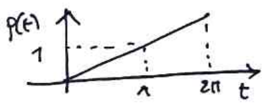
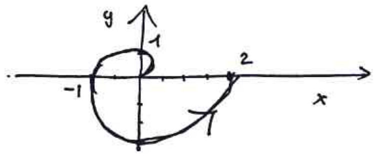
Exo 1 $\gamma(t) = \rho(t)\, e^{it}$, $\rho(t) = \frac{t}{\pi}$, $t \in [0, 2\pi]$

a)  $\Longrightarrow$ 

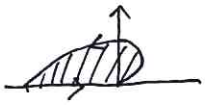
b) $\gamma(t) = \frac{t}{\pi} e^{it} \Rightarrow \gamma'(t) = \frac{1}{\pi}(1+it)e^{it} \Rightarrow \|\gamma'(t)\| = \frac{1}{\pi}\sqrt{1+t^2} \neq 0 \quad \forall t \in [0,2\pi] \Rightarrow \gamma$ régulière partout.

c) $$L_0^{2\pi}(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\, dt = \frac{1}{\pi}\int_0^{2\pi} \sqrt{1+t^2}\, dt \underset{\substack{t = \operatorname{sh} u \\ dt = \operatorname{ch} u\, du}}{=} \frac{1}{\pi}\int_{\operatorname{argsh} 0 = 0}^{\operatorname{argsh}(2\pi)} \operatorname{ch} u \cdot \operatorname{ch} u\, du \underset{(1)}{=} \frac{1}{2\pi}\Big[\operatorname{ch} u \operatorname{sh} u + u\Big]_0^{\operatorname{argsh}(2\pi)} \underset{(2)}{=}$$

$$= \frac{1}{2\pi}\left(2\pi\sqrt{1+4\pi^2} + \operatorname{argsh}(2\pi)\right) = \sqrt{1+4\pi^2} + \frac{\ln\left(2\pi + \sqrt{1+4\pi^2}\right)}{2\pi}.$$

(1) par parties: $\int \operatorname{ch}^2 u\, du = \operatorname{ch} u \operatorname{sh} u - \int \operatorname{sh}^2 u\, du = \operatorname{ch} u \operatorname{sh} u + u - \int \operatorname{ch}^2 u\, du$

(2) si $\operatorname{sh} u = 2\pi$ alors $\operatorname{ch} u = \sqrt{1+4\pi^2}$

d)  $D \subset \mathbb{R}^2$ t.q. $\partial D = \gamma|_{[0,\pi]} \cup \left([-1,0] \times \{0\}\right)$

$$\text{Aire } D = \iint_D dx\, dy = \left|\int_{\partial D^+} x\, dy\right| = \left|\int_{\gamma|_{[0,\pi]}} x\, dy + \int_{[-1,0]\times\{0\}} x\, dy\right| = \left|\int_{\gamma|_{[0,\pi]}} x\, dy\right| = \left|\int_0^{\pi} \frac{t\cos t}{\pi} \cdot \frac{(\sin t + t\cos t)}{\pi}\, dt\right|$$

$y = 0 \Rightarrow dy = 0$

$x = \frac{t}{\pi}\cos t$

$y = \frac{t}{\pi}\sin t \Rightarrow dy = \frac{1}{\pi}(\sin t + t\cos t)\, dt$

$$= \frac{1}{\pi^2}\left|\int_0^{\pi}\left(t\cos t + t^2\cos^2 t\right) dt\right| \underset{(3)}{=} \frac{1}{\pi^2}\left|\left[\frac{t^3}{6} + \frac{t^2 \sin t \cos t}{2}\right]_0^{\pi} + \int_0^{\pi} t\cos t \sin t\, dt - \int_0^{\pi} t\cos t \sin t\, dt\right| = \frac{1}{\pi^2}\left(\frac{\pi^3}{6} + 0\right) = \frac{\pi}{6}.$$

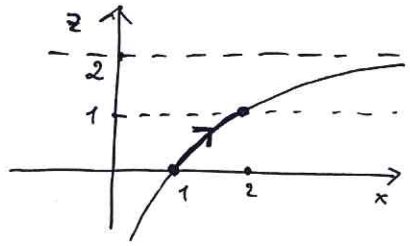
(3) $$\int t^2 \cos^2 t\, dt = \frac{t^3 + t^2 \sin t\cos t}{2} - \int\left(t^2 + t\cos t \sin t\right) dt = \underbrace{\frac{t^3}{2} - \frac{t^3}{3}}_{= \frac{t^3}{6}} + \frac{t^2 \sin t \cos t}{2} - \int t\cos t \sin t\, dt$$

$u = t^2$ $\quad v' = \cos^2 t$

$u' = 2t$ $\quad v = \frac{\cos t \sin t + t}{2}$

Exo 2 a) $\alpha(t) = \left(t+1, 0, \frac{2t}{t+1}\right) \in \mathbb{R}^3$, $t \in [0,1]$

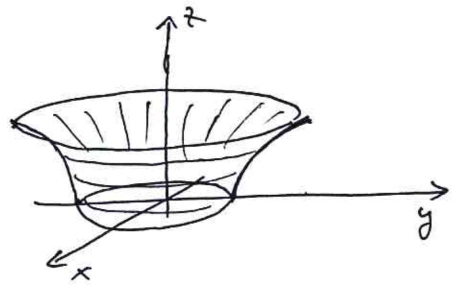
$$\begin{cases} x = t+1 \\ y = 0 \\ z = \frac{2t}{t+1} \end{cases} \Rightarrow \begin{cases} t = x-1 \\ y = 0 \\ z = \frac{2(x-1)}{x} = 2 - \frac{2}{x} \end{cases} \Rightarrow \begin{cases} y = 0 \\ x\left(\frac{z}{2} - 1\right) = -1 \end{cases} \quad \text{hyperbole}$$



$t \in [0,1] \Rightarrow x \in [1,2]$

b) $f(t,\varphi) = R_{\varphi}^{\partial z}\, \alpha(t) = \left((t+1)\cos\varphi, (t+1)\sin\varphi, 2\left(1 - \frac{1}{t+1}\right)\right)$

$S = \{ f(t,\varphi) \mid t \in [0,1], \varphi \in [0,2\pi] \}$



c) $$\begin{cases} \dfrac{\partial f}{\partial t} = \left(\cos\varphi, \sin\varphi, \dfrac{2}{(t+1)^2}\right) \\ \dfrac{\partial f}{\partial \varphi} = \left(-(t+1)\sin\varphi, (t+1)\cos\varphi, 0\right) \end{cases}$$

$$\frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial \varphi} = \left(-\frac{2(t+1)}{(t+1)^2}\cos\varphi, -\frac{2(t+1)}{(t+1)^2}\sin\varphi, t+1\right) = \frac{1}{t+1}\left(-2\cos\varphi, -2\sin\varphi, (t+1)^2\right)$$

$$\left\|\frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial \varphi}\right\| = \frac{1}{t+1}\sqrt{4 + (t+1)^4} \neq 0 \quad \forall t \in [0,1], \ \forall \varphi \in [0,2\pi] \Rightarrow S \text{ est régulière partout.}$$

2)

d) $\text{Aire } S = \iint_{[0,1]\times[0,2\pi]} \left\|\frac{\partial f}{\partial t}\wedge\frac{\partial f}{\partial \varphi}\right\| dt\,d\varphi = \int_0^{2\pi} d\varphi \int_0^1 \frac{\sqrt{4+(t+1)^4}}{t+1}\,dt = (*)$

$u(0)=\sqrt5$
$u(1)=\sqrt{4+16}=\sqrt{20}=2\sqrt5$

$u=\sqrt{4+(t+1)^4} \Rightarrow du = \frac{4(t+1)^3}{2\sqrt{4+(t+1)^3}}\,dt = \frac{2(t+1)^3}{u}\,dt$

$$(*) = \cancel{2}\pi \int_{\sqrt5}^{2\sqrt5} \frac{u^2}{\cancel{2}(t+1)^4}\,du \underset{(t+1)^4=u^2-4}{=} \pi \int_{\sqrt5}^{2\sqrt5} \frac{u^2-4+4}{u^2-4}\,du = \pi\int_{\sqrt5}^{2\sqrt5} du + \pi\int_{\sqrt5}^{2\sqrt5}\frac{4}{u^2-4}\,du \underset{(4)}{=} (**)$$

(4) $\frac{4}{u^2-4} = \frac{a}{u-2}+\frac{b}{u+2} = \frac{au+2a+bu-2b}{u^2-4} \Leftrightarrow \begin{cases} a+b=0 \\ 2(a-b)=4\end{cases} \Leftrightarrow \begin{cases} b=-a \\ 4a=4\end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=-1\end{cases}$

$$(**) = \pi\int_{\sqrt5}^{2\sqrt5}\left(1+\frac{1}{u-2}-\frac{1}{u+2}\right)du = \pi\left[u+\ln\left(\frac{u-2}{u+2}\right)\right]_{\sqrt5}^{2\sqrt5} = \pi\left(2\sqrt5-\sqrt5+\ln\left(\frac{\overset{2(\sqrt5-1)}{2\sqrt5-2}}{\underset{2(\sqrt5+1)}{2\sqrt5+2}}\cdot\frac{\sqrt5+2}{\sqrt5-2}\right)\right) =$$

$$= \pi\left(\sqrt5+\ln\left(\frac{5+\sqrt5-2}{5-\sqrt5-2}\right)\right) = \pi\left(\sqrt5+\ln\left(\frac{3+\sqrt5}{3-\sqrt5}\right)\right).$$

e) $\begin{cases} x=(t+1)\cos\varphi \\ y=(t+1)\sin\varphi \\ z=2\left(1-\frac{1}{t+1}\right)\end{cases} \Rightarrow \begin{cases} x^2+y^2=(t+1)^2 \\ \frac{z}{2}=1-\frac{1}{t+1} \Rightarrow t+1=\frac{1}{1-z/2}=\frac{2}{2-z}\end{cases} \Rightarrow x^2+y^2=\frac{4}{(2-z)^2}.$

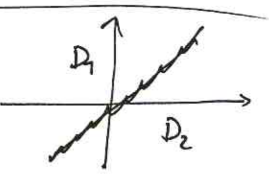
f) $D=\left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le\frac{4}{(2-z)^2},\ 0\le z\le 1\right\}$

$\mathrm{Vol}\,D = \iiint_D dx\,dy\,dz = \int_0^{2\pi}d\theta\int_0^1 dz\int_0^{\frac{2}{2-z}}\rho\,d\rho = \cancel{2}\pi\cdot\int_0^1 \frac{1}{\cancel{2}}\left[\rho^2\right]_0^{\frac{2}{2-z}}dz =$

$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta\end{cases} \Rightarrow dx\,dy=\rho\,d\rho\,d\theta,\quad \rho\le\frac{2}{2-z}$

$$= \pi\int_0^1\frac{4}{(2-z)^2}dz = \pi\left[\frac{4}{2-z}\right]_0^1 = \pi\left(\frac41-\frac42\right) = \pi(4-2) = 2\pi.$$

Exo 3



a) $\omega(x,y) = -\frac{2y}{(x-y)^2}dx + \frac{2x}{(x-y)^2}dy \Rightarrow \omega\in\Omega^1(D)$ où $D=\{(x,y)\mid y\ne x\}=D_1\sqcup D_2$

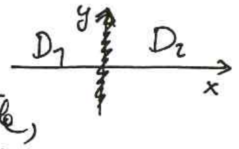
$$d\omega(x,y) = \left[\frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)-\frac{\partial}{\partial y}\left(-\frac{2y}{(x-y)^2}\right)\right]dx\wedge dy = \left(\frac{2(x-y)^{\cancel{2}}-4x\cancel{(x-y)}}{(x-y)^{\cancel{4}\,3}}+\frac{2(x-y)^{\cancel{2}}+4y\cancel{(x-y)}}{(x-y)^{\cancel{4}\,3}}\right)dx\wedge dy$$

$$= \frac{2x-2y-4x+2x-2y+4y}{(x-y)^3}dx\wedge dy = 0 \quad \text{donc } \omega \text{ est fermée.}$$

Le domaine D de ω est l'union disjointe de deux demi-plans D_1, D_2 qui sont simpl. convexes. Par le lemme de Poincaré, ω est exacte sur D_i, $i=1,2$: calculons une primitive f tq. $df=\omega|_{D_i}$

$\begin{cases}\frac{\partial f}{\partial x} = -\frac{2y}{(x-y)^2} \\ \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}\end{cases} \Rightarrow f(x,y)=\int\frac{-2y}{(x-y)^2}dx+g(y) = \frac{2y}{x-y}+g(y)$

$\frac{\partial f}{\partial y} = \frac{2(x-y)-2y(-1)}{(x-y)^2}+g'(y) = \frac{2x-\cancel{2y}+\cancel{2y}}{(x-y)^2}+g'(y) \overset{!}{=} \frac{2x}{(x-y)^2} \Rightarrow g'(y)=0 \Rightarrow g(y)=c.$



b) $\omega(x,y) = \left(xy^2-\frac{y}{x^2}\right)dx\wedge dy \Rightarrow \omega\in\Omega^2(D)$ où $D=\{(x,y)\mid x\ne0\}=D_1\sqcup D_2$

ω est fermée car 2-forme sur $\mathbb{R}^2$. Puisque chaque demi-plan D_i est contractile, ω est exacte sur chaque D_i. Cherchons une primitive $\eta\in\Omega^1(D_i)$ tq $d\eta=\omega|_{D_i}$:

$\eta(x,y) = a(x,y)\,dx + b(x,y)\,dy$ tq. $\frac{\partial b}{\partial x}-\frac{\partial a}{\partial y} = xy^2-\frac{y}{x^2}$.

ex : $\begin{cases}\frac{\partial b}{\partial x}=xy^2 \\ \frac{\partial a}{\partial y}=\frac{y}{x^2}\end{cases} \Leftarrow \begin{cases} b(x,y)=\frac12x^2y^2 \\ a(x,y)=\frac12\frac{y^2}{x^2}\end{cases}$ alors $\eta(x,y)=\frac12\frac{y^2}{x^2}dx+\frac12x^2y^2dy$ est une primitive de ω sur chaque D_i.