.1 An elliptic parametrisation to prove a theorem of Steiner

In 1864, Clebsch published an article where he showed how to parametrise any non-singular cubic curve with elliptic functions, his aim being to prove a theorem stated almost 20 years before by Jacob Steiner [Clebsch 1864b].[10]

Clebsch crucially drew on a result that Siegmund Aronhold developed in a paper where he tackled a question coming from analysis and dealing with elliptic integrals [Aronhold 1862]. Specifically, Aronhold proved that if a non-singular cubic curve and one of its points $(\alpha_1 : \alpha_2 : \alpha_3)$ are given, there exist three polynomials P_1, P_2, P_3 and two constants S, T such that the formulas

$$x_1 = P_1(\lambda) + \alpha_1\sqrt{\frac{1}{6}(2T + 3S\lambda - \lambda^3)}$$
$$x_2 = P_2(\lambda) + \alpha_2\sqrt{\frac{1}{6}(2T + 3S\lambda - \lambda^3)}$$
$$x_3 = P_3(\lambda) + \alpha_3\sqrt{\frac{1}{6}(2T + 3S\lambda - \lambda^3)}$$

describe the homogeneous coordinates of the points of the cubic as λ runs through all complex numbers.[11] As can be seen these parametrisation formulas involve rational functions and the square root of a thrid-degree polynomial in the parameter λ. Clebsch transformed them by introducing elliptic functions.

[10]Clebsch's approach and sources are studied in [Lê 2018a]. Here I only repeat what is important to see the links with his and his successors' research.

[11]Homogeneous coordinates $(x_1 : x_2 : x_3)$ allow to describe the points of the projective plane, with those situated at infinity corresponding to $x_3 = 0$. The equation of a curve given in non-homogeneous coordinates is then associated with an homogeneous version in this projective framework. For instance, the unit circle $x^2 + y^2 = 1$ has $x_1^2 + x_2^2 = x_3^3$ as an homogeneous equation, so that its points at infinity are given by $x_1^2 + x_2^2 = 0$.

At the time, a common approach was to define these functions as the reciprocals of elliptic integrals, i.e. of primitive complex functions

$$u(x) = \int \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

where k^2 is a constant different from 0 and 1.[12] In this integral, the integration is made along a path in the complex plane, starting from a fixed point and ending in x. Due to the nature of the integrand, different integration paths joining the same lower and upper bounds generally lead to different values of $u(x)$. More precisely, there exist two numbers ω_1, ω_2, which are themselves special values of the integral, such that any change of the integration path transforms $u(x)$ into $u(x) + m_1\omega_1 + m_2\omega_2$, for some integers m_1, m_2.[13]

To this property corresponds the double periodicity of the elliptic function sn, defined as the reciprocal of the previous elliptic integral.[14] In other words, if $\operatorname{sn} u$ is defined by

$$\operatorname{sn} u = x \iff u(x) = \int \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

one has

$$\operatorname{sn}(u + m_1\omega_1 + m_2\omega_2) = \operatorname{sn} u$$

for any integers m_1, m_2.

Starting from a non-singular cubic curve, Clebsch managed to define a number k^2, and thus an elliptic function sn, in such a way that Aronhold's formulas can be transformed into

$$x_i = \varphi_i(\operatorname{sn}^2 u) + m\alpha_i \frac{\mathrm{d}\operatorname{sn}^2 u}{\mathrm{d}u},$$

where the φ_i are polynomials, m is a constant and u is the new parameter. In itself, this elliptic parametrisation was not enough to prove Steiner's theorem: Clebsch used it to express geometrical properties of the points of the cubic. In particular, he proved that three points of parameters $u^{(1)}, u^{(2)}, u^{(3)}$ are aligned if and only if

$$u^{(1)} + u^{(2)} + u^{(3)} \equiv 0,$$

[12] See for instance [Houzel 2002, Ch. VIII; Bottazzini and Gray 2013, Ch. 4].

[13] Clebsch did not explicitly adopt the viewpoint of complex integration paths in the article discussed here, but he would do so when tackling Abelian functions. This viewpoint was not original at the time: it is for instance used in [Briot and Bouquet 1859, 95 sq.].

[14] Although Clebsch did not use Christoph Gudermann's notation sn and favoured Carl Gustav Jacob Jacobi's $\sin \operatorname{am}$, I will use it for the sake of simplicity. More generally, in this subsection I am slightly adjusting Clebsch's notations and constant values so as to facilitate the understanding of the technique and to highlight the analogy with the future work on Abelian functions.

in the sense that there exist integers m_1, m_2 such that the left-hand side is equal to $m_1\omega_1 + m_2\omega_2$.[15] As Clebsch explained, this theorem is a particular case of a much more general one, which is a consequence of Abel's theorem on the addition of Abelian functions and which Clebsch would make central in the next 1864 paper, devoted to the application of Abelian functions to geometry.

1.2 Turning to Abelian functions...

In this paper, Clebsch moved to curves of any order n [Clebsch 1864a]. To deal with these curves, he resorted to general Abelian functions, which are primitive integrals

$$u = \int R(x, y)\,\mathrm{d}x,$$

where R is a rational function and y is an algebraic function of x, that is, a function defined implicitly by a polynomial equation $f(x, y) = 0$. As in the case of elliptic integrals, the integration is made along a path in the complex plane joining a given starting point and x.[16]

In 1864, the theory of Abelian functions was much less advanced than that of elliptic functions. Bernhard Riemann had published his fundamental 1857 memoir on the topic [Riemann 1857], but mathematicians were still struggling to assimilate his ideas [Bottazzini and Gray 2013, pp. 311–339]. While Clebsch rejected the use of the surfaces that Riemann had introduced to study Abelian functions (and that would soon be called "Riemann surfaces"), he did draw on many of Riemann's other results. In particular, he saw the importance of a number p attached to any Abelian function defined via the same grounding equation $f(x, y) = 0$. Riemann had defined this number by topological considerations, and had shown that it is equal to the maximal number of linearly independent integrals of the first kind, of which a set is denoted by $u_1, \ldots, u_p$.[17]

In his work, Clebsch's first move was to import p in the framework of projective geometry: starting from an algebraic curve C of order n, he attached to it a certain equation $f(x, y) = 0$ and thus a class of Abelian functions. He then proved that the associated number p is given by

$$p = \frac{(n-1)(n-2)}{2} - d,$$

[15] Here I am distorting Clebsch's presentation to quite a great extent for the pedagogical reasons explained above. In his original version, the right-hand side of the congruence is a constant that could be non-zero, and the congruence is not taken modulo ω_1, ω_2 but modulo $\omega_1/2, \omega_2$, which are the "periods of the curve".

[16] With $R(x, y) = 1/y$ and $f(x, y) = y^2 - (1-x^2)(1-k^2x^2)$, one finds the elliptic integral noted $u(x)$ above. Hence the vocabulary of the time was quite confusing: elliptic *integrals* are particular cases of Abelian *functions*.

[17] Integrals of the first kind are those which remain finite for all x. In current terms, they are holomorphic Abelian functions and their differentials form a space of dimension p, a basis of which corresponds to the integrals $u_1, \ldots, u_p$.

where d is the number of double points of C, under the assumption that this curve has only double points as possible singularities.

It is important to note that in Clebsch's association of the number p with a curve C, the latter was not seen as the locus of the points whose coordinates satisfy $f(x, y) = 0$. In fact, his proof showed that these coordinates are rational functions of x, y, which means that these quantities can be seen as parameters related to one another by $f(x, y) = 0$.[18] This parametrisation possibility was actually quite discreet in Clebsch's memoir and, at any rate, was not applied to any thing. Nevertheless, Clebsch would return to it in a paper published one year later, as will be seen in the next subsection.

Among the other results Clebsch borrowed from Riemann are those related to the periodicity of Abelian functions. As in the elliptic case, the nature of the integrand implies that different integration paths between the same two points can lead to different values of the integral. Hence each integral u_j is generally replaced by

$$u_j + m_1\omega_{j,1} + m_2\omega_{j,2} + \cdots + m_{2p}\omega_{j,2p},$$

where the m_j are integers and the ω_{jk} are complex numbers called the "periodicity modules" of u_j. Here again, these modules are obtained as particular values of the integral. Further, Riemann had demonstrated the possibility of choosing the p integrals of the first kind in such a way that their periodicity modules have a specific form. To be more precise, for such a choice, the periodicity modules are $i\pi$ and a series of numbers a_{jk}, so that a change of integration path transforms (simultaneously) the integrals $u_1, \ldots, u_p$ into

$$\begin{aligned} &u_1 + m_1 i\pi + q_1 a_{11} + \cdots + q_p a_{1p} \\ &u_2 + m_2 i\pi + q_1 a_{21} + \cdots + q_p a_{2p} \\ &\quad\vdots \\ &u_p + m_p i\pi + q_1 a_{p1} + \cdots + q_p a_{pp} \end{aligned}$$

for some integers $m_1, \ldots, m_p, q_1, \ldots, q_p$.[19]

Clebsch's central theorem was what he presented as a geometric consequence of Abel's theorem on the addition of Abelian integrals.[20] For its statement, and thus for the rest of his paper, Clebsch supposed that the curve C is non-singular. He considered mn points $x^{(1)}, \ldots, x^{(mn)}$ on it and denoted by $u_j^{(1)}, \ldots, u_j^{(mn)}$ the values taken by the integral u_j in these points. The theorem then states

[18] For more details on Clebsch's approach, see [Lê 2020].

[19] Such integrals of the first kind and such periodicity modules would later be qualified as "normal" [Clebsch and Gordan 1866, pp. 107–109]. Although this is not essential to understand what follows, let me note that the normal periodicity modules a_{jk} form a symmetric system, i.e. $a_{jk} = a_{kj}$ for every pair of indices j, k.

[20] On this theorem, see [Cooke 1989; Kleiman 2004].

that the points $x^{(k)}$ are the intersections of C with some curve of order m if and only if

$$\begin{aligned} u_1^{(1)} + \cdots + u_1^{(mn)} &\equiv 0 \\ u_2^{(1)} + \cdots + u_2^{(mn)} &\equiv 0 \\ \vdots \qquad\qquad & \quad \vdots \\ u_p^{(1)} + \cdots + u_p^{(mn)} &\equiv 0, \end{aligned}$$

the jth congruence being considered modulo $i\pi, a_{j1}, \ldots, a_{jp}$.[21]

As a first consequence of this theorem, Clebsch proposed to solve what he called the problem of "contact curves". This problem, of which several variations were proposed, can be stated as follows. The given data are a non-singular curve C of order n; two integers m, r such that $m \geqslant n-2$ and $\lambda := mn - pr \geqslant 0$; and λ points on C. One must then determine the curves of order m passing through these λ points and touching C in p other points with multiplicity r.[22]

Clebsch's approach consisted first in counting the solutions of the problem, their existence being guaranteed by a constant counting. He explained that if $u_j^{(1)}, \ldots, u_j^{(\lambda)}$ are the values of the integrals associated with the given points, the remaining ones $u_j^{(\lambda+1)}, \ldots, u_j^{(mn)}$ must coincide r by r since they represent points where there is an r-fold tangency. Thus, setting $v_j = u_j^{(\lambda+1)} + \cdots + u_j^{(mn)}$, the equations of the above theorem become

$$\begin{aligned} rv_1 &\equiv -(u_1^{(1)} + \cdots + u_1^{(\lambda)}) \\ rv_2 &\equiv -(u_2^{(1)} + \cdots + u_2^{(\lambda)}) \\ &\vdots \\ rv_p &\equiv -(u_p^{(1)} + \cdots + u_p^{(\lambda)}), \end{aligned}$$

which amounts to the existence of integers $m_1, \ldots, m_p, q_1, \ldots, q_p$ such that

$$v_1 = -\frac{u_1^{(1)} + \cdots + u_1^{(\lambda)}}{r} + \frac{m_1 i\pi + q_1 a_{11} + \cdots + q_p a_{1p}}{r}$$

[21] Let me recall that Bezout's theorem states that two curves of orders n and m (without common component) intersect in mn points, counted with multiplicity, provided complex points and points at infinity are considered. Clebsch's theorem goes the other way round: given mn points on a curve of order n, it gives a necessary and sufficient condition for these points to belong to a curve of order m.

[22] Bezout's theorem assures the coherence of the data with the problem: if a contact curve exists, its intersection points with C are made of the λ points (each being simply counted) and of the p tangency points (each counted r times), so that there are $\lambda + pr = mn$ intersection points.

$$v_2 = -\frac{u_2^{(1)} + \cdots + u_2^{(\lambda)}}{r} + \frac{m_2 i\pi + q_1 a_{21} + \cdots + q_p a_{2p}}{r}$$

$$\vdots$$

$$v_p = -\frac{u_p^{(1)} + \cdots + u_p^{(\lambda)}}{r} + \frac{m_p i\pi + q_1 a_{p1} + \cdots + q_p a_{pp}}{r}.$$

Since adding multiples of r to the m_j and q_j leads to the same values of the v_j up to the periods, these conditions provide in all r^{2p} possibilities for all the v_j, each of them being associated with the values of the m_j and q_j modulo r. From this finally follows that there are r^{2p} contact curves.[23]

Another component of Clebsch's approach of the contact curves problem was to investigate the grouping properties of the obtained configurations of curves and points, always using Abelian integrals and the consequence of Abel's theorem. To take an example, by definition, any contact curve is associated with a system of p points of r-fold contact with the given curve C; Clebsch proved that for any $r-1$ such systems of points, there exists a curve of order m passing through them and the λ fixed points, with the property that the remaining intersection points with C form yet another system of contact points.[24]

Clebsch also investigated particular cases for m or n, which would lead to further consequent properties. For instance, he counted $2^{\frac{n(n-3)}{2}}(2^{\frac{(n-1)(n-2)}{2}}-1)$ curves of order $n-3$ that are tangent to the given curve in $\frac{n(n-3)}{2}$ points. He also showed that there are 3^{20} curves of order 5 that touch a given curve of order 6 in ten points with multiplicity 3. Then, a grouping property of this configuration is that the 2×10 contact points associated with any two solutions belong to a curve of order 5 such that its 10 remaining intersection points with the sextic are themselves contact points of another solution, etc.

1.3 … and returning to (rational and) elliptic functions

While the number p played a crucial technical role in the 1864 memoir, it would also serve as the basis for the new classification of algebraic curves in genera when Clebsch proposed, in 1865, to gather in one and the same genus all the curves having the same number p – shortly afterwards, the number p itself would be called the genus of the curves [Lê 2020, 2023]. If the genera were thus

[23] Clebsch's proof of this last point involves θ-functions and the inversion of Abelian functions to make the link between the v_j and the existence of corresponding contact curves. Under other hypotheses, Clebsch showed that an infinite number of contact curves exist, but that they can be grouped into a finite number of *systems*, of which the notion was borrowed from Otto Hesse [1855].

[24] Given $\lambda + p(r-1)$ fixed points, there always exist a curve of order m passing through them. By Bezout's theorem, this curve intersects C in $mn - \lambda - p(r-1) = p$ points, and Clebsch's result states that these p points are not arbitrary: they are contact points as in the initial problem.

defined simultaneously through the order and the number of double points of curves, on the one hand, and the Abelian functions associated with them, on the other hand, Clebsch also underlined their link with the parametrisations that curves admit:

> Curves that belong to the same genus can also be defined by the fact that their homogeneous coordinates can be represented as integral rational functions of two parameters s, z, between which there is an algebraic equation $f(s, z) = 0$, which founds the corresponding class of Abelian functions. In particular, for $p = 0$ one can express s by z rationally; for $p = 1$, this representation demands the square root of an expression of the third of fourth degree, etc.[25] [Clebsch 1865a, pp. 43–44]

This assertion was supported by a reference to the memoir on Abelian functions where Clebsch only alluded to this, as I explained above.

Clebsch devoted two papers to curves with $p = 0$ and $p = 1$, respectively [Clebsch 1865a,b]. Here I focus on the latter, which will be referred to by Clebsch's successors.[26]

The theorem contained in the previous quotation (implicitly) states that any curve of genus 1 can be parametrised by formulas of the form

$$x_i = P_i(\lambda) + \Pi_i(\lambda)\sqrt{Q(\lambda)},$$

where the P_i, Π_i and Q are polynomials, Q being of degree 4. In the article dedicated to the case $p = 1$, Clebsch demonstrated this result avoiding any recourse to Abelian functions. Indeed, the proof of the existence of polynomials P_i, Π_i, Q (of unspecified degree) was based on the fact that, by definition of p, curves for which $p = 1$ possess $\frac{(n-1)(n-2)}{2} - 1 = \frac{n(n-3)}{2}$ double points. Roughly speaking, these points allow to define two pencils of curves (that is, two one-dimensional families of curves) whose parameters λ, μ, related by an algebraic equation $\Omega(\lambda, \mu) = 0$, can be used to express the coordinates x_i as desired. Further, proving that Q is of degree 4 makes use of arguments dealing with the so-called Jacobian of three curves.[27] As will be seen, this part of the

[25] "Die zu demselben Geschlechte gehörigen Curven kann man [...] auch dadurch definiren, dass ihre homogene Coordinaten sich als ganz rationale Functionen von zwei Parametern s, z darstellen lassen, zwischen denen eine algebraische Gleichung $f(s, z) = 0$ besteht, die entsprechende Classe Abelscher Functionen begründet. Insbesondere für $p = 0$ kann man s durch z rational ausdrücken; für $p = 1$ erfordert diese Darstellung die Quadratwurzel aus einem Ausdruck dritten oder vierten Grades, u.s.w."

[26] This focus should not mask that the two papers were twins in their aims and overall techniques, the first one dealing with curves that admit a rational parametrisation. In that sense, their very existence participates to the general framework of analogies I am describing.

[27] If $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ are homogeneous equations of three curves, their Jacobian is defined by the vanishing of the determinant made of the first partial derivatives of the f_i. See [Berzolari 1906, pp. 337–338] for further explanations. In particular, this notion is not the same as the current usual one, related to one algebraic curve.

proof would be described as quite involved by D'Esclaibes, who proposed to simplify it.

In the image of what he did in the 1864 paper on Steiner's theorem, Clebsch then rewrote the previous formulas as

$$x_i = F_i(\operatorname{sn}^2 u) + G_i(\operatorname{sn}^2 u)\frac{\mathrm{d}\operatorname{sn}^2 u}{\mathrm{d}u},$$

where the F_i and G_i are new polynomials. However, this was not the definitive elliptic parametrisation.

Immediately after, Clebsch cited Hermite's *Note sur le calcul differentiel et le calcul intégral*, originally a supplement to the sixth edition of Sylvestre-François Lacroix's famous treatise on differential calculus and devoted to an exposition of elliptic function theory [Hermite 1862].[28] Clebsch used a theorem from this text to transform the previous formulas into

$$x_i = c_i\frac{\mathrm{H}(u-\alpha_i')\cdots\mathrm{H}(u-\alpha_i^{(n)})}{\Theta^n(u)}.$$

In this expression, the c_i and $\alpha_i^{(k)}$ are constants, while H and Θ are functions which are closely related to elliptic functions and had already been considered in Jacobi's famous *Fundamenta nova theoriae functionum ellipticarum* [Jacobi 1829]. As Jacobi did at some occasions, Hermite used these "theta" functions to *define* the elliptic function sn [Houzel 2002, pp. 118–124]. They themselves were defined in two alternative ways, as infinite products or convergent sums such as

$$\Theta(u) = \sum_{n=0}^{+\infty}(-1)^n e^{2i\pi n^2\omega/\omega'}\cos\frac{4\pi n u}{\omega},$$

where ω, ω' are given constants such that ω/ω' has a positive imaginary part. The functions Θ, H are not (ω,ω')-periodic, but they satisfy the equations

$$\Theta(u+\omega) = \Theta(u) \quad \text{and} \quad \Theta(u+\omega') = -\Theta(u)e^{-2i\pi(2u+\omega')/\omega},$$
$$\mathrm{H}(u+\omega) = \mathrm{H}(u) \quad \text{and} \quad \mathrm{H}(u+\omega') = -\mathrm{H}(u)e^{-2i\pi(2u+\omega')/\omega},$$

so that the quotient H/Θ is (ω,ω')-periodic. In fact, if one poses $k = \frac{\mathrm{H}^2(\omega/4)}{\Theta^2(\omega/4)}$, one can set

$$\operatorname{sn} u = \frac{1}{\sqrt{k}}\frac{\mathrm{H}(u)}{\Theta(u)},$$

and this function is the same as the one obtained by the inversion of the elliptic integral, as seen above.

[28] On Hermite and elliptic functions, see [Belhoste 1996; Archibald 2024].

The parametrisation with the functions Θ and H, combined with their usual properties (such as those bearing on their zeros and poles[29]), allowed Clebsch to prove a theorem on the intersection of a curve C of order n and genus $p = 1$ with any curve of order m. This theorem was of the same flavour as the consequence of Abel's theorem for Abelian functions, but differed from the latter since singular curves could be considered. Indeed, it states that mn points on C with parameters $u^{(1)}, \dots, u^{(mn)}$ are the intersections of C with a curve of order m if and only if

$$u^{(1)} + \cdots + u^{(mn)} \equiv 0,$$

to which are added a number of conditions whose existence is tied to the fact that C may have double points. These conditions, of which I omit the (involved) expression, would later be referred to as "Clebsch's equations" by Humbert.

Clebsch used this theorem to tackle the problem of contact curves. He himself emphasised the resemblance with his previous works:

> One can treat all the problems that I have solved for curves without double points in my memoir on the application of Abelian functions to geometry [Clebsch 1864a], and for curves with $\frac{(n-1)(n-2)}{2}$ double points in this volume [Clebsch 1865a]. Since the method is completely the same, I will content myself to mention the following theorems for the present case.[30] [Clebsch 1865b, p. 245]

The statement of the problems and the results, indeed, looked just alike the previous ones. For example, Clebsch fixed two integers r, m (with $m \geqslant n-2$), as well as $\frac{n(n-3)}{2} - \mu$ double points and $mn - n(n-3) + 2\mu - r(\mu+1)$ other points on C. Then there exist $r^{\mu+2}$ curves of order m passing through these points and touching additionally the given curve in $\mu + 1$ points with multiplicity r.[31]

Similarly to the previous papers, Clebsch also investigated special cases of curves, as well as various grouping properties of the solutions. For instance, he counted $2^{\mu+1}$ curves of order $n-3$ which pass through $\frac{n(n-3)}{2} - \mu$ double points of C and touch it in μ additional points. Or, for binodal quartics (i.e. curves of order 4 with two double points), that there are 16 conics passing through the two double points and touching the quartic in another point with

[29] Although Clebsch did not state it explicitly, it seems that a decisive advantage of the use of Θ and H is that they make it easier to deal with the zeros and the poles of the parametrising functions x_i.

[30] "Man kann [...] alle diejenigen Probleme behandeln, welche in meinem Aufsatz über die Anwendung der Abelschen Functionen in der Geometrie, Band 63, p. 189 dieses Journals für Curven ohne Doppelpunkte, und welche ich in diesem Bande, pag. 43 für Curven mit $\frac{(n-1)(n-2)}{2}$ Doppelpunkte gelöst habe. Ich werde mich, da die Methode ganz dieselbe bleibt, begnügen, für den vorliegenden Fall die folgenden Sätze anzuführen."

[31] Clebsch also tackled the case where C has cusps in addition to double points. For the sake of brevity, I will not discuss this case and simply note that Clebsch's successors would include it in their own works.

multiplicity 4. As an example of grouping property, Clebsch remarked that these conics can be gathered into 4 groups such that for any two conics of the same group, there exists a third one that passes through the two contact points of the first two and through the two double points of the quartic.

1.4 Clebsch–Lindemann

As Felix Klein explained, following Clebsch's sudden death at the end of 1872, a number of his friends and students soon endeavoured to edit some of the lectures he held, especially those related to geometry. This resulted in the *Vorlesungen über Geometrie*, edited by Ferdinand Lindemann and published in 1876 [Clebsch and Lindemann 1876].[32] The book was translated into French by Adolphe Benoist a few years later, giving rise to three volumes published in 1879, 1880 and 1883 [Clebsch and Lindemann 1879/1883]. As will be seen, Humbert cited this translation as one of his sources for both his doctoral thesis and his paper on Fuchsian functions.

Introducing the book, Lindemann emphasised that the edition work had been more than a mere compilation of the content of Clebsch's lecture manuscripts and of his student's notes. In fact, Klein also asserted that:

> This work has thus become [Lindemann's] own, especially in its later parts; several sections had to be drafted by him from scratch, based on the original works. Yet Clebsch's name should stand at the forefront, not only because Clebsch provided the impetus for the book's creation, and because the scope of its content was always measured in accordance with his lectures, but because the inner substance of the book derives entirely from him [...].[33] [Clebsch and Lindemann 1876, pp. iii–iv]

In particular, the section on the application of Abelian functions to the study of curves was presented by Lindemann as "an entirely independent work of the material given therein".[34] This does not mean, of course, that the results and proofs were necessarily absent from Clebsch's works, quite the contrary: the elliptic parametrisation of non-singular cubics was taken over (and reworked) from [Clebsch 1864b]; many pages were devoted to the link between Abelian functions and curves of any genus, including the consequence of Abel's theorem and its application to contact problems; and, still in line with Clebsch's publications, specific subsections were dedicated to curves with

[32]The book included a preface by Klein where its genesis was described.

[33]"Es ist das Werk dadurch, zumal in den späteren Partieen, namentlich auch *sein eigen* geworden; verschiedene Abschnitte mussten von ihm, auf Grund der Originalarbeiten, überhaupt erst entworfen werden. Aber der Name Clebsch durfte voranstehen – nicht bloss, weil Clebsch die Veranlassung zum Entstehen des Buches abgegeben, und weil die Umgränzung des Stoffes doch immer seinen Vorlesungen conform bemessen wurde – sondern weil der innere Gehalt des Buches durchaus auf ihn zurückgeht [...]."

[34]"Die sechste Abtheilung ist eine vollständig selbstständige Bearbeitung des in ihr gegebenen Stoffe" [Clebsch and Lindemann 1876, p. viii].

$p = 0$ and $p = 1$. Moreover, authors such as Humbert actually tended to erase Lindemann's name when referring to the book, which contributed to credit Clebsch with the paternity of notions, theorems and proofs these authors extracted from it.

For general Abelian functions, a major difference between Clebsch's 1864 memoir and Lindemann's book is that the latter extended the formulation of the contact problem so as to include singular curves: a curve C of order n, genus p, and with d double points being considered, one fixes $m \geqslant n - 2$ and $mn - 2d - pr$ points on the curve. The problem is then to find the curves of order m that pass through these points, have a contact of order r in p other points and are *adjoint* curves, which means that they contain all the double points of C.[35]

This extension of Clebsch's original contact problem was in fact treated in the almost exact same way, since imposing to search for adjoint curves has the effect of involving only Abelian functions of the first kind in the analytic treatment: in particular, it leads to the same form of Abel's addition theorem as in the non-singular case. On the contrary, to solve the contact problem if the given curve is singular and the searched curves are not adjoint requires to deal with Abelian integrals of the third kind, which are more difficult to handle.[36] Lindemann did treat this case but he focussed almost exclusively on quartic curves with one double point (which are of genus 2).

Ten years later, Humbert would present his approach using Fuchsian functions as advantageous because it allowed to avoid Abelian functions of the third kind in the treatment of the general problem of contact curves. In the meantime, other mathematicians (including Lindemann) continued to work on the parametrisations of curves and their consequences.

2. Reworking and exploiting parametrisations

2.1 Hermite and his correspondents

In 1877 appeared in Crelle's journal a paper which consisted of two extracts of letters by Hermite to Lazarus Fuchs, dated June and December 1876 [Hermite 1877a]. In this paper, Hermite investigated the question of the parametrisation of curves, taking as his starting point the function defined by

$$\mathrm{Z}(x) = \frac{\mathrm{H}'(x)}{\mathrm{H}(x)},$$

[35]In fact, Lindemann dealt with curves with multiple points of any order: double points, triple points (around which the curve looks like a trefoil), etc. Specifically, if C contains α_i multiple points of order i, the contact curves are subject to pass through $mn - \sum \alpha_i i(i-1) - pr$ points of C. In this case, a curve C' is said to be adjoint if each multiple point of C of order i is a multiple point of C' of order $i - 1$.

[36]Abelian functions of the third kind are those of which the differentials are meromorphic with two simple poles.

where H is the theta function we already met – Hermite called Z a function "of the second kind".[37] The function Z allowed to express any (meromorphic) doubly periodic function F thanks to a theorem which Hermite presented as the elliptic counterpart of the partial fraction decomposition of rational functions. Specifically, for such a function F with poles $a, b \ldots, l$, there exist constants $A_i, B_i \ldots, L_i$ such that $A_i + B_i + \cdots + L_i = 0$ and

$$\begin{aligned} F(x) = \text{const.} &+ A_0 \mathrm{Z}(x-a) + A_1 \mathrm{D}_x \mathrm{Z}(x-a) + A_2 \mathrm{D}_x^2 \mathrm{Z}(x-a) + \cdots \\ &+ B_0 \mathrm{Z}(x-b) + B_1 \mathrm{D}_x \mathrm{Z}(x-b) + B_2 \mathrm{D}_x^2 \mathrm{Z}(x-b) + \cdots \\ &+ \quad \cdots \\ &+ L_0 \mathrm{Z}(x-l) + L_1 \mathrm{D}_x \mathrm{Z}(x-l) + L_2 \mathrm{D}_x^2 \mathrm{Z}(x-l) + \cdots , \end{aligned}$$

the number of terms in each row being finite. "It is this expression, which I have made use of on many occasions, that I shall now employ in the search for the coordinates of a plane cubic curve as an explicit function of a parameter", Hermite announced.[38]

Accordingly, Hermite considered the curve whose (non-homogeneous) coordinates are parametrised as

$$\begin{aligned} x &= x_0 + A\mathrm{Z}(t-a) + B\mathrm{Z}(t-b) + C\mathrm{Z}(t-c) \\ y &= y_0 + A'\mathrm{Z}(t-a) + B'\mathrm{Z}(t-b) + C'\mathrm{Z}(t-c), \end{aligned}$$

with $A + B + C = 0$ and $A' + B' + C' = 0$. The key point was that the nine quantities $x, y, x^2, \ldots, xy^2, y^3$ are all expressed linearly as functions of eight quantities derived from Z and its derivatives: this implies the existence of a linear equation between $x, y, \ldots, y^3$ and thus proves that the curve parametrised by the above formulas is of order 3. Furthermore, Hermite remarked that the equation between $x, \ldots, y^3$ necessarily contains 9 independent constants, which is the number of constants that appear in a general equation of degree 3 between two unknowns. From this follows that any cubic curve can be parametrised by formulas such as the previous ones.

After having treated similarly the case of space curves defined as the intersection of two quadric surfaces, Hermite tackled "Clebsch's curves, whose

[37]The terminology is recurrent in Hermite's papers. See for instance [Hermite 1862, p. 78]. Moreover, let me note that the articles discussed here are among the very few of Hermite's contributions on questions related to algebraic curves.

[38]"C'est cette expression, dont j'ai fait usage dans bien des circonstances, que je vais employer à la recherche des coordonnées d'une cubique plane en fonction explicite d'un paramètre." [Hermite 1877a, p. 343]. The theorem was proved (and applied) in [Hermite 1862]. It was also used in later papers, such as [Hermite 1877b, 1880]. On Hermite's practice of recycling theorems and formulas, and the links with his view on the unity of mathematics, see [Goldstein 2024].

coordinates are elliptic functions of one parameter”,[39] which he took as

$$x = x_0 + A\mathrm{Z}(t-a) + B\mathrm{Z}(t-b) + \cdots + L\mathrm{Z}(t-l)$$
$$y = y_0 + A'\mathrm{Z}(t-a) + B'\mathrm{Z}(t-b) + \cdots + L'\mathrm{Z}(t-l),$$

with $A + \cdots + L = 0$ and $A' + \cdots + L' = 0$. In this case, Hermite noted that the technique he used for the cubic case no longer applied, since it seemed to lead to a curve of order $2m-3$ instead of m. Nevertheless, he proposed to show “the existence, at least, of an equation of degree m”[40] between x and y. For this, he considered an arbitrary line $\alpha x + \beta y + \gamma = 0$ and remarked that the parameters of its intersection points with the proposed curve are the solutions of

$$\alpha(x_0 + A\mathrm{Z}(t-a) + \cdots + L\mathrm{Z}(t-l)) + \beta(y_0 + A'\mathrm{Z}(t-a) + \cdots + L'\mathrm{Z}(t-l)) + \gamma = 0.$$

Now, the left-hand side of the equation being a doubly periodic function with m poles (which are $a, \ldots, l$), it has the same number of zeros “according to a well-known theorem of elliptic function theory”.[41] Thus the line intersects the curve in m points, which proves (according to Bezout’s theorem) that the curve is of order m. Applying the same method to the first polar of the parametrised curve, Hermite also showed that this curve contains $\frac{m(m-3)}{2}$ double points, as expected for a curve of genus 1.[42]

The technique for determining the degree of a parametrised curve by intersecting it with a line and applying known analytic results on the resulting equation was not new. For instance, Clebsch had employed it to show that a curve parametrised by polynomials of degree n is of order n [Clebsch 1865a, p. 45]. It also appeared in Clebsch’s paper on elliptic curves, although it was employed the other way round, to obtain information on the parametrising functions, knowing the degree of the curve [Clebsch 1865b, p. 218]. On the other hand, Hermite’s wording highlights that, contrary to his approach of cubics, this technique did not allow to form the curve equation, whose existence only was proved. However, the explicit computation of the equation for the cubic was not carried out either: it seems that Hermite meant that one could

[39] “Je reviens à la géométrie plane pour considérer les courbes de Clebsch, dont les coordonnées sont des fonctions elliptiques d’un paramètre” [Hermite 1877a, p. 344].

[40] “Mais l’existence, au moins, d’une équation de ce degré m se prouve très-facilement.” [Hermite 1877a, p. 345].

[41] “[Cette fonction] ne peut donc s’annuler, d’après un théorème connu de la théorie des fonctions elliptiques, que pour m valeurs de t, à l’intérieur du rectangle des périodes”. [Hermite 1877a, p. 345]. The counted zeros and poles are those contained in the parallelogram delimited by the periods.

[42] Given a point $y = (y_1 : y_2 : y_3)$ in the plane and an nth-order curve $f(x_1, x_2, x_3) = 0$, its first polar (with respect to y) is the curve defined by $y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} + y_3 \frac{\partial f}{\partial x_3} = 0$. The intersections of a curve and its first polar include its double points, whence the link with Hermite’s result.

do it, since his process amounted to eliminate 8 quantities between 9 linear equations.

The next issues of Crelle's journal do not contain any paper by Fuchs which would correspond to an answer to Hermite. Instead, two extracts of letters written by Lindemann to Hermite, together with the latter's answer to the first letter, were published only year later, in 1878 [Lindemann 1878; Hermite 1878].

Lindemann first explained that he had taken great attention to Hermite's results contained in the letter to Fuchs and he asserted that he "eventually generalised [these] results in applying the same method to curves of any genus."[43]

Responding to Hermite's theorem of the representation of doubly periodic functions by the second-kind functions Z, Lindemann's own starting point was a theorem which he attributed to Gustav Roch and for which he referred to his own edition of Clebsch's lectures [Clebsch and Lindemann 1876]. This theorem states that any algebraic function with m poles is the sum of a constant and a number of Abelian integrals of the second kind,[44] also denoted by Z. Accordingly, Lindemann considered a curve $f(x, y) = 0$ of order n and genus p, as well as a change of coordinates given by

$$\xi = \xi_0 + A_1 \mathrm{Z}_1 + A_2 \mathrm{Z}_2 + \cdots + A_m \mathrm{Z}_m$$
$$\eta = \eta_0 + B_1 \mathrm{Z}_1 + B_2 \mathrm{Z}_2 + \cdots + B_m \mathrm{Z}_m,$$

where the constants $A_1, \ldots, B_m$ satisfy additional conditions and the Z_i are Abelian functions of the second kind. His aim was to show that the degree of the transformed curve is m and that its genus is the same as that of $f(x, y) = 0$. For the two points, he imitated Hermite's approach in intersecting the curve with an arbitrary line and computing the number of its double points via the first polar, and in applying known results from Abelian function theory.

In his answer, Hermite asked in particular if the number of the constants involved in the equation of a curve described by the above formulas for ξ, η equals the number of constants required to describe a general curve of order m and genus p. While Hermite checked himself that it is the case if $p = 0$ and $p = 1$, Lindemann then answered affirmatively for any p, which thus proves that any curve can be parametrised with the help of Abelian functions of the second kind. As general as this result may appear, it did not mark the end of the question, even around Hermite.[45]

Indeed, two years later, in 1880, Hermann Amandus Schwarz published an article titled: "An attempt of a proof of a Geometry theorem, written

[43]"J'ai fini par généraliser vos résultats, en appliquant la même méthode aux courbes d'un genre quelconque. [Lindemann 1878, p. 294].

[44]Abelian integrals of the second kind are those of which the differential is meromorphic with one single pole.

[45]An important point to note is that Lindemann's parametrisation is multivalued, contrary to Poincaré's future parametrisation by Fuchsian functions, which are uniform.

at Mr. Charles Hermite's invitation"[46] [Schwarz 1880]. It concerned curves $F(x, y) = 0$ of order n having $\frac{(n-1)(n-2)}{2} - 2$ double points – in other words, curves of order n and genus 2. The theorem in question stated the possibility of parametrising such curves with functions that involve, apart from rational functions, the square root of a polynomial of degree 5 or 6. Although Schwarz did not evoke Clebsch's works in any way, the theorem was thus the exact counterpart to what Clebsch did for curves of genus 1, and the proof was extremely resemblant to Clebsch's, since it was also based on the consideration of adequate pencils of curves.

Hermite did not publish any text taking over this theorem in one way or the other. However, two other mathematicians cited Schwarz's article as soon as in 1881. On the one hand, Picard interpreted Schwarz's theorem as the possibility to transform birationally any curve of genus 2 into an hyperelliptic curve defined by an equation $y^2 - (x - a_1) \dots (x - a_6) = 0$ [Picard 1881] – further details will be given in subsection 3.2. On the other hand, D'Esclaibes proposed a simpler proof of Schwarz's theorem, a proof using the same method he developed in his 1880 thesis, which was devoted to the applications of elliptic functions to the study of the curves of the first genus [D'Esclaibes 1880, 1881].

2.2 D'Esclaibes thesis

This doctoral thesis was defended on 2 May 1880 in front of the jury presided over by Victor Puiseux and completed by the two examiners Hermite and Charles Bouquet.[47]

D'Esclaibes introduced his thesis by recalling the definition of the genus p of a curve of order n, which he took as $\frac{(n-1)(n-2)}{2}$ minus the number of double points (and cusps). He also mentioned Riemann's result that p is equal to the maximal number of linearly independent Abelian integrals of the first kind, before insisting on Clebsch's merit of having "opened a new path for the works of the geometers of [his] time" thanks his memoir on the applications of Abelian functions to geometry.[48] In the case of $p = 1$, Clebsch was also credited

[46] "Essai d'une démonstration d'un théorème de Géométrie, rédigé sur l'invitation de M. Charles Hermite."

[47] According to Roland Brasseur's website, Robert D'Esclaibes (1848–1918) had studied in Polytechnique before joining the École des Mines, from which he resigned in 1870 to become a Jesuit. He then taught mathematics at the Sainte-Geneviève *classes préparatoires* between 1875 and 1880, when the Jesuits were removed from the school. D'Esclaibes spent his later career teaching in various schools and universities. See https://sites.google.com/site/rolandbrasseur/5-dictionnaire-des-professeurs-de-mathématiques-spéciales (visited on 13 March 2026).

[48] "C'est en partant de ce fait que Clebsch a été conduit à des résultats très remarquables sur les propriétés des courbes algébriques, et son Mémoire sur les applications des fonctions abéliennes a ouvert une voie nouvelle aux travaux des géomètres de notre époque." [D'Esclaibes 1880, p. 2].

for his contributions published either in the original papers [Clebsch 1864b, 1865b] or in Lindemann's edition of the *Vorlesungen über Geometrie* [Clebsch and Lindemann 1876]. D'Esclaibes also cited, although more discreetly, some research by Axel Harnack [1876, 1877] and Gustav Westphal [1878] on plane cubic curves and space quartics defined as the intersection of two quadric surfaces.[49] D'Esclaibes then explained that although he would "reproduce" most of the results obtained by Clebsch, he had tried to complete them and expose them in a simpler way. This simplification, which was also underscored in Hermite's thesis report[50] concerned essentially two points.

The first one was about the parametrisation involving the square root of a polynomial: just as Clebsch did in his paper on curves with $p = 1$ [Clebsch 1865b], D'Esclaibes first showed that any curve of order n with $\frac{n(n-3)}{2}$ double points admits such a parametrisation.[51] The proof that the polynomial is of degree 4 is what was presented as a simplification of Clebsch's research and an original contribution by D'Esclaibes. The simplification consisted in avoiding Clebsch's use of the Jacobian of three curves, and in replacing it by the examination of how the curve is intersected by certain lines. As mentioned above, the next year, D'Esclaibes tackled by the same method the analogous question for a curve of genus 2 and presented his proof as a simplification of Schwarz's [D'Esclaibes 1881].

In his thesis, after having proved that any curve parametrised by formulas involving the square root of a fourth-degree polynomial is of genus 1, D'Esclaibes presented, sometimes with slight modifications and always in non-homogeneous coordinates, all the other parametrisations that his predecessors had found, and that involved the functions sn, H, Θ and Z. For instance, the one with H was given as

$$x = Ae^{-i\pi ht/K}\frac{H(t-\alpha_1)\cdots H(t-\alpha_n)}{H(t-a)\cdots H(t-l)}$$
$$y = Be^{-i\pi h't/K}\frac{H(t-\beta_1)\cdots H(t-\beta_n)}{H(t-a)\cdots H(t-l)}.$$

But D'Esclaibes also showed how to parametrise a curve of genus 1 with the function p that Karl Weierstrass had introduced in his lectures, and that

[49]These two mathematicians, indeed, made use of parametrisations to study these curves. Their papers will not be discussed here since D'Esclaibes did not make any significant link between them and his works.

[50]This report, as well as the report of the defence, are kept at the French Archives nationales (file number AJ/16/5533). My warmest thanks go to Jenny Boucard, who sent me the pictures of these reports.

[51]A notable difference between the two is that D'Esclaibes worked mostly with non-homogeneous coordinates, and turned to homogeneous ones only for the applications of the general theory.

D'Esclaibes defined as the reciprocal of the elliptic integral[52]

$$\int \frac{\mathrm{d}z}{\sqrt{4z^3 - Iz - J}}.$$

This intervention of the function p was presented as D'Esclaibes' second main simplification, especially because it allowed to use Ludwig Kiepert's "remarkable formula"[53] linking $p(nu)$ and $p(u)$, of which D'Esclaibes would take advantage for several applications.

Interestingly, to expose the properties of Weierstrass' function was the subject of the questions posed by the jury as the "second thesis", and this was mentioned in glowing terms in the report:

> To the merit of the geometer, M. D'Esclaibes adds that of the professor; the exposition he made of the properties of M. Weierstrass' $p(u)$ transcendental function was extremely remarkable by the clarity and the elegance of the proofs, and it provided new evidence of his profound knowledge of the theory of elliptic functions and the most recent related works.[54]

At the same time, D'Esclaibes did not use systematically Weierstrass' function throughout the thesis. For instance, he privileged the parametrisation with the function H to establish the conditions on the elliptic parameters of intersection points of the curve of genus 1 with another curve.

With these conditions at his disposal, D'Esclaibes turned to "geometric consequences". At this point, he asserted that "the many theorems that Clebsch has deduced [from these conditions] being but their immediate translation, we will content ourselves to state the most important ones."[55] These theorems were related to contact curves, and they were indeed exactly the same as Clebsch's. For instance, given a curve C of order n and genus 1, there exist $r^{\mu+2}$ curves of order m that pass through $\frac{n(n-3)}{2} - \mu$ double points and $n(m-n+3)+2\mu-r(\mu+1)$

[52]D'Esclaibes talked of "the functions $p(u)$, $p'(u)$, which are often used by M. Weierstrass" [D'Esclaibes 1880, p. 2], without citing any paper or lecture. As explained in [Bottazzini and Gray 2013, pp. 428–431], Weierstrass had defined this function in a 1863 lecture as the solution of the differential equation $y'^2 = 4y^3 - g_2 y - g_3$, which amounts to the definition given by D'Esclaibes, with $g_2 = I$ and $g_3 = J$. However, his approach of the theory via this function was not widely known in Europe before the mid-1880s, with the publication of [Schwarz 1885].

[53]"Cette transformation permet d'appliquer aux problèmes dans lesquels intervient la multiplication de l'argument la formule remarquable due à M. Kiepert." [D'Esclaibes 1880, p. 3]. The paper in question is [Kiepert 1873].

[54]"Au mérite du géomètre M. D'Esclaibes joint celui du professeur ; l'exposition qu'il a faite des propriétés de la transcendante $p(u)$ de M. Weierstrass, a été extrêmement remarquable par la clarté et l'élégance des démonstrations et a donné une nouvelle preuve de ses connaissances approfondies dans la théorie des fonctions elliptiques et les travaux les plus récents dont elle a été l'objet."

[55]"Conséquences géométriques. [...] Les nombreux théorèmes qui en ont été déuits par Clebsch n'étant que la traduction immédiate de ces équations, nous nous contenterons d'énoncer les plus importants." [D'Esclaibes 1880, pp. 44–45].

simple points of C, and are tangent to C in $\mu + 1$ points with multiplicity r. Similarly, there exist $2^{\mu+1}$ curves of order $n - 3$ that pass through $\delta - \mu$ double points of C and are simply tangent to C in μ points, and D'Esclaibes investigated the grouping properties of these curves, just as Clebsch did.

The last parts of the thesis were devoted to particular curves of genus 1: binodal quartics, non-singular cubics and space quartics that are the intersection of two quadric surfaces. In each case, D'Esclaibes relied on the parametrisation formulas (the general versions of which he refined, based on the specificity of the treated cases) and their consequences to prove various theorems related to these curves. For example, D'Esclaibes made a great use of Weierstrass' function p and Kiepert's results to study cubics, and thus found again many results that Clebsch had proved in [Clebsch 1864b].

According to the *Jahrbuch über die Fortschritte der Mathematik*, D'Esclaibes did not publish any research paper afterwards, except for the short article [D'Esclaibes 1881]. His thesis, however, served in Humbert's doctoral work.

2.3 Enter Humbert

When he defended his doctoral thesis in 1885, Humbert had already published 13 papers, between 1880 and 1883. In fact, he had graduated from Polytechnique in 1879, and attended the École des Mines between 1879 and 1882 before becoming an engineer in Vesoul, in the East of France. As attests the following extract of a letter from Hermite to Poincaré (dated 10 March 1881), Humbert had been interested in the latter's works on Fuchsian functions, which began to be published in February 1881, already when attending this school:

> Allow me to seize this opportunity to ask for your interest and your kindness towards your young colleague, Mr Humbert, a student engineer at the École des Mines, whom I had the pleasure of knowing at the Collège Stanislas. His talent and dedication make him worthy of encouragement and guidance in his studies. I have seen him filled with admiration and enthusiasm for the work you have succinctly introduced on Fuchsian functions. Should you have the chance to meet him, I hope you might suggest some questions for him to explore within the vast field you have opened. In doing so, you would gain a grateful disciple who, I believe, will soon bring honour to your name.[56]

[56] "Permettez moi de saisir de cette occasion pour vous demander votre intérêt et votre bienveillance pour votre jeune collègue M^{r} Humbert élève ingénieur des Mines que j'ai connu au collège Stanislas et qui mérite par le talent et le zèle d'être encouragé et dirigé dans ses études. Je l'ai vu rempli d'admiration et d'enthousiasme pour les travaux dont vous avez donné une idée succincte sur les fonctions Fuchsiennes, et s'il vous conenait lorsque vous aurez l'occasion de le voir, de lui indiquer dans le champ si vaste que vous avez ouvert quelques questions à traiter, vous auriez un disciple reconnaissant et qui bientôt, je le crois, pourrait vous faire honneur." [Nabonnand 2024, p. 367]. Letters from Hermite to Mittag-Leffler dated 1889 and 1891 indicate that Hermite gave oral examinations at the Collège Stanislas during

Poincaré's answer is lost, and the following letters by Hermite do not evoke Humbert's situation any more. Moreover, Humbert's publications and letters do not contain anything that would suggest a personal connection with Poincaré at the time, especially on the topic of Fuchsian functions.

However, his first publications do include a series of papers which announce the doctoral work, being centred on particular cases of what would be dealt with in the thesis. The first one was titled "On Clebsch's curves, whose coordinates can be expressed as elliptic functions of a parameter" [Humbert 1881], the second one bore the same title as the dissertation, "On the curves of genus one" [Humbert 1883a], while the third one was "On the curve of the fourth degree with two double points" [Humbert 1883b]. All these papers made a central use of the elliptic parametrisation of the curves in question.[57]

Humbert came back to the French capital in 1884, becoming a *répétiteur auxiliaire* at Polytechnique, after which he became an engineer in 1885, the year when he defended his thesis.[58]

The thesis, which was dedicated to Camille Jordan, was defended on 18 July 1885 before the jury made of Hermite (the president), Gaston Darboux and Poincaré (the two examiners). Hermite's report, which I already quoted in part in the introduction of the present paper, was very encouraging and recalled both the role Clebsch had in the development of the subject and D'Esclaibes' earlier contribution:

> All the analysts have read with awe the memoirs in which Clebsch gave the interpretation of Abel's theorem on the sums of integrals of algebraic differentials, those which he devoted to plane cubics, to the [quartic] which is the intersection of two surfaces of the second order, finally to the curves of any order which are of genus one.

these years [Hermite and Stieljes 1905, pp. 4, 171]. It is thus likely that this is in this context that Hermite and Humbert met, when the latter was a student at Stanislas before entering Polytechnique in 1877.

[57]Interestingly, in 1884, Henry Picquet, who was then a *répétiteur titulaire* of descriptive geometry at Polytechnique and had been a *répétiteur auxiliaire* of analysis between 1874 and 1882 [Vincent 2019, pp. 319, 322], published an extensive paper called "Applications of the representation of third-degree curves with the help of elliptic functions" [Picquet 1884]. Clebsch's parametrisation of curves of genus one was praised as "one of the most important advances that have been made recently in the study of algebraic curves" (p. 31). Picquet used this parametrisation to study the polygons that are both inscribed in, and circumscribed about a cubic curve. On such theorems, see [Lê 2018b].

[58]Humbert would become professor of constructions and material resistance at the École des Mines and *répétiteur adjoint* of analysis at Polytechnique in 1886. A decade later, in 1895, he became professor of analysis at Polytechnique. In 1901, he was elected at the Academy of Sciences, in succession of Hermite, and obtained the chair of mathematics at the Collège de France in 1912, thus succeeding to Jordan after having been the latter's deputy since 1904. During his career, Humbert was awarded thrice by the Academy of Sciences: with the Poncelet prize in 1891 "for all his works", with the Bordin prize in 1892 for his works on hyperelliptic surfaces, and with the "petit prix d'Ormoy" in 1921, once again "for all his works".

> It is in these beautiful works that Mr Humbert, after another young doctor in science, the father d'Esclaibes, whose thesis obtained the most favourable reception from the Faculty, found the starting point of his researches and the principle of the methods he uses. Many results, new and worthy of interest, have been obtained by Mr Humbert [...].[59]

These results, Hermite specified, concerned mainly special cases of curves of genus one, such as those called "cyclics". Moreover, although he noted that the thesis' "form sometimes leaves to be desired as simplicity and expository clarity", Hermite asserted that it "showed an incontestable analytic merit and allows to hope that the author continues his research and will give new evidence of his talent."[60]

In the introduction of his thesis, Humbert also rooted straight away his work in Clebsch's research. He then announced the three main questions he would tackle in the first part of the dissertation. The first one was to determine whether any curve having a parametrisation by doubly periodic functions is of genus 1 and to find its degree. The second question was to deduce the equation of the curve from its parametric representation. This question, Humbert recalled, had been successfully solved by Hermite only in the particular case of a non-singular cubic – the implicit reference here is Hermite's 1877 paper discussed above [Hermite 1877a]. As for the third question, it aimed at determining, still from the parametric representation, the equations of the adjoints of degree $n-3$ and $n-2$. Humbert insisted that he solved these questions thanks to "a new parametric representation, in homogeneous coordinates, where only sums of Θ functions appear."[61]

The second part of the thesis was devoted to the intersection of a curve of genus one with any curve, and to the contact problems. Once again, Humbert explained his intention to go beyond Clebsch's works, by providing the equations of the contact curves, while the latter only studied their number and groupings. Finally, the third and last part defined the notions of conjugated systems and of correspondences, and used them, together with the results obtained in the previous parts, to study binodal quartics. Cyclics, which correspond to the case where the two nodes are at infinity, were eventually thoroughly investigated.

[59]"Tous les analystes ont lu avec admiration les mémoires dans lesquels Clebsch a donné l'interprétation du théorème d'Abel sur les sommes d'intégrales de différentielles algébriques, ceux qu'il a consacrés aux cubiques planes, à la quadrique [sic] intersection de deux surfaces du second ordre, enfin aux courbes d'un ordre quelconque qui sont de genre un. C'est dans ces beaux travaux que M. Humbert, après un autre jeune docteur ès sciences, le P. d'Esclaibes dont la thèse a reçu le plus favorable accueil de la Faculté, a trouvé le point de départ de ses recherches et le principe des méthodes dont il fait usage. Beaucoup de résultats nouveaux et dignes d'intérêt ont été obtenus par M. Humbert [...]." Cited from [Gispert 2015, p. 222].

[60]"Son travail dont la forme laisse parfois à désirer comme simplicité et clarté d'exposition, montre un incontestable mérite analytique et permet d'espérer que l'auteur poursuive ses recherches et donnera de nouvelles preuves de son talent."

[61]"A cet effet, nous avons employé une représentation paramétrique nouvelle, en coordonnées homogènes, où ne figurent que des sommes de fonctions Θ."

These curves had been notably researched by Darboux, whose book on this was cited as one of Humbert's general references [Darboux 1873; Humbert 1885, p. VIII]. These references also comprised "Clebsch's well known memoirs" of 1864 and 1865, the French translation of his lectures on geometry – with Lindemann's name being completely erased – and D'Esclaibes' thesis.

To give the flavour of Humbert's proofs and to evidence the similarity with the future paper on Fuchsian functions, let me first explain how he proceeded to find what he announced as a new parametric representation. Considering a positive integer n and two complex numbers ω, ω' whose quotient has a positive imaginary part, he defined[62]

$$\theta_3(t) = \sum_{m=-\infty}^{+\infty} e^{m^2 i\pi \frac{\omega'}{\omega} + 2m\frac{i\pi t}{\omega}}.$$

Like the theta functions we already met, this one is not doubly periodic, since

$$\theta_3(t+\omega) = \theta_3(t) \quad \text{and} \quad \theta_3(t+n\omega') = \theta_3(t)e^{-2ni\frac{\pi}{\omega}t - n^2 i\pi\frac{\omega'}{\omega}}$$

for any positive integer n. In fact, Humbert proved that the function θ_3 allows to express any function with the same property, in the sense that any such function is a linear combination of the functions $P_k(t) = \theta_3(t + k\frac{\omega}{n})$, for $1 \leqslant k \leqslant n$. Further, he showed that such linear combinations have exactly n zeros in the fundamental parallelogram delimited by $\omega, n\omega'$, and that the sum of these zeros is congruent to $\frac{n}{2}(\omega + n\omega')$. Conversely, given n points in the fundamental parallelogram whose sum is congruent to this quantity, there exists a unique linear combination of $P_1, \dots, P_n$ that vanishes in these points.

Now, Humbert recalled D'Esclaibes' parametrisation, which he wrote

$$x = Ae^{-i\pi h t_1/K}\frac{H(t_1-\alpha_1)\cdots H(t_1-\alpha_n)}{H(t_1-\gamma_1)\cdots H(t_1-\gamma_n)}$$
$$y = Be^{-i\pi h_1 t_1/K}\frac{H(t_1-\beta_1)\cdots H(t_1-\beta_n)}{H(t_1-\gamma_1)\cdots H(t_1-\gamma_n)},$$

the parameter being denoted by t_1 and the function H being defined so as to be 'almost' $(\omega, n\omega')$-periodic. Humbert then introduced the new variable $t = t_1 - \theta$, the constant θ being chosen so that the sum $\gamma_1 + \cdots + \gamma_n - n\theta$ of the zeros of denominator equals $\frac{n}{2}(\omega + n\omega')$. According to the previous theorem, this implies that the denominator is of the form $C_1P_1 + \cdots + C_nP_n$ for some constants C_i. Then, because the α_i and γ_i are not independent from one

[62] Humbert did not cite any reference dealing specifically with elliptic functions. I note, however, that his notational and numerical conventions align with Briot and Bouquet's book on elliptic functions [Briot and Bouquet 1875]. As explained in [Fricke 1913, p. 219], Jacobi had also defined this function θ_3 in a lecture that was later edited by Carl Borchardt and published in Jacobi's complete works. I finally remark that the function θ_3 is not exactly the same as the function Θ seen above.

another, the sum of the zeros of the first numerator is also equal to $\frac{n}{2}(\omega + n\omega')$, with the consequence that this numerator is of the form $A'_1P_1 + \cdots + A'_nP_n$ for some constants A'_i.

Finally, Humbert remarked that the functions x and $\frac{A'_1P_1+\cdots+A'_nP_n}{C_1P_1+\cdots+C_nP_n}$ have the same periods, the same zeros and the same poles, from which he deduced that they differ only by a constant, since their ratio is an entire, bounded function. In other words, there exist constants $A_1, \ldots, A_n$ such that $x = \frac{A_1P_1+\cdots+A_nP_n}{C_1P_1+\cdots+C_nP_n}$. Acting similarly on the function y eventually proves that the homogeneous coordinates of the curve are of the form

$$\begin{aligned} x_1 &= A_1P_1 + \cdots + A_nP_n \\ x_2 &= B_1P_1 + \cdots + B_nP_n \\ x_3 &= C_1P_1 + \cdots + C_nP_n. \end{aligned}$$

This is Humbert's new form of the parametrisation, upon which he would rest for the following proofs.

Before that, Humbert proved that if a curve is defined by such a parametrisation, it is indeed of genus 1. More precisely, he showed that if the x_i have k common zeros and if (in current terms) the application $t \mapsto (x_1(t) : x_2(t) : x_3(t))$ is injective on the fundamental parallelogram, the curve is of order $n - k$ and genus 1. For this, he considered, as his predecessors did, the intersection of the curve with an arbitrary line and used arguments of complex analysis – including results on the monodromy of functions – to count the number of intersection points. He also transformed birationally the curve into a non-singular cubic, thus proving that the original curve is of genus 1 "according to Riemann's well known theorem".[63]

Afterwards, to study the intersection of the given curve C with another one, defined by an mth-degree equation $f = 0$, Humbert introduced the function $f(x_1(t), x_2(t), x_3(t))$, whose zeros correspond to the sought intersection points. The results he presented above, on zeros of complex functions, then yielded that mn points on C correspond to the intersection with a curve of order m if and only if their sum is congruent to $\frac{mn}{2}(\omega + n\omega')$ and supplementary conditions, whose existence reflect the presence of double points on C, are satisfied. These conditions, which Clebsch had established under another form in [Clebsch 1865b], were called "Clebsch's equations" by Humbert.

Further, just like in Clebsch's paper, an important application was the solution of the contact problem. Humbert generalised it a little bit, since he considered a curve C of degree n and genus 1, and integers m, k_1, k_2, r, l such that $m \geqslant n - 2$ and $mn = 2k_1 + k_2 + rl$. The problem is then to study the curves of order m that pass through k_1 double points and k_2 simple points of

[63]"Les deux courbes [...] sont donc de même genre, d'après le théorème bien connu de Riemann, et, par suite, [la courbe originale] est de genre un." [Humbert 1885, p. 33]. The mentioned theorem states that birationally equivalent curves have the same genus.

C and that, additionally, have a contact of order r in l points.[64] If s_1 is the sum of the elliptic arguments which correspond to the fixed points, and s that which correspond to the l contact points, the first condition is that $s_1 + rs$ be congruent to $\frac{mn}{2}(\omega + nw')$ modulo the periods ω, $n\omega'$. With arguments similar to Clebsch's, Humbert concluded that there are r^2 systems of curves satisfying the condition, and the inspection of "Clebsch's equations" then implied that each system is made of $r^{\delta - k_1}$ groups of curves. In all, Humbert proved that there are $r^{\delta - k_1 + 2}$ groups of curves that are solutions of the contact problem.

As announced, Humbert also distinguished himself from Clebsch by establishing the equations of the contact curves and of the adjoints. Since describing this in full would take me too far afield, I will limit myself to making two remarks on the subject. On the one hand, the establishment of these equations relied on work that primarily involved a deep understanding, from the perspective of theta function analysis, of functions such as $t \mapsto (x_1(t) : x_2(t) : x_3(t))$. On the other hand, these equations subsequently enabled Humbert to uncover new properties of the curves they represent. As we shall see later, the same pattern recurs in the article on Fuchsian functions. I now turn to these functions: although Humbert's thesis still contains extensive developments on binodal quartics, I shall pass over these aspects in silence to focus on the connections with Clebsch and with Poincaré.

3. Picard, Poincaré and Fuchsian functions

Poincaré introduced the functions he named in honour of Fuchs primarily to study linear differential equations whose coefficients are algebraic functions of a complex variable. This research was first published from February 1881 in a series of notes in the *Comptes rendus* of the French Academy of Sciences (eleven notes in 1881, six in 1882 and three in 1883), which were supplemented between 1882 and 1884 by extensive articles appearing in the *Mémoires* of the Academy of Caen, *Mathematische Annalen* and, above all, *Acta Mathematica*.[65]

Although these works have already been the subject of historical research, I briefly outline some of their key elements in order to establish the necessary mathematical foundations for understanding Humbert's works. I also emphasise the analogy between Fuchsian and elliptic functions and trace the early chronology of the parametrisation theorem set-up, which includes epistolary

[64]In fact, Humbert first stated an even more general problem, where the searched curves must touch C in l_1 points with multiplicity r_1, in l_2 points with multiplicity $r_2, \dots$, and in l_q points with multiplicity r_q. However, he presented some "general principles" on this case before studying more closely the particular case for which $q = 1$. Even in this case, Humbert's was more general than Clebsch's because the latter had restricted the value of k_2 to specific forms. As can be easily checked, their solutions to the contact problem coincide when the hypotheses are the same.

[65]These data are gathered from the second volume of Poincaré's collected works, which is devoted to Fuchsian functions and includes later papers [Poincaré 1916].

exchanges between Poincaré and Picard.[66]

3.1 Fuchsian functions: definition and analogy with elliptic functions

The notion of a Fuchsian function rests upon that of a Fuchsian group, which Poincaré defined as (a conjugate of) a group G of homographic substitutions $z \mapsto \frac{\lambda z+\mu}{\nu z+\varpi}$ with real coefficients satisfying $\lambda\varpi - \mu\nu = 1$.[67] A Fuchsian function, then, is a function f defined on the unit disk D and invariant under the action of G, that is,

$$f\left(\frac{\lambda z + \mu}{\nu z + \varpi}\right) = f(z)$$

for all z in D and all $z \mapsto \frac{\lambda z+\mu}{\nu z+\varpi}$ in G. As from his very first note on the topic, Poincaré repeatedly emphasised the analogy between Fuchsian and elliptic functions [Poincaré 1881a, p. 333]. To the invariance of Fuchsian functions under Fuchsian substitutions thus corresponds the double periodicity of elliptic functions, interpreted as their invariance under substitutions such as $z \mapsto z + m_1\omega_1 + m_2\omega_2$.

Further, the periods ω_1, ω_2 of elliptic functions provide a fundamental parallelogram with vertices 0, ω_1, ω_2, $\omega_1 + \omega_2$, of which the transforms by substitutions $z \mapsto z+m_1\omega_1+m_2\omega_2$ form a tessellation of the complex plane, and on which any elliptic function is entirely determined because of its periodicity. Similarly, given a Fuchsian group G, there exists a fundamental (hyperbolic) polygon with $4p$ sides, of which the transforms by substitutions of G tessellate the unit disk, and on which any Fuchsian function is entirely determined. The number $4p$ thus defined happens to be the double of the number of generators $S_1, \dots, S_{2p}$ of G.[68]

As seen in the previous sections, elliptic functions have close ties with the functions Η and Θ, which are representatives of elliptic theta functions. In the same way, Poincaré qualified as "thetafuchsian" any function $\Theta : D \to \mathbf{C}$ for

[66] On the genesis of the theory of Fuchsian functions, including the polemic with Felix Klein around the qualifier "Fuchsian", see in particular [Freudenthal 1955; Dieudonné 1982; Gray 2000]. The parametrisation theorem, which is a particular case of the uniformisation theorem, is studied in these references, as well as in [Gray 1994; Saint-Gervais 2016]. As for the links between Poincaré and Picard, see the comments in [Nabonnand 2024, pp. 626–640].

[67] The group also needs to be discontinuous, which means that the image of any z cannot be infinitely close to z. In current terms, a Fuchsian group is a subgroup of $\mathrm{PSL}(2, \mathbf{R})$ which acts discontinuously on the upper half plane. Appropriate complex conjugates of such subgroups, which Poincaré still called Fuchsian groups, act on the open unit disk [Poincaré 1882c, pp. 58–61]. Since Humbert favoured this version of Fuchsian groups and functions, I will stick to it in the present section. More generally, I will avoid anything not essential to understand Humbert's work, such as Poincaré's classification of Fuchsian groups and topological considerations, for which I refer to the previously cited literature.

[68] On Poincaré's tacit assumption of considering only finitely generated Fuchsian groups, and for current explanations between generators and fundamental polygons, see [Saint-Gervais 2016, pp. 152–159].

which there is an integer m, called the degree of the function, such that

$$\Theta\left(\frac{\lambda z+\mu}{\nu z+\varpi}\right)=(\nu z+\varpi)^{2m}\Theta(z),$$

for any z and any $z \mapsto \frac{\lambda z+\mu}{\nu z+\varpi}$. This definition echoes the property that elliptic theta functions are 'almost' doubly periodic and, just as in the elliptic case, Fuchsian functions are quotients of two thetafuchsian functions of the same degree.[69]

3.2 Towards the Fuchsian parametrisation of curves

Already in the note dated 4 April 1881, Poincaré announced that for a given group G, there exist two Fuchsian functions x, y such that any other one can be expressed as a rational function of them. Further, such functions x, y are related by an algebraic equation $f(x, y) = 0$, of which the genus, as Poincaré proved later, is equal to the number p introduced above [Poincaré 1882a, pp. 228, 255].

This result did not go unnoticed. At the beginning of the 1880s, Picard was investigating various questions of complex analysis, among which one revolved around the parametrisation of quantities x, y satisfying an algebraic relation $F(x, y) = 0$ [Picard 1880a]. Picard knew that if the genus of the curve represented by such an equation is 0 or 1, the quantities x, y can be expressed as rational or elliptic functions of a parameter, which are functions having only poles as singularities. His aim was then to prove that a parametrisation of x, y by such functions is impossible if the genus of $F = 0$ is greater than 1. Picard did it in the hyperelliptic case, where $F(x, y) = y^2 - (x - a_1) \dots (x - a_n)$, and indicated how to proceed in general, before completing the proof a few months after [Picard 1880b]. All this eventually gave rise to a larger article published in Darboux's *Bulletin des sciences mathématiques et astronomiques*, where Picard also provided applications to differential equations [Picard 1880c].

It is in this context that Picard wrote his first letter to Poincaré, on 17 May 1881. This letter was an answer to a letter by Poincaré, who had requested some clarification on a proof contained in the paper just mentioned.[70] After having fulfilled this task, Picard took the opportunity to express his "great

[69]The analogy is even strengthened by Poincaré's proof of the existence of thetafuchsian functions (and thus of Fuchsian functions) by considering convergent series, and his introduction of zetafuchsian functions, presented as the counterpart of the elliptic function Z.

[70]Poincaré mentioned Picard's result in one of his notes in the *Comptes rendus* [Poincaré 1881a, pp. 860–861]. Let me observe that Poincaré's first version of the theorem on the algebraic dependency of Fuchsian functions underlined the role of the possible essential singularities of these functions [Poincaré 1881a, p. 395]. Later, in 1883, Picard would generalise his theorem to the case where the parametrising functions have a finite number of essential singularities. For this, he drew on a proposition he presented as "resulting from M. Poincaré's research on Fuchsian functions" [Picard 1883a,b]. All these clues clearly display Poincaré's and Picard's common complex-analytic framework.

interest"[71] in Poincaré's research on Fuchsian functions, and to announce that he had proved that curves of genus 2 can be parametrised by uniform functions, although he admitted that he could not yet understand the link between these functions and Fuchsian functions.[72]

In the flow of Poincaré's notes on Fuchsian functions, that of 30 May 1880 contained the first formulation of the possibility of parametrising curves of any genus by Fuchsian functions:

> A final remark: I have shown that the coordinates of a point on infinitely many algebraic curves can be expressed by Fuchsian functions of a single parameter (just as the coordinates of a point on a curve of genus 0 can be expressed by rational functions, and those of a point on a curve of genus 1 by elliptic functions): among the curves that enjoy this property, there are curves of all possible genera; but I do not yet know whether this property belongs to any algebraic curve.[73] [Poincaré 1881a, p. 1276]

The next week, on 6 June 1881, Picard wrote a note in the *Comptes rendus* on the very same question:

> M. Poincaré [...] establishes that between two Fuchsian functions corresponding to one group, there exists an algebraic relation. Taking, in a way, the reciprocal question, I would like to indicate the steps to take to recognise if one can express the coordinates u and v of a given algebraic curve $F(u,v) = 0$ by Fuchsian functions of a parameter corresponding to a given Fuchsian group.[74] [Picard 1881, p. 1332]

Picard's proof was based on a result he developed in his 1880 research. He first treated the case of a curve of genus 2, which he reduced to the hyperelliptic case $v^2 = (u - a_1) \dots (u - a_6)$ according to Schwarz's 1880 theorem that has been described above. Then, he asserted that hyperelliptic curve of any genus could "obviously" be treated similarly. As for the most general case, Picard actually contented himself to indicate how to proceed, admitting that he had left aside special cases of the equations involved in his computations.

[71] "J'ai suivi avec grand intérêt vos recherches sur les fonctions Fuchsiennes." [Nabonnand 2024, p. 636].

[72] As I evoked earlier in my comment on Lindemann's parametrisation, uniform functions are functions that are not multivalued, as are Abelian functions.

[73] "Une dernière remarque : j'ai fait voir que les coordonnées d'un point d'une infinité de courbes algébriques s'expriment par des fonctions fuchsiennes d'un même paramètre (de même que les coordonnées d'un point d'une courbe de genre 0 s'expriment par des fonctions rationnelles et celles d'un point d'une courbe de genre 1 par des fonctions elliptiques) : parmi les courbes qui jouissent de cette propriété, il y en a de tous les genres possibles ; mais je ne sais pas encore si cette propriété appartient à une courbe algébrique quelconque."

[74] "M. Poincaré [...] établit qu'entre deux fonctions fuchsiennes correspondant à un même groupe existe une relation algébrique. Prenant en quelque sorte la question inverse, je voudrais indiquer la marche à suivre pour reconnaître si l'on peut exprimer les coordonnées u et v d'une courbe algébrique donnée $F(u,v) = 0$ par des fonctions fuchsiennes d'un paramètre correspondant à un groupe fuchsien donné."

While the parametrisation theorem was thus not proved by Picard, it was stated – along with a sketch of the proof typical of the *Comptes rendus* format – in Poincaré's note of 8 August 1881: "[T]he coordinates of the points of any algebraic curve can be expressed by Fuchsian functions of an auxiliary variable."[75] The theorem then reappeared in the later notes and memoirs published in 1882 and 1884, together with more complete proofs.[76] Still it circulated among mathematicians already in the days that followed Poincaré's note: Hermite, for example, highly praised it in a letter he wrote to Gösta Mittag-Leffler on 18 August 1881:

> What do you think of the latest article that Mr Poincaré has just published in the *Comptes Rendus*? It is far too succinct for me to fully grasp its content, but I cannot help but be struck by the result that, in any algebraic equation between two variables, these quantities can be expressed by Fuchsian functions of one indeterminate.[77]

Hermite's vocabulary, which refers to quantities related by an algebraic equation rather than to coordinates of a point of an algebraic curve, is telling. Although Picard, Poincaré and he himself, in his 1877 and 1878 publications, did talk about algebraic curves, they did not propose any significant application of the parametrisation theorems to the study of these curves.[78] On the contrary, Humbert's perspective was exactly this one.

4. Humbert and the Fuchsian parametrisation

As previously explained, although Humbert had been interested in Poincaré's research on Fuchsian functions already in 1881, he did not publish anything on them before 1886, one year after his thesis. The corresponding paper, titled "Application of the Fuchsian function theory to the study of algebraic curves", appeared in the *Journal de Mathématiques pures et appliquées* [Humbert 1886].

Humbert opened the paper recalling that Poincaré had proved that the coordinates of an algebraic curve can be expressed as Fuchsian functions of a parameter, and he added that "it is this crucial result which served as a

[75] "[L]es coordonnées des points d'une courbe algébrique quelconque s'epxriment par des fonctions fuchsiennes d'une variable auxiliaire." [Poincaré 1881b, p. 303].

[76] Nevertheless, the proofs were still questionable, as described in the historical literature cited above.

[77] "Que pensez-vous du dernier article que Mr Poincaré vient de publier dans les *Comptes Rendus* ? C'est beaucoup trop succinct pour que je puisse m'en rendre bien compte, mais il m'est impossible de ne pas être bien frappé de ce résultat que, dans toute équation algébrique entre deux variables, ces quantités peuvent s'exprimer par des fonctions fuchsiennes d'une indéterminée." [Hermite 1984, p. 126].

[78] Among the three, only Hermite tackled questions related to double points and inflection points of curves [Hermite 1877a, pp. 346–347, 1878, p. 299]. Even in these cases, it seems that such questions served primarily as an opportunity for Hermite to discuss issues related to equation and elimination theory.

starting point for the present memoir, and allowed us to apply to curves of any genus the principles and the methods we have used, in another work, to study the curves of genus one."[79]

He then insisted on the fact that Clebsch was the one who evidenced "the intimate relation which exists between the geometry on an algebraic curve and the theory of Abelian functions",[80] citing both the memoir on these functions [Clebsch 1864a] and the lectures edited by Lindemann – as in his thesis, the citations he made all concerned the French version of the book [Clebsch and Lindemann 1879/1883] and never mentioned Lindemann's name. In accordance with this erasure, Humbert attributed to Clebsch only several results present in this book: first, he explained that Clebsch had investigated the intersection of a curve with any adjoint curve, and thus solved the contact problem in the case of adjoint contact curves. Further, he asserted that Clebsch had partially dealt with the general case thanks to Abelian functions of the third kind. On this point, Humbert announced that his own approach based on Fuchsian functions allowed him to solve the problem completely. He eventually claimed to differentiate himself from Clebsch by establishing the equations of the contact curves, while the latter had only counted these curves and their systems.

The similarity with the thesis is also apparent in the structure of the paper. The first five sections dealt with the theory of Fuchsian functions, while the next seven ones were devoted to the parametrisation of curves, the intersection of a curve with any other curve, as well as to the contact problem. Finally, the last two sections were dedicated to applications to particular curves, among which quartic, quintic and hyperelliptic curves.

4.1 Reworking the theory of Fuchsian functions

In the first sections, Humbert did not content himself to merely copy Poincaré's theory of Fuchsian functions. He proposed his own presentation of the parts he was interested in, he extended some theorems and established new ones, and all this involved arguments with algebraic curves which cannot be found in Poincaré.

Let me illustrate this, as well as the resemblance with the thesis, by considering the very first technical lines of the paper, which were devoted to holomorphic thetafuchsian functions of degree 1. To start with, this focus

[79] "[C]'est ce résultat capital qui a servi de point de départ au présent Mémoire, et nous a permis d'appliquer aux courbes de genre quelconque les principes et la méthode dont nous avons fait usage, dans un autre travail, pour étudier les courbes de genre un." [Humbert 1886, p. 239]. In the whole paper, Humbert never cited precise reference by Poincaré on Fuchsian functions.

[80] "On connaît la relation intime qui exite entre la Géométrie sur une courbe algébrique et la théorie des intégrales abéliennes qui appartiennent à cette courbe : c'est Clebsch qui a mis le premier cette relation en évidence [...]" [Humbert 1886, p. 239].

on thetafuchsian instead of Fuchsian functions – a focus that appears clearly throughout the paper – echoes Humbert's aims in projective geometry: roughly speaking, since Fuchsian functions f are quotients of thetafuchsian functions, the parametrisation of the non-homogeneous coordinates of a curve point $(x, y) = (f_1(z), f_2(z))$ corresponds to a homogeneous version $(x_1 : x_2 : x_3) = (\Theta_1(z) : \Theta_2(z) : \Theta_3(z))$.[81]

Now, starting from a Fuchsian group G and three thetafunctions x_1, x_2, x_3 of degree m and with k common zeros (in the fundamental polygon), Humbert considered the two Fuchsian functions $x = \frac{x_2}{x_1}$, $y = \frac{x_3}{x_1}$, between which there exists an algebraic relation $f(x, y) = 0$ according to Poincaré. This implies the existence of an homogeneous polynomial, also denoted by f, such that $f(x_1, x_2, x_3) = 0$. To determine the degree of f, Humbert evaluated the number of intersection points of the curve $f = 0$ with an arbitrary line in studying the resulting equation with tools from function theory – a technique we encountered several times now. His result is that if the function $z \mapsto (x_1(z) : x_2(z) : x_3(z))$ is injective, the curve $f = 0$ is of degree $n = 2m(p-1) - k$, where p is the number defined by G, as seen above.

Humbert also needed to work with holomorphic thetafuchsian functions of degree 1, while Poincaré had focussed on thetafuchsian functions of degree $m > 1$. Nevertheless, Humbert asserted that the latter's result and proof that these functions have $2m(p-1)$ zeros held for $m = 1$. Next, he demonstrated the existence of thetafuchsian functions of degree 1 by defining (non-homogeneously and homogeneously)

$$\theta(z) = \frac{P(x, y)}{f'_y}\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{P(x_1, x_2, x_3)}{f'_{x_3}}\left(x_1\frac{\mathrm{d}x_2}{\mathrm{d}z} - x_2\frac{\mathrm{d}x_1}{\mathrm{d}z}\right),$$

where f'_y and f'_{x_3} designate partial derivatives of f, and "where $P(x, y)$ is the first member of the equation of a curve of degree $n-3$ [...], adjoint to the curve $f = 0$."[82] Since the curve $f = 0$ is of genus p, Humbert explained, there exist p linearly independent adjoint curves $P_1 = 0, \ldots, P_p = 0$ of degree $n - 3$ and thus p linearly independent holomorphic thetafunctions $\theta_1, \ldots, \theta_p$ of degree 1.

This kind of mixture between thetafuchsian functions and the geometry of algebraic curves was steady in Humbert's paper, even before he tackled issues such as the contact problems, where algebraic curves were at the heart of the statements of problems and theorems, and not only in their proofs. Moreover, as will be seen a bit later, Humbert did see two distinct mathematical domains

[81] In [Lê 2020], the use of homogeneous coordinates is a feature used to differentiate Clebsch's works on Abelian functions, which were concentrated on projective-geometric questions, from Riemann's. Let me note that the passage from $(x, y) = (f_1(z), f_2(z))$ to $(x_1 : x_2 : x_3) = (\Theta_1(z) : \Theta_2(z) : \Theta_3(z))$ is not as obvious as it may appear, if one wants to control the degrees, zeros and poles of functions involved. Humbert dealt with this in the sixth section of his article.

[82] "[...] où $P(x, y)$ est le premier membre de l'équation d'une courbe de degré $n - 3$ [...], adjointe à la courbe $f = 0$." [Humbert 1886, pp. 242–243].

in contact here, as attest his use of qualifiers such as "geometric" on the one hand, and "analytic" and "algebraic" on the other hand.

4.2 Abel's theorem revisited

In order to emulate Clebsch's theory of Abelian functions, Humbert first needed to define what the periods of thetafuchsian functions are. To do so, he took the previously defined functions $\theta_1, \ldots, \theta_p$ and the generators $S_1, \ldots, S_{2p}$ of the group G, and he set

$$\omega_k^{(i)} = \int_z^{S_i(z)} \theta_k.$$

These complex numbers, of which the value does not depend on z, are the sought periods. As Humbert remarked, they are exactly the periods of the Abelian functions of the first kind $\int \frac{P_k(x,y)}{f'_y}\,\mathrm{d}x$. In the Fuchsian framework, their status as periods expresses the fact that if S is a substitution of G, there exist integers $h^{(1)}, \ldots, h^{(2p)}$ such that

$$\int_z^{S(z)} \theta_k = h^{(1)}\omega_k^{(1)} + \cdots + h^{(2p)}\omega_k^{(2p)}$$

for any $1 \leqslant k \leqslant p$. This corresponds to how the value of an Abelian integral of the first kind changes when the integration path is modified.

Then, to Abel's addition theorem, which was extensively used by Clebsch, corresponds the following one, which Humbert also called "Abel's theorem". Let $\alpha_1, \ldots, \alpha_\mu$ be complex numbers and $\beta_1, \ldots, \beta_\mu$ be the zeros of a thetafuchsian function of a given degree. Then the α_i are the zeros of a thetafuchsian function of the same degree if and only if there exist integers $h^{(1)}, \ldots, h^{(2p)}$ such that

$$\int_{\alpha_1}^{\beta_1} \theta_k + \cdots + \int_{\alpha_\mu}^{\beta_\mu} \theta_k = h^{(1)}\omega_k^{(1)} + \cdots + h^{(2p)}\omega_k^{(2p)}$$

for any $1 \leqslant k \leqslant p$. This pivotal theorem was presented by Humbert as a novelty for the theory of thetafuchsian functions and was qualified as "purely algebraic".[83] To connect it with questions related to curves and their intersections, indeed, Humbert would need first to establish links between the zeros of thetafuchsian functions and curves satisfying some properties.

For this, he first mimicked his doctoral work by rewriting the parametrisation of the curve as

$$x_i = a_1^{(i)}\Theta_1 + \cdots + a_{(2m-1)(p-1)}^{(i)}\Theta_{(2m-1)(p-1)},$$

where the Θ_k form (in anachronistic terms) a basis of the space of holomorphic thetafuchsian functions of degree $m > 1$, the dimension of which Humbert had

[83] "[E]n particulier, on démontre [...] une proposition purement algébrique, relative aux zéros et aux infinis d'une fonction fuchsienne, proposition identique, au fond, au théorème d'Abel [...]" [Humbert 1886, p. 240].

computed before thanks to Abel's theorem. Humbert then proved that if C is a curve parametrised by three thetafuchsian functions, the existence of an adjoint curve of a given degree amounts to the existence of a thetafuchsian function whose (a part of the) zeros correspond to the intersection points of C and the adjoint.

This result and Abel's theorem were the two pillars on which Humbert developed the theory of groups of points on a curve, that is, of the intersection points of the curve with an adjoint that passes through fixed points.[84] As he explained:

> The study of the systems of points that are common to an algebraic curve and its adjoint curves constitutes one of the most important theories in the Geometry on a curve; as Clebsch demonstrated, it is based on Abel's theorem concerning integrals of the first kind.
>
> The use of thetafuchsian functions will enable us to simply rediscover most of the known results of this theory and to prove new ones.[85] [Humbert 1886, p. 265]

In particular, Humbert tackled the remainder theorem and the Riemann–Roch theorem,[86] for which he referred to Clebsch–Lindemann's lectures both for their statements and for notions he used to prove them in his own Fuchsian way. As already mentioned, Humbert would only make appear Clebsch's name when citing this work (as well as the French translator Benoist), which explains his attribution to Clebsch of the theory of groups of points. However, in a paper published one year after, he corrected this attribution and clarified that the theory was founded and developed by Alexander Brill and Max Noether [Humbert 1887, p. 327].

Afterwards, Humbert tackled the issue of the intersection of a curve of genus p and an arbitrary curve. As in the case of the intersection with an adjoint, the idea was to interpret the parameters of the intersection points as the zeros of a certain thetafuchsian function. In the present case of a non-adjoint curve, following his thesis, Humbert found some relations between the parameters, relations of which he gave "a very important form from the viewpoint of the applications, as Clebsch did in the case of curves of genus 0 and of genus 1"[87]

[84] From a current point of view, groups of points on a curve correspond to divisors on this curve.

[85] "L'étude des systèmes de points communs à une courbe algébrique et à ses courbes adjointes constitue une des théories les plus importantes de la Géométrie sur une courbe ; elle repose, comme l'a fait voir Clebsch, sur le théorème d'Abel relatif aux intégrales de première espèce. L'emploi des fonctions thêtafuchsiennes va nous permettre de retrouver simplement la plupart des résultats connus de cette théorie, et d'en démontrer de nouveaux."

[86] The remainder theorem is a sort of transitivity property for groups of points. As for the Riemann–Roch theorem, it yields the (dimension) number of adjoint curves of degree $n-3$ which pass through particular groups of points. On the latter theorem, see [Gray 1998].

[87] "On peut mettre les relations dont il s'agit sous une forme très importante au point de vue des applications, comme l'a fait Clebsch dans le cas des courbes de genre 0 et de genre 1." [Humbert 1886, p. 282].

– the corresponding relations for $p = 1$ are what Humbert called "Clebsch's equations" in his thesis. The proof that they represent necessary and sufficient conditions for some parameters to be the arguments of intersection points rested on Noether's theorem, another emblematic result from algebraic curve theory, for which Clebsch–Lindemann's lectures were cited once more.[88]

4.3 Geometric applications

Only after having constructed all this would Humbert turn to what he called the "geometric applications", among which the most important one was the contact curves problem. As in the case of the genus 1, one considers a curve C of order n and genus p, as well as integers $m \geqslant n-2$, k_1, k_2, r and l such that $mn = 2k_1 + k_2 + rl$. The problem is to find and study the curves of order m which pass through k_1 double points and k_2 simple points of C, and which touch this curve in l points with multiplicity r.

As in the thesis, Humbert first used Abel's theorem to group the solutions into systems and enumerate them. Indeed, if the parameters of the contact points of a given contact curve are denoted by $\beta_1, \ldots, \beta_l$ and those related to another contact curve are $\alpha_1, \ldots, \alpha_l$, then, knowing that these contacts are of multiplicity r, Abel's theorem gives

$$r\int_{\alpha_1}^{\beta_1} \theta_k + \cdots + r\int_{\alpha_l}^{\beta_l} \theta_k = h^{(1)}\omega_k^{(1)} + \cdots + h^{(2p)}\omega_k^{(2p)}$$

for any $1 \leqslant k \leqslant p$. Thus

$$\int_{\alpha_1}^{\beta_1} \theta_k + \cdots + \int_{\alpha_l}^{\beta_l} \theta_k = \frac{1}{r}(h^{(1)}\omega_k^{(1)} + \cdots + h^{(2p)}\omega_k^{(2p)}),$$

and the periodicity property shows that the contact curves can be grouped in r^{2p} systems, which correspond to the possible values of the $h^{(i)}$ modulo r – the resemblance with Clebsch's proofs in [Clebsch 1864a] appears clearly here. Further, using the supplementary conditions developed above proves that the curves of each system can be gathered into r^{d-k_1} groups, where d is the number of double points of C. Finally, and again as in the thesis, these conditions also allowed to establish the form of the equations of the contact curves, a point that Humbert emphasised several times.

Humbert then investigated the properties of the systems of contact points. For instance, he proved that the fixed points and the contact points of r curves belonging to one group are all on a curve of degree m having particular features. These properties were of the exact same vein as those found in the thesis, but a new disciplinary configuration occurred now: "This research will evidence the existence of a class of interesting uniform functions, the study of which is

[88] This theorem states that if a curve goes through all the intersection points of two curves $F = 0$, $G = 0$, there exist polynomials A, B such that its equation is $AF + BG = 0$.

linked to that of Fuchsian functions," asserted Humbert.[89] Specifically, these functions were functions $\mathscr{F}$ defined on the unit disk and satisfying the relations

$$\mathscr{F}\left(\frac{\lambda_j z + \mu_j}{\nu_j + \varpi_j}\right) = \mathscr{F}(z)e^{2h_j\frac{i\pi}{r}},$$

where the substitutions $z \mapsto \frac{\lambda_j z+\mu_j}{\nu_j+\varpi_j}$, for $1 \leqslant j \leqslant 2p$, are the fundamental substitutions of G, and the h_j are some integers. I do not know if these functions were later integrated into any research, but still it is interesting to observe the movement between geometry and function theory that accompanies their genesis: Humbert insisted that the theorem stating the existence of these functions was "algebraic" and that it originated from the "preceding geometric considerations" on curves and their systems [Humbert 1886, p. 296].

The last sections of the paper concerned particular cases: contact curves of degree $n - 3$, given curves of degree 4, 5, or of an hyperelliptic nature. In each situation, Humbert proposed a thorough study of the underlying geometric configuration. For example, for curves of degree 5 with one double point (thus of genus 5), Humbert counted 1023 conics touching the quintic in five points; for curves of degree 4 with one double point (thus of genus 2), he proved that there are 16 systems of conics tangent to the quartic in four points, and that all these systems but one can be divided into two groups of conics, the exceptional system being exclusively made of doubly counted lines.[90] And these systems and groups enjoyed properties such as the following ones: the contact points of two conics belonging to the same system and the same group are situated on a conic; through the contact points of two conics belonging to the same system but not to the same group, one can draw a pencil of cubics having the double point of the given quartic as a common point, etc.

5. Fame and legacy

Humbert's 1886 paper was not the only one where he used the Fuchsian parametrisation of algebraic curves and its consequences. For instance, in the next years he devoted several articles to applications of Abel's theorem – in its Fuchsian version – to the study of plane and space curves [Humbert 1887, 1889, 1890].

[89]"Ces recherches mettront en évidence l'existence d'une classe de fonctions uniformes intéressantes, et dont l'étude est liée à celle des fonctions fuchsiennes." [Humbert 1886, p. 293].

[90]These groups of conics had also been studied in Clebsch–Lindemann's lectures, but Humbert noted that these lectures mistakenly counted 31 of them – albeit this was not stated explicitly, Humbert seemingly used the form of the contact curves equations to find that the odd one out in Clebsch–Lindemann's list was made doubly counted lines instead of proper conics. See [Humbert 1886, p. 306]. A similar correction was proposed in another case [Humbert 1886, p. 307]. On the issue of conceiving of adequately conics and their degenerations to count them rightly, see [Michel 2023].

More generally, the idea of parametrising geometric objects gave structure to his approach of geometry during the part of his career extending from his first academic years to the end of the 19th century, when he turned to number theory: as Henri Lebesgue asserted in his 1922 inaugural lecture at the Collège de France, where he succeeded to Humbert:

> In any true work, there emerges a continuity of thought that naturally allows the various memoirs to be grouped around a few guiding ideas. [...] The works of the first period, which are devoted to studies on algebraic curves and surfaces, are always, and this is what characterises them, carried out using what are now called *uniformising functions*. Humbert repeatedly noted that he had been deeply struck by seeing the order, clarity and generality that had been obtained in the study of curves of genus 1 once the coordinates of any point on these curves were expressed using elliptic functions, and that it is this observation which had always guided him.[91] [Lebesgue 1922, p. 221]

Quite characteristically, indeed, Humbert's fundamental contributions to algebraic surfaces were dedicated to hyperelliptic surfaces, defined by a global parametrisation by meromorphic two-variable functions having four pairs of independent periods [Houzel 2002, p. 235].

Lebesgue also insisted that one of Humbert's great merit in all these works was his ability to craft geometric results from his deep knowledge of function theory [Lebesgue 1922, pp. 223–224]. The analysis I proposed in this article confirms this statement and shows how such a crafting was realised effectively within the technique.

It also allows to complete Lebesgues' assertion on two points. The first one is that reciprocal disciplinary movements also existed, for instance in Humbert's use of geometric arguments to prove results on Fuchsian functions, or his discovery of a new class of functions as a consequence of his work on contact curves.

The second point bears on the joint dynamics of the works of Humbert, but also Clebsch, Hermite, D'Esblaibes and Lindemann, on curve parametrisations on the one hand, and of elliptic, Abelian and Fuchsian functions on the other hand. All these mathematicians clearly had a very good basic knowledge of complex analysis and elliptic functions, which, in D'Esclaibes' and Humbert's cases, certainly results from their training at Polytechnique [Gispert 1994].

[91] "Dans toute œuvre véritable se révèle une continuité de pensée qui permet de grouper tout naturellement les divers Mémoires autour de quelques idées directrices. [...] [Les] travaux de la première époque, qui sont consacrés à des études sur les courbes et les surfaces algébriques, sont faits toujours, et c'est ce qui les caractérisent, à l'aide de ce qu'on appelle maintenant des *fonctions uniformisantes*. Humbert signale à plusieurs reprises qu'il avait été très frappé en constatant quel ordre, quelle clarté, quelle généralité avaient été obtenus dans l'étude des courbes de genre 1 à partir du moment où l'on avait exprimé les coordonnées d'un point quelconque de ces courbes à l'aide des fonctions elliptiques, et que c'est cette constatation qui l'a toujours guidé."

But it is striking to see that some pieces of elliptic function theory were quite new when they were applied to geometric aims. For example, in his paper on curves of genus 1, published in 1865 but written in 1864, Clebsch referred to Hermite's 1862 note on elliptic functions for the theorem he needed to find the parametrisation with the functions Θ and H; and D'Esclaibes' 1880 thesis integrated Weierstrass' function p at a time when its use to tackle elliptic function theory was far from standard. Similarly, Clebsch produced his 1864 memoir on the applications of Abelian functions to geometry in a context of assimilation and clarification of Riemann's works,[92] to which he contributed shortly afterwards, together with Paul Gordan, with the book *Theorie der Ableschen Functionen* [Clebsch and Gordan 1866]. As for Poincaré's theory of Fuchsian functions, it was obviously still in its youth when Humbert reworked some of its pieces to better apply it to the study of algebraic curves. In other words, geometers did not wait for the theories of elliptic, Abelian and Fuchsian functions to be stabilised before integrating them into research on curves; they even participated to their collective constructions.

Beyond the issue of the interfaces between analysis and geometry, we have seen that Clebsch's results and methods were effectively taken over – in variegated ways – in almost all the texts of the corpus, with the exceptions of Picard's and Poincaré's contributions, which aimed different kinds of results on curves as the others. Thus, in their work on the parametrisations of curves of genus 1 and 2 by formulas that involve the square root of a polynomial, D'Esclaibes and Schwarz drew on the basic idea of Clebsch's proof, although the former explicitly proposed a simplification of one of its parts. Further, Humbert's Fuchsian version of Abel's addition theorem and the proof structure of its applications to the contact problem appear as completely analogous to Clebsch's approach; and while Humbert presented his theorem as both a simplification and a generalisation of Clebsch's, his use of the phrase "Clebsch's equations" clearly display his intellectual rooting. Finally, the very inclusion of the contact curves problem and its treatment by Humbert and D'Esclaibes in their texts emerges as yet another strong continuity link with Clebsch. All these elements therefore appear as so many traces of Clebsch's technical legacy.

These traces undoubtedly participated to the fame of Clebsch and his works on the connections between Abelian functions and algebraic curves, a fame which was reinforced by other features. Indeed, many of the protagonists we met credited Clebsch for having first established these connections in explicit laudatory sentences, often included in the introductions of their papers. This credit was also manifest in other, more indirect textual devices, such as the genitive in the expression "Clebsch's curves, whose coordinates are elliptic functions of one parameter", which Hermite and Humbert used.

At the same time, Humbert's way of citing the Clebsch–Lindemann book

[92]See [Lê 2020] for a more details of this point, and [Bottazzini and Gray 2013, pp. 311–339] for a description of the various responses to Riemann's work.

and his related mistaken attribution to Clebsch of the theory of point groups raise another question. Obliterating Lindemann's name had the obvious effect of attaching anything found in the book directly and exclusively to Clebsch, and can be interpreted as follows. At a moment when, for any reason, Humbert had already identified Clebsch as the founder of the topic, it was all the more natural to him to leave Lindemann (as well as Brill and Noether for the theory of point groups) out of the picture, with the effect of strengthening Clebsch's fame in the eyes of Humbert's readers. Moreover, not contradicting this interpretation, nor can we rule out that Humbert simply did not read Lindemann's introduction and Klein's preface, where the role of the former as being more than just a compiler was specified.

Thus the issue of the actual reading of Clebsch's original writings looms on the horizon. Caroline Ehrhardt [2011, 2012] has demonstrated that, by the end of the 19th century, in parallel with various memorial events which participated to the construction of his immense renown, the association of Évariste Galois's name with a theorem and a theory concealed both his research in its original formulation and the extensive appropriation work produced by other mathematicians. In the early 20th century, among the scholars who credited Galois with the paternity of the abstract group concept, some had surely read his memoirs and deliberately interpreted them so, but it is also certain that others had not, and merely perpetuated an attribution pattern that was already commonplace.

It is likely that this phenomenon of gradual change in the process of crediting a mathematician for any achievement – the demonstration of a theorem, the definition of an object or the setting up of a method – is the general rule, even if significant differences obviously exist regarding the power of a renown and the longevity of a name in the collective memory of mathematicians (to borrow Maurice Halbwachs's concept, as Ehrhardt does), and the historical phenomena that underlie their creation, perpetuation and possible oblivion.[93]

To decipher this process in Clebsch's case remains an open question. The elements I have presented are but a very partial small step in this direction, yet they evidence that thorough investigations are necessary to understand with the required nuances how technical elements and meta comments may be knit together to build a fame and an actual technical legacy. From this viewpoint, one might eventually rightly wonder what was commonplace and what was based on genuine reading when, almost sixty years after Clebsch published his grounding works on the parametrisations of algebraic curves, Lebesgue wrote:

> By this method, indeed, Clebsch and his successors had managed to group together and extend numerous geometric propositions – seemingly particular and disparate – which had previously been valued solely for their aesthetic character. Moreover, rather than admiring their true

[93]The case of Carl Friedriech Gauss' *Disquisitiones arithmeticae* offers an interesting comparative point of view. See [Goldstein, Schappacher, and Schwermer 2007].

beauty, one was seduced by their mystery; now, we still admire these propositions, but our admiration is grounded on our knowledge rather than our ignorance.[94] [Lebesgue 1922, pp. 221–222]

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[94] "Par cette méthode, en effet, Clebsch et ses continuateurs avaient réussi à grouper et à étendre de nombreuses propositions géométriques, particulières et disparates, semblait-il, que l'on avait estimé jusque-là que pour leur seul caractère esthétique. D'ailleurs, au lieu d'admirer leur véritable beauté, on était plutôt séduit par leur mystère ; maintenant, nous admirons encore ces propositions, mais notre admiration est fondée sur nos connaissances et non plus sur notre ignorance."

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