Géométrie et calcul diff   CC5   14 juin 2012

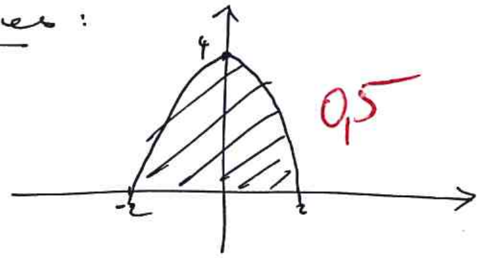
pts

Exo 3 D = domaine compris entre $\vec{Ox}$ et $y = 4 - x^2$

1. Dessiner D et la décrire en coord. cartésiennes :

1pt $D = \{ (x,y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2 \ , \ 0 \leq y \leq 4 - x^2 \}$ 0,5

car $\begin{cases} y = 4 - x^2 \\ y = 0 \end{cases} \iff x^2 = 4 \quad x = \pm 2$.



0,5

2. Calculer l'aire de D.

pt

$$\text{Aire } D = \iint_D dx\,dy = \int_{-2}^{2} dx \int_0^{4-x^2} dy = \int_{-2}^{2} [y]_0^{4-x^2} dx = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{1}{3} x^3 \right]_{-2}^{2}$$

$$= 8 - \frac{1}{3} \cdot 8 + 8 - \frac{1}{3} 8 = 2 \cdot 8 \left(1 - \frac{1}{3}\right) = 16 \cdot \frac{3-1}{3} = \frac{32}{3}.$$

3. Calculer l'intégrale $\iint_D 2xy \, dx dy$.

1pt

$$\iint_D 2xy \, dx\,dy = \int_{-2}^{2} x \left(\int_0^{4-x^2} 2y \, dy \right) dx = \int_{-2}^{2} x \cdot [y^2]_0^{4-x^2} dx = \int_{-2}^{2} x (16 - 8x^2 + x^4)\,dx =$$

$$= \int_{-2}^{2} (16x - 8x^3 + x^5)\,dx = \left[8x^2 - 2x^4 + \frac{1}{6} x^6 \right]_{-2}^{2} = 8 \cdot 4 - 2 \cdot 16 + \frac{1}{6} \cdot 64 - 8 \cdot 4 + 2 \cdot 16 - \frac{1}{6} 64 =$$

4. Montrer que la 2-forme $\omega = 2xy \, dx \wedge dy$ est exacte sur D et calculer une primitive η

pts

$d\omega = 0$ car $\omega \in \Omega^2(D)$ et $D \subset \mathbb{R}^2$, donc ω exacte car fermée et D contractile. 0,5pt

$\eta = a\,dx + b\,dy$, $d\eta = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy = \omega = 2xy \, dx \wedge dy \iff$

$$\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 2xy$$

a) supposons $\frac{\partial a}{\partial y} = 0$ alors $\frac{\partial b}{\partial x} = 2xy \implies b = \int 2xy \, dx + f(y)$
$= x^2 y + f(y)$

supp. $a = 0$ et $f(y) = 0$, alors $b = x^2 y$ et $\boxed{\eta = x^2 y \, dy}$ 1,5pt

preuve : $d\eta = 2xy \, dx \wedge dy$ ok.

b) si on suppose $\frac{\partial b}{\partial x} = 0$ alors $\frac{\partial a}{\partial y} = -2xy \implies a = -\int 2xy \, dy + f(x)$

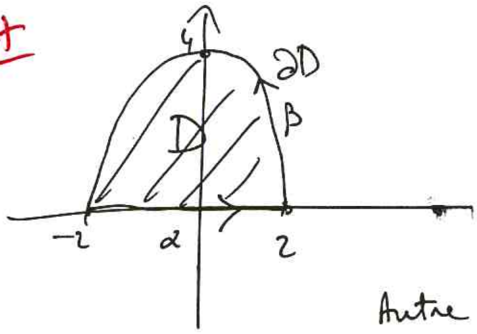
si $f(x) \equiv 0$ on a $a = -xy^2$ et si $b \equiv 0$ on a $\boxed{\eta = -xy^2 \, dx.}$

c) mélange : $\eta = \cancel{\frac{1}{2}\left(-xy^2 dx + x^2 y \, dy\right)}$

$$\eta = -\frac{xy^2}{2} dx + \frac{x^2 y}{2} dy$$

5. Paramétrer le bord de D comme union de deux courbes orientées dans le sens antihoraire.

1pt



$\partial D = \alpha \cup \beta$ avec

(1) $\alpha(t) = (t, 0)$ pour $t \in [-2,2]$ 0,5

$\beta(t) = (-t, 4-t^2)$ $t \in [-2,2]$ 0,5

Autre choix : (2) $\alpha(t) = (t-2, 0)$ $t \in [0,4]$

$\alpha(t) = (4t-2, 0)$ $t \in [0,1]$

(2) $t \in [0,4]$ $\beta(t) = \left(2-t, 4-(2-t)^2\right)$ ~~$t \in [0,4]$~~ $\beta = (2-t, 4t-t^2)$

$t \in [0,1]$ $\beta(t) = \left(2-4t, 4-(2-4t)^2\right) = \left(2-4t, 16t-16t^2\right)$

6. Calculer l'intégrale $\iint_D 2xy\,dx\,dy$ en utilisant Green-Riemann.

2pts

GR : $\iint_D \omega = \int_{\partial D^+} \eta$ si $\omega = d\eta$ sur D.

$$\iint_D 2xy\,dx\,dy = \int_{\partial D^+} x^2 y\,dy = \int_{\alpha \; \substack{x=t \\ y=0}} x^2y\,dy + \int_{\beta \; \substack{x=-t \\ y=4-t^2}} x^2 y\,dy \quad dy = -2t\,dt$$

1

(1) $= \int_{-2}^{2} t^2 \cdot 0 \cdot 0 + \int_{-2}^{2} (-t)^2(4-t^2)\cdot(-2t)\,dt = 0 + \int_{-2}^{2} -2t^3(4-t^2)\,dt$

$= \int_{-2}^{2} (-8t^3 + 2t^5)\,dt = \left[-2t^4 + \frac{2}{6}t^6\right]_{-2}^{2} = 0$,

1

(2) $= \int_0^4 (t-2)^2 \cdot 0 \cdot 0 + \int_0^4 (2-t)^2 \cdot (4t-t^2) \cdot \underbrace{(4-2t)}_{2(2-t)}\,dt = \int_0^4 2\underbrace{(2-t)^3}_{(8-12t+6t^2-t^3)}(4t-t^2)\,dt$

$= \int_0^4 2\left(36t - 8t^2 - 48t^2 + 12t^3 + 24t^3 - 6t^4 - 4t^4 + t^5\right) dt$

$= \int_0^4 2\left(36t - 56t^2 + 36t^3 - 10t^4 + t^5\right) dt = 2\left[18t^2 - \frac{56}{3}t^3 + 9t^4 - 2t^5 + \frac{1}{6}t^6\right]_0^4$

~~$= 2\left(18 \cdot 16 - \frac{56}{3} \cdot 64 + 9 \cdot 256 - 2 \cdot 1024 + \frac{1}{6} 4096\right)$~~

$= \cancel{2(288} = 2 \cdot 16 \left(18 - \overbrace{\frac{56}{3} \cdot 4}^{224} + 9 \cdot 16 - \underbrace{2 \cdot 64}_{-8 \cdot 16} + \frac{1}{\cancel{6}_3} \overset{128}{\cancel{256}}\right)$

$= 2 \cdot 16 \left(\underbrace{18 + 16}_{34} - \frac{\overset{32}{\cancel{96}}}{\cancel{3}}\right) = 2 \cdot 16 \cdot 2$

$\begin{array}{r} 4 \\ 18 \\ 16 \\ \hline 108 \\ 18 \\ \hline 288 \end{array}$ $\begin{array}{r} 2 \\ 56 \\ 4 \\ \hline 224 - \\ 128 \\ \hline 96 \,|\, 3 \\ 32 \end{array}$ $256 \,|\, 16$