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Bivariate Distributions with Given Extreme Value Attractor

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A new class of bivariate distributions is introduced and studied, which encompasses Archimedean copulas and extreme value distributions as special cases. Its dependence structure is described, its maximum and minimum attractors are determined, and an algorithm is given for generating observations from any member of this class. It is also shown how it is possible to construct distributions in this family with a predetermined extreme value attractor. This construction is used to study via simulation the small-sample behavior of a bivariate threshold method suggested by H. Joe, R. L. Smith, and I. Weissman (1992, *J. Roy. Statist. Soc. Ser. B* **54**, 171–183) for estimating the joint distribution of extremes of two random variates.

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1. INTRODUCTION

These past few years, a growing awareness of risk management issues has led the statistical community to develop new procedures for modeling multivariate extreme events and values. The state of the art was summarized by Smith *et al.* (1990). Notable contributions have since been made by Coles and Tawn (1991, 1994), Einmahl *et al.* (1993, 1997), Joe *et al.* (1992, 1996), Capéraà *et al.* (1997a), and Ghoudi *et al.* (1998), as well as Abdous *et al.* (in press).

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To a large extent, recent developments have exploited the point process representation of extremes, as described by Galambos (1987) and Resnick (1987). For example, Joe *et al.* (1992) show how it is possible to make inference for maximum extreme values by fitting a stochastic model to the observed distribution of bivariate data, subject to some function of the components exceeding a high threshold. While this and similar techniques have been reported to perform satisfactorily in applications, very little is known to date about their behavior in small and large samples.

The present paper was originally motivated by the desire to document the performance of multivariate threshold methods through extensive Monte Carlo simulations. The intent was to compare variants of this technique in terms of bias and efficiency. This required the selection of a few bivariate extreme value distributions for maxima and, for each of them, the application of the various methods on a series of random samples of different sizes from distributions belonging to its domain of attraction.

Upon reviewing the literature, however, it soon became apparent that a general scheme for constructing flexible parametric families with a given attractor was lacking. In fact, it turns out that independence is the attractor of quite a few classical bivariate systems of distributions, as may be seen using the techniques developed by Marshall and Olkin (1983) or Yun (1997). Such is the case (except under very restrictive conditions) for the parametric families of Ali *et al.* (1978), Clayton (1978), Farlie–Gumbel–Morgenstern (e.g., Farlie, 1960), Frank (1979), and Plackett (1965), among others. For normal vectors, asymptotic independence is only avoided if the correlations between all the components are equal to one (Sibuya, 1960) or, as in Hüsler and Reiss (1989), in situations where these correlations are allowed to vary with the sample size. Circumstances where the extreme value distribution of the maximum belongs to Gumbel's family were described by Genest and Rivest (1989).

The main purpose of this paper, therefore, is to propose a scheme for building large families of analytically tractable bivariate distributions that are reasonably convenient to simulate and that belong to the domain of attraction of a predetermined extreme value distribution for maxima. A class of distributions that meets this requirement is introduced in Sect. 2. It can accomodate arbitrary marginals and encompasses all bivariate maximum extreme value distributions, the frailty models of Oakes (1989) as well as the Archimedean system of copulas introduced by Genest and MacKay (1986a, 1986b), which itself includes many well-known families of bivariate distributions (cf., e.g., Chap. 4 of Nelsen, 1999). Examples of this new class of so-called Archimax distributions are exhibited in Sect. 3. In Sect. 4, the maximum and minimum attractors of class members are determined, under fairly general conditions. In Sect. 5, a distributional result is presented which leads to effective simulation algorithms; it also provides insight into the dependence structure of Archimax distributions, as described in Sect. 6. Finally, this tool is exploited in Sect. 7 to explore briefly the small-sample behavior of the threshold method of Joe *et al.* (1992) in a variety of simulated conditions.

2. DEFINITION AND INTEREST OF THE NEW FAMILY

A bivariate function is said to be an *Archimax copula* if and only if it can be expressed in the form

$$C_{\phi, A}(x, y) = \phi^{-1}\left[\{\phi(x) + \phi(y)\} A \left\{ \frac{\phi(x)}{\phi(x) + \phi(y)} \right\} \right] \tag{1}$$

for all $0 \leqslant x, y \leqslant 1$ in terms of

(i) a convex function $A: [0, 1] \to [1/2, 1]$ such that $\max(t, 1-t) \leqslant A(t) \leqslant 1$ for all $0 \leqslant t \leqslant 1$;

(ii) a convex, decreasing function $\phi: (0, 1] \to [0, \infty)$ verifying $\phi(1) = 0$, with the convention that $\phi(0) \equiv \lim_{t \to 0^+} \phi(t)$ and $\phi^{-1}(s) = 0$ when $s \geqslant \phi(0)$.

Appendix A contains a proof that $C_{\phi, A}$ is a bivariate distribution function with uniform marginal for arbitrary choices of ϕ and A meeting conditions (i) and (ii). In other words, $C_{\phi, A}$ is a "copula," in the sense given to that term by Sklar (1959). Note that $C_{k\phi, A} \equiv C_{\phi, A}$ for all $k > 0$. Replacing x and y by $F(x)$ and $G(y)$ everywhere in (1) produces a bivariate model with marginal distribution functions F and G.

The name "Archimax" was chosen to reflect the fact that the new family includes both the maximum extreme value distributions (cf., e.g., Pickands, 1981, or Chap. 6 of Joe, 1997) and the Archimedean copulas (cf., e.g., Genest and MacKay, 1986a, 1986b, or Chap. 4 of Nelsen, 1999). On one hand, it is easy to see that when $\phi(t) = \log(1/t)$, one has

$$C_{\phi, A}(x, y) = C_A(x, y) \equiv \exp[\log(xy)\, A\{\log(x)/\log(xy)\}]$$

for all $0 \leqslant x, y \leqslant 1$, which is the general form of bivariate extreme value copulas, in the formulation used by Capéraà *et al.* (1997a), which differs slightly from that employed, e.g., by Tawn (1988). On the other hand,

$$C_{\phi, A}(x, y) = C_\phi(x, y) \equiv \phi^{-1}\{\phi(x) + \phi(y)\}$$

when $A \equiv 1$ which is the general form of Archimedean copulas.

In the sequel, functions A satisfying (i) are called *dependence functions* and functions ϕ satisfying (ii) are referred to as *Archimedean generators*. The notations C_A and C_ϕ are used wherever no risk of confusion arises.

Since Archimedean and extreme value distributions may sometimes have singular components, the same is true for Archimax copulas. However, it is easy to see that $C_{\phi, A}$ is absolutely continuous when ϕ and A are twice differentiable and $\phi(t)/\phi'(t) \to 0$ as $t \to 0$.

The main interest of construction (1) is that through appropriate choices of ϕ and A, it yields bivariate distributions belonging to the domain of attraction of any predetermined attractor C_{A*}. To be specific, suppose that a function ϕ satisfying condition (ii) above is such that $\phi(1-1/t)$ is regularly varying at infinity with degree $-m$ for some $m \geqslant 1$. Recall that a function $\psi: (0, \infty) \to (0, \infty)$ is said to be regularly varying at infinity with degree ρ, denoted $\psi \in RV_\rho$, if and only if $\lim_{t\to\infty} \psi(st)/\psi(t) = s^\rho$ for all $s > 0$ (e.g., Resnick, 1987, p. 13). It is shown in Prop. 4.1 below that the Archimax copula $C_{\phi, A}$ then belongs to the maximum domain of attraction of C_{A*}, where

$$A^*(t) = \{t^m + (1-t)^m\}^{1/m} A^{1/m} \left\{\frac{t^m}{t^m + (1-t)^m}\right\}, \qquad 0 \leqslant t \leqslant 1. \tag{2}$$

Thus if $A = A^*$ and if $\phi(1-1/t) \in RV_{-1}$, $C_{\phi, A}$ is a bivariate copula that belongs to the domain of attraction of C_{A*}. This condition on ϕ is verified for the generators of many well-known families of Archimedean copulas, such as those of Clayton (1978), Frank (1979), and Ali *et al.* (1978).

However, A need not necessarily be taken equal to A^*. Starting from (2) and arguing backwards, it is clear that $C_{\phi, A}$ belongs to the domain of attraction of C_{A*}, provided that $\phi(1-1/t) \in RV_{-m}$ and

$$A(t) = \{t^{1/m} + (1-t)^{1/m}\}^m \left[A^* \left\{\frac{t^{1/m}}{t^{1/m} + (1-t)^{1/m}}\right\}\right]^m. \tag{3}$$

Now introduce $D(t) = d\log\{A^*(t)\}/dt$ and let $H^*(t) = t + t(1-t)\, D(t)$ stand for the distribution function of $\log(X)/\log(XY)$ when (X, Y) is a random pair from copula C_{A*} (cf., Prop. 2.1, Capéraà *et al.*, 1997a). If A^* is twice differentiable on $(0, 1)$, so that the density h^* of H^* exists, it is shown in Appendix B that equation (3) then defines a dependence function A if and only if

$$\frac{h^*(t)}{H^*(t)\{1 - H^*(t)\}} \geqslant \frac{m}{t(1-t)} \tag{4}$$

for all $0 \leqslant t \leqslant 1$. This condition is easy to check graphically and is obviously verified for all $1 \leqslant m' \leqslant m$ whenever it holds true for m. Thus when (4) is valid and A is defined by (3), $C_{\phi, A}$ has C_{A*} for attractor provided that $\phi(1-1/t)$ is regularly varying at infinity with degree $-m'$ for some $1 \leqslant m' \leqslant m$. Observe in passing that for arbitrary A and ϕ such that $\phi(1-1/t) \in RV_{-m}$, one has

$$\lim_{u \to 0} \frac{1-2u+C_{\phi, A}(u, u)}{1-u} = 2 - \{2A(1/2)\}^{1/m} \leqslant 1.$$

Accordingly, Archimax copulas whose generator ϕ satisfies this condition exhibit upper tail dependence in the sense of Joe (1993) unless $m=1$ and $A \equiv 1$, in which case the above limit is zero.

Before looking at examples of Archimax copulas, it may be instructive to exhibit contexts in which copulas of the form (1) arise. Suppose for example that X and Y are survival times with survivor functions $\bar{F}$ and $\bar{G}$ respectively, and that a model for their joint behavior is sought. One popular option (see, for example, Oakes 1989, 1994, or Bandeen-Roche and Liang, 1996) consists in assuming that, conditional on a frailty Z with distribution function M on $[0, \infty)$, the joint survivor function can be expressed as $\{S(x)\, T(y)\}^z$ in terms of baseline survivor functions S and T. The joint survival function of X and Y is then given by

$$\mathrm{pr}(X > x, Y > y) = \int \{S(x)\, T(y)\}^z \, dM(z). \tag{5}$$

Let ϕ^{-1} denote the Laplace transform of M and observe that $\bar{F} = \phi^{-1}\{-\log(S)\}$ and $\bar{G} = \phi^{-1}\{-\log(T)\}$. The joint survivor function may then be written as

$$\phi^{-1}\{\phi(\bar{F}) + \phi(\bar{G})\},$$

which is an Archimedean copula with generator ϕ. Now it has been pointed out by Marshall and Olkin (1988) that other conditional distributions could be used to model dependence between survival times X and Y. That is, one could extend formula (5) to

$$\mathrm{pr}(X > x, Y > y) = \int J\{S^z(x), T^z(y)\} \, dM(z),$$

where $J(u, v)$ is an arbitrary bivariate copula. When $J = C_A$, an Archimax distribution $C_{\phi, A}$ emerges with ϕ^{-1} standing for the Laplace transform of M. An application of Theorem 2.1 of Marshall and Olkin (1988) thus shows that formula (1) yields a bivariate distribution in that special case.

3. EXAMPLES OF ARCHIMAX DISTRIBUTIONS

Extreme value copulas, C_A, and Archimedean copulas, C_ϕ, are the simplest examples of Archimax distributions. The reader may refer to Tawn (1988) or Joe *et al.* (1992) for lists of extreme value generators, A, and to Genest and MacKay (1986a, 1986b), Genest and Rivest (1993), or Nelsen (1999, Chap. 4) for classical Archimedean generators, ϕ. Gumbel's extreme value copulas, whose dependence function is defined for all $\ell \geqslant 1$ by

$$G_\ell(t) = \{t^\ell + (1-t)^\ell\}^{1/\ell}, \qquad 0 \leqslant t \leqslant 1 \tag{6}$$

have a distinguished role in that they are the only copulas that are both Archimedean and extreme (Genest and Rivest, 1989).

Archimax distributions that are neither Archimedean nor extreme may be constructed at will. Any functions A and ϕ verifying conditions (i) and (ii) will do, provided that A is not identically equal to 1 and $\phi(t) \neq \log^\ell(1/t)$ for every $\ell \geqslant 1$. For example, the generator of Tawn's mixed model (1988), defined for all $0 \leqslant \theta \leqslant 1$ by

$$A_\theta(t) = \theta t^2 - \theta t + 1, \qquad 0 \leqslant t \leqslant 1 \tag{7}$$

may be combined with the generator of Clayton's family,

$$\phi_{1,\alpha}(t) = (t^{-\alpha} - 1)/\alpha, \qquad \alpha > 0 \tag{8}$$

or with

$$\phi_{2,\alpha}(t) = (1 - t^\alpha)^{1/\alpha}, \qquad 0 < \alpha \leqslant 1 \tag{9}$$

which generates the copula of Genest and Ghoudi (1994). These Archimax distributions are used for simulations reported in Sect. 7.

Formula (9) provides an example of an Archimedean generator ϕ for which $\phi(1-1/t) \in RV_{-m}$ with $m = 1/\alpha$ possibly different from 1. Thus an Archimax copula $C_{\phi,A}$ with this choice of ϕ does not have C_A as its attractor, but rather belongs to the domain of attraction of C_{A^*} with A^* defined as in (2). Observe in passing that if $\phi(1-1/t) \in RV_{-1}$, then $\phi^m(1-1/t) \in RV_{-m}$; this process, which is at the root of Oakes' interior power families (1994), is useful for producing Archimedean generators with the desired degree of regular variation.

As a final example, suppose that one wishes to construct a copula in the domain of attraction of

$$\begin{aligned} A^*_{\alpha,\beta}(t) = {} & \{\alpha^\ell t^\ell + \beta^\ell (1-t)^\ell\}^{1/\ell} \\ & + \{(1-\alpha)^\ell t^\ell + (1-\beta)^\ell (1-t)^\ell\}^{1/\ell}, \qquad 0 \leqslant t \leqslant 1 \end{aligned}$$

in which $0 \leqslant \alpha, \beta \leqslant 1$ and $\ell \geqslant 1$. This dependence function corresponds to extreme value copula

$$C_{G_\ell}(x^\alpha, y^\beta)\, C_{G_\ell}(x^{1-\alpha}, y^{1-\beta})$$

obtained from a mixture of two Gumbel copulas with parameter $\ell \geqslant 1$ through the asymmetrization process described by Genest *et al.* (1998). Using a software such as Matlab, it is easy to check that condition (4) is verified when $\alpha = 1/10$, $\beta = 3/10$, $m = 3/2$ and $\ell = 1.6$ or 5, say. However, function A defined by (3) is not a dependence function when $\ell = 16$. It is interesting to note that when $\alpha = \beta$, $A^* = G_\ell$ and condition (4) obtains for all $1 \leqslant m \leqslant \ell$, but that in this case, $A = G_{\ell/m}$ and $C_{\phi, A}$ is then an Archimedean copula with generator $\phi^{\ell/m}$.

4. DETERMINATION OF THE MAXIMUM AND MINIMUM ATTRACTORS

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from Archimax copula $C_{\phi, A}$ and put

$$X_n^* = \max(X_1, \ldots, X_n) \qquad \text{and} \qquad Y_n^* = \max(Y_1, \ldots, Y_n). \tag{10}$$

An extreme value copula C_{A^*} is said to be the maximum attractor of $C_{\phi, A}$ if it is the unique copula associated with the limiting distribution function

$$\lim_{n \to \infty} \operatorname{pr}\{n(X_n^* - 1) \leqslant x, n(Y_n^* - 1) \leqslant y\},$$

where the normalization of X_n^* and Y_n^* ensures that their limiting distribution is Weibull, since the X_i's and the Y_j's are uniformly distributed on $[0, 1]$. From Theor. 5.2.3 in Galambos (1987), C_{A^*} is then the limit of the sequence $(C_n)_{n \geqslant 1}$ of copulas defined by

$$C_n(x, y) = C^n_{\phi, A}(x^{1/n}, y^{1/n}), \qquad 0 \leqslant x, y \leqslant 1. \tag{11}$$

The following proposition identifies this attractor under conditions on $C_{\phi, A}$ which involve the notion of regular variation, recalled in Sect. 2. Observe that if an increasing function ψ belongs to RV_ρ for some $0 < \rho < \infty$ and is such that $\lim_{t \to \infty} \psi(t) = \infty$, then $\psi^{-1} \in RV_{1/\rho}$. This result (Prop. 0.8, p. 22 of Resnick, 1987) is used in the sequel.

PROPOSITION 4.1. *If $C_{\phi, A}$ is a bivariate Archimax copula with generators A and ϕ such that $\phi(1 - 1/t) \in RV_{-m}$ for some $m \geqslant 1$, then $C_{\phi, A}$ belongs to the maximum domain of attraction of an extreme value distribution with* *generator* (2). *This attractor may be regarded alternatively as an Archimax distribution with generators* A^* *and* $\phi(t) = \log(1/t)$ *or with generators* A *and* $\phi^*(t) = \log^m(1/t)$. *Furthermore,* A *and* A^* *coincide if and only if* $m = 1$.

Proof. If $C_{\phi, A}$ is an Archimax copula with generators A and $\phi(1 - 1/t) \in RV_{-m}$, it must be shown that copula C_n defined by (11) converges pointwise to C_{A^*}. Fix $0 < x, y < 1$ and note that

$$C_n(x, y) = \left(1 - \frac{a_n}{n}\right)^n$$

with

$$a_n = \frac{1 - \phi^{-1}\{s_n \phi(1 - 1/n)\}}{1/n}$$

and

$$s_n = \frac{\phi(x^{1/n}) + \phi(y^{1/n})}{\phi(1 - 1/n)} A \left\{ \frac{1}{1 + \phi(y^{1/n})/\phi(x^{1/n})} \right\}.$$

Now

$$\frac{\phi(x^{1/n})}{\phi(1 - 1/n)} = \frac{\phi(1 - t_n/n)}{\phi(1 - 1/n)}$$

with $t_n = n(1 - x^{1/n})$ and $t_n \to \log(1/x)$. Since $\phi(1 - 1/t) \in RV_{-m}$ by hypothesis, one must have

$$\lim_{n \to \infty} \frac{\phi(x^{1/n})}{\phi(1 - 1/n)} = \log^m(1/x)$$

by Prop. 0.5, p. 17 of Resnick (1987). Repeating the argument with y and using the continuity of A, one finds

$$\lim_{n \to \infty} s_n = s = \{(-\log x)^m + (-\log y)^m\} \; A \left[\frac{1}{1 + \{\log(y)/\log(x)\}^m} \right].$$

Exploiting once again Prop. 0.5 of Resnick (1987) and the fact that $1 - \phi^{-1}(1/t) \in RV_{-1/m}$ allows one to conclude that $a_n \to s^{1/m}$, and the proof is complete.

An analogous result concerning the minimum domain of attraction is given below. Of interest here is the determination of the copula associated with the limiting distribution of the pair $(\min_{1 \leqslant i \leqslant n} X_i, \min_{1 \leqslant i \leqslant n} Y_i)$, once appropriately normalized. As the proof of this result is similar to that of Prop. 4.1, it is relegated to Appendix C. While the argument in the case of the maximum involves a condition on ϕ near 1, it is its behavior in the neighborhood of 0 which matters for the treatment of the minimum.

PROPOSITION 4.2. *If $C_{\phi, A}$ is a bivariate Archimax copula with generators A and ϕ such that $\lim_{t \to 0} \phi(t) = \infty$ and $\phi(1/t) \in RV_{1/m}$ for some $m \geqslant 0$, then $C_{\phi, A}$ belongs to the minimum domain of attraction of the copula $C_{*}(x, y) = x + y - 1 + (1-x)(1-y)/C_{\phi^*, A}(1-x, 1-y)$, where $\phi^*(t) = \log^{-1/m}(1/t)$.*

Note that in the above proposition, $C_{\phi^*, A}$ is defined as in Eq. (1) but is not a copula, because ϕ^* does not satisfy condition (ii) of Sect. 2 and hence is not an Archimedean generator.

5. A SIMULATION ALGORITHM

To generate pseudo-observations from a bivariate Archimax copula $C_{\phi, A}(x, y)$, a natural way to proceed would be to draw X uniformly from the interval $[0, 1]$ and to generate Y from the conditional distribution $\partial C_{\phi, A}(x, y)/\partial x$. In general, however, the latter distribution will not be invertible. An alternative algorithm is suggested by the following result, in which $K_\phi(t) = t - \lambda_\phi(t)$ is defined for all $0 \leqslant t \leqslant 1$ in terms of the function $\lambda_\phi = \phi/\phi'$. (Here, ϕ' stands for the right derivative wherever ϕ is not differentiable.) The proof of this proposition may be found in Appendix D.

PROPOSITION 5.1. *Let (X, Y) be a random pair with distribution function $C_{\phi, A}$ and define $Z = \phi(X)/\{\phi(X) + \phi(Y)\}$ and $W = C_{\phi, A}(X, Y)$. The joint distribution of the pair (Z, W) is given by*

$$\mathrm{pr}(Z \leqslant z, W \leqslant w) = K_\phi(w) \left\{ z + z(1-z) \frac{A'(z)}{A(z)} \right\} + \lambda_\phi(w) \int_0^z \frac{t(1-t)}{A(t)} \, dA'(t)$$

for all $0 \leqslant z, w \leqslant 1$. In particular, one has

$$\mathrm{pr}(Z \leqslant z) \equiv H(z) = z + z(1-z)\, A'(z)/A(z), \qquad 0 \leqslant z \leqslant 1.$$

Now assume that A'' exists and is continuous everywhere on $(0, 1)$. Denote $h = H'$ and let

$$p(z) = \frac{z(1-z)\, A''(z)}{h(z)\, A(z)}, \qquad 0 \leqslant z \leqslant 1.$$

The conditional distribution of W given $Z = z$ may then be written as

$$\mathrm{pr}(W \leqslant w \mid Z = z) = p(z)\, w + \{1 - p(z)\}\, K_\phi(w),$$

where $0 \leqslant p(z) \leqslant 1$, as pointed out by Ghoudi *et al.* (1998). Therefore, the distribution of W given z is a mixture of the univariate distribution K_ϕ and of a uniform distribution on $[0, 1]$. To generate an observation from $C_{\phi, A}$, one may thus proceed as follows:

(i) generate z from distribution H;

(ii) given z, draw u from a uniform distribution on $[0, 1]$;

(iii) if $u \leqslant p(z)$, take w from the uniform distribution on $[0, 1]$; otherwise, generate w from distribution K_ϕ;

(iv) set $x = \phi^{-1}\{z\phi(w)/A(z)\}$ and $y = \phi^{-1}\{(1-z)\,\phi(w)/A(z)\}$.

Acceptance-rejection methods may be used to simulate H, when the latter is noninvertible. Remark that if $D = A'/A$, then h may be written simply as

$$h(z) = 1 + (1-2z)\,D(z) + z(1-z)\,D'(z) \tag{12}$$

and is bounded, as long as A'' is continuous. To generate observations from K_ϕ, it is often convenient to use the fact that it is the distribution of $V = C_\phi(X, Y)$, where (X, Y) is a random pair from Archimedean copula C_ϕ. A general algorithm for simulating such copulas is given by Genest and MacKay (1986a).

6. DEPENDENCE STRUCTURE OF THE FAMILY

Through Prop. 5.1, it is also possible to gain insight into the dependence structure of Archimax distributions. For, the authors have shown elsewhere (Capéraà *et al.*, 1997b) that random pairs (X_i, Y_i) with copulas C_i can be meaningfully ordered by comparing the distribution functions K_i of the probability integral transforms $W_i = C_i(X_i, Y_i)$. Since $4E(W_i) - 1 = \tau_i$, the value of Kendall's tau for the pair (X_i, Y_i), the condition

$$K_1(w) \geqslant K_2(w), \qquad 0 \leqslant w \leqslant 1$$

denoted $C_1 \prec C_2$, implies that $\tau_1 \leqslant \tau_2$. This ordering is neither stronger nor weaker than the concordance ordering $C_1 \leqslant C_2$, also referred to as positive quadrant dependence (PQD, Yanagimoto and Okamoto, 1969). As pointed out by Capéraà *et al.* (1997b), $C_1 \leqslant C_2 \Rightarrow C_1 \prec C_2$ for extreme value copulas, but the reverse implication is not true. It is also easy to check that the $\prec$ ordering is stronger than PQD within the Archimedean class.

When (X, Y) is from copula $C_{\phi, A}$, Prop. 5.1 yields

$$K_{\phi, A}(w) \equiv \mathrm{pr}(W \leqslant w) = K_\phi(w) + \lambda_\phi(w)\,\tau_A, \qquad 0 \leqslant w \leqslant 1 \tag{13}$$

and

$$\tau_{\phi, A} = 4E(W) - 1 = \tau_A + (1 - \tau_A)\,\tau_\phi, \tag{14}$$

where τ_A and τ_ϕ are the values of Kendall's tau associated with copulas C_A and C_ϕ, respectively. The latter quantities are given explicitly by

$$\tau_A = \int_0^1 \frac{t(1-t)}{A(t)}\, dA'(t) \qquad \text{and} \qquad \tau_\phi = 1 + 4 \int_0^1 \lambda_\phi(t)\, dt.$$

Derivations of these formulas can be found in Ghoudi *et al.* (1998) and in Genest and MacKay (1986a, 1986b), respectively. Note that since extreme value distributions are associated (Marshall and Olkin, 1983), one always has $0 \leqslant \tau_A \leqslant 1$, so that $\tau_{\phi, A}$ is a convex combination of τ_ϕ and 1, by Eq. (14). In addition, the following implications are obvious from Eq. (13),

$$\tau_{A_1} \geqslant \tau_{A_2} \Leftrightarrow C_{A_1} \prec C_{A_2} \Leftrightarrow C_{\phi, A_1} \prec C_{\phi, A_2},$$
$$C_{\phi_1, A} \prec C_{\phi_2, A} \Leftrightarrow C_{\phi_1} \prec C_{\phi_2} \Rightarrow C_{\phi_1} \leqslant C_{\phi_2}$$

and

$$C_{A_1} \prec C_{A_2} \quad \text{and} \quad C_{\phi_1} \prec C_{\phi_2} \quad \Rightarrow \quad C_{\phi_1, A_1} \prec C_{\phi_2, A_2}.$$

These relations are useful in checking that parametric families of Archimax copulas are ordered by $\prec$.

It is also instructive to compare the degree of dependence of $C_{\phi, A}$ to that of its attractor C_{A^*}. As mentioned earlier, Sibuya (1960) showed that unless the components X and Y of a bivariate normal pair are perfectly positively correlated, the maxima of the X_i's and of the Y_j's in a random sample of size n are asymptotically independent. Although the bivariate normal distribution is not Archimax, it might be expected that as $n \to \infty$, the dependence between variables X_n^* and Y_n^* defined by (10) would always be weaker than between X_1 and Y_1. As it happens, however, the degree of dependence in an extreme value distribution C_{A^*} may be smaller, equal or even greater than in the Archimax copula $C_{\phi, A}$ which belongs to its domain of attraction. To see this, note that A always dominates A^* when the two are related by equation (3). This inequality, which is obtained by showing that the dependence function A so defined is increasing in m, may be combined with (14) to see that $\tau_{\phi, A} < \tau_{A^*}$ when $\tau_\phi < 0$. In fact, a simple calculation confirms that when $A^* = G_\ell$ and ϕ is the generator of a Genest–Ghoudi copula with parameter $\alpha = 1/m$, $1 \leqslant m \leqslant \ell$, one has $C_{\phi, A} \prec C_{A^*}$, while when ϕ is the generator of a Clayton copula with parameter $\alpha \geqslant 0$, one has rather $C_{A^*} \prec C_{\phi, A}$.

7. A CLOSER LOOK AT THE THRESHOLD METHOD OF JOE *ET AL.*

In recent years, several authors have been concerned with studying the dependence structure of multivariate extreme events, based on a sample of data that are merely in the domain of attraction of an extreme value distribution. In that context, a result of de Haan (1985) has been exploited by Joe *et al.* (1992) to extend the threshold method to the bivariate case in order to estimate the dependence function A^* associated with the maxima of a random sample. However, the behavior of this estimation procedure as a function of the threshold value does not appear to have been examined to date, whether in large or small samples. As Archimax distributions allow to simulate distributions having a given maximum attractor, they provide a handy tool for filling this gap. Before illustrating the effect of the threshold value on the method of Joe *et al.*, a brief description of their technique is given in the following paragraph.

Assume that $(X_1, Y_1), \ldots, (X_n, Y_n)$ form a random sample from some distribution belonging to the maximum domain of attraction of an extreme value distribution with dependence function A^*. Following Joe *et al.* (1992), assume without loss of generality that the marginal distribution of the X_i's satisfies the condition $\mathrm{pr}(X_i > x) \approx 1/x$, and likewise for the Y_i's. This may be achieved by replacing each coordinate by $1/\log\{n/(R_i - 1/2)\}$, where R_i is the rank of the ith observation from that variable. Under these conditions, it was shown by de Haan (1985) that the limit of the point process associated with the $(X_i, Y_i)/n$'s converges to a non-homogeneous Poisson process on $\mathbb{R}^2_+ \setminus \{(0, 0)\}$ with intensity measure μ^*. To estimate A^*, Joe *et al.* exploit the fact that on $(0, 1)$, $A(t) = t\mu\{1, t/(1-t)\}$ in terms of $\mu(u, v) = \mu^*\{([0, u] \times [0, v])^c\}$, where c denotes complementation. Now it may be seen that for arbitrary $\omega = t/(1-t)$, $v\{(z, \infty)\} = \mu(1, \omega)/z$ is also the intensity measure of the Poisson process on $(0, \infty)$ to which converges the point process associated with the $Z_i = \max(X_i, Y_i/\omega)/n$, $1 \leqslant i \leqslant n$. Joe *et al.* simply suggest that $\mu(1, \omega)/z$ be estimated by the number of Z_i's that are superior to z. Of course, this depends on the threshold value z, which should be taken large enough in order for the asymptotic approximation to be reasonable. Thus if $z = Z_{(n-k+1)}$ is the kth *largest* order statistic, one might estimate $\mu(1, \omega)$ by $kZ_{(n-k+1)}$. To reduce the variability of this estimation, which again depends on the choice of k, Joe *et al.* recommend averaging these quantities over those values of k for which $kZ_{(n-k+1)}$ is approximately constant. Practically speaking, this implies that k should be neither too small, nor too large. In the simulations reported here, proportions $\pi = 1$, 4 and 8% of the sample were selected, and for each value of $\omega = t/(1-t)$, the integer k was allowed to run from $n \times (1-\pi)$ to $n - 16$, inclusively.

TABLE I

Values of Parameter α Used to Simulate Archimax Distributions with Predetermined Kendall's Tau

	Values of $\tau_{\phi, A}$			
	-0.5	-0.3	0.3	0.5
$\phi^{m}_{1,\alpha}$	·	·	0.1455	1
$\phi^{m}_{3,\alpha}$	·	-13	0.6125	·
$\phi^{m\alpha}_{2,\alpha}$	1	0.85	·	·

Note. The value of m is equal to 6/5 for the three generators.

To examine the effect of the choice of π on the quality of the estimation of A^*, pseudo-random samples of various sizes were generated from three families of Archimax distributions having the same dependence function A and different Archimedean generators ϕ. More precisely, A was taken of the form (7) with $\theta = 0.28$ and the ϕ's were rigged to have same degree of regular variation $-m = -6/5$, so that the value of Kendall's tau of their common maximum attractor A^* be approximately equal to 1/4. Specifically, ϕ was successively taken to be $\phi^{m}_{1,\alpha}$, $\phi^{m\alpha}_{2,\alpha}$, and $\phi^{m}_{3,\alpha}$ where $\phi_{1,\alpha}$ and $\phi_{2,\alpha}$ are given respectively by (8) and (9), and

$$\phi_{3,\alpha}(t) = -\log\left\{\frac{1-\exp(-t\alpha)}{1-\exp(-\alpha)}\right\}, \qquad 0 < t \leqslant 1$$

is the generator of Frank's family of copulas (Genest, 1987) with parameter $\alpha \in \mathbb{R}$. Finally, for each of these families, the two values of α given in Table I were used in order to achieve predetermined degrees of dependence in $C_{\phi, A}$, as measured by Kendall's tau, $\tau_{\phi, A}$.

For each of the six Archimax distributions whose Archimedean generator is specified in Table I, 500 samples of size $n = 500$, 2,500 and 10,000 were generated. For each such sample, the extreme value copula A^* was then estimated by the method of Joe *et al.* (1992) using 3 different proportions $\pi = 1, 4, 8\,\%$ (note however that because of the range selected for k in the averaging process described above, the case $\pi = 1\,\%$ and $n = 500$ could not be treated). The mean of the resulting estimations $A^*_{n,1}, ..., A^*_{n,500}$ was then computed, as well as the corresponding value of Kendall's tau. The results are presented in Table II and show clearly that the estimation of $\tau_{A^*} = 0.25$ is generally biased downward but that this bias gets smaller as n increases, as expected.

The results also make it obvious that the performance of the method is sensitive to the difference $\Delta = |\tau_{\phi, A} - \tau_{A^*}|$ between the degree of dependence

TABLE II

Value of Kendall's Tau of the Average of 500 Estimations of the Maximum Attractor of the Six Archimax Distributions Specified in Table I

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.241	0.244	0.246
4%	0.249	0.238	0.243
1%	—	0.233	0.238

$\phi=\phi^m_{1,\alpha}$, $\tau_{\phi,A}=0.3$

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.239	0.253	0.257
4%	0.244	0.243	0.250
1%	—	0.238	0.243

$\phi=\phi^m_{1,\alpha}$, $\tau_{\phi,A}=0.5$

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.207	0.230	0.234
4%	0.228	0.233	0.239
1%	—	0.238	0.237

$\phi=\phi^m_{3,\alpha}$, $\tau_{\phi,A}=-0.3$

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.229	0.246	0.250
4%	0.241	0.241	0.247
1%	—	0.239	0.238

$\phi=\phi^m_{3,\alpha}$, $\tau_{\phi,A}=0.3$

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.207	0.230	0.234
4%	0.226	0.234	0.239
1%	—	0.236	0.239

$\phi=\phi^{m\alpha}_{2,\alpha}$, $\tau_{\phi,A}=-0.5$

	$n=500$	$n=2{,}500$	$n=10{,}000$
$\pi=8\%$	0.205	0.232	0.236
4%	0.223	0.234	0.240
1%	—	0.236	0.236

$\phi=\phi^{m\alpha}_{2,\alpha}$, $\tau_{\phi,A}=-0.3$

Note. Values are a function of the sample size and of the proportion π of observations used in the method of Joe *et al.* (1992).

in the original distribution and its attractor. When Δ is small to moderate, that is when $\tau_{\phi,A}=0.3$ or 0.5, the bias is comparatively smaller, for given n, and values of $\pi=4$ and 8% seem preferable to $\pi=1\%$. When Δ is large, however, it would appear that $\pi=1$ and 4% should be preferred to $\pi=8\%$. Over all, therefore, it seems that $\pi=4\%$ is an acceptable compromise for the situations considered, although this tentative conclusion would need to be confirmed with further experimentation under a greater variety of conditions and criteria. This will be the object of future investigations.

APPENDIX A

Proof that Archimax Functions Are Copulas

Let $C=C_{\phi,A}$ be a bivariate function of the form (1). To show that C is a bivariate distribution with uniform marginals, one must check (e.g., Sklar, 1959) that

(i) $C(x, y)=0$ when x or $y=0$;

(ii) $C(x, 1)\equiv x$ and $C(1, y)\equiv y$;

(iii) $C(x_2, y_2)+C(x_1, y_1)\geqslant C(x_2, y_1)+C(x_1, y_2)$ for all $0\leqslant x_1\leqslant x_2\leqslant 1$ and $0\leqslant y_1\leqslant y_2\leqslant 1$.

The first two conditions are easily verified, using the fact that $\max(t, 1-t)\leqslant A(t)\leqslant 1$ and the conventions $\phi(0)\equiv \lim_{t\to 0}\phi(t)$ and $\phi^{-1}(s)=0$ when $s\geqslant \phi(0)$. To establish the third condition, first let $X_i=\phi(x_i)$, $Y_j=\phi(y_j)$ and $S_{ij}=X_i/(X_i+Y_j)$ for $i, j=1, 2$, so that $S_{21}\leqslant S_{ii}\leqslant S_{12}$ for $i=1, 2$ in view of the fact that ϕ is decreasing. Using the fact that $A(u)/u$ and $A(u)/(u-1)$ are nonincreasing (e.g., Deheuvels, 1991), one has also

$$\frac{X_2}{S_{22}}A(S_{22})\leqslant\frac{X_1}{S_{12}}A(S_{12}),\qquad \frac{X_2}{S_{21}}A(S_{21})\leqslant\frac{X_1}{S_{11}}A(S_{11}). \tag{15}$$

What will be shown is that

$$\frac{X_2}{S_{22}}A(S_{22})+\frac{X_1}{S_{11}}A(S_{11})\leqslant\frac{X_2}{S_{21}}A(S_{21})+\frac{X_1}{S_{12}}A(S_{12}) \tag{16}$$

or, equivalently, that

$$\phi\{C(x_2, y_2)\}+\phi\{C(x_1, y_1)\}\leqslant\phi\{C(x_2, y_1)\}+\phi\{C(x_1, y_2)\}$$

whatever the choice of $0\leqslant x_1\leqslant x_2\leqslant 1$ and $0\leqslant y_1\leqslant y_2\leqslant 1$. Condition (iii) will then follow immediately from (15) and from the fact that ϕ^{-1} is both convex and decreasing.

To prove (16), one needs to distinguish two cases, depending on the sign of $A(S_{21})-A(S_{12})$. Assuming, for example, that this difference is positive, it would suffice to see that the inequality holds with A replaced by $\tilde{A}(t)=A(t)$ for $t\leqslant \inf\{s: A(s)\leqslant A(S_{12})\}$ and equal to $A(S_{12})$ thereafter. Now, using for example Lemma 2.1 of Cambanis and Simons (1982), it is sufficient to prove (16) for convex functions of the form $\max(1-\delta t, 0)+A(S_{12})$ with $\delta>0$. As this assertion is readily verified, the proof is complete.

APPENDIX B

Derivation of Condition (4) *and Related Matters*

It is clear that if A is defined by relation (3), then $A(0)=A(1)=1$ and $A(t)\geqslant\max(t, 1-t)$ for all $0\leqslant t\leqslant 1$, because A^* already enjoys these properties. Furthermore, A will be bounded above by 1 if it is convex. To show that the latter occurs when condition (4) holds, first write $A(t) = \{A^*(s)/G_m(s)\}^m$ in terms of Gumbel's generator G_m, defined by (6), and the bijection

$$s = s(t) = \frac{t^{1/m}}{t^{1/m} + (1-t)^{1/m}},$$

which is strictly increasing in t on $[0, 1]$.

Letting $D(t) = d \log\{A^*(t)\}/dt$, long but pleasant algebraic manipulations then lead to express condition $A'' \geqslant 0$ in the form

$$\begin{aligned} D^2(t) + D'(t)/m + \{Q_m(t) - 2L'(t)\}\ D(t) \\ \geqslant -\{L'(t)\}^2 + Q_m(t)\ L'(t) + L''(t)/m, \end{aligned} \tag{17}$$

where $L(t) = \log\{G_m(t)\}$ and

$$Q_m(t) = \frac{s''\{s^{-1}(t)\}}{m[s'\{s^{-1}(t)\}]^2}.$$

Further calculations show that the right-hand side of inequality (17) reduces to $q_m(t) = (m-1)/\{mt(1-t)\}$ and that $Q_m(t) - 2L'(t) = (2t-1)\ q_m(t)$. As a result, A is a dependence function if and only if

$$D^2(t) + D'(t)/m + (2t-1)\ q_m(t)\ D(t) \geqslant q_m(t) \tag{18}$$

for all $0 \leqslant t \leqslant 1$. Letting $H^*(t) = t + t(1-t)\ D(t)$ and h^* stand for its derivative, it is now a simple matter to convert condition (18) into condition (4).

APPENDIX C

Proof of Proposition 4.3

Let $C = C_{\phi, A}$ stand once again for a bivariate function of the form (1). Since the copula associated with $(\min_{1 \leqslant i \leqslant n} X_i, \min_{1 \leqslant i \leqslant n} Y_i)$ can be expressed as $x + y - 1 + \bar{C}^n\{(1-x)^{1/n}, (1-y)^{1/n}\}$ with

$$\bar{C}(u, v) = u + v - 1 + C(1-u, 1-v),$$

the problem reduces to finding $\lim_{n\to\infty} \bar{C}^n(x^{1/n}, y^{1/n})$ for arbitrary $0 \leqslant x, y \leqslant 1$. Writing the latter as $(1+a_n/n)^n$ with $a_n = t_n - n(1-x^{1/n}) - n(1-y^{1/n})$ and $t_n = nC_{\phi, A}(1-x^{1/n}, 1-y^{1/n})$, it suffices to show that

$$\lim_{n\to\infty} t_n = t = -\log C_{\phi^*, A}(x, y)$$

with $\phi^*(t) = \log^{-1/m}(1/t)$. To this end, write

$$t_n = \frac{\phi^{-1}\{s_n \phi(1/n)\}}{1/n}$$

and verify that $s_n \to t^{-1/m}$, using the hypothesis that $\phi(1/t) \in RV_{1/m}$. The conclusion then follows by uniform convergence and the fact that $1/\phi^{-1} \in RV_m$, which results from the hypothesis and Prop. 0.8, p. 22 of Resnick (1987).

APPENDIX D

Proof of Proposition 5.1

Fix $0 < z, w \leqslant 1$ and write

$$\begin{aligned}\mathrm{pr}(Z \leqslant z, W \leqslant w) &= \mathrm{pr}[Z \leqslant z, \phi^{-1}\{\phi(X)\, A(Z)/Z\} \leqslant w] \\ &= \mathrm{pr}[\, g(Z) \leqslant \min\{g(z), \phi(X)/\phi(w)\}\,],\end{aligned}$$

where $g(z) = z/A(z)$ is nondecreasing (e.g., Deheuvels, 1991). Using the fact that the marginal distribution of X is uniform, this may be rewritten as

$$\mathrm{pr}[Z \leqslant z, X \leqslant \phi^{-1}\{g(z)\, \phi(w)\}\,] + \int_{\gamma(z, w)}^{1} \mathrm{pr}[Z \leqslant g^{-1}\{\phi(x)/\phi(w)\} \mid X = x]\, dx, \tag{19}$$

where $\gamma(z, w) = \phi^{-1}\{g(z)\, \phi(w)\}$, so that both summands may be evaluated from the joint distribution function B of the pair (Z, X). The latter is given by

$$\begin{aligned}B(z, x) &= \int_0^x \mathrm{pr}[\, Y \leqslant \phi^{-1}\{\phi(t)(1-z)/z\} \mid X = t]\, dt \\ &= \int_0^x C_1'[t, \phi^{-1}\{\phi(t)(1-z)/z\}\,]\, dt,\end{aligned}$$

where C_1' denotes the partial derivative of $C_{\phi,A}(x, y)$ with respect to x. A simple calculation shows that the integrand is equal to

$$\frac{\phi'(t)}{\phi'[\phi^{-1}\{\phi(t)\,A(z)/z\}]}\{A(z)+(1-z)\,A'(z)\},$$

so that

$$B(z, x)=\{z+z(1-z)\,A'(z)/A(z)\}\,\phi^{-1}\,\{\phi(x)\,A(z)/z\}.$$

Accordingly, the first summand in Eq. (19) equals

$$\{z+z(1-z)\,A'(z)/A(z)\}\,w.$$

As for the second summand, the change of variable $s=g^{-1}\{\phi(x)\,\phi(w)\}$ allows one to express it in the form

$$-\int_0^z B_2'\{s, \gamma(s, w)\}\,\frac{g'(s)\,\phi(w)}{\phi'\{\gamma(s, w)\}}\,ds,$$

where B_2' stands for $\partial B(z, x)/\partial x$. Upon substitution, one gets

$$-\int_0^z \{A(s)+(1-s)\,A'(s)\}\;g'(s)\,\lambda_\phi(w)\,ds.$$

Using the fact that $g'(s)=\{A(s)-sA'(s)\}/A^2(s)$, this reduces to

$$-z\lambda_\phi(w)+\lambda_\phi(w)\left[\int_0^z s(1-s)\left\{\frac{A'(s)}{A(s)}\right\}^2 ds-\int_0^z (1-2s)\,\frac{A'(s)}{A(s)}\,ds\right].$$

Finally, observing that the term in square brackets is equal to

$$\int_0^z \frac{s(1-s)}{A(s)}\,dA'(s)-z(1-z)\,\frac{A'(z)}{A(z)},$$

one may conclude.

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