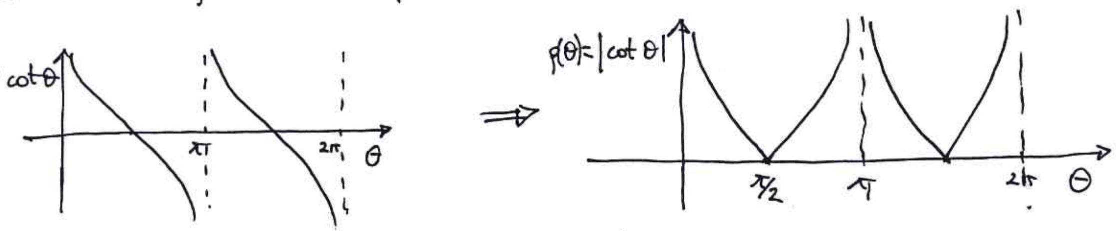
UE Géométrie et Calcul Diff. - Corrigé du CC1 du 27 mars 2014 - A. Frabetti (1

Exo : $\gamma(\theta) = \rho(\theta)\, e^{i\theta}$, $\rho(\theta) = |\cot\theta|$, $\theta \in]0,\pi[\cup]\pi, 2\pi[$.

1.



Puisque $\lim_{\theta \to \frac{\pi}{2}^-} \rho'(\theta) = \lim_{\theta \to \frac{\pi}{2}^-} \left(\frac{\cos\theta}{\sin\theta}\right)' = \lim_{\theta \to \frac{\pi}{2}^-} -\frac{1}{\sin^2\theta} = -1$

et $\lim_{\theta \to \frac{\pi}{2}^+} \rho'(\theta) = \lim_{\theta \to \frac{\pi}{2}^+} \left(-\frac{\cos\theta}{\sin\theta}\right)' = +1$

on voit que $\rho(\theta)$ n'est pas dérivable en $\theta = \frac{\pi}{2}$ (et de même en $\theta = \frac{3\pi}{2}$).

La courbe γ est donc C^∞ sauf aux points $\theta = \frac{\pi}{2}$ et $\frac{3\pi}{2}$.

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2. La paramétrisation $\gamma(\theta) = \rho(\theta) e^{i\theta}$ n'est sûrement pas régulière aux points où $\rho(\theta)$ n'est pas dérivable : en $\theta = \frac{\pi}{2}$ et $\frac{3\pi}{2}$. Pour $\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$, on a :

$\gamma(\theta) = \pm \cot\theta\, e^{i\theta} = \pm\left(\frac{\cos^2\theta}{\sin\theta}, \cos\theta\right)$ où $\pm$ dépend de l'intervol où on prend θ.

$$\gamma'(\theta) = \pm\left(\frac{-2\cos\theta\sin^2\theta - \cos^3\theta}{\sin^2\theta}, -\sin\theta\right) = \pm\left(-\frac{\cos\theta(2+3\cos^2\theta)}{\sin^2\theta}, -\sin\theta\right)$$

puisque $\sin\theta \neq 0$ pour $\theta \neq 0, \pi, 2\pi$, on a $\gamma'(\theta) \neq (0,0)$ $\forall \theta \in]0, 2\pi[$, $\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$.

Donc γ est régulière sauf en $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

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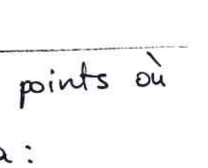
3. Pour tout $\theta \in]0, 2\pi[$, $\theta \neq \pi$, cherchons $u(\theta)$ et $v(\theta)$ t.q. $\begin{cases} T_{Ox}(\gamma(\theta)) = \gamma(u(\theta)) & (1) \\ T_{Oy}(\gamma(\theta)) = \gamma(v(\theta)) & (2) \end{cases}$

$(1) \Leftrightarrow \rho(\theta)(\cos\theta, -\sin\theta) = \rho(u(\theta))\left(\cos(u(\theta)), \sin(u(\theta))\right)$

$\Leftrightarrow \begin{cases} \cos u(\theta) = \cos\theta \\ \sin u(\theta) = -\sin\theta \end{cases}$ (et $\cot u(\theta) = \pm\cot\theta$) $\Leftrightarrow u(\theta) = 2\pi - \theta$.



$(2) \Leftrightarrow \rho(\theta)(-\cos\theta, \sin\theta) = \rho(v(\theta))\left(\cos(v(\theta)), \sin(v(\theta))\right)$

$\Leftrightarrow \begin{cases} \cos v(\theta) = -\cos\theta \\ \sin v(\theta) = \sin\theta \end{cases}$ (et $\cot v(t) = \pm\cot\theta$) $\Leftrightarrow v(\theta) = \pi - \theta$

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4. $\operatorname{Supp}\gamma = \left\{ \gamma(\theta) = \pm\left(\frac{\cos^2\theta}{\sin\theta}, \cos\theta\right) \,\middle|\, \theta \in]0,\pi[\cup]\pi, 2\pi[\right\}$

Si on pose $\gamma(\theta) = (x(\theta), y(\theta))$, on a $y(\theta) = \pm\cos\theta \in [-1, 1]$ $\forall\theta$,

donc $\operatorname{Supp}\gamma \subset \mathbb{R} \times [-1, 1]$.

Trouvons les droites asymptotes pour $\theta \to 0^+, \pi^-, \pi^+$ et $2\pi^-$: posons

$\gamma(\theta) = \left|\frac{\cos\theta}{\sin\theta}\right| \cdot (\cos\theta, \sin\theta) = (x(\theta), y(\theta))$, alors :

- pour $\theta \to 0^+$ on a : $\left.\begin{array}{l} x(\theta) = \frac{\cos^2\theta}{\sin\theta} \xrightarrow[\theta \to 0^+]{} \frac{1}{0^+} = +\infty \\ y(\theta) = \cos\theta \xrightarrow[\theta \to 0^+]{} 1 \end{array}\right\} \Rightarrow$ droite $y = 1$ pour $x \to +\infty$

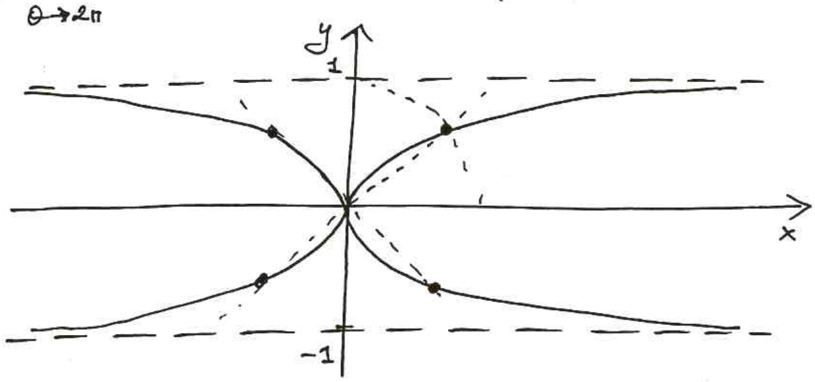
2) · pour $\theta \to \pi^-$ on a $x(\theta) = -\frac{\cos^2\theta}{\sin\theta} \xrightarrow[\theta\to\pi^-]{} -\left(\frac{1}{0^+}\right) = -\infty$
$y(\theta) = -\cos\theta \xrightarrow[\theta\to\pi^-]{} -(-1) = 1$ $\Big\}$ droite $y=1$ pour $x \to -\infty$

· pour $\theta \to \pi^+$ on a $x(\theta) = \frac{\cos^2\theta}{\sin\theta} \xrightarrow[\theta\to\pi]{} \frac{1}{0^-} = -\infty$
$y(\theta) = \cos\theta \longrightarrow -1$ $\Big\}$ droite $y=-1$ pour $x \to -\infty$

· pour $\theta \to 2\pi^-$ on a $x(\theta) = -\frac{\cos^2\theta}{\sin\theta} \xrightarrow[\theta\to2\pi^-]{} -\frac{1}{0^-} = +\infty$
$y(\theta) = -\cos\theta \xrightarrow[\theta\to2\pi]{} -1$ $\Big\}$ droite $y=-1$ pour $x \to +\infty$

Pour dessiner le graphe :

$$\gamma\left(\frac{\pi}{4}\right) = \frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{2}}\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



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5. $\gamma(\theta) = (x,y)$ avec $\begin{cases} x = \pm \frac{\cos^2\theta}{\sin\theta} \\ y = \pm\cos\theta \end{cases} \Rightarrow \begin{cases} x^2 = \frac{\cos^4\theta}{\sin^2\theta} \\ y^2 = \cos^2\theta \end{cases}$

$$x^2+y^2 = \cos^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}+1\right) = \cos^2\theta \cdot \frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta}$$

$\Rightarrow y^2(x^2+y^2) = \frac{\cos^4\theta}{\sin^2\theta} = x^2 \Rightarrow$ éq. cartesienne : $y^2(x^2+y^2) = x^2$.

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6. Courbure : $k_\gamma(\theta) = \dfrac{|x'(\theta)y''(\theta) - x''(\theta)y'(\theta)|}{(x'(\theta)^2 + y'(\theta)^2)^{3/2}}$

$$x(\theta) = \frac{\cos^2\theta}{\sin\theta} = \frac{1-\sin^2\theta}{\sin\theta} = \frac{1}{\sin\theta} - \sin\theta \Rightarrow x'(\theta) = -\frac{\cos\theta}{\sin^2\theta} - \cos\theta = \frac{-\cos\theta\,(1+\sin^2\theta)}{\sin^2\theta}$$

$$x''(\theta) = \frac{\sin^3\theta + \cos\theta\cdot 2\sin\theta\cos\theta}{\sin^4\theta} + \sin\theta = \frac{\sin^2\theta + 2(1-\sin^2\theta)}{\sin^3\theta} + \sin\theta = \frac{2}{\sin^3\theta} - \frac{1}{\sin\theta} + \sin\theta$$

$$y(\theta) = \cos\theta \Rightarrow y'(\theta) = -\sin\theta \Rightarrow y''(\theta) = -\cos\theta$$

$$\|\gamma'(\theta)\|^2 = \frac{\cos^2\theta\,(1+\sin^2\theta)^2}{\sin^4\theta} + \sin^2\theta = \frac{(1-\sin^2\theta)(1+\sin^2\theta)^2 + \sin^6\theta}{\sin^4\theta} = \frac{(1-\sin^4\theta)(1+\sin^2\theta)+\sin^6\theta}{\sin^4\theta} =$$

$$= \frac{1+\sin^2\theta - \sin^4\theta - \sin^6\theta + \sin^6\theta}{\sin^6\theta} = \frac{1+\sin^2\theta-\sin^4\theta}{\sin^6\theta}$$

$$k_\gamma(\theta) = \left|\frac{-\cos\theta(1+\sin^2\theta)\cdot(-\cos\theta)}{\sin^2\theta} + \frac{2\cancel{\sin\theta}}{\sin^3\theta} - \frac{\cancel{\sin\theta}}{\cancel{\sin\theta}} + \sin^2\theta\right| \cdot \frac{(\sin^4\theta)^{3/2}}{(1+\sin^2\theta-\sin^4\theta)^{3/2}}$$

$$= \frac{|\overbrace{\cos^2\theta}^{1-\sin^2\theta}(1+\sin^2\theta) + 2 - \sin^2\theta + \sin^4\theta|}{\sin^2\theta} \cdot \frac{\sin^6\theta}{(1+\sin^2\theta-\sin^4\theta)^{3/2}}$$

$$= \frac{|1-\cancel{\sin^4\theta} + 2 - \sin^2\theta + \cancel{\sin^4\theta}|\cdot\sin^4\theta}{(1+\sin^2\theta-\sin^4\theta)^{3/2}} = \frac{(3-\sin^2\theta)\sin^4\theta}{(1+\sin^2\theta-\sin^4\theta)^{3/2}} \neq 0 \quad \forall\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Donc γ est birégulière sauf en $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.