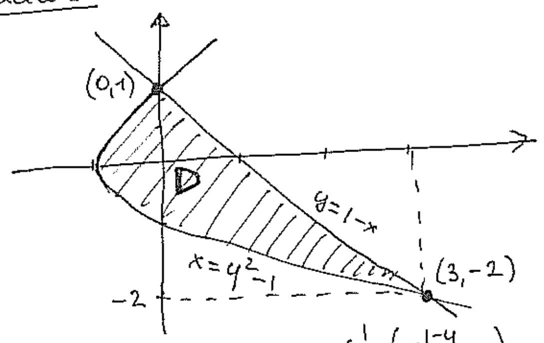
Exercice 1 :

1.

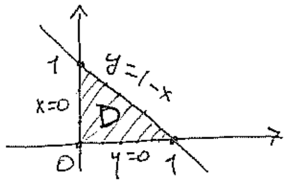


$$\begin{cases} y=1-x \\ x=y^2-1 \end{cases} \Rightarrow y^2-1=1-y \quad y^2+y-2=0 \quad y=\frac{-1\pm\sqrt{9}}{2} = \begin{cases} -2 \\ 1 \end{cases}$$

$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; -2\le y\le 1 \;,\; y^2-1\le x\le 1-y\right\}$$

$$\text{Aire}(D)=\iint_D dx\,dy = \int_{-2}^{1}\left(\int_{y^2-1}^{1-y} dx\right)dy = \int_{-2}^{1}(1-y-y^2+1)\,dx = \int_{-2}^{1}(2-y-y^2)\,dy = \left[2y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\right]_{-2}^{1} = \boxed{\frac{9}{2}}.$$

2.

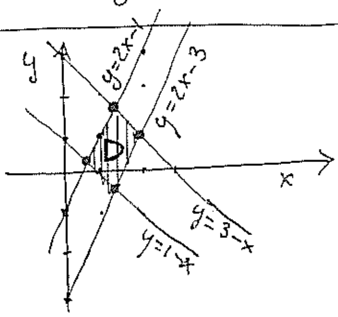


$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; 0\le x\le 1 \;,\; 0\le y\le 1-x\right\}$$

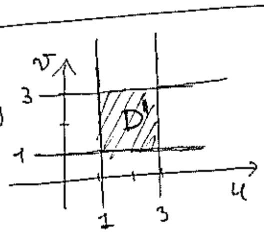
$$\iint_D x^2(y-1)\,dx\,dy = \int_0^1 x^2\left(\int_0^{1-x}(y-1)\,dy\right)dx = \int_0^1 x^2\left[\tfrac{1}{2}y^2-y\right]_0^{1-x}dx = \int_0^1 x^2(1-x)\left(\frac{1-x}{2}-1\right)dx$$

$$= \int_0^1 -\tfrac{1}{2}x^2(1-x)(x+1)\,dx = \int_0^1 -\tfrac{1}{2}x^2(1-x^2)\,dx = -\tfrac{1}{2}\int_0^1 (x^2-x^4)\,dx = -\tfrac{1}{2}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = -\tfrac{1}{2}\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = \boxed{-\frac{1}{15}}.$$

3.



$$\begin{cases} u=x+y \\ v=2x-y \end{cases} \Longrightarrow$$



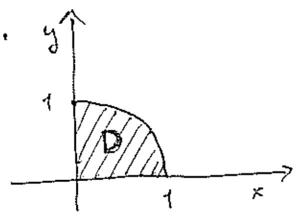
$$D'=\left\{(u,v) \;\middle|\; 1\le u\le 3 \;,\; 1\le v\le 3\right\}$$

$$h:(u,v)\longmapsto(x,y) \quad \text{avec} \quad \begin{cases} x=\frac{u+v}{3} \\ y=\frac{2u-v}{3} \end{cases}$$

$$dh=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \Rightarrow |\det(dh)| = \left|-\tfrac{1}{9}-\tfrac{2}{9}\right| = \tfrac{1}{3}$$

$$\iint_D (2x-y)\,dx\,dy = \iint_{D'} v\cdot\tfrac{1}{3}\,du\,dv = \tfrac{1}{3}\int_1^3 du\cdot\int_1^3 v\,dv = \tfrac{1}{3}\left[u\right]_1^3\cdot\left[\tfrac{1}{2}v^2\right]_1^3 = \tfrac{1}{3}(3-1)\cdot\frac{9-1}{2} = \boxed{\frac{8}{3}}.$$

4.



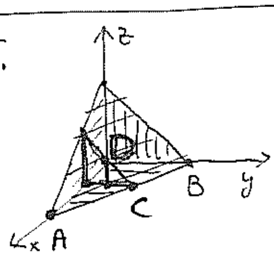
$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; x^2+y^2\le 1 \;,\; x\ge 0 \;,\; y\ge 0\right\}$$

$$D'\xrightarrow{h}D \qquad (\rho,\theta)\longmapsto(x,y) \quad \text{avec} \quad \begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \qquad dh=\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \Rightarrow |\det(dh)|=\rho$$

$$D'=\left\{(\rho,\theta)\in\mathbb{R}^+\times[0,2\pi[\;\middle|\; 0\le\rho\le 1 \;,\; 0\le\theta\le\tfrac{\pi}{2}\right\} \quad , \quad x^2+y^2=\rho^2$$

$$\iint_D e^{x^2+y^2}\,dx\,dy = \iint_{D'} e^{\rho^2}\rho\,d\rho\,d\theta = \int_0^1 e^{\rho^2}\rho\,d\rho\cdot\int_0^{\pi/2}d\theta = \tfrac{1}{2}\left[e^{\rho^2}\right]_0^1\cdot\left[\theta\right]_0^{\pi/2} = \tfrac{1}{2}(e-1)\cdot\tfrac{\pi}{2} = \boxed{\frac{\pi}{4}(e-1)}.$$

5.



$$A:\begin{cases} y=0 \\ z=0 \\ x+y+z=1 \end{cases} \Rightarrow x=1 \Rightarrow A=(1,0,0) \Longrightarrow 0\le x\le 1$$

$$AB:\begin{cases} z=0 \\ x+y+z=1 \end{cases} \Rightarrow x+y=1 \Rightarrow C=(x,1-x,0) \Longrightarrow \forall x \text{ fixé} : 0\le y\le 1-x$$

$$\forall x,y \text{ fixé} : 0\le z\le 1-(x+y) \Longrightarrow D=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} 0\le x\le 1 \\ 0\le y\le 1-x \\ 0\le z\le 1-(x+y) \end{array}\right\}$$

$$\text{Vol}(D)=\iiint_D dx\,dy\,dz = \int_0^1 dx\int_0^{1-x}dy\int_0^{1-(x+y)}dz = \int_0^1 dx\int_0^{1-x}\left[z\right]_0^{1-(x+y)}dy = \int_0^1 dx\int_0^{1-x}(1-x-y)\,dy = \int_0^1\left[(1-x)y-\tfrac{1}{2}y^2\right]_0^{1-x}dx$$

$$= \int_0^1\left[(1-x)^2-\tfrac{1}{2}(1-x)^2\right]dx = \tfrac{1}{2}\int_0^1(1-x)^2\,dx = -\tfrac{1}{2}\int_1^0 u^2\,du = \tfrac{1}{2}\left[\tfrac{1}{3}u^3\right]_0^1 = \tfrac{1}{2}\cdot\tfrac{1}{3} = \boxed{\frac{1}{6}}.$$

2) <u>Exercice 2 :</u>

1. $f(x,y,z) = xy^2 + yz \Longrightarrow df(x,y,z) = y^2 dx + (2xy+z)\,dy + y\,dz$.

$\alpha(x,y,z) = y\,dx + z\,dy + x\,dz \Longrightarrow d\alpha = dy\wedge dx + dz\wedge dy + dx\wedge dz$
$= -dx\wedge dy - dy\wedge dz + dx\wedge dz$.

$\omega(x,y,z) = x^2 dx\wedge dy + xy\, dy\wedge dz + z^2 dx\wedge dz \Longrightarrow d\omega = 0 + y\, dx\wedge dy\wedge dz + 0$.

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2. $df\wedge\alpha = \left[y^2 dx + (2xy+z)\,dy + y\,dz\right]\wedge\left[y\,dx + z\,dy + x\,dz\right]$

$= y^2 z\, dx\wedge dy + xy^2\, dx\wedge dz + y(2xy+z)\, dy\wedge dx + x(2xy+z)\, dy\wedge dz + y^2\, dz\wedge dx + yz\, dz\wedge dy$

$= y(yz - 2xy - z)\, dx\wedge dy + y^2(x-1)\, dx\wedge dz + (2x^2y + xz - yz)\, dy\wedge dz$.

$\alpha\wedge\omega = \left[y\,dx + z\,dy + x\,dz\right]\wedge\left[x^2 dx\wedge dy + xy\, dy\wedge dz + z^2 dx\wedge dz\right]$

$= xy^2\, dx\wedge dy\wedge dz + z^3\, dy\wedge dx\wedge dz + x^3\, dz\wedge dx\wedge dy$

$= (xy^2 - z^3 + x^3)\, dx\wedge dy\wedge dz$.

<u>Exercice 3</u>

1. $\omega(x,y) = xy^2 dx + x^2 y\, dy$ $\quad U = \mathbb{R}^2$

- <u>fermée</u> ? $d\omega = 2xy\, dy\wedge dx + 2xy\, dx\wedge dy = 0 \Longrightarrow \omega$ fermée.
- <u>exacte</u> ? puisque $\mathbb{R}^2$ est contractile, par le lemme de Poincaré ω est exacte.

$\omega = df \iff \begin{cases} \dfrac{\partial f}{\partial x} = xy^2 \Longrightarrow f(x,y) = \int xy^2 dx + g(y) = \frac{1}{2}x^2y^2 + g(y) \\ \dfrac{\partial f}{\partial y} = x^2 y \end{cases}$

$\Downarrow$

$\dfrac{\partial f}{\partial y} = x^2 y + g'(y) = x^2 y \iff g'(y) = 0$ i.e. $g(y) = $ constant.

$\iff \boxed{f(x,y) = \frac{1}{2}x^2y^2 + c}$ bien déf $\forall (x,y)\in U = \mathbb{R}^2$.

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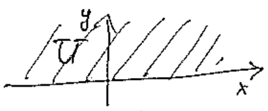
2. $\omega(x,y) = \frac{1}{y^2}dx - \frac{2x}{y^3}dy$ $\quad U = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$ $\quad D_\omega = \{(x,y)\in\mathbb{R}^2 \mid y\neq 0\}$

- <u>fermée</u>? $d\omega = -\frac{2}{y^3}dy\wedge dx - \frac{2}{y^3}dx\wedge dy = 0 \Longrightarrow \omega$ fermée.
- <u>exacte sur U</u> ? puisque U est contractile : 

, par le lemme de Poincaré ω est exacte sur U.

$\omega = df \iff \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{1}{y^2} \Longrightarrow f(x,y) = \int \frac{1}{y^2}dx + g(y) = \frac{x}{y^2} + g(y) \\ \dfrac{\partial f}{\partial y} = -\dfrac{2x}{y^3} \end{cases}$

$\Downarrow$

$\dfrac{\partial f}{\partial y} = -\dfrac{2x}{y^3} + g'(y) = -\dfrac{2x}{y^3} \iff g'(y) = 0$ i.e. $g(y) = c$

$\iff \boxed{f(x,y) = \frac{x}{y^2} + c}$ bien déf. sur $U = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$.

- <u>ω exacte sur $D_\omega = \{(x,y)\in\mathbb{R}^2 \mid y\neq 0\}$</u> ? ω est exacte sur chaque demi-plan $y>0$ et $y<0$, et sa primitive sur chaque demi-plan est $f(x,y) = \frac{x}{y^2} + c$ qui est bien définie sur tout D_ω, donc ω est exacte sur D_ω avec cette primitive globale.